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# Computing the Sullivan Milnor–Moore S.S. and the rational LS category of certain spaces

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**Abstract**

Let $(\Lambda V, d)$ be the Sullivan model of an elliptic space $S$ and $(\Lambda V, d_\sigma)$ be the associated pure model. We give an algorithm, based on Groebner basis computations, that computes the stage $l_\sigma = l_0(\Lambda V, d_\sigma)$ at which the (Sullivan version of the) Milnor–Moore spectral sequence of $(\Lambda V, d_\sigma)$ collapses. When $(d - d_\sigma)V \subset \Lambda^{>l_\sigma} V$ we call $S$ a Ginsburg space. We show that the rational LS category of any Ginsburg space $S$, $\mathrm{cat}_0(\Lambda V, d)$, coincides with that of the associated pure space $\mathrm{cat}_0(\Lambda V, d_\sigma)$. A previous algorithm due to the author computes $\mathrm{cat}_0(\Lambda V, d_\sigma)$. So we obtain an algorithm that determines whether a space is Ginsburg and which in this case computes its rational LS category.


*Keywords:* Rational homotopy; Groebner basis; LS category

## 1. Introduction

The Lusternik–Schirelmann category [13] of a space is the least number of open sets, less one, which cover and are contractible in it. It is an important invariant which for a manifold gives a lower bound for the number of critical points any function must have. The computation of the LS category of spaces is a very subtle matter in general. Even when very explicit data, such as Sullivan minimal models, are available, its determination remains difficult and much effort has been spent in the last 20 years in the pursuit of good estimates for the LS category. In this paper we first prove a theorem which provides an algorithm for computing the Ginsburg invariant, another measure of the complexity of a

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space which is a lower bound for the LS category. Then, we show that for spaces (we call them Ginsburg spaces), that are elliptic and whose minimal model satisfies certain conditions (that are easy to check once has a Sullivan model in hand), the rational LS category of these spaces may be computed via a much simpler model, its "associated pure" model. A previous algorithm due to this author computes the rational category of any pure elliptic space. So we obtain an algorithm that determines whether a space is Ginsburg and which in this case computes its rational LS category.

## 2. Basic facts

Ours results rely heavily on the algebraic machinery of Sullivan models and on Groebner basis theory. We recall here some basic facts and notation we shall need from Sullivan's theory of minimal models, for which [6,12,18] are standard references.

The (Sullivan) minimal model of $S$ is a commutative graded differential algebra $(\Lambda V, d)$ over the rational field which algebraically models the rational homotopy type of $S$. We denote by $\Lambda V$ the tensor product of the exterior algebra on $V^{\mathrm{odd}}$, the elements of odd degree, and the symmetric algebra on $V^{\mathrm{even}}$, the elements of even degree. The differential is a graded derivation which satisfies $d^2 = 0$ and $d(V) \subset \Lambda^{\geqslant 2} V$, where for any $k$, $\Lambda^k V$ is the subspace of all products of length $k$ of elements of $V$.

The algebra generators of the minimal model are identified, as a graded vector space, with the rational homotopy groups of the space. Moreover, the cohomology of the minimal model is isomorphic to that of the space.

A simply connected space $S$ such that $\dim H^*(S; \mathbb{Q}) < \infty$ is called rationally elliptic if $\dim \pi_*(S) \otimes \mathbb{Q} < \infty$, otherwise $S$ is called rationally hyperbolic. For an elliptic space with model $(\Lambda V, d)$ the formal dimension $N$, i.e., the largest $n$ for which $H^n(\Lambda V, d) \neq 0$, is given by [11, p. 188]

$$N = \dim V^{\mathrm{even}} - \sum_{i=1}^{\dim V} (-1)^{|x_i|} |x_i|.$$

An element $0 \neq w \in H^N(\Lambda V, d)$ is called a fundamental or top class.

**Definition 1.** We define the length of $\alpha$ to be $l(\alpha) = \max\{k \mid \alpha \in \Lambda^{\geqslant k} V\}$.

Consider a general Sullivan algebra $(\Lambda V, d)$ in which $d = d_0 + d_1 + \cdots$, with $d_i : V \to \Lambda^{i+1} V$. Filter $(\Lambda V, d)$ by the decreasing sequence of ideals $F^p = \Lambda^{\geqslant p} V$ and set $F^0 = \Lambda V$. This is called the word length filtration. It determines a first quadrant spectral sequence $(E_i, d_i)$, that is called the Milnor–Moore spectral sequence of the Sullivan algebra.

We recall the Sullivan version of the invariant $l_0 = l_0(\Lambda V, d)$ introduced by Ginsburg [10]. Let $(E_i, d_i)$ be the Milnor–Moore spectral sequence for $(\Lambda V, d)$, arising from the filtration $\Lambda^{\geqslant k} V$ of $\Lambda V$. Then $l_0(\Lambda V, d) = \max\{j \mid d_j \neq 0\}$ where $d_j$ is the $j$th differential in the Milnor–Moore spectral sequence. It is an easy exercise in spectral sequences to prove the following

**Lemma 2.** *$l_0(\Lambda V, d)$ is the least integer $l$ such that for any coboundary $f$ there exists $b$ with $db = f$ and $l(b) \geqslant l(f) - l$.*

The Lusternik–Schirelmann category, $\mathrm{cat}_0(S)$, of a topological space $S$ is the least integer $m$ such that $S$ is the union of $m+1$ open sets, each contractible in $S$. If $S$ is a simple connected CW complex, the rational LS category, $\mathrm{cat}_0(S)$, introduced by Félix and Halperin in [5], satisfies $\mathrm{cat}_0(S) = \mathrm{cat}_0(S_{\mathbb{Q}}) \leqslant \mathrm{cat}_0(S)$.

We recall the following Sullivan algebra version of a theorem of Ginsburg [10], for which a simple proof was later given by Ganea [8] and by Jessup [9]. Suppose $(\Lambda V, d)$ is a minimal Sullivan algebra and $V = \{V^i\}_{i \geqslant 2}$. If $\mathrm{cat}_0(\Lambda V, d) = m$ then $l_0(\Lambda V, d) \leqslant \mathrm{cat}_0(\Lambda V, d)$.

Let $S$ a simply connected (rationally) elliptic space and $(\Lambda V, d)$ be its minimal model. If $(E_i, d_i)$ is the Milnor–Moore spectral sequence of $(\Lambda V, d)$ then by [5], the rational Toomer invariant $\mathrm{e}_0(\Lambda V, d)$ is the largest $p$ such that $E_r^{p,*} \neq 0$. By [5, Lemma 10.1] $\mathrm{e}_0(\Lambda V, d)$ is the largest integer $k$ such that the top class can be represented by a cocycle in $\Lambda^{\geqslant k} V$. In [7, Theorem 3] it is proven that $\mathrm{cat}_0(S) = \mathrm{e}_0(S)$. Hence $\mathrm{cat}_0(S) = \mathrm{cat}_0(\Lambda V, d) = \sup\{l(w) \mid [w]$ is a top class of $(\Lambda V, d)\}$.

### *2.1. Pure spaces*

Henceforth, if $S$ is a space with minimal model $(\Lambda V, d)$ we shall denote $X = V^{\mathrm{even}}$, $Y = V^{\mathrm{odd}}$, $n = \dim X$, $m = \dim Y$. The integer $\chi_\pi = n - m$ is called the Euler homotopy characteristic of $S$, and $\sum_i (-1)^i \dim(H^i(S;\mathbb{Q}))$ is the Euler characteristic of $S$.

A pure space $S$ is a space whose minimal model $(\Lambda V, d) = \Lambda X \otimes \Lambda Y$ satisfies $dX = 0$ and $dY \subset \Lambda X$. Spheres and compact homogeneous spaces are examples of pure spaces. If $\dim V < \infty$ then $S$ is called a finite pure space. We shall henceforth also use the terms "pure" and "elliptic" when referring to a minimal model of such a space.

A bigradation on $(\Lambda V, d)$ is given by $\Lambda V = \sum_{n, j \geqslant 0} (\Lambda_j V)^n$ where $(\Lambda_j V)^n = (\Lambda X \otimes \Lambda^j Y)^n$. When $(\Lambda V, d)$ is pure, $d(\Lambda_j V)^n \subset (\Lambda_{j-1} V)^{n+1}$, the differential $d$ has bidegree $(1, -1)$ and this induces a bigradation in cohomology.

The following is proved in [11]. If $(\Lambda V, d)$ is pure and elliptic then $H(\Lambda V, d)$ is a Poincaré duality algebra and if $k = -\chi_\pi$, then it is verified that $H_k(\Lambda V) \neq 0$ and $H_{k+p}(\Lambda V) = 0$, $p \geqslant 1$. Hence, if $n = m$ then $H(\Lambda V, d) = H_0(\Lambda V, d)$. As an immediate consequence of these properties we obtain:

**Lemma 3.** *Let $(\Lambda V, d)$ be a pure elliptic space. Then there is a cocycle $w_1$ in $\Lambda_{m-n} V$ that represents the top class and such that $w_1 \in \Lambda^{\geqslant k} V$ with $k = \mathrm{cat}_0(\Lambda V, d)$.*

Let $d_\sigma$ be the linear map defined by $d_\sigma X = 0$, $d_\sigma Y \subset \Lambda X$ and such that $dv - d_\sigma v \in \Lambda^+ Y \otimes \Lambda X$ for $v \in Y$. If we extend $d_\sigma$ to a derivation of $(\Lambda V, d)$, then $d_\sigma^2 = 0$ and $(\Lambda V, d_\sigma)$ is then called the associated pure model for $(\Lambda V, d)$. The odd spectral sequence is obtained from a filtration of $(\Lambda V, d)$ by $F^{p,q} = \sum_{j+q \geqslant 0} \Lambda_j^{p+q}$. This defines a spectral sequence of algebras of the first and second quadrant.

**Proposition 4** [11]. *Let $(E_i, d_i)$ be the odd spectral sequence for $(\Lambda V, d)$. Then,*

(i) *$(E_i, d_i)$ converges to $H^*(\Lambda V, d)$,*
(ii) *its $(E_0, d_0)$-term is precisely $(\Lambda V, d_\sigma)$,*
(iii) *each $(E_i, d_i)$ is a Poincaré duality algebra,*
(iv) *each $(E_i, d_i)$ and $(\Lambda V, d)$ have the same formal dimension, and finally,*
(v) $\dim H(\Lambda V, d) < \infty \Leftrightarrow \dim H(\Lambda V, d_\sigma) < \infty$.

In [17] there is a formula for computing a cocycle representing the fundamental class of a pure elliptic space $(\Lambda X \otimes \Lambda Y, d)$. A slight modification of this formula gives the following algorithm.

**Proposition 5** [16]. *Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ be homogeneous bases of $X$ and $Y$ respectively and let $\overline{X} = sX$ denote the suspension of $X$ with $d\bar{x}_i = x_i$. Choose elements $\Psi_j \in \Lambda X \otimes \Lambda^1 \overline{X}$ for which $d\Psi_j = dy_j$, $j = 1, \dots, m$. If $w$ is the coefficient of*

$$\prod_{i=1}^{n} \bar{x}_i \quad \text{in the development of} \quad \prod_{j=1}^{m} (y_j - \Psi_j),$$

*then $w$ is a cocycle in $\Lambda_{m-n} V$ that represents the fundamental class of $(\Lambda V, d)$.*

Observe that to construct $\Psi_j$ it suffices to replace in each term of $dy_j$ one $x_k \in \{x_1, \dots, x_n\}$ by its suspension $\bar{x}_k$.

*2.2. Groebner bases for ideals*

Here we recall some standard facts and definitions on Groebner bases for which [1,4] are standard references.

First, we recall that the set of monomials in $\Lambda X = \mathbb{Q}[x_1, \dots, x_n]$ is denoted by $\mathbb{T}^n = \{x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \mid \beta_i \in \mathbb{N},\ i = 1, \dots, n\}$.

**Definition 6.** By a term order on $\mathbb{T}^n$ we mean a total order $\leqslant$ on $\mathbb{T}^n$ satisfying the following conditions:

(1) $1 \leqslant x^\alpha$ for all $\alpha \in \mathbb{N}^n$.
(2) if $x^\alpha \leqslant x^\beta$ then $x^\alpha \cdot x^\gamma \leqslant x^\beta \cdot x^\gamma$, for all $\gamma \in \mathbb{N}^n$.

The total degree of $x^\beta \in \mathbb{T}^n$ is $\|\beta\| = \sum_{i=1}^n \beta_i$ and we will write $\mathrm{hdeg}(f)$ for the homological degree of a homogeneous element $f \in (\Lambda V, d)$.

**Definition 7.** The graded lexicographical order $\leqslant_{\mathrm{glex}}$ on $\mathbb{T}^n$ with $x_1 > x_2 > \cdots > x_n$ is defined by $x^\alpha \leqslant_{\mathrm{glex}} x^\beta$ if and only if $\alpha = \beta$ or $\|\alpha\| < \|\beta\|$ or $\|\alpha\| = \|\beta\|$ and $\alpha_i < \beta_i$ for the first $i$ with $\alpha_i \neq \beta_i$.

The bijection $\alpha \to x^\alpha$ show that a term order (say) "$\leqslant_1$" induces a compatible total order on $\mathbb{N}^n$ via $\alpha \leqslant \beta$ iff $x^\alpha \leqslant_1 x^\beta$.

**Definition 8.** Let $f = \sum_{\alpha \in A} a_\alpha x^\alpha \in \mathbb{Q}[x_1, \ldots, x_n]$, with $\forall \alpha \in A,\ a_\alpha \neq 0$, and let $\leqslant$ be a total order on $\mathbb{N}^n$. Then

(1) The total degree of $f$ is $\operatorname{tdeg}(f) = \max_\alpha(\|\alpha\|)$.
(2) The multidegree of $f$ is $\operatorname{multideg}(f) = \max(\alpha,\ \alpha \in A)$.
(3) The leading coefficient of $f$ is $\operatorname{lc}(f) = a_{\operatorname{multideg}(f)} \in \mathbb{K}$.
(4) The leading monomial of $f$ is $\operatorname{lm}(f) = x^{\operatorname{multideg}(f)}$.
(5) The leading term of $f$ is $\operatorname{lt}(f) = \operatorname{lc}(f) \cdot \operatorname{lm}(f)$.

**Definition 9.** Fix a term order. Given $f, g, h$ in $\mathbb{Q}[x_1, \ldots, x_n]$ with $g \neq 0$, we say that $f$ reduces to $h$ modulo $g$ in one step, written $f \xrightarrow{g} h$, if and only if $\operatorname{lt}(g)$ divides a non-zero term $Z$ that appears in $f$ and $h = f - \frac{Z}{\operatorname{lt}(g)} g$.

Let $f, h$, and $f_1, \ldots, f_s$ be polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$, with $f_i \neq 0$, and let $F = \{f_1, \ldots, f_s\}$. Fix a term order, we say that $f$ reduces to $h$ modulo $F$, denoted $f \xrightarrow{F}_+ h$, if and only if there exists a sequence of indices $i_1, i_2, \ldots, i_t \in \{1, \ldots, s\}$ and a sequence of polynomials $h_1, \ldots, h_{t-1}$ such that

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h.$$

A polynomial $r$ is called reduced with respect to a set of non-zero polynomials $F = \{f_1, \ldots, f_s\}$ if $r = 0$ or no monomial that appears in $r$ is divisible by any one of the $\operatorname{lm}(f_i)$, $i = 1, \ldots, s$.

If $f \xrightarrow{F}_+ r$ and $r$ is reduced with respect to $F$, then we call $r$ a remainder for $f$ with respect to $F$. Note that $r$ is not unique in general. The reduction process allows us to define a division algorithm that mimics the usual division algorithm in one variable. Given $f$ and a family of non-zero polynomials $\{f_i \in \mathbb{Q}[x_1, \ldots, x_n] \mid f_i \neq 0\}_{i=1}^s$, this algorithm returns quotients $u_1, \ldots, u_s \in \mathbb{Q}[x_1, \ldots, x_n]$ and a remainder $r \in \mathbb{Q}[x_1, \ldots, x_n]$, such that $f = u_1 f_1 + \cdots + u_s f_s + r$. We shall call $r$ a remainder of $f$ after division by $\{f_i\}$.

For a subset $A$ of $\mathbb{Q}[x_1, \ldots, x_n]$, we define the leading term ideal of $A$ to be the ideal $\operatorname{lt}(A) = \langle \operatorname{lt}(a \mid a \in A) \rangle$, where $\langle B \rangle$ denotes the ideal generated by the set $B$. We recall that a set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal $I$, is called a Groebner (or standard) basis for $I$ if and only if $\operatorname{lt}(G) = \operatorname{lt}(I)$. A set $G$ of non-zero polynomials is called a Groebner or standard basis if it is a Groebner basis of $\langle G \rangle$. If $f$ is a polynomial in $\mathbb{Q}[x_1, \ldots, x_n]$ and $G$ is a Groebner basis for some ideal, then the remainder of $f$ after division by $G$ is unique. It is called the normal form of $f$ with respect to $G$ and we denote it by $\mathrm{NF}_G(f)$. Moreover, [1, Theorem 1.6.2] a crucial property of the reduction is that $\operatorname{lm}(\mathrm{NF}_G(f)) \leqslant \operatorname{lt}(f)$. If $G$ is a Groebner basis and $f \in \langle G \rangle$, the division algorithm provides an expansion in the form $f = \sum_{i=1}^t u_i g_i$ with $\operatorname{lm}(f) = \max_{1 \leqslant i \leqslant t}(\operatorname{lm}(u_i) \cdot \operatorname{lm}(g_i))$. When for all $g_i \in G$, $\operatorname{lc}(g_i) = 1$ and $g_i$ is reduced with respect to $G \setminus \{g_i\}$, we call $G$ a reduced Groebner basis.

*2.3. Groebner bases for modules*

For $S$ a pure space with Sullivan model $\Lambda V = \Lambda X \otimes \Lambda Y$, we regard $\Lambda V$ as a module over $\Lambda X$. We fix the basis $\{e_i \mid 1 \leqslant i \leqslant 2^m\}$ of $\Lambda V$ consisting of all non-zero products of elements in $\{1, y_1, \ldots, y_m\}$. Because $dX = 0$, we observe that the vector spaces of $d$-coboundaries and $d$-cocycles are both $\Lambda X$-submodules of $\Lambda V$. By Hilbert basis theorem, $\Lambda X$ and $\Lambda V$ are Noetherian. By a monomial in $\Lambda V$ we mean a vector of the type $Ze_i$ $(1 \leqslant i \leqslant q)$ where $Z$ is a monomial in $\Lambda X$. If $Re_i$ and $Ze_j$ are monomials in $\Lambda V$, we say that $Re_i$ divides $Ze_j$ provided that $i = j$ and $R$ divides $Z$. In this case we define $\frac{Ze_i}{Re_i} = \frac{Z}{R}$. Similarly, by a term, we mean a vector of the type $cM$ where $c \in k$, and $M$ is a monomial.

**Definition 10.** By a term order on the monomials of $\Lambda V$ we mean a total order $\leqslant$ on these monomials satisfying the following two conditions:

(i) $M < ZM$, for every monomial $M$ of $\Lambda V$ and monomial $Z \neq 1$ of $\Lambda X$.
(ii) If $M < N$, then $ZM < ZN$ for all monomials $M, N \in \Lambda V$ and every monomial $Z \in \Lambda X$.

Fix a term order $\leqslant$ on the monomials of $\Lambda V$. Then for all $f \in \Lambda V$, with $f \neq 0$, we may write $f = a_1M_1 + a_2M_2 + \cdots + a_rM_r$, where $a_i \neq 0$ for $1 \leqslant i \leqslant r$ are scalars and the $M_i$ are monomials in $\Lambda V$ satisfying $M_1 > M_2 > \cdots > M_r$. We recall that $\mathrm{lm}(f) = M_1$ is the leading monomial, $\mathrm{lc}(f) = a_1$ is the leading coefficient of $f$, and $\mathrm{lt}(f) = a_1M_1$ is the leading term of $f$. We define $\mathrm{lt}(0) = 0$, $\mathrm{lm}(0) = 0$, and $\mathrm{lc}(0) = 0$.

For a subset $W$ of $\Lambda V$, the leading term module of $W$ is the submodule of $\Lambda V$ given by $\mathrm{lt}(W) = \langle \mathrm{lt}(w) \mid w \in W \rangle$, where $\langle B \rangle$ here denotes the $\Lambda X$ module generated by $B$.

A set on non-zero vectors $G = \{g_1, g_2, \ldots, g_t\}$ contained in the submodule $\Lambda V$ is called a Groebner basis for $\Lambda V$ if and only if $\mathrm{lt}(G) = \mathrm{lt}(\Lambda V)$. We say that the set $G$ is a Groebner basis provided $G$ is a Groebner basis for the submodule, $\langle G \rangle$, it generates. The definitions of: reduction, reduced, and the division algorithm are word for word as above.

**Definition 11.** For monomials $M = Re_i$ and $N = Ze_j$ of $\Lambda V$, we define $M \leqslant_{\mathrm{top}} N$ iff $M = N$ or $\mathrm{hdeg}\, M < \mathrm{hdeg}\, N$ or $\mathrm{hdeg}(M) = \mathrm{hdeg}(N)$ and $Z <_{\mathrm{glex}} R$ or $(R = Z$ and $i < j)$.

That is, in the order $\leqslant_{\mathrm{top}}$, we order first by homological degree, then we refine this ordering by the opposite of the graded lexicographic order and finally we refine this ordering so that $Re_i < Re_j$ when $i < j$. Clearly, the $\leqslant_{\mathrm{top}}$ order is a term order.

The proof of the following are straightforward.

**Lemma 12.** *Let $G \subset \Lambda V$ be a Groebner basis with respect to $\leqslant_{\mathrm{top}}$. If $\lambda \in \Lambda X$ and $v, w \in (\Lambda_p V)^q$ for some $p$ and $q$, then it is verified that*

(i) $l(v) = l(\mathrm{lt}(v))$.
(ii) $l(\lambda \cdot v) = l(\lambda) + l(v)$.
(iii) *if* $\mathrm{lt}(v) = Ze_j$, *with* $Z \in \Lambda X$ *and* $e_j \in \Lambda_p V$ *then* $l(a) = \|Z\| + p$.

(iv) $\mathrm{lt}(v) \leqslant_{\mathrm{top}} \mathrm{lt}(w)$ *then* $l(v) \geqslant l(w)$.

(v) $(v \xrightarrow{G}_{+} v_0) \Longrightarrow l(v) \leqslant l(v_0)$.

**Theorem 13.** *Let $(\Lambda V, d)$ be a pure elliptic space and $G_1 = \{g_1, \ldots, g_k\}$ be the reduced Groebner basis of the module of boundaries $B$ with respect to $\leqslant_{\mathrm{top}}$. Consider $A = \{\alpha_1, \ldots, \alpha_k\}$ where $\alpha_i$ satisfies $d(\alpha_i) = g_i$. Let $G_2$ be the reduced Groebner basis of module of $d$-cocycles with respect to $\leqslant_{\mathrm{top}}$. Now let $\Gamma = \{\gamma_i = \mathrm{NF}_{G_2}(\alpha_i)\} \mid i = 1, \ldots, k\}$. Then the rational Ginsburg invariant of $(\Lambda V, d)$ is given by $l_0(\Lambda V, d) = \max\{l(g_i) - l(\gamma_i)\colon i = 1, \ldots, k\}$.*

**Proof.** Observe that the graduation of $(\Lambda V, d)$ by lower degree induces a graduation in both $\ker d$ and $\mathrm{Im}\, d$. This implies that the elements in $G_1$ and those in $G_2$ are homogeneous with respect to the lower degree.

Denote $l_0 = l_0(\Lambda V, d)$ and let $q \in \{1, \ldots, k\}$ be such that $t = l(g_q) - l(\gamma_q) = \max\{l(g_i) - l(\gamma_i)\}_{i=1}^{k}$. By Lemma 2 there exists $\beta$ such that $d(\beta) = g_q$ and $l(\beta) \geqslant l(g_q) - l_0$. Since $\gamma_q - \alpha_q$ is a cocycle, we have $d\gamma_q = d\alpha_q = g_q = d\beta$ so that $\gamma_q - \beta$ is a cocycle. Hence $\mathrm{NF}_{G_2}(\beta) = \mathrm{NF}_{G_2}(\gamma_q) = \gamma_q$. Thus, $\mathrm{lt}(\gamma_q) \leqslant_{\mathrm{top}} \mathrm{lt}(\beta)$ and by Lemma 12, $l(\gamma_q) \geqslant l(\beta) \geqslant l(g_q) - l_0$. So that $l_0 \geqslant l(g_q) - l(\gamma_q)$. This proves $l_0 \geqslant t$.

We proceed to prove the reverse inequality. Let $f$ be a coboundary. Recall that the Groebner basis division algorithm provides $f = \sum_{j \in J} \lambda_j g_j$, with $\mathrm{lt}(f) = \max_j\{\mathrm{lt}(\lambda_j g_j)\}$. By Lemma 12, for each $j \in J$ it is verified that $l(\lambda_j g_j) = l(\lambda_j) + l(g_j) \geqslant l(f)$. Subtracting $l(g_j) - l(\gamma_j)$ from both sides we obtain $l(\lambda_j \gamma_j) \geqslant l(f) - (l(g_j) - l(\gamma_j))$. Thus, $u = \sum_{j \in J} \lambda_j \gamma_j$ satisfies $l(u) \geqslant l(f) - t$ and $du = f$. By Lemma 2 this proves $t \geqslant l_0$. We conclude that $l_0 = t = \max\{l(g_i) - l(\gamma_i) \mid 1 \leqslant i \leqslant k\}$ as claimed. □

As an immediate consequence we obtain:

**Proposition 14.** *Let $(\Lambda V, d)$ be a pure elliptic model, then the following algorithm yields $l_o(\Lambda V, d)$.*

For each $j = 0, \ldots, m - 1$.
Compute $S_j = \{d(y_{i_1} \cdots y_{i_{j+1}})\colon 1 \leqslant i_1 < \cdots < i_{j+1} \leqslant m\}$.
Apply Buchberger's algorithm to obtain a Groebner basis $G_{1j}$ for $S_j$ with respect to $\leqslant_{\mathrm{top}}$.
Compute (by standard techniques as in [1, Proposition 3.7.2 and Algorithm 3.5.2]) the syzygy module $T_j$ of $\{d(y_{i_1} \cdots y_{i_j})\colon 1 \leqslant i_1 < \cdots < i_j \leqslant m\}$.
Compute a Groebner basis $G_{2j}$ of $T_j$ with respect to $\leqslant_{\mathrm{top}}$.
For each $g_{ij} \in G_{1j}$ apply the division algorithm to obtain $\alpha_{ij}$ such that $d\alpha_{ij} = g_{ij}$.
For each $\alpha_{ij}$ compute the normal form $\gamma_{ij} = \mathrm{NF}_{G_2}(\alpha_{ij})$.
Compute $t_j = \max\{l(g_{ij}) - l(\gamma_{ij})\}$. Then $t = \max\{t_j\colon j = 0, \ldots, m - 1\}$ is the rational Ginsburg invariant of $(\Lambda V, d)$.

We recall [14] that if $\dim X = \dim Y$ then $\mathrm{cat}_0(\Lambda V, d)$ is the index of nilpotency [19] of $I = \langle dy_1, \ldots, dy_n\rangle$ minus one. The following example is a version of Kollar [2] and shows that on a model with $dV \subset \Lambda^{\leqslant 4} V$, the rational category of $(\Lambda V, d)$ can grow exponentially in $\dim X$. And it suggests that upper bounds for $l_0(\Lambda V, d)$ based on $l(dy_i)$ should be very close to upper bounds on $\mathrm{cat}_0(\Lambda V, d)$ based on $l(dy_i)$, that is upper bounds for $l_0(\Lambda V, d)$ should grow exponentially in $\dim X$.

**Example 15.** Let $(\Lambda V, d)$ be such that $X = \langle x_1, x_2, x_3, x_4, x_5\rangle$, with $|x_i| = 64 \times 2^{-i}$ and $Y = \langle y_1, y_2, y_3, y_4, y_5\rangle$, with the differential given by:

$$dy_1 = x_1^3, \qquad dy_2 = x_1^2 - x_2^4, \qquad dy_3 = x_2^2 - x_3^4,$$
$$dy_4 = x_3^2 - x_4^4, \qquad dy_5 = x_4^2 - x_5^4.$$

We illustrate the use of a computer algebra program such as CoCoA [3] to perform the computation of $l_0(\Lambda V, d)$. The following program, written in the CoCoA programming language, computes Ginsburg's invariant of any pure elliptic space. The input to this program are the values of $NX = \dim X$ and $Dyi = [dy_1, \ldots, dy_m]$.

```
NX:=5;
Use R:= Q[x[1..NX], Ord(-DegLexMat(NX)),ToPos;
Dyi := [x[1]^3, x[1]^2 - x[2]^4, x[2]^2 - x[3]^4,
x[3]^2 - x[4]^4,x[4]^2 - x[5]^4];

I := Ideal(ListOfDyi);
MEMORY.I :=I;
MM := Len(Gens(I));

Define D(Q)
F:= Gens(MEMORY.I);
M:= Len(F);
S:= NewVector(2^M);
Signus:=1;
P:=Q-1;
For I:=M -1 To 0 Step -1 Do
If P >= 2^I Then
P:= P - 2^I; S:= S + Signus *F[I+1] * E_(Q-2^I,2^M);
Signus:= - Signus;
End;
End;
Return(S);
End;

ListD := [D(X) | X In 2..(2^MM - 1)];
Boundary := Module(ListD);
G1 := GBasis(Boundary);
G2 := Syz(ListD);
Alpha := [0 | I In 1..(Len(G1))];
Gamma := Alpha;
```

```
For K:=1 To Len(G1) Do
Alpha[K] := GenRepr(G1[K],Boundary);
Gamma[K] := NF(Cast(Alpha[K],VECTOR),G2);
End;
T := [Deg(LT(G1[K]))
- Deg(LT(Gamma[K])) | K In 1..(Len(G1))];
L0 := Max(T) -1;
Print 'The rational invariant of Ginsburg is', L0;
```

We obtain $l_0(\Lambda V, d) = 62$. The exact value of $\mathrm{cat}_0(\Lambda V, d)$, obtained applying Proposition 5 and Theorem 19, is $\mathrm{cat}_0(\Lambda V, d) = l(x_2 x_3 x_4 x_5^{63}) = 66$.

The order just defined in the above example is not a term order. We have dropped the requirement of ordering first by homological degree. Since the elements we are working with are homogeneous with respect to the homological degree, both orders agree when comparing homogeneous elements. Clearly, when we reduce or perform the computation of a Groebner basis of a set of homogeneous elements, both orders produce the same results.

**Theorem 16.** *Let $(\Lambda V, d)$ be an elliptic space such that $(d - d_\sigma)V \subset \Lambda^{>l_\sigma} V$ where $(\Lambda V, d_\sigma)$ is the associated pure model and $l_\sigma = l_0(\Lambda V, d_\sigma)$. Then,* $\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d_\sigma)$.

Before giving the proof, we recall that the following is proven in [14].

**Proposition 17.** *Let $(\Lambda V, d)$ be an elliptic model and $(\Lambda V, d_\sigma)$ its associated pure model. If $d_\sigma Y \subset \Lambda^l X$ and $(d - d_\sigma)V \subset \Lambda^{\geqslant l} V$, then*

$$\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d_\sigma) = n(l-2) + m.$$

This proposition corresponds to the trivial case for the computation of the Ginsburg invariant, because the condition $d_\sigma Y \subset \Lambda^l V$ easily implies that $l_0(\Lambda V, d_\sigma) = l - 1$ and so, by Theorem 16 $\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d_\sigma)$. The proof of Theorem 16 is similar to that of Proposition 17 and we include it for completeness. First, we need a preliminary

**Proposition 18.** *Let $(\Lambda V, d)$ be an elliptic space. Then the following procedure* [16, Proposition 6] *computes a cocycle that represents the fundamental class of $(\Lambda V, d)$.*

Let $(\Lambda V, d_\sigma)$ the associated pure model of $(\Lambda V, d)$. Observe that $w_0$, the top class of the associated pure model lives in $(\Lambda X \otimes \Lambda^{m-n} Y)^N$ in which $N$ is the formal dimension. Recall the bigrading $\Lambda V_j^i = (\Lambda X \otimes \Lambda^j Y)^i$ then $dw_0 = \alpha_1^0 + \alpha_3^0 + \cdots + \alpha_k^0$, with $\alpha_i^0 \in \Lambda V_{m-n+i}^{N+1}$ and there is $\beta_1$ such that $d_\sigma \beta_1 = \alpha_1^0$. If $w_1 = w_0 - \beta_1$ then $dw_1 = \alpha_3^1 + \alpha_5^1 + \cdots + \alpha_k^1$ and again there is $\beta_2$ such that $d_\sigma \beta_2 = \alpha_3^1$. Hence we inductively define elements $w_j$ and $\beta_j$ satisfying $w_j = w_{j-1} - \beta_j$ and $dw_j \in \sum_{i=2j+1}^{k} \Lambda V_{m-n+i}^{N+1}$. Then, for the first $j_0$ such that $2j_0 + 1 > n$ this process stops and $w_{j_0}$ is a cocycle representing the fundamental class of $(\Lambda V, d)$.

**Proof of Theorem 16.** Let $(\Lambda V, d)$ be a pure elliptic model and $w_\sigma \in \Lambda^{\geqslant k} V$ be a cocycle that represents a top class of $(\Lambda V, d_\sigma)$. Then apply Proposition 18 to compute the cocycle $w$ with $[w]$ a top class of $(\Lambda V, d)$ and note that the assumption on $(d - d_\sigma)$ shows that we can choose the $\beta$'s of Proposition 23 so that $w$ still lives in $\Lambda^{\geqslant k} V$. Hence $\mathrm{cat}_0(\Lambda V, d_\sigma) \leqslant \mathrm{cat}_0(\Lambda V, d)$. To prove the reverse inequality, let $w$ be a cocycle representing the top class of $(\Lambda V, d)$, with $w \in \Lambda^{\geqslant k} V$ and $k = \mathrm{cat}_0(\Lambda V, d)$. Then, for some $p$, we may write $w = \alpha_1^0 + \alpha_2^0$ with $\alpha_1^0 \in \Lambda_p V$ and $\alpha_2^0 \in \Lambda_{>p} V$. Now, we apply Proposition 4 and the fact that if $p \neq m - n$ then $H^N(\Lambda_p V, d_\sigma) = 0$. It follows that $\alpha_2^0$ is a $d_\sigma$ boundary, so there is $\beta_1$ such that $d_\sigma \beta_1 = \alpha_2^0$, and the assumption on $(d - d_\sigma)$ show that we may choose $\beta_1$ so that $w_1 = w - d\beta \in \Lambda^{\geqslant k}_{\geqslant p+1} V$ still represents the top class of $(\Lambda V, d)$. Iterating this process shows that $p \leqslant m - n$ (otherwise $[w] = 0$) and so we obtain $w_i \in \Lambda^{\geqslant k} V$ such that $\alpha_1^i \in \Lambda_{m-n} V$ represents the top class of $(\Lambda V, d_\sigma)$. Hence $\mathrm{cat}_0(\Lambda V, d) \leqslant \mathrm{cat}_0(\Lambda V, d_\sigma)$. This proves $\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d_\sigma)$. $\square$

**Theorem 19** [15]. *Let $(\Lambda V, d)$ be a pure elliptic model, $G$ be a Groebner basis for the coboundary module $B$ with respect to $\leqslant_{\mathrm{top}}$, $w \in \Lambda_{m-n} V$ be a cocycle that represents the top class, and $w_0 = \mathrm{NF}_G(w)$. Then $\mathrm{cat}_0(\Lambda V, d) = l(w_0)$.*

As an immediate consequence of the above results we obtain:

**Proposition 20.** *The following algorithm computes the rational category of any elliptic space in which $(d - d_\sigma)V \subset \Lambda^{>l_\sigma} V$ where $l_\sigma$ is Ginsburg's invariant of $(\Lambda V, d_\sigma)$.*

Consider the associated pure model $(\Lambda V, d_\sigma)$ of $(\Lambda V, d)$.
Apply Proposition 5 to the model $(\Lambda V, d_\sigma)$. This provides $w \in \Lambda_{m-n} V$ with $[w]$ the top class of $(\Lambda V, d_\sigma)$.
Compute $S = \{d_\sigma(y_{i_1} \cdots y_{i_{m-n+1}})\colon 1 \leqslant i_1 < \cdots < i_{m-n+1} \leqslant m\}$.
Apply Buchberger's algorithm to obtain a Groebner basis $G$ for $S$ with respect to $\leqslant_{\mathrm{top}}$.
Obtain the normal form $w_0$ of $w$ with respect to $G$ by the division algorithm.
Compute $l = \|\mathrm{lt}(w_0)\|$.
Then $k = l(w_0) = m - n + l$ is the rational category of $(\Lambda V, d)$.

**Proof.** By Theorem 16, $\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d_\sigma)$ and by [15], this algorithm provides $\mathrm{cat}_0(\Lambda V, d_\sigma)$. $\square$

**Example 21.** Let $(\Lambda V, d)$ be a model such that $X$ is spanned by $\{x_1, x_2, x_3\}$ and $Y$ by $\{y_1, y_2, y_3, y_4, y_5\}$, with the graduation $|x_1| = |x_2| = |x_3| = 2$, $|y_1| = |y_2| = 3$, $|y_3| = |y_4| = 5$, and $|y_5| = 15$ and with differential given by

$$\begin{aligned} dy_1 &= x_1^2, \qquad dy_3 = x_3^3, \qquad dy_5 = -x_1^2 x_2 x_3^2 y_1 y_2 + x_1 x_2^2 x_3 y_1 y_4 - x_1^3 x_3 y_2 y_4, \\ dy_2 &= x_2^2, \qquad dy_4 = x_1 x_2 x_3. \end{aligned}$$

In [16, Proposition 6] is proven that the cocycle

$$x_1^2x_2^2y_3y_5 - x_1x_2x_3^2y_4y_5 + x_1^3x_2^3x_3y_1y_2y_3y_4$$

represents the top class of the non-pure elliptic space $(\Lambda V, d)$. Thus, $\mathrm{cat}_0(\Lambda V, d) \geqslant l(w) = 6$. Now we are going to prove that $\mathrm{cat}_0(\Lambda V, d) = 6$.

The models $(\Lambda V, d_\sigma)$ and $(\Lambda V, d)$ have both the same differential except that $d_\sigma y_5 = 0$. We apply the algorithm of Proposition 14 to $(\Lambda V, d_\sigma)$ and obtain $l_\sigma = l_0(\Lambda V, d_\sigma) = 2$. Then, $(d - d_\sigma)V \subset \Lambda^{\geqslant 6}V \subset \Lambda V^{>l_\sigma}V$ shows that $(\Lambda V, d)$ is the model of a Ginsburg space. Thus, by Proposition 16, $\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d_\sigma)$. Now we proceed to obtain $\mathrm{cat}_0(\Lambda V, d_\sigma)$. First, we apply Proposition 5 and obtain the cocycle $w = x_1^2x_2^2y_3y_5 - x_1x_2x_3^2y_4y_5$ that represents the top class of $(\Lambda V, d_\sigma)$. Then, we compute a Groebner basis $G$ of the module of $d_\sigma$-coboundaries. Finally, the reduction of $w$ with respect to $G$ yields

$$w_0 = NF(w, G_1) = x_1^2x_3^3y_2y_5 - x_1x_2x_3^2y_4y_5.$$

Thus, $\mathrm{cat}_0(\Lambda V, d) = \mathrm{cat}_0(\Lambda V, d_\sigma) = l(w_0) = 6$.

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