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Orthogonal double covers of $K_{n,n}$ by small graphs

Ramadan El-Shanawany, Hans-Dietrich O.F. Gronau, Martin Grüttmüller

Fachbereich Mathematik, Universität Rostock, 18051 Rostock, Germany

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Abstract

An orthogonal double cover (ODC) of K_n is a collection of graphs such that each edge of K_n occurs in exactly two of the graphs and two graphs have precisely one edge in common. ODCs of K_n and their generalizations have been extensively studied by several authors (e.g. in: J.H. Dinitz, D.R. Stinson (Eds.), Contemporary Design Theory, Wiley, New York, 1992, pp. 13–40 (Chapter 2); Design Codes Cryptography 27 (2002) 49; Graphs Combin. 13 (1997) 251; V. Leck, Orthogonal double covers of K_m, Ph.D. Thesis, Universität Rostock, 2000). In this paper, we investigate ODCs where the graph to be covered twice is $K_{n,n}$ and all graphs in the collection are isomorphic to a given small graph G. We prove that there exists an ODC of $K_{n,n}$ by all proper subgraphs G of $K_{n,n}$ for $1 \leqslant n \leqslant 9$, with two genuine exceptions.

Keywords: Orthogonal double cover; ODC; Graph decompositions

1. Introduction

Let $\mathscr{G} = \{G_1, \ldots, G_n\}$ be a collection of n simple spanning graphs on an n-element vertex set and let K_n denote the complete graph on this vertex set. We call $\mathscr{G}$ an *Orthogonal Double Cover* (*ODC*) of K_n, if the following properties hold:

(i) Every edge of K_n is covered by exactly two of the graphs $G_1, \ldots, G_n$.
(ii) For every choice of integers i, j with $1 \leqslant i < j \leqslant n$, the graphs G_i and G_j have exactly one common edge.

E-mail address: gronau@mathematik.uni-rostock.de (H.-D.O.F. Gronau).

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The graphs $G_1, \dots, G_n$ are called *pages*. If all pages are isomorphic to a graph G, we speak about an *ODC of K_n by G*.

The original motivation for investigating ODCs comes from a problem of Demetrovics et al. [4] on minimal databases, and a problem of Hering and Rosenfeld [13] on the organization of statistical test plannings. In our notation they were interested in ODCs of K_n. Many researchers have been involved in investigating ODCs of K_n by G for several classes of graphs G for example in [1,2,5,6,8–10,12,14–19], a survey on the topic is given in [7].

A generalization of the notion of an ODC to arbitrary underlying graphs is as follows. Let H be an arbitrary graph with n vertices and let $\mathcal{G} = \{G_1, \dots, G_n\}$ be a collection of n spanning subgraphs of H. $\mathcal{G}$ is called an ODC of H if there exists a bijective mapping $\varphi : V(H) \to \mathcal{G}$ such that:

(i) Every edge of H is contained in exactly two of the graphs $G_1, \dots, G_n$.
(ii) For every choice of distinct vertices u, w of H,

$$|E(\varphi(u)) \cap E(\varphi(w))| = \begin{cases} 1 & \text{if } \{u, w\} \in E(H), \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

If all of the G_i are isomorphic to a graph G, we call $\mathcal{G}$ an ODC of H by G.

In this paper, we consider the case that H is $K_{n,n}$ the complete bipartite graph on $2n$ vertices. It seems natural to extend the concept of ODC of $H = K_n$ to other graphs. According to the obvious properties of ODCs by a graph G, the underlying graph H has to be $|E(G)|$-regular. Therefore, $H = K_{n,n}$ is of special interest. On the other hand, the idea of constructing an ODC of K_{2n} by two ODCs of K_n requires an ODC of $K_{n,n}$ (see [7, Section 2.4]). ODCs of other graphs H have been considered in [11].

We will label the vertices of $K_{n,n}$ by the elements of $\Gamma \times \mathbb{Z}_2$, where $\Gamma = \{\gamma_0, \dots, \gamma_{n-1}\}$ is an additive group of order n, such that $\{a_i, b_i\} \notin E(K_{n,n})$ for $a, b \in \Gamma$ and fixed $i \in \mathbb{Z}_2$. Here, we first use the groups Γ and $\mathbb{Z}_2$ just for labeling the vertices. It will turn out later that these groups can be used for the construction of ODCs. For the construction we will need an order of the elements of Γ. In this paper, we consider Abelian groups only represented by Cartesian products of cyclic groups. Therefore, we take the lexicographic order, i.e. in case of $\Gamma = \mathbb{Z}_n$ it is just the natural order $0, 1, \dots, n-1$.

If there is no chance of confusion we will write (a, b) instead of $\{a_0, b_1\}$ for the edge between the vertices a_0 and b_1. We denote $\varphi(a_i)$ by $G_{a,i}$ for all $a_i \in \Gamma \times \mathbb{Z}_2$. In Fig. 1 an ODC of $K_{3,3}$ by $G = S_2 \cup K_2$ (the vertex disjoint union of a star with two edges and a single edge) is exhibited.

As an immediate consequence of the double cover property (i) it follows that G contains exactly n edges. Moreover, the orthogonality property (ii) forces the graphs $G_{a,0}$ for all $a \in \Gamma$ and the graphs $G_{a,1}$ for all $a \in \Gamma$ to form two orthogonal edge decompositions of $K_{n,n}$.

To each of these two edge decompositions we may associate an $n \times n$-square with entries belonging to Γ denoted by $S_i = S_i(k, l),\ i = 0, 1; k, l \in \Gamma$ with $S_i(k, l) = m$ if and only if $\{k_0, l_1\} \in E(G_{m,i})$. For example, the ODC given in Fig. 1 is associated with

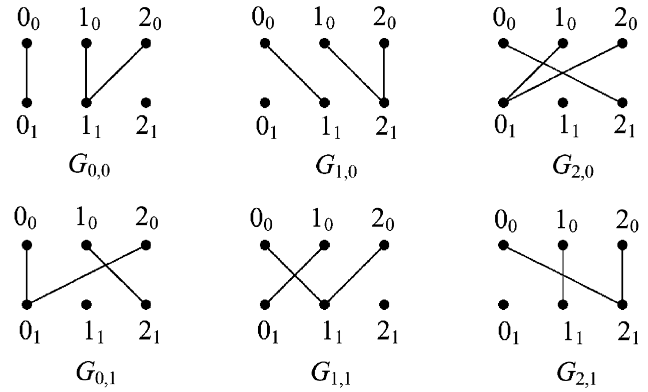

Fig. 1. ODC of $K_{3,3}$ by $G = S_2 \cup K_2$ with $\Gamma = \mathbb{Z}_3$.

Table 1
Survey of results and construction methods used

n	1	2	3	4	5	6	7	8	9
Number of graphs	1	2	4	9	18	43	96	239	595
Number of symmetric starters	1	1	2	5	14	40	92	235	589
Number of direct constructions	0	0	2	4	4	2	4	4	6
Number of graphs with no solution	0	1	0	0	0	1	0	0	0

the squares:

$$S_0 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

The orthogonality condition (ii) reads for the squares as usually as $|\{(S_0(k,l), S_1(k,l)) : k, l \in \Gamma\}| = n^2$. Obviously, $G = nK_2$ corresponds bijectively to Latin squares. According to the existence of mutually orthogonal Latin squares, see [3], we have the following.

Theorem 1. *There exists an ODC of $K_{n,n}$ by nK_2 if and only if $n \geqslant 1$, except for $n = 2$ or 6.*

In this paper, we will prove the following result.

Theorem 2. *There exists an ODC of $K_{n,n}$ by all spanning subgraphs G of $K_{n,n}$ with n edges for $1 \leqslant n \leqslant 9$, with two genuine exceptions, namely $G = 2K_2$ or $G = 6K_2$.*

Our main construction tool (called symmetric starter) for that result will be investigated in Section 2. In some cases (see Table 1) we need to find direct constructions which are considered in Section 3.

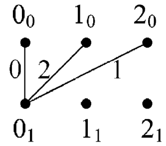

Fig. 2. Half starter S_3 with every edge labeled by its length.

2. Symmetric starters

Let G be a spanning subgraph of $K_{n,n}$ and $x \in \Gamma$. Then the graph $G + x$ with $E(G + x) = \{(a + x, b + x) : (a, b) \in E(G)\}$ is called the *x-translate* of G. The *length* of an edge $e = (a, b) \in E(G)$ is defined by $d(e) = b - a$. G is called *half starter* with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are mutually different, i.e. $\{d(e)\colon e \in E(G)\} = \Gamma$. It is easy to see that the following lemma is true.

Lemma 3. *If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e. $\bigcup_{x\in\Gamma} E(G + x) = E(K_{n,n})$.*

In what follows, we will represent a half starter G by the vector $v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$, where $v_{\gamma_j} \in \Gamma$ and $(v_{\gamma_j})_0$ is the unique vertex in $\Gamma \times \{0\}$ that belongs to the unique edge of length γ_j. For example, the graph in Fig. 2 (a star S_3 with the root at the vertex 0_1) is a half starter with respect to $\mathbb{Z}_3$ represented by $(0, 2, 1)$ (e.g. $\{\mathbf{2}_0, 0_1\}$ is the unique edge of length 1, thus $v_1 = \mathbf{2}$).

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be *orthogonal* if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 4. *If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $\mathcal{G} = \{G_{a,i} : a_i \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = (G_i + a)$ is an ODC of $K_{n,n}$.*

Proof. Firstly, it is clear from Lemma 3 that if $(a, b) \in E(K_{n,n})$, then there are exactly two graphs $G_{x,0}$ and $G_{y,1}$ with $(a, b) \in E(G_{x,0})$ and $(a, b) \in E(G_{y,1})$. Thus, the double cover property (i) is satisfied.

Secondly, let $a, b \in \Gamma$ and $k, l \in \mathbb{Z}_2$. We want to prove that $|E(G_{a,k}) \cap E(G_{b,l})| = 0$ if $k = l$ and $a \neq b$, and $|E(G_{a,k}) \cap E(G_{b,l})| = 1$ if $k \neq l$. The first case follows immediately from construction. If otherwise $k \neq l$ (w.l.o.g. $k = 0, l = 1$), then there is exactly one γ_i with $v_{\gamma_i}(G_0) - v_{\gamma_i}(G_1) = b - a$. Thus, $v_{\gamma_i}(G_0) + a = v_{\gamma_i}(G_1) + b$, but this means that there is exactly one edge e with $e = (v_{\gamma_i}(G_0) + a, v_{\gamma_i}(G_0) + a + \gamma_i) = (v_{\gamma_i}(G_1) + b, v_{\gamma_i}(G_1) + b + \gamma_i)$. Therefore, $e \in E(G_0 + a) = E(G_{a,0})$ and $e \in E(G_1 + b) = E(G_{b,1})$. We conclude $|E(G_{a,0}) \cap E(G_{b,1})| = 1$. □

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{b_0, a_1\} : \{a_0, b_1\} \in E(G)\}$ is called the *symmetric graph* of G. Note that if G is a half starter, then G_s is also a half starter. A half starter G is called a *symmetric starter* with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 5. *Let n be a positive integer and let G be a half starter represented by* $v(G)=(v_{\gamma_0},v_{\gamma_1},\ldots,v_{\gamma_{n-1}})$. *Then G is a symmetric starter if and only if* $\{v_\gamma - v_{-\gamma}+\gamma:\gamma\in\Gamma\}=\Gamma$.

Proof. We know that G_s is a half starter and representable by $v(G_s)$. Since $(v_{\gamma_i}(G_s), v_{\gamma_i}(G_s)+\gamma_i)$ is an edge in $E(G_s)$ we have $(v_{\gamma_i}(G_s)+\gamma_i, v_{\gamma_i}(G_s))$ is an edge in $E(G)$ of length $-\gamma_i$. Therefore, $v_{-\gamma_i}(G)=v_{\gamma_i}(G_s)+\gamma_i$ and thus $v_{\gamma_i}(G_s)=v_{-\gamma_i}(G)-\gamma_i$. Consequently, $v(G)$ and $v(G_s)$ are orthogonal if and only if $\{v_\gamma(G)-v_\gamma(G_s)=v_\gamma-(v_{-\gamma}-\gamma)=v_\gamma-v_{-\gamma}+\gamma:\gamma\in\Gamma\}=\Gamma$. □

Example 6.

(i) *Let* $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_2$. *Then there is an ODC of* $K_{4,4}$ *by* $4K_2$, *since G with* $v(G)=(00,10,11,01)$ *is a symmetric starter.*

(ii) *Let* $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. *Then there is an ODC of* $K_{8,8}$ *by* C_8 (*the circle with 8 vertices*), *since G with* $v(G)=(000,000,001,011,111,011,111,001)$ *is a symmetric starter.*

For our investigations, we need to have a list with all non-isomorphic spanning subgraphs of $K_{n,n}$ which have exactly n edges. This list $L(n)$ (with a canonical order of the graphs) was created using the program MAKEBG of Brendan McKay [20]. For our purpose, we need to define that graphs G_0 and G_1 are isomorphic if there is a bijection $\psi:\Gamma\times\mathbb{Z}_2\mapsto\Gamma\times\mathbb{Z}_2$ such that $E(G_0)=\{\{\psi(a_0),\psi(b_1)\}:(a,b)\in E(G_1)\}$. Therefore, we performed an isomorphism check on the graphs from $L(n)$ using the canonical output of MAKEBG to compute a partition of $L(n)$ into classes $L_1(n),L_2(n),\ldots,L_m(n)$ of isomorphic graphs. We apply Theorem 5 as described in Algorithm 1 to find a list of graphs which are symmetric starters. Note that $G_{\pi_1}^{\pi_2}$ is defined by $E(G_{\pi_1}^{\pi_2}):=\{(\pi_1(a),\pi_2(b)):(a,b)\in E(G)\}$ where π_1 and π_2 are permutations of the elements of Γ.

Algorithm 1. Algorithm for constructing symmetric starters

Require: Positive integer n, additive group Γ of order n
Ensure: list of symmetric starters, list of graphs which do not have a symmetric starter with respect to Γ

Compute a list $L(n)$ with all subgraphs of $K_{n,n}$ with n edges using MAKEBG
Compute a partition of the graphs of $L(n)$ into isomorphism classes $L_1(n)$, $L_2(n),\ldots,L_m(n)$
  for all isomorphism classes $L_k(n)$ **do**
    for all graphs G in $L_k(n)$ **do**
      for all permutations $\pi_1,\pi_2\in S_\Gamma$ **do**
        if $G_{\pi_1}^{\pi_2}$ is a half starter **do**
          Compute $v(G_{\pi_1}^{\pi_2})$
            if $\{v_\gamma-v_{-\gamma}+\gamma:\gamma\in\Gamma\}=\Gamma$ **do**

```
                Output: v(G^{π2}_{π1})                {found a symmetric starter}
                Goto NEXT                             {next isomorphism class}
                end if
            end if
        end for                                       {all π1, π2 ∈ S_Γ}
    end for                                           {all G ∈ L_k(n)}
    Output: No symmetric starter for graphs in L_k(n)
NEXT                                                  {label: NEXT}
end for                                               {all L_k(n)}
```

Remark 7. We remark that we do not need to consider all graphs in an isomorphism class $L_k(n)$ since $G \in L_k(n)$ has a symmetric starter if and only if $G_s \in L_k(n)$ has a symmetric starter. But note that there are graphs G, G' in some $L_k(n)$ such that G has a symmetric starter and G' does not. For example, let G and G' be isomorphic to the vertex disjoint union of three stars S_2, where in G one star has its root in $\mathbb{Z}_6 \times \{0\}$ and two stars have their roots in $\mathbb{Z}_6 \times \{1\}$ while in G' all three stars have their roots in $\mathbb{Z}_6 \times \{0\}$. Then $(0, 2, 2, 4, 3, 1)$ is a symmetric starter for G but there is no one for G'.

The list of all symmetric starters derived that way with $\Gamma = \mathbb{Z}_n$ is shown in the appendix. Graphs which do not admit a symmetric starter with $\Gamma = \mathbb{Z}_n$ are considered in the following section.

3. Direct constructions

In this section, we will investigate all remaining graphs which do not have a symmetric starter. In the sequel, we will use the notation $G^{n,k}$ to denote a subgraph of $K_{n,n}$ that is element of the class $L_k(n)$. We first give three results that yield infinitely many values of n for which there is an ODC of $K_{n,n}$ by graphs of a special type which do not admit symmetric starters.

Lemma 8. *Let $n \geqslant 5$, $\Gamma = \mathbb{Z}_n$ and $G = K_2 \cup S_{n-1}$. Then there exists an ODC of $K_{n,n}$ by G.*

Proof. We define the pages as follows. (Here, $x \bmod y$ always means the smallest nonnegative residue class modulo y.) For $j \in \{0, \ldots, n-4\}$ let $E(G_{j,0}) = \{(j, 0)\} \cup \{((j+1) \bmod (n-3), i)\colon i \in \{1, \ldots, n-1\}\}$. For $j \in \{n-3, n-2, n-1\}$ let $E(G_{j,0}) = \{(j, n-3)\} \cup \{(((j-n+4) \bmod 3)+n-3, i)\colon i \in \{0, \ldots, n-4, n-2, n-1\}\}$. Moreover, for $j \in \{0, \ldots, n-4\}$ let $E(G_{j,1}) = \{(n-1, (j+1) \bmod (n-3))\} \cup \{(i, j)\colon i \in \{0, \ldots, n-2\}\}$ and for $j \in \{n-3, n-2, n-1\}$ let $E(G_{j,1}) = \{(1, ((j-n+4) \bmod 3)+n-3)\} \cup \{(i, j)\colon i \in \{0, 2, 3, \ldots, n-1\}\}$. A graphical representation of the pages is given in Fig. 3.

For the sake of brevity we define sets $A = \{0, \ldots, n-4\}$ and $B = \{n-3, n-2, n-1\}$. To see that conditions (i) and (ii) of the ODC definition are satisfied, note that an

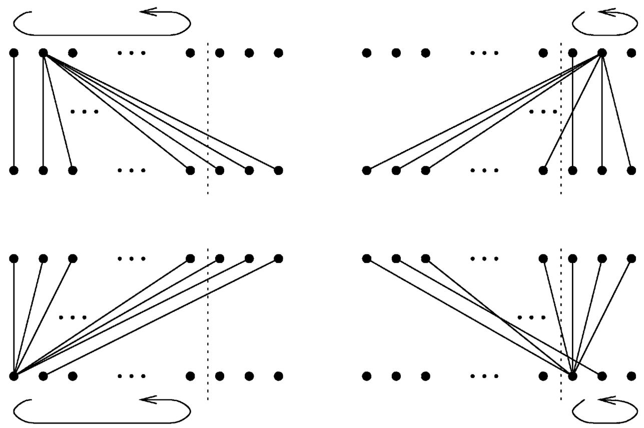

Fig. 3. Graphs $G_{0,0}$, $G_{n-3,0}$, $G_{0,1}$ and $G_{n-3,1}$ (top left to bottom right).

arbitrary edge (a,b) is contained exactly in the graphs $G_{j,0}$ and $G_{j',1}$, where

$$j = \begin{cases} (a-1) \bmod (n-3) & \text{if } b \neq 0 \text{ and } a \in A, \\ ((a-n+2) \bmod 3) + n - 3 & \text{if } b \neq n-3 \text{ and } a \in B, \text{ or} \\ a & \text{otherwise} \end{cases}$$

and

$$j' = \begin{cases} (b-1) \bmod (n-3) & \text{if } a = n-1 \text{ and } b \in A, \\ ((b-n+2) \bmod 3) + n - 3 & \text{if } a = 1 \text{ and } b \in B, \text{ or} \\ b & \text{otherwise.} \end{cases}$$

Furthermore, arbitrary graphs $G_{j,0}$ and $G_{j',1}$ share a unique edge. More precisely,

$$E(G_{j,0}) \cap E(G_{j',1})$$
$$= \begin{cases} \{(j,0)\} & \text{if } j \in A \text{ and } j' = 0, \\ \{((j+1) \bmod (n-3), j')\} & \text{if } j \in A \text{ and } j' \in A \setminus \{0\}, \text{ or} \\ & j \in A \setminus \{0\} \text{ and } j' \in B, \\ \{(1, ((j'-n+4) \bmod 3) + n - 3)\} & \text{if } j = 0 \text{ and } j' \in B, \\ \{(n-1, (j'+1) \bmod (n-3))\} & \text{if } j = n-2 \text{ and } j' \in A, \\ \{(((j-n+4) \bmod 3) + n - 3, j')\} & \text{if } j \in \{n-3, n-1\} \text{ and } j' \in A, \text{ or} \\ & j \in B \text{ and } j' \in \{n-2, n-1\}, \\ \{(j, n-3)\} & \text{if } j \in B \text{ and } j' = n-3. \end{cases}$$ □

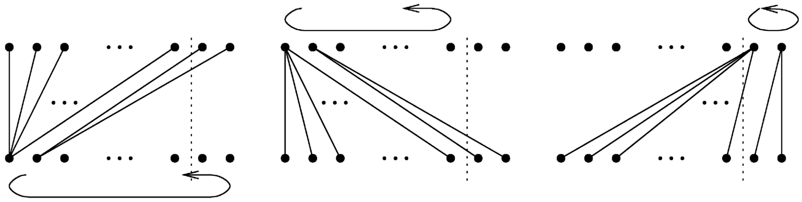

Fig. 4. Graphs $G_{0,0}$, $G_{0,1}$ and $G_{n-2,1}$ (left to right).

Corollary 9. *There exists an ODC of* $K_{5,5}$ *by* $G^{5,3}$, *an ODC of* $K_{6,6}$ *by* $G^{6,3}$, *an ODC of* $K_{7,7}$ *by* $G^{7,3}$, *an ODC of* $K_{8,8}$ *by* $G^{8,3}$, *and an ODC of* $K_{9,9}$ *by* $G^{9,3}$.

Remark 10. Notice that for none of the graphs in Lemma 8 there exists a symmetric starter since for an arbitrary group Γ and any labeling of the vertices there is an edge in the star of the same length as the length of the single edge.

Lemma 11. *Let* $n \geqslant 5$ *be an odd integer*, $\Gamma = \mathbb{Z}_n$ *and* $G = S_2 \cup S_{n-2}$. *Then there exists an ODC of* $K_{n,n}$ *by* G.

Proof. We define the pages as follows. For $j \in \mathbb{Z}_n$ let $E(G_{j,0}) = \{(i,j)\colon\ i \in \{0,\dots,n-3\}\} \cup \{(n-2,j+1),(n-1,j+1)\}$. Moreover, let $E(G_{j,1}) = \{(j,i)\colon\ i \in \{0,\dots,n-3\}\} \cup \{((j+1) \bmod (n-2), i)\colon\ i \in \{n-2,n-1\}\}$ for $j \in \{0,\dots,n-3\}$ and $E(G_{j,1}) = \{(j,i)\colon\ i \in \{0,\dots,n-3\}\} \cup \{(((j-n+3) \bmod 2)+n-2, i)\colon\ i \in \{n-2,n-1\}\}$ for $j \in \{n-2,n-1\}$. See Fig. 4.

Now, let $A = \{0,\dots,n-3\}$, $B = \{n-2,n-1\}$, $C = \{n-3,n-2\}$ and $D = \{0,\dots,n-4,n-1\}$. An arbitrary edge (a,b) is contained exactly in the graphs $G_{j,0}$ and $G_{j',1}$, where

$$j = \begin{cases} b & \text{if } a \in A, \text{ or} \\ b-1 & \text{otherwise} \end{cases}$$

and

$$j' = \begin{cases} ((a-n+3) \bmod 2)+n-2 & \text{if } a,b \in B, \\ a & \text{if } b \in A, \\ (a-1) \bmod (n-2) & \text{otherwise.} \end{cases}$$

Furthermore, arbitrary graphs $G_{j,0}$ and $G_{j',1}$ share a unique edge:

$$E(G_{j,0}) \cap E(G_{j',1}) = \begin{cases} \{(j',j)\} & \text{if } j,j' \in A, \\ \{((j'+1) \bmod (n-2), j)\} & \text{if } j' \in A \text{ and } j \in B, \\ \{(((j'-n+3) \bmod 2)+n-2, j+1)\} & \text{if } j' \in B \text{ and } j \in C, \text{ or} \\ \{(j',j+1)\} & \text{if } j \in D \text{ and } j' \in B. \end{cases} \qquad \square$$

Corollary 12. *There exists an ODC of* $K_{5,5}$ *by* $G^{5,6}$, *an ODC of* $K_{7,7}$ *by* $G^{7,6}$, *and an ODC of* $K_{9,9}$ *by* $G^{9,6}$.

Remark 13. If $G = S_2 \cup S_{n-2}$, then there is for even n always an ODC of $K_{n,n}$ created by the symmetric starter $(0, n/2, \ldots, n/2, v_{n/2} = 0, n/2, \ldots, n/2,)$ with $\Gamma = \mathbb{Z}_n$. On the other hand, if n is odd, then for any group Γ and any labeling of the vertices the stars S_2 and S_{n-2} have edges of the same length. Thus, there is no symmetric starter for odd n.

Lemma 14. *Let* $n \geqslant 5$ *be an odd integer*, $\Gamma = \mathbb{Z}_n$ *and* $G = S_2 \cup (n-2)K_2$. *Then there exists an ODC of* $K_{n,n}$ *by* G.

Proof. For $j \in \mathbb{Z}_n$ define $E(G_{j,0}) = \{(n-1, j)\} \cup \{(i, j+i)\colon\ i \in \{0, \ldots, n-2\}\}$ and $E(G_{j,1}) = \{(n-1, j+n-1)\} \cup \{(i, j+2i)\colon\ i \in \{0, \ldots, n-2\}\}$. Here, an arbitrary edge (a, b) is contained exactly in the graphs $G_{(b-a),0}$ and $G_{(b-2a),1}$ if $a \neq n-1$, or in $G_{(b-a+1),0}$ and $G_{(b-a),1}$ if $a = n-1$. Thus, condition (i) is satisfied. Furthermore, graphs $G_{j,0}$ and $G_{j',1}$ share a unique edge which is $(n-1, j)$ if $j = j'-1$, or $(j-j', 2j-j')$ otherwise. Therefore, condition (ii) is satisfied. □

Corollary 15. *There exists an ODC of* $K_{5,5}$ *by* $G^{5,17}$, *an ODC of* $K_{7,7}$ *by* $G^{7,95}$, *and an ODC of* $K_{9,9}$ *by* $G^{9,594}$.

As a consequence of Theorem 1 we have the following.

Corollary 16. *There exists an ODC of* $K_{3,3}$ *by* $G^{3,4}$, *an ODC of* $K_{8,8}$ *by* $G^{8,239}$, *and an ODC of* $K_{9,9}$ *by* $G^{9,595}$.

Lemma 17. *There exists an ODC of* $K_{3,3}$ *by* $G^{3,3}$.

Proof. See Fig. 1. □

Lemma 18. *There exists an ODC of* $K_{4,4}$ *by* $G^{4,3} = S_3 \cup K_2$.

Proof. Let $E(G_{0,0}) = \{(0,0)\} \cup \{(1,i)\colon\ i \in \{1,2,3\}\}$, $E(G_{1,0}) = \{(1,0)\} \cup \{(0,i)\colon\ i \in \{1,2,3\}\}$, $E(G_{2,0}) = \{(2,2)\} \cup \{(3,i)\colon\ i \in \{0,1,3\}\}$, $E(G_{3,0}) = \{(3,2)\} \cup \{(2,i)\colon\ i \in \{0,1,3\}\}$, $E(G_{0,1}) = \{(3,1)\} \cup \{(i,0)\colon\ i \in \{0,1,2\}\}$, $E(G_{1,1}) = \{(3,0)\} \cup \{(i,1)\colon\ i \in \{0,1,2\}\}$, $E(G_{2,1}) = \{(1,3)\} \cup \{(i,2)\colon\ i \in \{0,2,3\}\}$, and $E(G_{3,1}) = \{(1,2)\} \cup \{(i,3)\colon\ i \in \{0,2,3\}\}$. □

Lemma 19. *There exists an ODC of* $K_{4,4}$ *by* $G^{4,5} = P_5$ (*the path with* 5 *vertices*).

Proof. The ODC required can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0,0,3,3)$ and $v(G_1) = (0,3,1,0)$. □

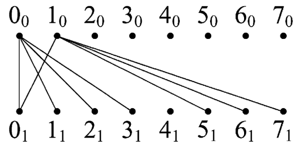

Fig. 5. Graph $G^{8,14}$.

Lemma 20. *There exists an ODC of* $K_{4,4}$ *by* $G^{4,8} = S_2 \cup 2K_2$.

Proof. An ODC of $K_{4,4}$ by $S_2 \cup 2K_2$ can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0,3,1,2)$ and $v(G_1) = (0,2,3,3)$. □

Lemma 21. *There exists an ODC of* $K_{6,6}$ *by* $G^{6,26} = C_6$.

Proof. The pages of the first edge decomposition can be defined as follows. Let $E(G_{0,0}) = \{(0,0),\ (0,1),\ (3,0),\ (3,5),\ (4,1),\ (4,5)\}$, $E(G_{1,0}) = \{(1,1),\ (1,2),\ (3,2),\ (3,4),\ (5,1),\ (5,4)\}$, $E(G_{2,0}) = \{(2,0),\ (2,2),\ (4,0),\ (4,3),\ (5,2),\ (5,3)\}$, $E(G_{3,0}) = \{(0,3),\ (0,4),\ (2,1),\ (2,4),\ (3,1),\ (3,3)\}$, $E(G_{4,0}) = \{(0,2),\ (0,5),\ (1,4),\ (1,5),\ (4,2),\ (4,4)\}$, $E(G_{5,0}) = \{(1,0),\ (1,3),\ (2,3),\ (2,5),\ (5,0),\ (5,5)\}$. While the pages of the second edge decomposition are obtained by taking the symmetric graphs $G_{j,1} = (G_{j,0})_s$ for $j \in \{0,\ldots,5\}$. □

Lemma 22. *There exists an ODC of* $K_{7,7}$ *by* $G^{7,10} = S_3 \cup S_4$.

Proof. For $j \in \mathbb{Z}_7$ let $E(G_{j,0}) = \{(i,j)\colon i \in \{0,1,2,3\}\} \cup \{(i,j+1)\colon i \in \{4,5,6\}\}$. Furthermore, for $j \in \{0,1,2,3\}$ let $E(G_{j,1}) = \{(j,i)\colon i \in \{0,1,2,3\}\} \cup \{((j+1) \bmod 4, i)\colon i \in \{4,5,6\}\}$ and for $j \in \{4,5,6\}$ let $E(G_{j,1}) = \{(j,i)\colon i \in \{0,1,2,3\}\} \cup \{(((j-2) \bmod 3)+4, i)\colon i \in \{4,5,6\}\}$. □

Lemma 23. *There exists an ODC of* $K_{8,8}$ *by* $G^{8,10} = S_3 \cup S_5$.

Proof. The pages of an ODC of $K_{8,8}$ by $S_3 \cup S_5$ can be defined as follows. For $j \in \mathbb{Z}_8$ let $E(G_{j,0}) = \{(i,j)\colon i \in \{0,\ldots,4\}\} \cup \{(i,j+1)\colon i \in \{5,6,7\}\}$. Furthermore, for $j \in \{0,\ldots,4\}$ let $E(G_{j,1}) = \{(j,i)\colon i \in \{0,\ldots,4\}\} \cup \{((j+1) \bmod 5, i)\colon i \in \{5,6,7\}\}$ and for $j \in \{5,6,7\}$ let $E(G_{j,1}) = \{(j,i)\colon i \in \{0,\ldots,4\}\} \cup \{(((j-3) \bmod 3)+5, i)\colon i \in \{5,6,7\}\}$. □

Lemma 24. *There exists an ODC of* $K_{8,8}$ *by* $G^{8,14}$ (*see* Fig. 5).

Proof. An ODC of $K_{8,8}$ by $G^{8,14}$ can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0,0,0,0,1,1,1,1)$ and $v(G_1) = (0,7,6,5,5,4,3,2)$. □

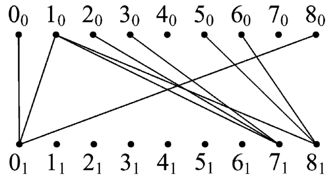

Fig. 6. Graph $G^{9,113}$.

Lemma 25. *There exists an ODC of $K_{9,9}$ by $G^{9,15} = S_4 \cup S_5$.*

Proof. Define the pages as follows. For $j \in \mathbb{Z}_9$ let $E(G_{j,0}) = \{(i,j)\colon i \in \{0,\dots,4\}\} \cup \{(i,j+1)\colon i \in \{5,\dots,8\}\}$. For $j \in \{0,\dots,4\}$ let $E(G_{j,1}) = \{(j,i)\colon i \in \{0,\dots,4\}\} \cup \{((j+1) \bmod 5, i)\colon i \in \{5,\dots,8\}\}$, and for $j \in \{5,\dots,8\}$ let $E(G_{j,1}) = \{(j,i)\colon i \in \{0,\dots,4\}\} \cup \{(((4-j) \bmod 4)+5, i)\colon i \in \{5,\dots,8\}\}$. □

Lemma 26. *There exists an ODC of $K_{9,9}$ by $G^{9,113}$* (*see* Fig. 6).

Proof. An ODC of $K_{9,9}$ by $G^{9,113}$ can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0,8,6,5,3,2,1,1,1)$ and $v(G_1) = (0,0,3,3,6,6,6,3,0)$. □

Appendix.

We list all symmetric starter vectors for $1 \leqslant n \leqslant 9$. The order of the vectors corresponds to the order of the isomorphism classes $L_k(n)$. For brevity, we omit commas inside of a vector. The web site `ftp://ftp.math.uni-rostock.de/pub/members/mgruttm/odc_knn.html` provides additional information, e.g. a graph drawing for each starter vector.

$\mathbf{n=1}$: 1:(0)

$\mathbf{n=2}$: 1:(00) 2:no solution

$\mathbf{n=3}$: 1:(021) 2:(011) 3:direct construction 4:direct construction

$\mathbf{n=4}$: 1:(0321) 2:(0311) 3:direct construction 4:(0022) 5:direct construction 6:(0123) 7:(0232) 8:direct construction 9:(00,10,11,01)

$\mathbf{n=5}$: 1:(04321) 2:(04221) 3:direct construction 4:(03311) 5:(04211) 6:direct construction 7:(04111) 8:(04224) 9:(03121) 10:(03421) 11:(04114) 12:(04232) 13:(03232) 14:(03422) 15:(01314) 16:(01034) 17:direct construction 18:(03142)

$\mathbf{n=6}$: 1:(054321) 2:(054221) 3:direct construction 4:(053311) 5:(014543) 6:(051324) 7:(004422) 8:(044211) 9:(053211) 10:(034125) 11:(052221) 12:(005232) 13:(041321) 14:(045321) 15:(044111) 16:(053111) 17:(052201) 18:(053431) 19:(010441) 20:(042320) 21:(044120) 22:(034225) 23:(055231) 24:(022231) 25:(043101)26:direct construction 27:(015333) 28:(053115) 29:(052132) 30:(022431) 31:(013533) 32:(003532) 33:(034312) 34:(024315) 35:(051215) 36:(001443) 37:(024215) 38:(031220) 39:(034412) 40:(012354) 41:(042353) 42:(034512) 43:no solution

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436:(085313211)	437:(020587542)	438:(015845433)	439:(010587534)
440:(077423370)	441:(077420211)	442:(011845644)	443:(008623428)
444:(057413310)	445:(075314211)	446:(015865413)	447:(057244218)
448:(012087465)	449:(022867755)	450:(028576644)	451:(018475233)
452:(017845623)	453:(005524232)	454:(056252201)	455:(006553038)
456:(086312111)	457:(084252218)	458:(076232327)	459:(030167663)
460:(030868655)	461:(018535343)	462:(086313128)	463:(046451318)
464:(075745337)	465:(005623232)	466:(080745548)	467:(011565544)

468:(070445533)	469:(077312120)	470:(075625232)	471:(085723232)
472:(026067542)	473:(078835343)	474:(004742312)	475:(057232310)
476:(078866501)	477:(075763427)	478:(084745318)	479:(056242301)
480:(078363122)	481:(067353412)	482:(070834433)	483:(083443028)
484:(016834433)	485:(087412217)	486:(075340111)	487:(084612218)
488:(015665243)	489:(011708744)	490:(005846522)	491:(045552281)
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532:(088847534)	533:(005623400)	534:(046351118)	535:(074413307)
536:(078352202)	537:(066525131)	538:(001567343)	539:(074412380)
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548:(030208456)	549:(068346543)	550:(047134281)	551:(064873542)
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576:(042633117)	577:(078486633)	578:(026475534)	579:(074152800)
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584:(046875147)	585:(063725415)	586:(074682531)	587:(013571438)
588:(025144286)	589:(024153278)	590:(072268853)	591:(051638270)
592:(061423875)	593:(034782506)	594:direct construction	595:direct construction

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