Complemented uniform lattices

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Abstract

It is proved that the completion of a complemented modular lattice with respect to a Hausdorff lattice uniformity which is metrizable or exhaustive is a complemented modular lattice. It is then shown that a complete complemented modular lattice endowed with a Hausdorff order continuous lattice uniformity is isomorphic to the product of an arcwise connected complemented lattice and of geometric lattices of finite length each of which endowed with the discrete uniformity. These two results are used to prove a decomposition theorem for modular functions on complemented lattices. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Birkhoff proved in [6, §X.5] that the completion of a complemented lattice with respect to the metric induced by a strictly increasing real-valued valuation is a complete complemented modular lattice and used this result to construct continuous geometries discovered by von Neumann. We generalize this result showing in Theorem 4.2 that the completion of a complemented modular lattice with respect to an exhaustive Hausdorff lattice uniformity is a complete complemented modular lattice. It seems that the proof of Theorem 4.2 is much easier in the metrizable case, which we study in Section 2. The proof of Theorem 4.2 is based on Section 2 and on a result of [3], which says that the set of all exhaustive lattice uniformities on a complemented modular lattice forms a Boolean algebra. Theorem 4.2 was already announced without proof in [23] and is a basic tool there as well as in [2] and also in Section 6 of this article.

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In Section 5 we study complete complemented lattices endowed with an order continuous Hausdorff lattice uniformity. In particular we examine (arcwise) connectedness, total disconnectedness and compactness. The results of Section 5 about compact topological complemented lattices are related to results of Choe and Greechie [8], of Pulmannová and Riečanová [15,16] and of Pulmannová and Rogalewicz [17] about compact topological orthomodular lattices.

In Section 6 we present a decomposition theorem for modular functions on complemented lattices, which is new also for modular functions on orthomodular lattices and agrees in the special case of measures on algebras of sets with the Hammer–Sobczyk decomposition (see [5, §5.2], [18]). The proof of the decomposition theorem of Section 6 is based on the description of the uniform completion of complemented modular lattices obtained in Sections 4 and 5.

1. Preliminaries

Throughout let \( L \) be a lattice.

We denote by \( \Delta \) the diagonal \( \Delta := \{(x, x): x \in L\} \) of \( L^2 \). If \( L \) is bounded, i.e., if \( L \) has a smallest and a greatest element, we denote these elements by 0 and 1, respectively.

\( L \) is called relatively complemented if all its closed intervals \([a, b] := \{x \in L: a \leq x \leq b\} \) with \( a \neq b \) are complemented lattices. A complement \( x' \) of an element \( x \in [a, b] \) in the sublattice \([a, b]\) of \( L \) is also called a relative complement of \( x \) in \([a, b]\); it satisfies the conditions \( x' \wedge x = a \) and \( x' \vee x = b \). A sectionally complemented lattice is a lattice with a smallest element 0 such that all its closed intervals of the type \([0, a]\) are complemented. Examples for bounded relatively complemented lattices (hence for complemented and sectionally complemented lattices) are orthomodular lattices (see [6, III.14]).

\( L \) is called continuous if \( x \uparrow \mapsto x \) implies \( x \wedge \uparrow y \mapsto x \wedge y \) and \( x \downarrow \mapsto x \) implies \( x \vee \downarrow y \mapsto x \vee y \) in \( L \). A continuous complemented modular lattice is called a von Neumann lattice. As the terminology in the literature is not unique, we point out that “continuous lattice” is here understood in the sense of von Neumann, and not of Scott.

We now summarize some facts about lattice topologies and lattice uniformities; for more information see [22]. A topological lattice is a lattice with a topology which makes the lattice operations \( \wedge \) and \( \vee \) continuous; its topology is called a lattice topology. A lattice topology on \( L \) is called locally convex if every point of \( L \) has a neighbourhood base of convex sets, i.e., of sets \( U \) such that \([a, b] \subseteq U\) for all \( a, b \in U \) with \( a \leq b \). A lattice topology \( \tau \) on \( L \) is called (\( \sigma \)-)order continuous if order convergence of a monotone – i.e., increasing or decreasing – net (sequence) implies topological convergence in \((L, \tau)\).

**Proposition 1.1.** If \( L \) admits a Hausdorff order continuous lattice topology, then \( L \) is continuous.

A lattice uniformity is a uniformity on a lattice which makes the lattice operations \( \vee \) and \( \wedge \) uniformly continuous; a lattice endowed with a lattice uniformity is called a uniform
A uniformity $u$ on $L$ is a lattice uniformity iff for every $U \in u$ there is a $V \in u$ with $V \vee V \subseteq U$ and $V \wedge V \subseteq U$ iff for every $U \in u$ there is a $V \in u$ with $V \vee A \subseteq U$ and $V \wedge A \subseteq U$. (Here $A \vee B := \{a \vee b: a \in A, b \in B\}$ and $A \wedge B := \{a \wedge b: a \in A, b \in B\}$.)

The topology induced by a lattice uniformity $u$, the $u$-topology, is a locally convex lattice topology. A lattice uniformity $u$ is called $\sigma$-order continuous if the $u$-topology is $\sigma$-order continuous.

It is well known that any uniformity on a set $X$ is generated by a system of semimetrics, i.e., by a system of symmetric functions $d: X \times X \to [0, +\infty]$ satisfying the triangle inequality and $d(x, x) = 0$ ($x \in X$). In some proofs it is here convenient even though not essential to use the corresponding fact for lattice uniformities.

**Theorem 1.2** [22, 1.4.1]. Any lattice uniformity $u$ on $L$ is generated by a system $D$ of semimetrics satisfying

$$d(x \vee z, y \vee z) \leq 2d(x, y) \quad \text{and} \quad d(x \wedge z, y \wedge z) \leq 2d(x, y)$$

for $x, y, z \in L$ and $d \in D$. If $u$ has a countable base, $D$ can be replaced by a single semimetric.

Any Hausdorff uniform lattice $(L, u)$ is a sublattice and dense subspace of a Hausdorff uniform lattice $(\tilde{L}, \tilde{u})$ which is complete as uniform space (see [13] or [22, 1.3]); $(\tilde{L}, \tilde{u})$ is called the completion of $(L, u)$. The following proposition can easily be proved.

**Proposition 1.3.** The completion of a Hausdorff uniform modular lattice is modular.

A lattice uniformity $u$ on $L$ is called exhaustive if every monotone sequence is Cauchy in $(L, u)$.

**Theorem 1.4.** If $u$ is a Hausdorff exhaustive lattice uniformity on $L$ and $(\tilde{L}, \tilde{u})$ the completion of $(L, u)$, then $(\tilde{L}, \leq)$ is a continuous complete lattice and $\tilde{u}$ is order continuous. (See [13] or [22, 6.15].)

**Theorem 1.5.** Let $u$ be a Hausdorff lattice uniformity on $L$. Then $u$ is exhaustive and complete iff $(L, \leq)$ is a complete lattice and $u$ is order continuous. In this case, $(L, \leq)$ is continuous. (See [22, 6.3 and 7.1.7].)

### 2. The completion of complemented modular metrizable lattices

In this section we prove:

**Theorem 2.1.** Let $L$ be a complemented or sectionally complemented or relatively complemented modular lattice and $u$ a metrizable lattice uniformity on $L$. Then the completion of $(L, u)$ is, respectively, a complemented or sectionally complemented or relatively complemented modular lattice.
In the proof of Theorem 2.1 we use the following facts:

**Proposition 2.2** [6, Theorem I.14]. Any complemented modular lattice is relatively complemented.

**Proposition 2.3.** Let \( L \) be a bounded modular lattice and \( y, z \in L \) with \( y \leq z \).

(a) If \( y' \) is a complement of \( y \) in \( L \), then \( y' \wedge z \) is a relative complement of \( y \) in \([0, z]\). 

(b) If \( z' \) is a complement of \( z \) in \( L \) and \( y^* \) a relative complement of \( y \) in \([0, z]\), then \( y^* \vee z' \) is a complement of \( y \) in \( L \).

**Proof.**

(a) See the proof of [6, Theorem I.14].

(b) \((y^* \vee z') \vee y = (y^* \vee y) \vee z' = z \vee z' = 1 \) and, by the modularity of \( L \), \((y^* \vee z') \wedge y \leq (y^* \vee z') \wedge z = y^* \wedge (z' \wedge z) = y^* \vee 0 = y^* \), hence \((y^* \vee z') \wedge y \leq y^* \wedge y = 0 \).

**Lemma 2.4.** Let \( L \) be a bounded modular lattice and \( d \) a semimetric on \( L \) satisfying \((*)\) of Theorem 1.2.

(a) If \( y, y' \in L \) and \( y' \) is a complement of \( y \), then \( d(y', 0) \leq 2d(y, 1) \).

(b) Let \( S \) be a sublattice of \( L \) such that any element of \( S \) has a complement in \( L \). If \( x, y \in S \) and \( x' \) is a complement of \( x \) in \( L \), then \( y \) has a complement \( y' \in L \) with \( d(x', y') \leq 16d(x, y) \).

**Proof.**

(a) \( d(y', 0) = d(1 \wedge y', y \wedge y') \leq 2d(1, y) \).

(b) (i) We first prove that for any \( y, z \in S \) with \( y \leq z \) and any complement \( z' \) of \( z \) in \( L \) there is a complement \( y' \) of \( y \) with \( d(y', z') \leq 4d(y, z) \): \( y \) has by Proposition 2.3(a) a relative complement \( y^* \) in \([0, z]\). Then \( y' := y^* \vee z' \) is a complement of \( y \) in \( L \) by Proposition 2.3(b), \( d(y^*, 0) \leq 2d(y, z) \) by (a) and

\[
d(y', z') = d(y^* \vee z', 0 \vee z') \leq 2d(y^*, 0) \leq 4d(y, z).
\]

(ii) By duality, for any \( x, z \in S \) with \( x \leq z \) and any complement \( x' \) of \( x \) in \( L \) there is a complement \( z' \) of \( z \) with \( d(x', z') \leq 4d(x, z) \).

(iii) Let \( x, y \in S \), \( x' \) a complement of \( x \) and \( z := x \vee y \). Then, by (ii), there is a complement \( z' \) of \( z \) with \( d(x', z') \leq 4d(x, z) \) and, by (i), a complement \( y' \) of \( y \) with \( d(z', y') \leq 4d(z, y) \). Therefore

\[
d(x', y') \leq d(x', z') + d(z', y') \leq 4d(x \vee x, x \vee y) + 4d(x \vee y, y \vee y) \leq 16d(x, y).
\]

**Proposition 2.5.** Let \( L \) be a bounded modular lattice and \( u \) a metrizable complete lattice uniformity on \( L \). If \( S \) is a sublattice of \( L \) such that any element of \( S \) has a complement in \( L \), then any element of the closure \( \overline{S} \) of \( S \) in \((L, u)\) has a complement in \( L \).

**Proof.** By Theorem 1.2, \( u \) is generated by a metric \( d \) satisfying \((*)\) of Theorem 1.2. Let \( x \in \overline{S} \) and \((x_n)\) be a sequence in \( S \) converging to \( x \) with \( d(x_n, x_{n+1}) \leq 2^{-n} \) \((n \in \mathbb{N})\). By Lemma 2.4, we can find inductively for any \( n \in \mathbb{N} \) a complement \( x_n' \) of \( x_n \) in \( L \) such that
With Lemma 2.4(a) we obtain
\[ d(x'_n, x'_{n+1}) \leq 16 \cdot d(x_n, x_{n+1}) \leq 16 \cdot 2^{-n}. \]
Then \( (x'_n) \) is a Cauchy sequence and its limit \( x' \) is a complement of \( x \). \( \Box \)

**Proof of Theorem 2.1.** (i) By Proposition 1.3, the completion \((\bar{L}, \bar{u})\) of \((L, u)\) is modular. If \( L \) is complemented, the the conclusion immediately follows from Proposition 2.5.

(ii) Let now \( L \) be sectionally complemented and \( d \) a metric, which generates \( \bar{u} \) and satisfies \((*)\) of Theorem 1.2 for \( x, y, z \in \bar{L} \). Let \( a \in \bar{L} \). We have to show that the interval \([0, a]\) is complemented. Let \((a_n)\) be a sequence in \( L \) converging to \( a \) with \( d(a_n, a_{n+1}) \leq 2^{-n} \). Then
\[
\sup_{t \leq a} d(s_n, s_{n+1}) = d(s_n \wedge a_n, s_n \wedge a_{n+1}) \leq 2d(a_n, a_{n+1}) \leq 2 \cdot 2^{-n}.
\]

Therefore \((s_n)\) is a Cauchy sequence converging to an element \( s \geq a \). By Proposition 2.2 we only need to show that \([0, s]\) is complemented. We will apply Proposition 2.5. Since \( S := \{ x \in L : x \leq s_n \text{ for some } n \in \mathbb{N} \} \cup \{ s \} \) is a dense sublattice of \([0, s, \bar{u}]\), it remains in view of Proposition 2.5 to prove that any element of \( S \) has a complement in \([0, s]\). Let \( x \in S \). We may assume that \( x < s \). Let \( m \in \mathbb{N} \) with \( x \leq s_m \), \( x^* \) a relative complement of \( x \) in \([0, s_m]\), \( t \) a relative complement of \( s_{m+1} \) in \([0, s_{m+1}]\) and \( x'_n := x^* \lor \sup_{i \leq i \leq n} t_i \). With Lemma 2.4(a) we obtain
\[
d(x'_n, x'_{n+1}) = d(x'_n \lor 0, x'_{n+1} \lor t_{n+1}) \leq 2d(0, t_{n+1}) \leq 4d(s_{m+n}, s_{m+n+1}) \leq 8 \cdot 2^{-(m+n)}.
\]
Therefore \((x'_n)\) is a Cauchy sequence and has a limit \( x' \) in \([0, s]\). Since \( x'_n \) is by Proposition 2.3(b) a complement of \( x \) in \([0, s_{m+n}]\), \( x' \) is a complement of \( x \) in \([0, s]\).

(iii) Let now \( L \) be relatively complemented. (Of course, also in the cases (i) and (ii) \( L \) is relatively complemented by Proposition 2.2.) We will show that for \( a, b \in \bar{L} \) with \( a \leq b \) the interval \([a, b]\) is complemented. By the same argument as in (ii) we may assume that there are sequences \((a_n)\) and \((b_n)\) in \( L \) converging to \( a \) and \( b \), respectively, with \( a_{n+1} \leq a_n \leq b_n \leq b_{n+1} \) for \( n \in \mathbb{N} \). We have already seen that the completion of a sectionally complemented modular lattice with respect to a metrizable lattice uniformity is sectionally complemented. By the dual statement we get that the completion \([a, b]\) of \([x \leq a \leq x \leq b]\) is complemented. Again applying (ii), we get that the completion \([a, b]\) of \([x \in \bar{L} : a \leq x \leq b_n \text{ for some } n \in \mathbb{N}]\) is complemented. \( \Box \)

3. The center of uniform lattices

The center \( C(L) \) of a bounded lattice \( L \) is the set of elements \( c \in L \) for which there is an element \( c' \in L \) such that \( \psi(x) = (x \land c, x \land c') \) defines a lattice isomorphism from \( L \) onto \([0, c] \times [0, c']\). \( C(L) \) is a Boolean sublattice of \( L \) (see [6, III.8.10]). \( C(L) \) can be characterized by means of the neutral elements of \( L \). An element \( a \in L \) is called neutral (see [12, III.2.1]) if
\[
(a \land x) \lor (x \land y) \lor (y \land a) = (a \lor x) \land (x \lor y) \land (y \lor a) \quad \text{for all } x, y \in L.
\]
By [12, III.2.4], an equivalent condition is that any triple in \( L \) including \( a \) generates a distributive sublattice of \( L \). Therefore Birkhoff’s definition of neutral elements [6, Section III.9] is equivalent to that of Grätzer [12]. The set \( N(L) \) of all neutral elements of \( L \) is a distributive sublattice of \( L \) (see [6, III.9.13] or [12, III.2.9]).

**Proposition 3.1** [6, Theorem III.9.12]. Let \( L \) be a bounded lattice. Then \( C(L) = \{ x \in N(L): x \text{ has a complement in } L \} \). In particular, \( C(L) = N(L) \) if \( L \) is complemented.

Easy examples show that the center of a Hausdorff uniform lattice \((L, u)\) need not be closed: Take for \( L \) the lattice of all countable and all cofinite subsets of an uncountable set \( X \) and for \( u \) the uniformity induced by the product topology of \( 2^X \) on \( L \). Then \( C(L) \) is the algebra of all finite and all cofinite subsets of \( X \) and therefore a dense proper subset of \((L, u)\).

In Propositions 3.2 and 3.3 we give conditions under which \( C(L) \) is closed.

**Proposition 3.2.** Let \((L, \tau)\) be a Hausdorff topological lattice. Then \( N(L) \) is a closed sublattice of \((L, \tau)\). If, moreover, \( L \) is complemented, then \( C(L) \) is closed.

**Proof.** Using the continuity of the lattice operations \( \wedge, \vee:(L, \tau) \times (L, \tau) \to (L, \tau) \) and the above given definition of neutral elements, \( N(L) \) is easily seen to be closed. If \( L \) is complemented, then \( C(L) = N(L) \) by Proposition 3.1.

**Proposition 3.3.** Let \( L \) be a bounded lattice and \( u \) a Hausdorff complete lattice uniformity on \( L \).

(a) If \( A \) is a Boolean sublattice of \( L \), then also its closure \( A \) in \((L, u)\) is a Boolean sublattice of \( L \).

(b) \( C(L) \) is closed.

**Proof.** (a) By the continuity of the lattice operations, \( A \) is a distributive sublattice of \( L \). It remains to prove that \( A \) is complemented. Let \( x \in A \) and \((x_y)_{y \in I} \) be a net in \( A \) converging to \( x \). For any \( z \in A \), denote by \( z' \) the complement of \( z \) in \( A \). Since by [22, 6.10], the complementation \( z \mapsto z' \) is a uniformly continuous map on \( A \), the net \((x_y')_y \) is Cauchy and has therefore by the completeness of \((L, u)\) a limit \( y \) in \( A \). By the continuity of the lattice operations, \( y \) is a complement of \( x \).

(b) By (a), \( C(L) \) is complemented. Moreover, \( C(L) \subseteq N(L) = N(L) \). Therefore \( C(L) \subseteq C(L) \) by Proposition 3.1.

**Theorem 3.4.** Let \((L, \leq)\) be a complete lattice and \( u \) be an order continuous Hausdorff lattice uniformity on \( L \).

(a) Then \((C(L), u)\) is a complete uniform space.

(b) \( C(L) \) is a complete sublattice of \( L \), i.e., \( \sup_L M, \inf_L M \in C(L) \) for \( M \subseteq C(L) \).
Proof. By Theorem 1.5, \((L, u)\) is complete, hence \(C(L)\) is closed in \((L, u)\) by Proposition 3.3 and therefore \((C(L), u)\) is complete. Obviously, any closed sublattice of \((L, u)\) is a complete sublattice of \((L, \leq)\) since \(u\) is order continuous and Hausdorff. \(\square\)

Disjoint subsets of the center correspond to decompositions of (uniform) lattices:

Proposition 3.5. Let \((L, \leq)\) be a complete lattice, \(u\) an order continuous Hausdorff lattice uniformity on \(L\) and \(D\) a disjoint subset of \(C(L)\) with \(\text{sup} D = 1\). Then \(\Phi : x \mapsto (x \wedge d)_{d \in D}\) defines a uniform lattice isomorphism from \((L, u)\) onto \(\prod_{d \in D} ([0, d], u)\), i.e., \(\Phi\) is a lattice isomorphism, and \(\Phi^{-1}\) are uniformly continuous.

Proof. \(\Phi\) is a lattice isomorphism since \((L, \leq)\) is complete and continuous. \(\Phi\) is uniformly continuous since for each \(d \in D\) the map \(x \mapsto x \wedge d\) is so. We now show that \(\Phi^{-1}\) is uniformly continuous. \(\Phi^{-1}\) is given by \((x_d) \mapsto \text{sup}_{d \in D} x_d\). Let \(U \subset u\) and \(V\) be a symmetric member of \(u\) with \(V \wedge \Delta \subset U\). Since \(u\) is order continuous, \(D\) contains a finite subset \(D_0\) with \((1, \sup D_0) \subset V\). Let \(n\) be the cardinality of \(D_0\). Choose \(W \subset u\) such that \((x_i, y_i) \in W\), \(i = 1, \ldots, n\), implies \((\sup_{i=1}^n x_i, \sup_{i=1}^n y_i) \in U\). Let \((x_d), (y_d) \in \prod_{d \in D_0} [0, d] \subset W\) for \(d \in D_0\). Then

\[
\left(\sup_{d \in D} x_d, \sup_{d \in D} x_d\right) = (1, \sup_{d \in D_0} x_d) \wedge \left(\sup_{d \in D} x_d, \sup_{d \in D} x_d\right) \in V \wedge \Delta \subset U,
\]

\[
\left(\sup_{d \in D_0} x_d, \sup_{d \in D_0} y_d\right) \in U \quad \text{and} \quad \left(\sup_{d \in D_0} y_d, \sup_{d \in D_0} y_d\right) \in V \wedge \Delta \subset U,
\]

hence

\[
\left(\sup_{d \in D} x_d, \sup_{d \in D} y_d\right) \in U \circ u \circ U. \quad \square
\]

4. The completion of complemented modular exhaustive uniform lattices

An essential tool for the description of the completion of (relatively or sectionally) complemented modular exhaustive uniform lattices is a result of [3, Section 5] summarized in Theorem 4.1. We here use the following notation. If \(u\) is a lattice uniformity on \(L\), then we denote by \(\mathcal{UL}(L, u)\) the set of all lattice uniformities on \(L\) coarser than \(u\). With the inclusion as partial ordering, \(\mathcal{UL}(L, u)\) is a complete lattice, \(u\) being its largest element and the trivial uniformity its smallest element.

Theorem 4.1 [3, Section 5]. Let \(u\) be a Hausdorff exhaustive lattice uniformity on \(L\) and \((\bar{L}, \bar{u})\) the completion of \((L, u)\). Suppose that \(L\) is sectionally complemented and modular or that \(\bar{L}\) is relatively complemented. Then there is a lattice isomorphism \(\Phi : C(\bar{L}) \rightarrow \mathcal{UL}(L, u)\) with the following property: If \(c \in C(\bar{L})\), \(c'\) the complement of \(c\), \(v = \Phi(c)\) and \(t\) the trivial uniformity on \(L\), then \(x \mapsto (x \wedge c, x \wedge c')\) defines a uniform lattice isomorphism from \((L, v)\) onto a dense sublattice of \((\{0, c\}, \bar{u}) \times (\{0, c'\}, t)\).
**Theorem 4.2.** Let $L$ be a complemented or sectionally complemented or relatively complemented modular lattice and $u$ be an exhaustive Hausdorff lattice uniformity on $L$. Then the completion $(\widetilde{L}, \tilde{u})$ of $(L, u)$ is a von Neumann lattice endowed with an order continuous lattice uniformity. More precisely, $(\widetilde{L}, \tilde{u})$ is (as uniform lattice) isomorphic to a product $\prod (\tilde{L}_a, \tilde{u}_a)$ of von Neumann lattices $\tilde{L}_a$ each of which endowed with an order continuous metrizable lattice uniformity $\tilde{u}_a$.

**Proof.** (i) We have already seen (Proposition 1.3, Theorem 1.4) that $\widetilde{L}$ is a modular continuous complete lattice and $\tilde{u}$ is order continuous.

(ii) We first consider the case that $L$ is sectionally complemented. We show that for any $c \in C(\widetilde{L}) \setminus \{0\}$ there is an element $d \in C(\widetilde{L})$ such that $0 < d \leq c$ and the restriction $\tilde{u}|\{0, d\}$ is metrizable: Since (with the notation of Theorem 4.1) $\Phi(c)$ is a nontrivial lattice uniformity, there is a nontrivial lattice uniformity $v$ on $L$ with countable base and coarser than $\Phi(c)$; then $d = \Phi^{-1}(v)$ has the desired property. Let $D$ be a maximal disjoint family in $C(\widetilde{L})$ with the property that $\tilde{u}|\{0, d\}$ is metrizable for any $d \in D$. With the statement proved above it is easily seen that $\sup D = 1$. Therefore $(\widetilde{L}, \tilde{u})$ is by Proposition 3.5 isomorphic to $\prod_{d \in D} (\{0, d\}, \tilde{u})$. It remains to prove (in case of $L$ being sectionally complemented) that $\{0, d\}$ is complemented for $d \in D$: $(\{0, d\}, \tilde{u}|\{0, d\})$ is the completion of $(L \land d, \tilde{u}|L \land d)$. Moreover, $L \land d$ is an epimorphic image of a sectionally complemented lattice and therefore itself sectionally complemented. It follows with Theorem 2.1 that $\{0, d\}$ is (sectionally) complemented since $\tilde{u}|\{0, d\}$ and therefore $\tilde{u}|L \land d$ have a countable base.

(iii) If $L$ is complemented, then $L$ is also sectionally complemented since $L$ is modular. Therefore the assertion follows from (ii).

(iv) Assume now that $L$ is a relatively complemented modular lattice. For any $a \in L$, $\{x \in L : x \geq a\}$ is sectionally complemented and therefore its uniform completion $[a, 1] = \{x \in \widetilde{L} : x \geq a\}$ is complemented as proved in (ii). Dually, $[0, a]$ is complemented for any $a \in L$. Therefore the lattice $\{x \in \widetilde{L} : x \leq a\}$ for some $a \in L$ is sectionally complemented and consequently its completion $\tilde{L}$ is complemented. To get the factorization of $\tilde{L}$, apply (ii) (for $\tilde{L}$ instead of $L$).

We will prove in Section 5 that the completion $(\widetilde{L}, \tilde{u})$—under the assumption of Theorem 4.2—is isomorphic to the product of an arcwise connected von Neumann lattice and of discrete irreducible modular geometric lattices of finite length (see Corollary 5.13).

**5. Decomposition of complemented uniform lattices**

In this section we give—under certain assumptions—a decomposition of a uniform complemented lattice into a connected factor and a totally disconnected factor. We then factorize the last one into irreducible factors each of which endowed with the discrete uniformity (see Corollary 5.13). Recall that a lattice is called irreducible if it is not isomorphic to a product of two lattices where both have more than one element. If $L$ is a bounded lattice, then $L$ is irreducible iff its center is trivial, i.e., $C(L) = \{0, 1\}$.
The following result of [24] links the topological notion of connected spaces with the algebraic density notion for lattices.

**Proposition 5.1** [24, 5.2 and 5.10]. Let $L$ be a complete lattice and $\tau$ an order continuous Hausdorff lattice topology on $L$.

(a) Then $(L, \tau)$ is connected iff $L$ is dense-in-itself (i.e., for every $a, b \in L$ with $a < b$ there is an $c \in L$ with $a < c < b$).

(b) If $\tau$ is induced by a metrizable lattice uniformity, then $(L, \tau)$ is arcwise connected iff $(L, \tau)$ is connected.

In the situations we are interested in, a lattice is dense-in-itself iff it is atomless:

**Proposition 5.2.** Let $L$ be a modular sectionally complemented lattice or a continuous sectionally complemented complete lattice or an orthomodular lattice. Then $L$ is dense-in-itself iff $L$ is atomless.

**Proof.** The implication $(\Rightarrow)$ obviously holds for any lattice with 0.

$(\Leftarrow)$ In any of these three cases, $L$ has the following property: For $p, q \in L$ with $p < q$ the set $c(p, q)$ of all relative complements of $p$ in $[0, q]$ contains minimal elements. In fact, in the first case, any element of $c(p, q)$ is minimal. In the second case, the infimum of a maximal chain in $c(p, q)$ is minimal. In the third case, the orthocomplement $p^\bot$ of $p$ in $[0, q]$ is minimal, since, for $x \leq p^\bot$, $p^\bot$ commutes with $x$, $p^\bot$ commutes with $p$ and therefore the lattice generated by $\{x, p, p^\bot\}$ is distributive (see [14, 1.3.1, 1.3.2, 1.3.11]).

Now suppose that $L$ is not dense-in-itself, i.e., that there are $p, q \in L$ and $q$ covers $p$. Then any minimal element of $c(p, q)$ is an atom. So $L$ is not atomless. 

**Proposition 5.3.** Let $(L, \tau)$ be a sectionally complemented topological lattice. Then $L$ is totally disconnected iff the component of 0 is $\{0\}$.

**Proof.** The components of a topological space constitute a decomposition of the space into disjoint connected closed subsets. This decomposition determines an equivalence relation $\simeq$, which is in any topological lattice a congruence relation. By [12, III.3.10 and III.3.5], $x \simeq y$ iff $(x \wedge y) \lor a = x \lor y$ for some $a$ of the component of 0. Therefore, the component of 0 is $\{0\}$ iff $x \simeq y$ is equivalent to $x = y$, i.e., iff $L$ is totally disconnected. 

**Theorem 5.4.** Let $L$ be a sectionally complemented complete lattice and $\tau$ an order continuous Hausdorff lattice topology on $L$.

(a) Then $(L, \tau)$ is connected iff $L$ is atomless.

(b) $(L, \tau)$ is totally disconnected iff $L$ is atomic (i.e., any non-zero element of $L$ contains an atom) iff $L$ is atomistic (i.e., any non-zero element is a join of atoms).

(c) If $L$ is relatively complemented, then $(L, \tau)$ is isomorphic and homeomorphic to the product of a connected and a totally disconnected lattice.

**Proof.** (a) follows from Propositions 1.1, 5.1 and 5.2.
(b) It is well known and easy to see that the component of \([0]\) is an ideal in \(L\). Therefore \((L, \tau)\) is by Proposition 5.3 totally disconnected iff for any \(a \in L \setminus [0]\) the interval \([0, a]\) is not connected, hence by (a), iff for any \(a \in L \setminus [0]\) the interval \([0, a]\) contains an atom, i.e., iff \(L\) is atomic. Any complete sectionally complemented lattice is atomic iff it is atomistic.

(c) The component \(C\) of \(0\) is a closed ideal of \((L, \tau)\), hence a principal ideal since \(\tau\) is order continuous. Since the ideal \(C\) determines a congruence relation, \(z := \text{sup} C\) is standard by [12, III.3.10 and III.3.3], therefore neutral by [12, III.2.5, III.2.6 and III.1.9]. Hence \(z \in C(L)\) by Proposition 3.1. Let \(z'\) be the complement of \(z\). Then \(x \mapsto (x \land z, x \land z')\) defines a lattice isomorphism and homeomorphism from \((L, \tau)\) onto \(([0, z], \tau) \times ([0, z'], \tau)\); the continuity of this map and of its inverse \((x_1, x_2) \mapsto x_1 \lor x_2\) follows from the continuity of the lattice operations. \([0, z]\) is connected and \([0, z']\) is totally disconnected by Proposition 5.3.

If in Theorem 5.4(c) the topology is induced by a uniformity, we will obtain more information about the connected and totally disconnected factors of the decomposition. For that, we use the following decomposition of uniform lattices. Such a decomposition was examined in [4, §6] for MV-algebras.

**Proposition 5.5.** Let \((L, \preceq)\) be a complete lattice and \(u\) an order continuous Hausdorff lattice uniformity on \(L\). Denote by \(\mathcal{A}(L)\) the set of all atoms of \(C(L)\), by \(\mathcal{A}_\infty(L)\) the set of atoms \(a\) of \(C(L)\) for which \([0, a]\) is dense-in-itself and put \(\mathcal{A}_f(L) := \mathcal{A}(L) \setminus \mathcal{A}_\infty(L)\). Let \(a_\infty := \text{sup} \mathcal{A}_\infty(L), a_f := \text{sup} \mathcal{A}_f(L)\) and \(c\) be the complement of \(a_\infty \lor a_f\) in \(C(L)\).

(Observe that \(a_f, a_\infty \in C(L)\) by Theorem 3.4(b).)

(a) Then \(x \mapsto (x \land c, x \land a_\infty, x \land a_f)\), \(x \mapsto (x \lor a)_{a \in \mathcal{A}_\infty(L)}\), \(x \mapsto (x \land a)_{a \in \mathcal{A}_f(L)}\), respectively, define uniform lattice isomorphisms from \(L\) onto \([0, c] \times [0, a_\infty] \times [0, a_f]\), from \([0, a_\infty]\) onto \(\prod_{a \in \mathcal{A}_\infty(L)} [0, a]\) and from \([0, a_f]\) onto \(\prod_{a \in \mathcal{A}_f(L)} [0, a]\).

(b) \([0, c]\) and its center are arcwise connected and not compact if \(c \neq 0\). For \(a \in \mathcal{A}_\infty(L)\), the intervals \([0, a]\) (and therefore \([0, c \lor a_\infty]\)) are connected. For \(a \in \mathcal{A}_f(L)\), the intervals \([0, a]\) (and therefore \([0, a_f]\)) are not connected.

(c) For any \(a \in \mathcal{A}(L)\), \([0, a]\) is an irreducible lattice.

**Proof.** (a) is proved in Proposition 3.5. (c) follows from the fact that
\[
C([0, z]) = C(L) \cap [0, z]
\]
for any \(z \in C(L)\).

(b) \(C([0, c])\) is an atomless Boolean algebra. \((C([0, c]), u)\) is exhaustive and by Theorem 3.4 a complete uniform space; moreover its topology is by [22, 6.10(b)] an FN-topology (= monotone Ringtopologie in the sense of [20, p. 465]). Therefore \((C([0, c]), u)\) is arcwise connected by [20, 1.4]. It follows that \(([0, c], u)\) is pathwise (and therefore arcwise) connected: In fact, if \(\gamma : I \to (C([0, c]), u)\) is a continuous function on the real closed unit interval \(I\) with \(\gamma(0) = 0\) and \(\gamma(1) = c\), then \(x \mapsto \gamma(x) \land b\) defines, for \(b \in [0, c]\), a \([0, c]\)-valued continuous function on \(I\) with \(\gamma(0) \land b = 0\) and \(\gamma(1) \land b = b\).
Since any compact Hausdorff topological ring with unit is totally disconnected [1, Theorem 2], the connected Boolean algebra $C([0, c])$ is not compact and therefore $[0, c]$ is not compact if $c \neq 0$.

The other statements of (b) follow from Proposition 5.1. 

In Proposition 5.5, $[0, c]$ is locally arcwise connected and $[0, a_\infty]$ is locally connected, since any (arcwise) connected locally convex topological lattice is locally (arcwise) connected.

If in Proposition 5.5 $c = 0$, then $(L, u)$ is isomorphic to a product of irreducible uniform lattices. So we obtain the following two corollaries.

**Corollary 5.6.** Any compact Hausdorff topological lattice is isomorphic and homeomorphic to a product of compact Hausdorff topological irreducible lattices.

This follows from Proposition 5.5 and the fact that any compact Hausdorff topological lattice is a uniform lattice and satisfies the assumption of Proposition 5.5 (see, e.g., [22, 6.5]).

**Corollary 5.7.** Let $(L, \leq)$ be a complete lattice and $u$ an order continuous totally disconnected Hausdorff lattice uniformity on $L$. Then $(L, u)$ is (as uniform lattice) isomorphic to the product of irreducible order continuous totally disconnected Hausdorff uniform lattices.

We now examine the irreducible factors of this decomposition in case of $L$ being sectionally complemented.

**Lemma 5.8.** Let $L$ be an atomic sectionally complemented bounded lattice such that any two atoms are perspective (i.e., have a common complement). Then the discrete uniformity is the only Hausdorff lattice uniformity on $L$.

**Proof.** Let $u$ be a Hausdorff lattice uniformity on $L$, $a$ an atom of $L$ and $U \in u$ with $(0, a) \notin U$. Let $V \in u$ such that $(V \vee \Delta) \land \Delta \subseteq U$. Suppose that $V(0) := \{y: (0, y) \in V\} \neq \{0\}$. Since $V(0)$ contains a convex $0$-neighborhood, it must contain an atom $b$. By assumption, $a$ and $b$ have a common complement $c$. Then

$$(0, a) = ((0, b) \lor (c, c)) \land (a, a) \in (V \lor \Delta) \land \Delta \subseteq U,$$

a contradiction. Therefore $V(0) = \{0\}$, so $[0]$ is a zero neighborhood. It follows that $u$ is discrete since any lattice uniformity on a sectionally complemented lattice is by [22, 6.10] uniquely determined by its zero neighborhood system. 

**Lemma 5.9.** The discrete uniformity on $L$ is exhaustive iff $L$ is chain-finite, i.e., any chain in $L$ is finite.

**Proof.** ($\Leftarrow$) is obvious. ($\Rightarrow$) Suppose that $L$ contains an infinite chain $C$. If $C$ is well-ordered, then $C$ contains a strictly increasing sequence; otherwise $C$ contains a strictly
decreasing sequence. So \( L \) contains a strictly monotone sequence and therefore the discrete uniformity is not exhaustive. 

A geometric lattice in the sense of [12, p. 179] is a complete atomistic semimodular lattice such that all its atoms are compact (in the lattice theoretical sense [12, Definition II.3.12]). By a theorem of Birkhoff, any geometric lattice is relatively complemented [12, IV.3.4].

**Corollary 5.10.** Let \( L \) be an irreducible geometric lattice.

(a) Then the only Hausdorff lattice uniformity on \( L \) is the discrete uniformity.

(b) If \( L \) admits a Hausdorff exhaustive lattice uniformity, then \( L \) has finite length (i.e., there is a natural number \( n \) such that any chain in \( L \) has less than \( n \) elements).

**Proof.** (a) follows from Lemma 5.8 and the fact [12, Theorem IV.3.6] that any two atoms of an irreducible geometric lattice are perspective.

(b) follows from (a), Lemma 5.9 and the fact [19, Theorem 1.1] that any chain-finite semimodular lattice has finite length. 

**Theorem 5.11.** Let \( L \) be a semimodular sectionally complemented complete lattice and \( u \) a totally disconnected order continuous Hausdorff lattice uniformity on \( L \). Then \( (L, u) \) is (as uniform lattice) isomorphic to the product of irreducible geometric lattices of finite length endowed with the discrete uniformity.

**Proof.** By Theorem 5.5(b), \( L \) is atomistic. Since \( L \) is continuous (see Proposition 1.1), any atom of \( L \) is compact. Therefore \( L \) is a geometric lattice. By Corollary 5.7, \( L \) is isomorphic to the product of irreducible lattices. Since \( u \) is Hausdorff and exhaustive, the irreducible factors of this decomposition are by Corollary 5.10 geometric lattices of finite length endowed with the discrete uniformity.

**Theorem 5.12.** Let \( L \) be a relatively complemented complete lattice and \( u \) an order continuous Hausdorff lattice uniformity on \( L \). Then \( (L, u) \) is (as uniform lattice) isomorphic to the product of an atomless arcwise connected uniform lattice and of irreducible atomistic totally disconnected uniform lattices.

**Proof.** We use the notation of Proposition 5.5. We first prove that for any \( a \in \mathcal{A}(L) \) the restriction \( u_a \) of \( u \) on \([0, a]\) is metrizable. Let \( v \) be a nontrivial lattice uniformity on \([0, a]\) with countable base and coarser than \( u_a \). Since \( L \mathcal{U}([0, a], u_a) \) is isomorphic to \( C([0, a]) \) by Theorems 4.1 and 1.5 and \( C([0, a]) = [0, a] \), we get \( v = u_a \). Therefore \( u_a \) has a countable base.

It now follows with Proposition 5.1 that the interval \([0, a]\) is arcwise connected for any \( a \in \mathcal{A}_\infty(L) \). Therefore \([0, c] \times [0, a] \) is arcwise connected as product of arcwise connected spaces. The other factors \([0, a], a \in \mathcal{A}_f(L) \), of the decomposition given in Proposition 5.5 are irreducible and not connected, hence totally disconnected by Theorem 5.4(c) and atomistic by Theorem 5.4(b).
Corollary 5.13. Let \( L \) be a modular complemented complete lattice and \( u \) an order continuous Hausdorff lattice uniformity on \( L \). Then \((L, u)\) is (as uniform lattice) isomorphic to the product of an atomless arcwise connected uniform lattice and of irreducible modular geometric lattices of finite length endowed with the discrete uniformity. Any of the discrete irreducible factors of this decomposition has length 2 or length 3 or is the lattice \( L(D, n) \) of all linear subspaces of the \( n \)-dimensional linear space \( D^n \) with a suitable division ring \( D \) and \( n \in \mathbb{N} \).

Proof. The first assertion follows from Theorems 5.12, 5.11. The second assertion follows from the Coordinatization Theorem of Projective Geometry [12, IV.5.16].

For the rest of this section we study compact modular complemented lattices. Choe and Greechie [8] proved that any compact Hausdorff topological orthomodular lattice is totally disconnected using the fact that any atom of a block is an atom of the orthomodular lattice. We here prove analogous results for modular complemented lattices.

Lemma 5.14. Let \( A \) be a maximal Boolean sublattice of a modular complemented lattice \( L \). Then any atom of \( A \) is an atom of \( L \).

Proof. Suppose that \( a \) is an atom of \( A \), but not of \( L \). Then there are disjoint elements \( a_1, a_2 \in L \setminus \{0\} \) with \( a = a_1 \lor a_2 \). Let \( B = \{x \in A : x \land a = 0\} \). We prove that \( \Phi : (x, y) \mapsto x \lor y \) defines a lattice homomorphism from the Boolean algebra \( C := \{0, a_1, a_2, a\} \times B \) into \( L \). Obviously \( \Phi \) is compatible with \( \lor \). We now prove that \( \Phi \) is compatible with \( \land \), i.e.,

\[
\Phi((x_1, y_1) \land (x_2, y_2)) = \Phi(x_1, y_1) \land \Phi(x_2, y_2)
\]

for \( x_1, x_2 \in \{0, a_1, a_2, a\} \) and \( y_1, y_2 \in B \). Put \( z_1 := a_1, z_2 := a_2, z_3 := y_1 \lor y_2, z_4 := y_1 \land y_2, z_5 := y_2 \setminus y_1 \). Then \( (z_1 \lor \cdots \lor z_i) \land z_{i+1} = 0 \) for \( i = 1, \ldots, 4 \). Therefore \( \{z_1, \ldots, z_5\} \) is independent, hence contained in a Boolean sublattice \( D \) of \( L \) (see [12, p. 167]). Since \( D \) contains the elements \( x_1, x_2, y_1, y_2 \) and \( x_1 \land y_1 = 0, \) we get \((x_1 \land x_2) \lor (y_1 \land y_2) = (x_1 \lor y_1) \land (x_2 \lor y_2) \). This proves (\(*\)).

It follows that \( \Phi(C) \) is a Boolean sublattice of \( L \) containing \( A \) and \( a_1 \), a contradiction to the maximality of \( A \).

Theorem 5.15. Any compact Hausdorff lattice topology on a modular complemented lattice \( L \) is totally disconnected.

Proof. First observe that any compact Hausdorff lattice topology is induced by an order continuous lattice uniformity and that the underlying lattice must be complete (see, e.g., [22, 6.5]). Therefore, by Theorem 5.4(b), it is enough to prove that \( L \) is atomic. Let \( a \in L \setminus \{0\} \) and \( A \) be a maximal Boolean sublattice of \( L \) containing \( a \). Then \( A \) is closed by Proposition 3.3(a), hence compact and therefore totally disconnected by [1]. It follows that \( a \) contains an atom \( b \) of \( A \) (see Theorem 5.4(b)). By Lemma 5.14, \( b \) is an atom of \( L \).
Theorem 5.15 improves Choe’s result [7, Theorem 1.11], where a sufficient condition is given for a complemented modular compact Hausdorff topological lattice to be totally disconnected.

**Corollary 5.16.** Let \( L \) be a modular complemented lattice endowed with a compact lattice topology. Then \( L \) is isomorphic to a product of finite irreducible modular geometric lattices endowed with the discrete topology.

**Proof.** This immediately follows from Theorem 5.15 and Corollary 5.13 observing that the assumptions of Corollary 5.13 are satisfied and a space compact with respect to the discrete uniformity is finite. \( \Box \)

Corollary 5.16 was proved for orthomodular lattices by Pulmannová and Riečanová [16, Corollary 2.5]. Earlier, Choe and Greechie [8, Theorem 2] proved an analogous result for compact profinite orthomodular lattices. In [17] is given an example of a compact irreducible orthomodular lattice which is not profinite and therefore not finite. This example shows that in Corollary 5.16 the assumption that \( L \) is modular is not superfluous.

### 6. The Hammer–Sobczyk decomposition for modular functions on complemented lattices

In this section let \( G \) be a complete Hausdorff topological commutative group.

We here give a decomposition theorem for \( G \)-valued modular functions on complemented lattices, which generalizes Hammer–Sobczyk’s decomposition [18] of a measure \( \mu : A \rightarrow [0, +\infty] \) defined on a Boolean algebra in the form \( \mu = \lambda + \sum_{n \in I} \mu_n \) where \( I \subset \mathbb{N} \), \( \mu_n \) are two-valued measures and \( \lambda \) is “strongly continuous” in the sense of [5], i.e., for any \( \varepsilon > 0 \) the maximal element 1 of \( A \) has a decomposition \( 1 = a_1 \lor \cdots \lor a_n \) with \( a_i \in A \) and \( \lambda(a_i) < \varepsilon \) \( (i = 1, \ldots, n) \). The proof of its generalization is based on a description of the completion of a modular complemented lattice (see Theorem 4.2 and Corollary 5.13) with respect to the lattice uniformity generated by an exhaustive modular function.

Let \( \mu : L \rightarrow G \) be a modular function, i.e.,

\[
\mu(a \lor b) + \mu(a \land b) = \mu(a) + \mu(b) \quad \text{for all } a, b \in L.
\]

Then the sets

\[
\{(a, b) \in L^2 : \mu(y) - \mu(x) \in U \text{ whenever } x, y \in [a \land b, a \lor b]\},
\]

where \( U \) is a 0-neighborhood in \( G \) form a base for the \( \mu \)-uniformity, i.e., the weakest lattice uniformity that makes \( \mu \) uniformly continuous (see [11], [24, §3.1]). \( \mu \) is called exhaustive if \( (\mu(a_n)) \) is Cauchy for every monotone sequence \( (a_n) \) in \( L \). It is easy to see that \( \mu \) is exhaustive iff the \( \mu \)-uniformity is exhaustive [24, 3.6]. Exhaustive measures are also called \( s \)-bounded.

A uniform space \( (X, u) \) is called chained if for every \( x, y \in X \) and every \( U \in u \) there is a finite sequence \( x_0, \ldots, x_n \in X \) with \( x_0 = x, x_n = y \) and \( (x_{i-1}, x_i) \in U \) for \( i = 1, \ldots, n \).
It is easy to see that a uniform lattice \((L, u)\) is chained iff for every \(a, b \in L\) with \(a < b\) and every \(U \in u\) there is a finite chain \(a = x_0 < x_1 < \cdots < x_n = b\) with \((x_{i-1}, x_i) \in U\) for \(i = 1, \ldots, n\) (see [24, 5.7]). If \(\mu : L \to G\) is a modular function on \(L\), we say that \(L\) is \(\mu\)-chained if \(L\) is chained with respect to the \(\mu\)-uniformity. This concept was introduced in [24, Section 5]. It is easy to see that a complemented lattice \(L\) is \(\mu\)-chained with respect to a modular function \(\mu : L \to G\), if \(\mu(0) = 0\) iff for every 0-neighborhood \(U\) in \(G\) there are finitely many elements \(a_1, \ldots, a_n \in L\) such that \(1 = a_1 \lor \cdots \lor a_n\) and \(\mu([0, a_i]) \subseteq U\) for \(i = 1, \ldots, n\) (cf. [2, 2.6]). The latter condition means for a positive measure on a Boolean algebra that \(\mu\) is “strongly continuous” in the sense of [5]; this concept was introduced as the finitely additive counterpart to atomless \(\sigma\)-additive (positive) measures.

**Lemma 6.1** (cf. [24, 5.6, 3.8]). Let \(L\) be a dense sublattice of a uniform lattice \((\tilde{L}, \tilde{u})\), \(\tilde{\mu} : (\tilde{L}, \tilde{u}) \to G\) a continuous modular function and \(\mu = \tilde{\mu}|L\) the restriction of \(\tilde{\mu}\) on \(L\).

(a) Then \(\tilde{u}\) is the \(\mu\)-uniformity iff \(u\) is the \(\mu\)-uniformity.

(b) \(\tilde{L}\) is \(\tilde{\mu}\)-chained iff \(L\) is \(\mu\)-chained.

(c) If \((L, u)\) is connected, then \(L\) is \(\mu\)-chained.

In the proof of Theorem 6.4 we will pass to a suitable quotient and use:

**Proposition 6.2** [24, 2.5]. Let \(\mu : L \to G\) be a modular function.

(a) Then

\[
N(\mu) := \{(x, y) \in L^2 : \mu \text{ is constant on } [x \land y, x \lor y]\}
\]

is a congruence relation and the quotient \(\tilde{L} := L/N(\mu)\) is a modular lattice. If \(L\) is complemented or sectionally complemented or relatively complemented, then \(\tilde{L}\) is relatively complemented.

(b) \(\tilde{\mu}(\tilde{x}) = \mu(x)\) for \(x \in \tilde{x} \subseteq \tilde{L}\) defines a modular function \(\tilde{\mu}\) on \(\tilde{L}\). The \(\tilde{\mu}\)-uniformity on \(\tilde{L}\) is Hausdorff.

The modularity of \(\tilde{L}\) in Proposition 6.2(a) was proved in [11] generalizing [6, Theorem X.2.2].

The decomposition of Theorem 6.4 is produced with a disjoint subset of the center of \(L\):

**Lemma 6.3.** Let \(u\) be an order continuous Hausdorff lattice uniformity on a complete lattice \(L\) and \(\mu : (L, u) \to G\) a continuous modular function with \(\mu(0) = 0\). A a disjoint subset of \(C(L)\), \(s = \sup A\) and \(t\) the complement of \(s\) (observe that \(s \in C(L)\) by Theorem 3.4). Put \(\mu_z(x) := \mu(x \land z)\) for \(z \in C(L)\) and \(x \in L\).

(a) Then \(\mu_z\) is a modular function for any \(z \in C(L)\).

(b) \((\mu_z(x))_{x \in A}\) is summable uniformly in \(x \in L\) and \(\mu = \mu_t + \sum_{a \in A} \mu_a\).

(c) \(\{\sum_{a \in A} \gamma_a : \gamma_a \in \mu_a(L)\}\) is the range of \(\sum_{a \in A} \mu_a\).

**Proof.** (a) Obviously, \(\mu_z\) is a modular function for \(z \in C(L)\) and \(\mu = \mu_t + \mu_s\).
(b) Let $V$ be a 0-neighborhood in $G$ and $U$ a convex 0-neighborhood in $L$ such that $\mu(U) \subset V$. Since $u$ is order continuous, there is a finite subset $F_0$ of $A$ such that $s \setminus \sup F_0 \in U$; the difference is taken in the Boolean algebra $C(L)$. Let $F$ be a finite subset of $A$ containing $F_0$, and put $z := s \setminus \sup F$. Then we have

$$\mu(x) - \left(\mu_U(x) + \sum_{a \in F} \mu_a(x)\right) = \mu(z \land x) \in V$$

for any $x \in L$.

(c) To prove the nonobvious inclusion of the last assertion, let $y_a \in \mu_a(L)$ and $x_a \in L$ with $\mu_a(x_a) = y_a$ for $a \in A$. Put $x := \sup_{a \in A} a \land x_a$. Then $\mu_a(x) = y_a$, hence $(y_a)_{a \in A}$ is sumtable by (b) and $\sum_{a \in A} y_a = \langle \sum_{a \in A} \mu_a \rangle(x)$ belongs to the range of $\sum_{a \in A} \mu_a$. \hfill \Box

In the following decomposition theorem, the summands $\mu_a$ are described by means of the height function. The height function $h : L \to \mathbb{N} \cup \{0, +\infty\}$ on a bounded lattice $L$ is defined by

$$h(x) = \sup\{|C| - 1 : C \text{ is a finite chain in } [0, x]|\},$$

where $|C|$ denotes the cardinality of $C$. Obviously, a lattice has finite length iff it is bounded and its height function is bounded. By [12, IV.2.3], a lattice of finite length is modular iff its height function is modular.

**Theorem 6.4.** Let $L$ be complemented or sectionally complemented or relatively complemented and $\mu : L \to G$ an exhaustive modular function.

(a) Then there are $G$-valued exhaustive modular functions $\lambda$ and $\mu_a$ $(a \in A)$ on $L$ and elements $g_a \in G$ $(a \in A)$ such that

1. $\langle \mu_a(x) \rangle_{a \in A}$ is sumtable uniformly in $x \in L$;
2. $\mu = \lambda + \sum_{a \in A} \mu_a$;
3. $L$ is $\lambda$-chained;
4. For any $a \in A$, the quotient $L_a := L / N(\mu_a)$ is an irreducible modular geometric lattice of finite length; $\mu_a(x) = h(\hat{x}) \cdot g_a$ where $x \in L$, $\hat{x}$ is the corresponding element of the quotient $L_a$ and $h(\hat{x})$ is the height of $\hat{x}$ in $L_a$.

(b) $\lambda(L)$ is dense in an arcwise connected subset of $G$, in particular, $\lambda(L)$ is connected. The range of $\sum_{a \in A} \mu_a$ is relatively compact.

(c) If $L$ is complete with respect to the $\mu$-uniformity, then $\lambda(L)$ is arcwise connected and the range of $\sum_{a \in A} \mu_a$ is compact. Moreover $\lambda$ can then be written as $\lambda = \nu + \sum_{b \in B} \lambda_b$ where $\nu$ and $\lambda_b$ are $G$-valued exhaustive modular functions on $L$, $\langle \lambda_b(x) \rangle_{b \in B}$ is sumtable uniformly in $x \in L$, $L / N(\nu)$ and $L / N(\lambda_b)$ are complemented complete modular atomless lattices, the center of $L / N(\nu)$ is atomless and $L / N(\lambda_b)$ are irreducible lattices.

**Proof.** Let $u$ be the $\mu$-uniformity. Passing to the quotient $L / N(\mu)$, we may assume that $u$ is Hausdorff. $L$ is then modular and relatively complemented by Proposition 6.2.

(i) We first consider the case that $(L, u)$ is complete. Then $(L, \preceq)$ is as lattice complete and $u$ is order continuous, see Theorem 1.4. We may assume furthermore that $\mu(0) = 0$: Actually it is enough to find a decomposition of $\mu - \mu(0)$ of the form $\mu - \mu(0) =$
\[ \lambda + \sum_{a \in A} \mu_a; \] then \( \mu = \lambda_0 + \sum_{a \in A} \mu_a \) with \( \lambda_0 := \lambda + \mu(0) \) is a decomposition for \( \mu \) according to the assertion.

With the notation of Proposition 5.5, let \( A = A_{\lambda}(L) \). Define \( s, t, \mu_t, \mu_a \) as in Lemma 6.3 and put \( \lambda := \mu_t \). Then (1) and (2) hold by Lemma 6.3. With the notation of Proposition 5.5, we have \( t = c \lor a_\infty \). So the interval \([0, t]\) is by Proposition 5.5(b) connected and therefore by Corollary 5.13 even arcwise connected. Therefore the continuous image \( \mu([0, t]) = \lambda(L) \) is arcwise connected. Since the \( \lambda \)-uniformity coincides on \([0, t]\) with \( u \) and is trivial on \([0, s]\), \( L \) is connected with respect to the \( \lambda \)-uniformity. Therefore \( L \) is \( \lambda \)-chained by Lemma 6.1(c).

For \( a \in A \), \( L_a \) is isomorphic to \([0, a]\) and therefore an irreducible modular geometric lattice of finite length (see Proposition 5.5, Corollary 5.13), \( \mu \) assumes on any atom of \([0, a]\) the same value \( g_a \); In fact, two atoms \( x_1, x_2 \) in \([0, a]\) have by [12, IV.3.6] a common complement \( y \); therefore \( \mu(x_1) + \mu(y) = \mu(a) = \mu(x_2) + \mu(y) \), hence \( \mu(x_1) = \mu(x_2) \). By [12, IV.3.3], any \( x \in [0, a] \) is the supremum of an independent set \( \{x_1, \ldots, x_n\} \) of atoms. We now have \( \mu(x) = \mu(x_1) + \cdots + \mu(x_n) = n \cdot g_a = h(x) \cdot g_a \) (cf. [12, IV.2.4]). This gives us the formula of (4) for \( \mu_a \).

Let \( C_a \) be a maximal chain in \([0, a]\); then \( \mu_a(L) = \mu_a(C_a) \) for \( a \in A \). Since \( C_a \) is finite for any \( a \in A \), \( K := \prod_{a \in A} C_a \) is a compact subspace of \( \prod_{a \in A} [0, a], u \). Therefore \( K_0 := \phi^{-1}(K) \) is compact where \( \phi \) is defined as in Proposition 3.5. Therefore the continuous image \( \mu(K_0) \) is compact. \( \mu(K_0) \) coincides with the range of \( \sum_{a \in A} \mu_a \) by the last assertion of Lemma 6.3.

One obtains the decomposition of \( \lambda \) given in (c) as follows: With the notation of Proposition 5.5, put \( B = A_{\lambda\infty}(L) \), \( \lambda_b = \mu_b \) and \( v = \mu_c \). Observe that \( L/N(\lambda_b) \) is isomorphic to \([0, b]\) and \( L/N(v) \) is isomorphic to \([0, c]\).

(ii) If \((L, u)\) is not complete, let \( \tilde{\mu} \) be the continuous extension of \( \mu \) on the completion \((\tilde{L}, \tilde{u})\) of \((L, u)\). Then \( \tilde{u} \) is the \( \tilde{\mu} \)-uniformity. Let \( \tilde{\mu} = \tilde{\lambda} + \sum_{a \in A} \tilde{\mu}_a \) be the decomposition of \( \tilde{\mu} \) according to (i) and \( \lambda, \mu_a \), respectively, be the restrictions of \( \tilde{\lambda}, \tilde{\mu}_a \) on \( L \). It is clear that then (1) and (2) are valid. Moreover, \( \sum_{a \in A} \tilde{\mu}_a \) and \( \tilde{\lambda} \) are continuous with respect to \( u \). Since \( L \) is \( \tilde{\lambda} \)-chained by (i) and \( L \) is dense in \((\tilde{L}, \tilde{u})\), \( L \) is \( \lambda \)-chained by Lemma 6.1(b). Moreover, \( \lambda(L) \) is dense in the arcwise connected set \( \hat{\lambda}(L) \).

Since the range of \( \sum_{a \in A} \tilde{\mu}_a \) is compact by (i), the range of \( \sum_{a \in A} \mu_a \) is relatively compact.

To prove (4) for \( \mu_a \), we use the corresponding property for \( \tilde{\mu}_a \). It is enough to show that \( j(\hat{x}) = x^* \) defines an isomorphism from \( L_a = L/N(\mu_a) \) onto \( \hat{L}_a = \hat{L}/N(\tilde{\mu}_a) \), where \( \hat{x} \) and \( x^* \), respectively, are the elements of \( L_a \) and \( \hat{L}_a \) corresponding to \( x \in L \). Let \( \tilde{u}_a \) be the \( \tilde{\mu}_a \)-uniformity. Then \( j \) is an isomorphism from \( L_a \) onto a dense subspace of \( (\hat{L}, \tilde{u}_a)/N(\tilde{\mu}_a) \). Since the uniformity of \( (\hat{L}, \tilde{u}_a)/N(\tilde{\mu}_a) \) is discrete by Corollary 5.10, we get \( j(L_a) = \hat{L}_a \). 

As mentioned in Corollary 5.13, by [12, IV.5.16] each of the lattices \( L_a \) has length 2 or length 3 or is for some \( n \in \mathbb{N} \) and a suitable division ring \( D \) isomorphic to the lattice \( \mathcal{L}(D, n) \) of all linear subspaces of the \( n \)-dimensional linear space \( D^n \). The height of a space belonging to \( \mathcal{L}(D, n) \) is its dimension.
In the case that $L$ is a Boolean algebra, Theorem 6.4 was proved in [21]. In this case $L_\alpha$ is the trivial Boolean algebra consisting of two elements. In [4, 6.2.3], a theorem analogous to Theorem 6.4 is proved for measures on MV-algebras. The lattices corresponding to $L_\alpha$ are there finite chains.

References