On Rings over Which Every Flat Left Module Is Finitely Projective

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INTRODUCTION

In [3], Bass has shown that a ring \( R \) is left perfect if and only if every flat left \( R \)-module is projective and if and only if \( R \) satisfies the descending chain condition on principal right ideals. Shortly thereafter, Björk [4] and Stenström [7, Proposition 5.6, VIII] showed that, in this case, every right \( R \)-module satisfies the descending chain condition on finitely generated submodules. Recently, Azumaya [2] generalized the concept of projectivity of modules to finite projectivity. A module \( M \) is called finitely projective if every epimorphism \( f: N \rightarrow M \) onto \( M \) is finitely split, i.e., for each finitely generated submodule \( M_0 \) of \( M \) there exists a homomorphism \( g: M_0 \rightarrow N \) such that \( gf \) is the identity map of \( M_0 \). By [2, Theorem 24], if every flat left \( R \)-module is finitely projective, then \( R \) satisfies the ascending chain condition for chains of annihilators in \( R \) of the form \( \text{Ann}(a_1) \subset \text{Ann}(a_1, a_2) \subset \text{Ann}(a_1, a_2, a_3) \subset \cdots \), where \( a_1, a_2, a_3, \ldots \) is a sequence of elements of \( R \). The converse was conjectured to be true in [2, Remark 1]. But this is not the case, since the first property on rings is Morita invariant, but the second is not (see Example 8).

The purpose of this article is to give characterizations of rings over which every flat left module is finitely projective. We approach this task by associating with each descending chain \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \) of finitely generated right ideals of \( R \) a multiplicable sequence of finite matrices—a sequence in which each successive pair is multiplicable as matrices in linear algebra—\( A_1, A_2, A_3, \ldots \) in such a fashion that the columns of the product \( A_1 A_2 \cdots A_k \) form a generating system for \( I_k \). If we let \( l(S) \) denote the set \( \{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^{n \times n} : (r_1, r_2, \ldots, r_n)A = 0 \text{ for all } A \in S\} \) for any set \( S \) of \( n \times m \) matrices, then the chain \( \text{Ann}(I_1) \subset \text{Ann}(I_2) \subset \text{Ann}(I_3) \subset \cdots \) of left annihilators in \( R \) terminates if and only if \( l(A_1) \subset l(A_1, A_2) \subset l(A_1, A_2, A_3) \subset \cdots \) terminates. This observation yields our main theorem:

**Theorem 5.** Let \( R \) be a ring and let \( n \) be a positive integer. Then the following are equivalent.
(1) Every flat left $R$-module is n-projective (see Definition 3).

(2) For each descending chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of finitely generated right ideals of $M_n(R)$, the ascending chain $\text{Ann}(I_1) \subseteq \text{Ann}(I_2) \subseteq \text{Ann}(I_3) \subseteq \cdots$ in $M_n(R)$ terminates.

(3) For each multiplicable sequence $A_1, A_2, A_3, \ldots$ of matrices over $R$, where the number of rows of $A_1$ equals $n$, the ascending chain $l(A_1) \subseteq l(A_1A_2) \subseteq l(A_1A_2A_3) \subseteq \cdots$ in $R^n$ terminates.

(4) Every flat left $M_n(R)$-module is singly projective.

In Section 1 we first introduce the notion of a multiplicable sequence of matrices and associate such a sequence with each descending chain of finitely generated right submodules of $R^n$. To illustrate the potential of this notion, we derive directly from Bass's construction the fact for rings whose flat left modules are projective, all right modules satisfy the descending chain condition on finitely generated submodules.

In Section 2 we apply the terminology developed in Section 1 to prove the main theorem and give an example which answers the conjecture of Azumaya in the negative. Also we prove that if $R$ satisfies the ascending chain condition on chains of the form $/(a_1) \supseteq /(a_1a_2) \supseteq /(a_1a_2a_3) \supseteq \cdots$, where $a_1, a_2, a_3, \ldots$ is a sequence of elements of $R$, then every cyclic flat left $R$-module is projective.

Throughout this paper $R$ denotes an associative ring with identity, and all modules are unital.

1. MULTIPLECTABLE SEQUENCES OF MATRICES

Let $A = (a_{ij})_{k \times l}$ and $B = (b_{ij})_{m \times n}$ be two matrices over $R$. When $l = m$ we say that $A$ and $B$ are multiplicable and define the product of $A$ and $B$ to be the $k \times n$ matrix $C = (c_{ij})_{k \times n}$, where $c_{ij} = \sum_{p=1}^{n} a_{ip} b_{pj}$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n$. A sequence $A_1, A_2, A_3, \ldots$ is defined as multiplicable if each successive pair $A_k$ and $A_{k+1}$ are multiplicable.

For each multiplicable sequence $A_1, A_2, A_3, \ldots$ of matrices over $R$ with $A_1$ being an $n \times m$ matrix, we have a descending chain $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$ of finitely generated right $R$-submodules of $R^n$ (treating the elements in $R^n$ as column vectors over $R$) by setting $M_k$ to be the submodule of $R^n$ generated by the columns of $A_1A_2\cdots A_k$ ($k = 1, 2, 3, \ldots$). We call this the chain of submodules defined by the sequence $A_1, A_2, A_3, \ldots$.

Conversely, given any descending chain $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$ of finitely generated right submodules of $R^n$, we can associate with it a multiplicable sequence $A_1, A_2, A_3, \ldots$ of matrices over $R$ in the following way:
If

$$M_1 = \begin{pmatrix} r_{11} \\ \vdots \\ r_{n1} \end{pmatrix} R + \begin{pmatrix} r_{12} \\ \vdots \\ r_{n2} \end{pmatrix} R + \cdots + \begin{pmatrix} r_{1n_1} \\ \vdots \\ r_{nn_1} \end{pmatrix} R$$

we set $A_1 = (r_{ij})_{n \times n}$. Suppose we have found a finite multiplicable sequence $A_1, A_2, \ldots, A_k$ with the property that the columns of $A_1A_2\cdots A_k$ form a generating system for $M_k$, that is, $A_1A_2\cdots A_k = (s_{ij})_{n \times n}$ such that

$$\begin{pmatrix} s_{11} \\ \vdots \\ s_{n1} \\ s_{12} \\ \vdots \\ s_{n2} \\ \vdots \\ s_{1n} \\ \vdots \\ s_{nn} \end{pmatrix} R + \begin{pmatrix} s_{12} \\ \vdots \\ s_{n2} \\ \vdots \\ s_{1n} \\ \vdots \\ s_{nn} \end{pmatrix} R + \cdots + \begin{pmatrix} s_{1n} \\ \vdots \\ s_{nn} \end{pmatrix} R = M_k.$$

If

$$\sum_{i=1}^{n_k} \begin{pmatrix} s_{1i} \\ \vdots \\ s_{ni} \end{pmatrix} t_{i,j}, \quad j = 1, 2, \ldots, n_k+1,$$

is a set of generators of $M_{k+1}$, then define $A_{k+1}$ to be the $n_k \times n_{k+1}$ matrix $(t_{i,j})$. In this way we obtain a multiplicable sequence $A_1, A_2, A_3, \ldots$ which defines the chain $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$. We call this sequence a defining sequence of matrices for the chain $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$.

Note that in general defining sequences of matrices for a chain $M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$ are not unique. However, for any other defining sequence $B_1, B_2, B_3, \ldots$ for this chain, the equality $l(A_1A_2\cdots A_k) = l(B_1B_2\cdots B_k)$ always holds, $k = 1, 2, 3, \ldots$.

Suppose that $A_1, A_2, \ldots, A_k, \ldots$ is an infinite multiplicable sequence of matrices with $A_k$ being an $n_k \times n_{k+1}$ matrix ($k \in \mathbb{N}$). By means of a construction similar to that of Bass's factor modules, we obtain a flat left $R$-module $F/G$ as follows:

Let $F$ be a free left $R$-module with a countably free basis $x_{1,1}, \ldots, x_{n,1}, x_{1,2}, \ldots, x_{n,2}, \ldots, x_{1,k}, \ldots, x_{n,k}, \ldots$. Write

$$X_k = \begin{pmatrix} x_{1k} \\ \vdots \\ x_{nk} \end{pmatrix}, \quad \text{and set} \quad Y_k = X_k - A_k X_{k+1} = \begin{pmatrix} y_{1k} \\ \vdots \\ y_{nk} \end{pmatrix}.$$

Let $G$ be the submodule of $F$ spanned by $y_{1,1}, \ldots, y_{n,1}, y_{1,2}, \ldots, y_{n,2}, \ldots, y_{1,k}, \ldots, y_{n,k}, \ldots$, which form a free basis of $G$ as can easily be seen. If we denote by $G_k$ the submodule of $G$ spanned by $y_{1,1}, \ldots, y_{n,1}, y_{1,2}, \ldots, y_{n,2}, \ldots, y_{1,k}, \ldots, y_{n,k}$, then $G_k$ is a direct summand of $F$, and $G \cap IF = (\bigcup_i G_n) \cap IF = \bigcup_i (G_n \cap IF) = \bigcup_i IG_n = IG$ for each right ideal $I$ of $R$. That
is, \( F/G \) is a flat left \( R \)-module. We call \( F/G \) the left Bass factor module over \( R \) belonging to the sequence \( A_1, A_2, A_3, \ldots \).

Using a method similar to that in [1] we derive the following well-known theorem of Björk and Stenström directly.

**Proposition 1.** If every flat left \( R \)-module is projective, then each right \( R \)-module satisfies the descending chain condition on finitely generated submodules.

**Proof.** It suffices to show the conclusion for the right \( R \)-module \( R^n \), \( n = 1, 2, 3, \ldots \).

Let \( M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots \) be a descending chain of finitely generated right submodules of \( R^n \), and let \( A_1, A_2, A_3, \ldots \) be a defining sequence of matrices for this chain with each \( A_k \) being an \( n_k \times n_{k+1} \) (where \( n_1 = n \)) matrix. Assume that \( F/G \) is the left Bass factor module belonging to \( A_1, A_2, A_3, \ldots \). As we have already observed, \( F/G \) is a flat left \( R \)-module and is thus projective by hypothesis; i.e., \( G \) is a direct summand of \( F \). Let \( \phi: F \to G \) be a projection. For each \( k \in \mathbb{N} \), write

\[
X_k \phi = \begin{pmatrix}
  x_{1,k} \\
  \vdots \\
  x_{n_k,k}
\end{pmatrix}
\]

and note that each \( X_k \phi \) has the form \( X_k \phi = \sum_l C_{k,l} Y_l \) for suitably \( n_k \times n_l \) matrices \( C_{k,l} \) over \( R \) such that \( C_{k,l} = 0 \) for all but finitely many \( l \). Then \( Y_k = Y_k \phi = (X_k - A_k X_{k+1}) \phi = \sum_l (C_{k,l} - A_k C_{k+1,l}) Y_l \), and so

\[
C_{k,l} - A_k C_{k+1,l} = \delta_{k,l} I_{n_k}.
\]

Now for some \( k \in \mathbb{N} \), \( C_{i,n} = 0 \) for all \( n \geq k \). So for each \( n \geq k \),

\[
A_1 A_2 \cdots A_{n-1} A_n (\cdot C_{n+1,n}) = A_1 A_2 \cdots A_{n-1} (I - C_{n,n})
= A_1 A_2 \cdots A_{n-1} - A_1 A_2 \cdots A_{n-1} C_{n,n}
= A_1 A_2 \cdots A_{n-1} - A_1 A_2 \cdots A_{n-2} C_{n-1,n}
\vdots
= A_1 A_2 \cdots A_{n-1} - C_{1,n}
= A_1 A_2 \cdots A_{n-1}.
\]

By construction, the columns of the product \( A_1 A_2 \cdots A_k \) are generators for \( M_k \) \((k \geq 1)\); in this light the above equations yields generators for \( M_{n-1} \) which are linear combinations of generators for \( M_n \), whence \( M_n = M_{n-1} \).
2. The Main Theorem

Before we can state our main theorem we need the results below.

**Definition 2.** Let $L$ and $M$ be left $R$-modules. An epimorphism $f: L \rightarrow M$ is called $n$-split ($n \in \mathbb{N}$) if for every $n$-generated submodule $M_0$ of $M$ there exists a homomorphism $g: M_0 \rightarrow L$ such that $gf$ is the identity map of $M_0$. A left $R$-module $M$ is said to be $n$-projective if every epimorphism onto $M$ is $n$-split.

Clearly, $M$ is finitely projective if and only if $M$ is $n$-projective for each $n \in \mathbb{N}$. Following the proof of [2, Proposition 12] one can easily check that an $R$-module $M$ is $n$-projective if and only if there exist a projective $R$-module $P$ and an $n$-split epimorphism $P \rightarrow M$, if and only if for each $n$-generated submodule $M_0$ of $M$ there exist a projective $R$-module $P$ and two homomorphisms $\phi: M_0 \rightarrow P$, $\psi: P \rightarrow M$ such that $\phi \psi$ is the identity map of $M_0$.

**Lemma 3** (cf. [2, Proposition 22]). Let $F/G$ be the left Bass factor module belonging to a multiplicable sequence $A_1$, $A_2$, $A_3$, ... of finite matrices over $R$. Then the following are equivalent.

1. There exists a homomorphism $\sum_{k=1}^{l} \sum_{j=1}^{n_k} Rx_{j,k} \rightarrow G$ which fixes $(\sum_{k=1}^{l} \sum_{j=1}^{n_k} Rx_{j,k}) \cap G$ elementwise.
2. There exists a homomorphism $\sum_{j=1}^{n} Rx_{j,1} \rightarrow G$ which fixes $(\sum_{j=1}^{n} Rx_{j,1}) \cap G$ elementwise.
3. The ascending chain $l(A_i) \subseteq l(A_i A_{i+1}) \subseteq l(A_i A_{i+1} A_{i+2}) \subseteq \cdots$ in $R^{(n)}(a)$ terminates.

**Lemma 4** [5, Proposition 2.2]. Let $R$ be a ring and $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ be an exact sequence of left $R$-modules, where $F$ is free. Then the following statements are equivalent.

a) $A$ is flat.

b) Given any $u \in K$, there exists a homomorphism $f: F \rightarrow K$ such that $(u)f = u$.

c) Given any $u_1, u_2, ..., u_n$ in $K$, there exists a homomorphism $f: F \rightarrow K$ such that $(u_i)f = u_i$, $i = 1, 2, ..., n$.

We are now ready for the promised theorem.

**Theorem 5.** Let $R$ be a ring and let $n$ be a positive integer. Then the following are equivalent.

1. Every flat left $R$-module is $n$-projective.
(2) For each descending chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of finitely generated right ideals of $M_n(R)$, the ascending chain $\text{Ann}(I_1) \subseteq \text{Ann}(I_2) \subseteq \text{Ann}(I_3) \subseteq \cdots$ of left annihilator ideals in $M_n(R)$ terminates.

(3) For each multiplicable sequence $A_1, A_2, A_3, \ldots$ of matrices over $R$, where the number of rows of $A_1$ equals $n$, the ascending chain $l(A_1) \subseteq l(A_1 A_2) \subseteq l(A_1 A_2 A_3) \subseteq \cdots$ in $R^{(n)}$ terminates.

(4) Every flat left $M_n(R)$-module is singly projective.

Proof. (1) ⇒ (3). Assume that $A_1, A_2, A_3, \ldots$ is a multiplicable sequence of matrices with the number of rows of $A_1$ equal to $n$. Let $F/G$ be the left Bass factor module belonging to it. Then $F/G$ is flat and thus $n$-projective. For the $n$-generated submodule $\sum_{j=1}^n R(x_j + G)$ of $F/G$, there exists a homomorphism $\lambda: \sum_{j=1}^n R(x_j + G) \rightarrow F$ such that $\lambda \pi$ is the identity map of $\sum_{j=1}^n R(x_j + G)$, where $\pi$ is the canonical projection $F \rightarrow F/G$. One can easily check that the homomorphism $\gamma$, setting $x$ in $\sum_{j=1}^n R x_j$ to $x - (x) \pi \lambda$, is a homomorphism into $\mathcal{G}$ which fixes $(\sum_{j=1}^n R x_j) \cap \mathcal{G}$ elementwise. By Lemma 3, it follows that $l(A_1) \subseteq l(A_1 A_2) \subseteq l(A_1 A_2 A_3) \subseteq \cdots$ terminates.

(3) ⇒ (2). Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a descending chain of finitely generated right ideals of $M_n(R)$. Let $A_1, A_2, A_3, \ldots$ be a defining sequence of matrices over $M_n(R)$ for the chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$; note that over $M_n(R)$ the matrix $A_1$ has only one row. If we reinterpret this viewing $R$ as the ring coefficients, we obtain a multiplicable sequence $A_1, A_2, A_3, \ldots$ of matrices over $R$ such that $A_1$ has precisely $n$ rows, and all occurring row and column sizes are multiples of $n$. By (2), $l(A_1) \subseteq l(A_1 A_2) \subseteq l(A_1 A_2 A_3) \subseteq \cdots$ terminates. (Observe that if the product $A_1 A_2 \cdots A_m$ is an $n \times (nk)$ matrix, meaning that it can be written as a row of $k n \times n$ blocks, then these blocks form a generating set for the right ideal $l_m$ of $M_n(R)$.)

Therefore $\text{Ann}(I_1) \subseteq \text{Ann}(I_2) \subseteq \text{Ann}(I_3) \subseteq \cdots$ in $M_n(R)$ terminates.

(2) ⇒ (4). It suffices to show the implication for the case $n = 1$.

Assume that $R$ is a ring in which every ascending chain of the form $\text{Ann}(I_1) \subseteq \text{Ann}(I_2) \subseteq \text{Ann}(I_3) \subseteq \cdots$, where $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a descending chain of finitely generated right ideals of $R$, terminates. Let $M$ be a flat left $R$-module and assume that $0 \neq M_0 = Rx_0$ is a cyclic submodule of $M$. Then there exist a set $J$ and an epimorphism $f: R^{(J)} \rightarrow M$. Take $K = \ker f$ and denote by $e_j$ the element $(\delta_{ij})_{i \in J}$ in $R^{(J)}(i \in J)$. Then $\{e_j\}_{i \in J}$ forms a free basis for $\mathcal{R}^{(J)}$, and we may assume that $(e_{j'}) f = x_0$ for some $j' \in J$. We claim that there exists an element $(k_j)_{j \in J}$ in $K$ such that $l(e_j - (k_j)) = l((e_j - (k_j))(I - (k_{j'})))$ (where $l(-)$ denotes the left annihilator in $R$) for each $(k_{j'}) \in M_{J \times J}(R)$ with the property that $(k_{j'})_{j \in J} \in K$ for each $i \in J$. Assume this is not the case. Then for any $(k_{j'})_{j \in J} \in K$ there exists a $(k_{j'})_{j \in J} \in M_{J \times J}(R)$ of which every row vector (over $R$) $(k_{j'})_{j \in J}$ is in $K$, such
that \( l(e_j - (k_j)) \subseteq l(e_j - (k_j))(I - (k_j)) \). Let \((k^0_{ij}) = 0\) and assume that \((k^1_{ij})\) is a matrix in \( M_{\infty}(R) \) with the property stated above. Since \((e_j - (k^0_{ij}))(I - (k^1_{ij})) = e_j - (k^1_{ij})\) for some \((k^1_{ij})\) in \( K \), our hypothesis yields some \((k^2_{ij})\) in \( M_{\infty}(R) \) such that \( l(e_j - (k^1_{ij})) \subseteq l((e_j - (k^2_{ij}))(I - (k^2_{ij}))) \) and \((k^2_{ij}), i, j \in K\) for each \( i, j \). By repeating this argument we obtain an infinite sequence \((k^n_0), (k^1_0), \ldots, (k^n_0)\) of elements in \( K \) and a sequence \((k^n_{ij})\), \( n = 1, 2, 3, \ldots \) in \( M_{\infty}(R) \) such that \((e_j - (k^n_{ij}))(I - (k^n_{ij} + 1)) = e_j - (k^n_{ij} + 1)\) and \( l(e_j - (k^n_{ij})) \subseteq l(e_j - (k^n_{ij} + 1))\). Since the multiplications of \( e_j - (k^0_{ij}), I - (k^1_{ij}), \ldots, I - (k^n_{ij})\) \((m = 1, 2, \ldots)\) deal with only finitely many nonzero entries, it follows that there exists a multiplicable sequence \( A_0, A_1, A_2, \ldots\) of finite matrices over \( R \) such that \( A_0 \) is a \( 1 \times n \) matrix and \( l(A_0) \subseteq l(A_0 A_1) \subseteq l(A_0 A_1 A_2) \subseteq \cdots \). Let \( I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \) be the descending chain of right ideals of \( R \) defined by \( A_0, A_1, A_2, \ldots \). Then it is easy to see that \( \text{Ann}(I_0) = l(A_0 A_1 \cdots A_k) \) \((k = 0, 1, 2, \ldots)\). Thus \( \text{Ann}(I_0) \supseteq \text{Ann}(I_1) \supseteq \text{Ann}(I_2) \supseteq \cdots \), which contradicts condition (3). Hence there exists an element \((k_{ij})_{i,j} \in K\) in \( K \) such that \( l(e_j - (k_j)) = l((e_j - (k_j))(I - (k_j))) \) for every \((k_j)\) in \( M_{\infty}(R) \) with the property that \((k_{ij})_{i,j} \in K\) for each \( i, j \).

Let \( a \in R \). If \( a \in l(e_j - (k_j)) \), then \( ae_j = a(k_j) \in K \). Conversely, suppose that \( ae_j \in K \). By Lemma 4 there exists an \( R \)-homomorphism \( g: R^J \to K \) such that \((a(e_j - (k_j))) g = a(e_j - (k_j))\). Write \( g = \sum_{i \in J} k_{ij} e_i \) for \( i \in J \). Then \((k_{ij})_{i,j} \in K\) for each \( i \in J \), and the above equation yields that \( a(e_j - (k_j))(I - (k_l)) = 0 \). By the choice of \((k_j)\), we have \( a \in l((e_j - (k_j))(I - (k_l))) \). Thus we have proved that \( a \in l(e_j - (k_j)) \) if and only if \( ae_j \in K \).

Define \( h: M_{\infty}(R) \to R^J \) to be the map which sets \( ax_0 \) to \((a(e_j - (k_j)))\). The previous argument shows that \( h \) is a well-defined homomorphism, and one can easily check that \( hf \) is the identity map of \( M_{\infty} \). Hence \( M \) is a singly projective left \( R \)-module.

(4) \( \Rightarrow \) (1). Let \( H \) be the category equivalence functor \( \text{Hom}_R(R^n, -): R\text{-Mod} \to M_n(R)\text{-Mod} \). Since for each \( n \)-generated \( R \)-module \( M \), \( H(M) \) is a cyclic \( M_n(R) \)-module, the implication is obvious.

**Corollary 6.** Let \( R \) be a ring. Then every flat left \( R \)-module is finitely projective if and only if \( M_n(R) \) \((n \in \mathbb{N})\) satisfies the ascending chain condition for chains of the form \( \text{Ann}(I_1) \subseteq \text{Ann}(I_2) \subseteq \text{Ann}(I_3) \subseteq \cdots \), where \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \) is a descending chain of finitely generated right ideals of \( M_n(R) \), if and only if for every multiplicable sequence \( A_1, A_2, A_3, \ldots \) of finite matrices over \( R \) the ascending chain \( l(A_1) \subseteq l(A_1 A_2) \subseteq l(A_1 A_2 A_3) \subseteq \cdots \) terminates.

**Corollary 7** [2, Note added in proof]. Let \( R \) be a ring. If every flat left \( R \)-module is \( n \)-projective, then the same holds for any subring of \( R \). In particular, every subring of a left perfect ring or a left Noetherian ring has the property that every flat left module is finitely projective.
EXAMPLE 8. Shepherdson [6] has constructed a domain \( R \) such that there exist \( A, B \) in \( M_2(R) \) with \( AB = I \) and \( BA \neq I \). Since \( R \) is a domain, for any nonzero element \( a \) in \( R \), \( \text{Ann}(a) = 0 \); thus \( R \) satisfies the ascending chain condition on chains of the form \( \text{Ann}(a_1) \subseteq \text{Ann}(a_1a_2) \subseteq \text{Ann}(a_1a_2a_3) \subseteq \cdots \), where \( a_1, a_2, a_3, \ldots \) is an infinite sequence of elements of \( R \). But in the ring \( M_2(R) \), there exists an infinite set of orthogonal idempotents \( E_n = B^n(I - BA)A^n \) \((n \in \mathbb{N})\). Let \( A_n \) be the \( 2 \times 2 \) matrix \( I - (E_1 + E_2 + \cdots + E_n) \). It is trivial that the ascending chain \( \text{Ann}(A_1) \subseteq \text{Ann}(A_1A_2) \subseteq \text{Ann}(A_1A_2A_3) \subseteq \cdots \) does not terminate. Thus by Theorem 5 there exists a flat left \( R \)-module which is not \( 2 \)-projective.

The above example shows that satisfying of the ascending chain condition on chains of the form \( \text{Ann}(a_1) \subseteq \text{Ann}(a_1a_2) \subseteq \text{Ann}(a_1a_2a_3) \subseteq \cdots \), where \( a_1, a_2, a_3, \ldots \) is an infinite sequence of elements of \( R \), is not sufficient for the finite projectivity of all flat left \( R \)-modules. However, we can show that it is sufficient for every cyclic flat left \( R \)-module to be projective.

PROPOSITION 9. Let \( R \) be a ring and let \( n \) be a positive integer. If \( M_n(R) \) satisfies the ascending chain condition for chains of the form \( \text{Ann}(A_1) \subseteq \text{Ann}(A_1A_2) \subseteq \text{Ann}(A_1A_2A_3) \subseteq \cdots \), where \( A_1, A_2, A_3, \ldots \) is an infinite sequence of matrices of elements in \( M_n(R) \), then every \( n \)-generated flat left \( R \)-module is projective.

Proof. Using the functor \( \text{Hom}_R(R^{(n)}, -): R\text{-Mod} \to M_n(R)\text{-Mod} \), we reduce the argument to the case \( n = 1 \).

Assume that \( R \) satisfies the ascending chain condition for chains of the form \( \text{Ann}(a_1) \subseteq \text{Ann}(a_1a_2) \subseteq \text{Ann}(a_1a_2a_3) \subseteq \cdots \), where \( a_1, a_2, a_3, \ldots \) is an infinite sequence of elements in \( R \). Let \( I \) be a left ideal of \( R \) such that \( R/I \) is flat. Then by hypothesis we can find an element \( e \) in \( I \) with the property that \( l(1-e) = l((1-e)(1-x)) \) for each \( x \) in \( I \). Obviously we have \( l(1-e) \subseteq I \).

On the other hand, suppose \( y \in I \); since \( R/I \) is flat, \( y(1-e) \subseteq y(1-e)R \cap I = y(1-e)I \), meaning that \( y(1-e) = y(1-e)x \) for some \( x \in I \); it follows that \( y \in l((1-e)(1-x)) = l(1-e) \). Hence \( I = l(1-e) \). Since \( e \in I \), we infer \( e^2 = e \). Thus \( I = Ie = Re \) is a direct summand of \( R \); i.e., \( R/I \) is a projective \( R \)-module.

We complete this section with a discussion on the existence of a certain epimorphism behaving like projective covers onto a module. Given a projective cover \( f: P \to M \) for a left \( R \)-module \( M \), we claim that for any projective module \( Q \) and any epimorphism \( g: Q \to M \), the diagram

\[
P \longrightarrow M \\
\downarrow h \downarrow \quad \downarrow g \\
\quad Q
\]

exists.
can be completed only by monomorphism. For, if $h$ is such a homomorphism, then by the projectivity of $Q$ there exists a homomorphism $h': Q \rightarrow P$ such that $h'f = g$; thus $hh'f = f$; i.e., $\text{Im}(1 - hh') \subseteq \ker \ll P$; so $1 - hh'$ is in the Jacobson radical of $\text{End}_R P$ and $hh' = 1 - (1 - hh')$ is invertible in $\text{End}_R P$, and hence $h$ is a monomorphism.

In connection with the previous observation, the following may be of interest.

**Proposition 10.** Let $R$ be an $n$-hereditary ring such that every flat left $R$-module is $n$-projective. Then for every $n$-generated left $R$-module $M$ there exist a projective module $P$ and an epimorphism $f: P \rightarrow M$ such that for any projective left $R$-module $Q$ and each epimorphism $g: Q \rightarrow M$ the diagram

$$\begin{array}{ccc}
P & \xrightarrow{f} & M \\
\downarrow{h} & & \\
Q & \xrightarrow{g} & \\
\end{array}$$

can be completed only by monomorphism $h$.

**Proof.** Let $R^{(n)} \rightarrow M$ be an epimorphism. We first claim that there exist a projective left $R$-module $P$ and epimorphisms $f: P \rightarrow M$, $h: R^{(n)} \rightarrow P$ completing the commutative diagram

$$\begin{array}{ccc}
R^{(n)} & \xrightarrow{h} & M \\
\downarrow{f} & & \\
P & \xrightarrow{} & \\
\end{array}$$

with the property that for any epimorphisms completing the commutative diagram

$$\begin{array}{ccc}
R^{(n)} & \xrightarrow{h} & M \\
\downarrow{h_1} & & \\
P_1 & \xrightarrow{h} & \\
\end{array}$$

where $P_1$ is a projective left $R$-module, $\ker hh_1 = \ker h$. Suppose the conclusion were not true. Then we have a commutative diagram
where each $P_i (i \in \mathbb{N})$ is projective and all the homomorphisms in the diagram are epimorphisms, such that $\ker h_1 h_2 \cdots h_k \subseteq \ker h_1 h_2 \cdots h_k h_{k+1}$, $k = 1, 2, \ldots$. Assume that $P_k \oplus Q_k = R^{(n_k)}$ and treat elements of $R^{(n_k)}$ as row vectors over $R$. Let $h'_i: R^{(n_k)} \rightarrow R^{(n_k)}$ be extensions of $h_i$ ($n_0 = n$, $k = 1, 2, \ldots$). Observe that each $h'_i$ corresponds to an $n_k \times n_k$ matrix $A_k$ such that $(x)h'_i = xA_k$ for $x = (r_1, r_2, \ldots, r_n) \in R^{(n_k)}$. It follows that $\ker h'_1 h'_2 \cdots h'_k = l(A_1 A_2 \cdots A_k)$, $k = 1, 2, 3, \ldots$. From the assumption on $h_k$ we conclude that the chain $l(A_1) \subseteq l(A_1 A_2) \subseteq l(A_1 A_2 A_3) \subseteq \cdots$ does not terminate, contrary to (3) of Theorem 5.

Let $P$ and $h: R^{(n)} \rightarrow P$ be as in our claim. Assume that $Q$ is a projective left $R$-module and $h_1, g$ are homomorphisms completing the diagram

Since $R$ is $n$-hereditary, the $n$-generated submodule $\text{Im} h_1$ of $Q$ is a projective module; by our choice of $h$ we have $\ker hh_1 = \ker h$. Note that $h$ is an epimorphism, thus $h_1$ is a monomorphism. Thus $f: P \rightarrow M$ has the property required in Proposition 10.

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