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# Exact traveling wave solutions to the Klein–Gordon equation using the novel $(G'/G)$-expansion method

M.G. Hafez [a,*], Md. Nur Alam [b], M. Ali Akbar [c]

[a] Department of Mathematics, Chittagong University of Engineering and Technology, Bangladesh
[b] Department of Mathematics, Pabna University of Science and Technology, Bangladesh
[c] Department of Applied Mathematics, University of Rajshahi, Bangladesh

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ABSTRACT

The novel $(G'/G)$-expansion method is one of the powerful methods that appeared in recent times for establishing exact traveling wave solutions of nonlinear partial differential equations. Exact traveling wave solutions in terms of hyperbolic, trigonometric and rational functions to the cubic nonlinear Klein–Gordon equation via this method are obtained in this article. The efficiency of this method for finding exact solutions and traveling wave solutions has been demonstrated. It is shown that the novel $(G'/G)$-expansion method is a simple and valuable mathematical tool for solving nonlinear evolution equations (NLEEs) in applied mathematics, mathematical physics and engineering.


## 1. Introduction

In the recent years, the exact solutions of nonlinear partial differential equations have been investigated by many researchers who are involved in nonlinear phenomena which exist in all fields including either the systematic works or engineering fields, such as, plasma physics, fluid mechanics, chemical physics, chemical kinematics, elastic media, optical fibers, solid state physics, biology, atmospheric and oceanic phenomena and so on. The research of traveling wave solutions of some nonlinear evolution equations derived from such fields played an important role in the analysis of these phenomena. To obtain traveling wave solutions, many effective methods have been presented in the literature, such as, the $\exp(-\varphi(\eta))$-expansion method [1,2], the $(G'/G,1/G)$-expansion method [3], the $(G'/G)$-expansion method [4–10], the inverse scattering transform method [11], the Exp-function method [12,13], the Cole–Hopf transformation method [14], the Adomian decomposition method [15], the homotopy perturbation method [16], the Kudryashov method [17], the new approach of generalized $(G'/G)$-expansion method [18–20], the improved $(G'/G)$-expansion method [21], the tanh-function method [22], the tanh-coth method [23] and so on.

Kudryashov [24] substantiated that the $(G'/G)$-expansion method together with the linear ordinary differential equation $G'' - \lambda G' - \mu G = 0, \lambda, \mu \in \Re$ is equivalent to the well known tanh-method. Recently, Alam et al. [25,26] established extremely valuable extension of the $(G'/G)$- expansion method, called the novel $(G'/G)$-expansion method to obtain exact traveling wave solutions of NLEEs. According to nonlinear ordinary differential equation $GG'' = \lambda GG' + \mu G^2 + \upsilon(G')^2$, the novel $(G'/G)$-expansion method constructs twenty five explicit solutions to the NLEEs and it can be shown that the novel $(G'/G)$-expansion method is not identical to the tanh-function method. The methods mention in refs. [4–10] are only special cases of the novel $(G'/G)$-expansion method.

The Klein–Gordon (KG) equations are an important class of NLEEs that arise in relativistic quantum mechanics and quantum field theory, which is also of great importance for the high energy particle physics and is used to model many types of phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles. There is an amount of paper [27–33], where the various types of nonlinear KG equations are studied. Chowdhury and Biswas [32] studied the singular solitons and numerical analysis of the Phi-four equation $q_{tt} - k^2 q_{xx} = aq + bq^3$ that appears in relativistic quantum mechanics. The Phi-four equation is a special case of the Klein–Gordon equations that is studied with several forms of nonlinearity that includes quadratic nonlinearity, power law nonlinearity, as well as log law nonlinearity. Biswas et al. [33] also studied the solitons and conservation law of the KG equation with power law and log law nonlinearities. It is primarily the perturbation theory, numerical simulation, and integrability issues that have been addressed thus far in such

* Corresponding author.
*E-mail address:* golam_hafez@yahoo.com (M.G. Hafez).

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models. If we set $k = 1$, $a = -\alpha$, $b = -\beta$, then the Phi-four equation can be reduced to the KG equation with cubic nonlinearity $u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0$ which is found in the literature [34,35]. The cubic nonlinear Klein–Gordon (KG) equation also appeared in relativistic quantum mechanics, field theory, and particle physics as physical model equation for describing many different phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles. The aim of this article is to explore a new study linking to the novel $(G'/G)$-expansion method for solving the famous cubic nonlinear Klein–Gordon equation to demonstrate the correctness and truthfulness of the method.

The advantage of the proposed method over the existing method is that it provides new exact traveling wave solutions together with additional free parameters. The exact solutions have great values to unveil the inner structure of the physical phenomena. Apart from the physical significance, the close-form solutions of nonlinear evolution equations help the numerical solvers to compare the correctness of their results and help them in the stability analysis. Algebraic manipulation of the proposed scheme with the help of Maple is much easier than the other methods.

The rest of the article is organized as follows: In Section 2, the description of the novel $(G'/G)$-expansion method is given. In Section 3, we apply this method to the nonlinear evolution equation pointed out above. The physical explanations and graphical representations of the obtained solutions are presented in Section 4. In Sections 5, we draw our conclusions.

## 2. Description of the method

Consider a general nonlinear partial differential equation of the form,

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0, \tag{1}$$

where, $u = u(x,t)$ is an unknown function, $P$ is a polynomial in $u(x,t)$ and its partial derivatives in which the higher order partial derivatives and the nonlinear terms are involved.

By combining the real variables $x$ and $t$ by a compound variable $\xi$, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x \pm Vt \tag{2}$$

where $V$ is the speed of the traveling wave. The transformation Eq. (2) transforms Eq. (1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$Q(u, u', u'', u''', \cdots) = 0, \tag{3}$$

where $Q$ is a function of $u(\xi)$ and its derivatives.

Suppose the solution of Eq. (3) can be expressed by a polynomial in $\psi(\xi)$:

$$u(\xi) = \sum_{j=-n}^{n} \alpha_j (\psi(\xi))^j \tag{4}$$

where

$$\psi(\xi) = d + \frac{G'(\xi)}{G(\xi)} \tag{5}$$

The unknown constants $\alpha_{-n}$ or $\alpha_n$ may be zero, but both of them could not be zero simultaneously. $\alpha_j$ $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$ and $d$ are constants to be determined later and $(G'/G)$ satisfies the second order nonlinear ODE:

$$GG'' = \lambda GG' + \mu G^2 + \upsilon (G')^2 \tag{6}$$

where prime denotes the derivative with respect $\xi$ and $\lambda$, $\mu$ and $\upsilon$ are real parameters.

The Cole–Hopf transformation $\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}$ reduces Eq. (6) to the following equation:

$$\Phi'(\xi) = \mu + \lambda \Phi(\xi) + (\upsilon - 1)\Phi^2(\xi) \tag{7}$$

Eq. (7) has individual twenty five solutions (see Zhu [36] for details).

The value of the positive integer $n$ can be determined by balancing the higher order linear terms with nonlinear terms of the highest order occurring in Eq. (3).

Substituting Eq. (4) along with Eqs. (5) and (6) into Eq. (3), we obtain polynomials in $\left(d + \frac{G'(\xi)}{G(\xi)}\right)$ and $\left(d + \frac{G'(\xi)}{G(\xi)}\right)^{-1}$, $(j = 0, 1, 2, \cdots, N)$. Collecting the coefficients of the resulted polynomials to zero, yields an over-determined set of algebraic equations for $\alpha_j$ $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$, $d$ and $V$. Solving the resulting algebraic system by using symbolic computation, such as, Maple, we obtained the value of the constants $\alpha_j$ $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$, $d$ and $V$. Substituting the values of the constants together with the solutions of Eq. (7), we obtain new and comprehensive exact traveling wave solutions of the nonlinear evolution Eq. (1).

**Remark 1.** It is worth mentioning to observe that if we replace $\lambda$ by $-\lambda$ and $\mu$ by $-\mu$ and put $\upsilon = 0$ in Eq. (6), then the novel $(G'/G)$-expansion overlaps with the Akbar et al.'s [8] generalized and improved $(G'/G)$-expansion method. On the other hand, if we put $d = 0$ in Eq. (5) and $v = 0$ in Eq. (6) then the method is identical to the improved $(G'/G)$-expansion method presented by Zhang et al. [7]. Again if we set $d = 0$, $v = 0$ and negative indices of $(G'/G)$ are zero in Eq. (4), then the method rotates into the basic $(G'/G)$-expansion method introduced by Wang et al. [4]. Finally, if we put $v = 0$ in Eq. (6) and $\alpha_j$ $(j = 1, 2, 3, \cdots, N)$ are functions of $x$ and $t$ instead of constants then the method is transformed into the generalized the $(G'/G)$-expansion method developed by Zhang et al. [9]. Thus the methods presented in the Refs. [4,7–9] are only special cases of the novel $(G'/G)$-expansion method.

## 3. Applications of the novel $(G'/G)$ -expansion method

In this section, we apply the novel $(G'/G)$-expansion method to obtain some new and more general exact traveling wave solutions of the cubic nonlinear Klein–Gordon equation.

Consider the cubic nonlinear Klein–Gordon equation [34,35]

$$u_{tt} - u_{xx} + \alpha u + \beta u^3 = 0. \tag{8}$$

Here, $u(x,t)$ represents the particle wave profile at any varied instances and $\alpha$, $\beta$ are nonzero real constants. Eq. (8) has appeared as a model equation for describing the propagation of dislocations within crystals, the Blochwall motion of magnetic crystals, the propagation of a splay wave along a lied membrane, the unitary theory for elementary particles and the propagation of magnetic flux on a Josephson line, etc.

Making use of the traveling wave transformation $\xi = x - Vt$, Eq. (8) is reducing into the following ODE:

$$(V^2 - 1)u'' + \alpha u + \beta u^3 = 0. \tag{9}$$

Inserting (4) in (9) and balancing the higher order derivative $u''$ with the nonlinear term of the highest order $u^3$, we obtain $n = 1$.

Therefore, the solution of Eq. (9) takes the form,

$$u(\xi) = \alpha_{-1}(\psi(\xi))^{-1} + \alpha_0 + \alpha_1(\psi(\xi)). \tag{10}$$

Inserting Eq. (10) into Eq. (9), the left hand side is transformed into polynomials of $\left(d + \frac{G'(\xi)}{G(\xi)}\right)$ and $\left(d + \frac{G'(\xi)}{G(\xi)}\right)^{-1}$. Equating the coefficients of like power of these polynomials to zero, we obtain a set of algebraic equations (for minimalism we leave out to display the equations) for $\alpha_0$, $\alpha_1$, $\alpha_{-1}$, $d$ and $V$. Solving the obtaining set of algebraic equations by using the symbolic computation software, such as, Maple 13, we obtain

**Set 1:**

$$\alpha_0 = \pm(\lambda - 2d(\upsilon - 1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}},\ \alpha_{-1} = 0,$$
$$d = d,\ V = \pm\sqrt{\frac{4\mu(\upsilon-1) - \lambda^2 - 2\alpha}{4\mu(\upsilon-1) - \lambda^2}},$$
$$\alpha_1 = \pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \tag{11}$$

where $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.

**Set 2**:

$$\alpha_0 = \pm(\lambda - 2d(\upsilon - 1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}},$$
$$\alpha_{-1} = \pm\frac{2(d^2(\upsilon-1) + \mu - \lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}},$$
$$d = d,\ a_1 = 0,$$
$$V = \pm\sqrt{\frac{4\mu(\upsilon-1) - \lambda^2 - 2\alpha}{4\mu(\upsilon-1) - \lambda^2}} \tag{12}$$

where $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.

**Set 3**:

$$V = \pm\sqrt{\frac{8\mu(\upsilon-1) - \alpha - 2\lambda^2}{8\mu(\upsilon-1) - 2\lambda^2}},$$
$$\alpha_0 = 0, \alpha_{-1} = \mp\frac{\sqrt{\alpha(4\mu(\upsilon-1) - \lambda^2)}}{4(\upsilon-1)\sqrt{\beta}},$$
$$d = \frac{\lambda}{2(\upsilon-1)}, \alpha_1 = \pm\frac{(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}}, \tag{13}$$

where $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.

**Set 4**:

$$V = \pm\sqrt{\frac{4\mu(\upsilon-1) + \alpha - \lambda^2}{4\mu(\upsilon-1) - \lambda^2}},\ \alpha_0 = 0,$$
$$\alpha_{-1} = \mp\frac{1}{2(\upsilon-1)\sqrt{\frac{-2\beta}{4\mu\alpha(\upsilon-1) - \lambda^2\alpha}}},$$
$$d = \frac{\lambda}{2(\upsilon-1)}\alpha_1 = \pm\frac{(\upsilon-1)\sqrt{-2\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}}, \tag{14}$$

where $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.
Substituting (11)–(14) into solution Eq. (10), we obtain

$$u_1(x,t) = \pm(\lambda - 2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}}(d + (G'/G)) \tag{15}$$

where $\xi = x \mp \sqrt{\frac{4\mu(\upsilon-1) - \lambda^2 - 2\alpha}{4\mu(\upsilon-1) - \lambda^2}}t$, and $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.

$$u_2(x,t) = \mp(\lambda - 2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \pm\frac{2(d^2(\upsilon-1) + \mu - \lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}}(d + (G'/G))^{-1} \tag{16}$$

where $\xi = x \mp \sqrt{\frac{4\mu(\upsilon-1) - \lambda^2 - 2\alpha}{4\mu(\upsilon-1) - \lambda^2}}t$, and $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.

$$u_3(x,t) = \pm\frac{(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}}\left(\frac{\lambda}{2(\upsilon-1)} + (G'/G)\right) \pm\frac{\sqrt{\alpha(4\mu(\upsilon-1) - \lambda^2)}}{4(\upsilon-1)\sqrt{\beta}}\left(\frac{\lambda}{2(\upsilon-1)} + (G'/G)\right)^{-1} \tag{17}$$

where $\xi = x \mp \sqrt{\frac{8\mu(\upsilon-1) - \alpha - 2\lambda^2}{8\mu(\upsilon-1) - 2\lambda^2}}t$, and $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.

$$u_4(x,t) = \pm\frac{(\upsilon-1)\sqrt{-2\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}}\left(\frac{\lambda}{2(\upsilon-1)} + (G'/G)\right) \mp\frac{1}{2(\upsilon-1)\sqrt{\frac{-2\beta}{4\mu\alpha(\upsilon-1) - \lambda^2\alpha}}}\left(\frac{\lambda}{2(\upsilon-1)} + (G'/G)\right)^{-1} \tag{18}$$

where $\xi = x \mp \sqrt{\frac{4\mu(\upsilon-1) + \alpha - \lambda^2}{4\mu(\upsilon-1) - \lambda^2}}t$, and $\alpha$, $\beta$, $d$, $\lambda$, $\mu$, and $\upsilon$ are arbitrary constants.

Substituting the value of $(G'/G)$ into Eq. (15) and simplifying, we obtained multiple explicit solutions of the Klein–Gordon as follows:

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu > 0$ and $\lambda(\upsilon - 1) \neq 0$ (or $\mu(\upsilon - 1) \neq 0$),

$$u_{1_1}(x,t) = \pm(\lambda - 2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \times\left\{d - \frac{1}{2(\upsilon-1)}\left(\lambda + \sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}. \tag{19}$$

$$u_{1_2}(x,t) = \pm(\lambda - 2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \times\left\{d - \frac{1}{2(\upsilon-1)}\left(\lambda + \sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}. \tag{20}$$

$$u_{1_3}(x,t) = \pm(\lambda - 2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \times\left[d - \frac{1}{2(\upsilon-1)}\left\{\lambda + \sqrt{\Omega}\left(\tanh\left(\sqrt{\Omega}\xi\right) \pm i\,\mathrm{sec}\,h\left(\sqrt{\Omega}\xi\right)\right)\right\}\right]. \tag{21}$$

$$u_{1_4}(x,t) = \pm(\lambda - 2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \times\left[d - \frac{1}{2(\upsilon-1)}\left\{\lambda + \sqrt{\Omega}\left(\coth\left(\sqrt{\Omega}\xi\right) \pm \mathrm{csc}\,h\left(\sqrt{\Omega}\xi\right)\right)\right\}\right]. \tag{22}$$

$$u_{1_5}(x,t) = \pm(\lambda - 2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1) - \lambda^2\beta}} \times\left[d - \frac{1}{4(\upsilon-1)}\left\{2\lambda + \sqrt{\Omega}\left(\tanh\left(\frac{1}{4}\sqrt{\Omega}\xi\right) + \coth\left(\frac{1}{4}\sqrt{\Omega}\xi\right)\right)\right\}\right]. \tag{23}$$

$$u_{1_6}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}-A\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\xi\right)}{A\sinh\left(\sqrt{\Omega}\xi\right)+B}\right\}\right]. \quad (24)$$

$$u_{1_7}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\frac{\pm\sqrt{\Omega(A^2+B^2)}+A\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\xi\right)}{A\sinh\left(\sqrt{\Omega}\xi\right)+B}\right\}\right], \quad (25)$$

where $A$ and $B$ are real non-zero constants.

$$u_{1_8}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{2\mu\cosh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)}{\sqrt{\Omega}\sinh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)-\lambda\cosh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)}\right\}. \quad (26)$$

$$u_{1_9}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{2\mu\sinh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)}{\sqrt{\Omega}\cosh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)-\lambda\sinh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)}\right\}. \quad (27)$$

$$u_{1_{10}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{2\mu\cosh\left(\sqrt{\Omega}\xi\right)}{\sqrt{\Omega}\sinh\left(\sqrt{\Omega}\xi\right)-\lambda\cosh\left(\sqrt{\Omega}\xi\right)\pm i\sqrt{\Omega}}\right\}. \quad (28)$$

$$u_{1_{11}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{2\mu\sinh\left(\sqrt{\Omega}\xi\right)}{\sqrt{\Omega}\cosh\left(\sqrt{\Omega}\xi\right)-\lambda\sinh\left(\sqrt{\Omega}\xi\right)\pm\sqrt{\Omega}}\right\}. \quad (29)$$

When $\Omega=\lambda^2-4\mu\upsilon+4\mu<0$ and $\lambda(\upsilon-1)\neq 0$ (or $\mu(\upsilon-1)\neq 0$),

$$u_{1_{12}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{1}{2(\upsilon-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}. \quad (30)$$

$$u_{1_{13}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}. \quad (31)$$

$$u_{1_{14}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\sqrt{-\Omega}\left(\tan\left(\sqrt{-\Omega}\xi\right)\pm\sec\left(\sqrt{-\Omega}\xi\right)\right)\right\}\right]. \quad (32)$$

$$u_{1_{15}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d-\frac{1}{2(\upsilon-1)}\left\{\lambda+\sqrt{-\Omega}\left(\cot\left(\sqrt{-\Omega}\xi\right)\pm\csc\left(\sqrt{-\Omega}\xi\right)\right)\right\}\right]. \quad (33)$$

$$u_{1_{16}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d+\frac{1}{4(\upsilon-1)}\left\{-2\lambda+\sqrt{-\Omega}\left(\tan\left(\frac{1}{4}\sqrt{-\Omega}\xi\right)-\cot\left(\frac{1}{4}\sqrt{-\Omega}\xi\right)\right)\right\}\right]. \quad (34)$$

$$u_{1_{17}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm 2\sqrt{\frac{\alpha(\upsilon-1)}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}-A\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\xi\right)}{A\sin\left(\sqrt{-\Omega}\xi\right)+B}\right\}\right]. \quad (35)$$

$$u_{1_{18}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}+A\sqrt{-\Omega}\cos\left(\sqrt{-\Omega}\xi\right)}{A\sin\left(\sqrt{-\Omega}\xi\right)+B}\right\}\right]. \quad (36)$$

where $A$ and $B$ are arbitrary constants such that $A^2-B^2>0$.

$$u_{1_{19}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{2\mu\cos\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)}{\sqrt{-\Omega}\sin\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)+\lambda\cos\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)}\right\}. \quad (37)$$

$$u_{1_{20}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{2\mu\sin\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)}{\sqrt{-\Omega}\cos\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)-\lambda\sin\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)}\right\}. \quad (38)$$

$$u_{1_{21}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{2\mu\cos\left(\sqrt{-\Omega}\xi\right)}{\sqrt{-\Omega}\sin\left(\sqrt{-\Omega}\xi\right)+\lambda\cos\left(\sqrt{-\Omega}\xi\right)\pm\sqrt{-\Omega}}\right\}. \quad (39)$$

$$u_{1_{22}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{2\mu\sin\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)}{\sqrt{-\Omega}\cos\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)-\lambda\sin\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\pm\sqrt{-\Omega}}\right\}. \quad (40)$$

When $\mu=0$ and $\lambda(\upsilon-1)\neq 0$,

$$u_{1_{23}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{\lambda k}{(\upsilon-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\right\}. \quad (41)$$

$$u_{1_{24}}(x,t)=\pm(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{\lambda\{\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}{(\upsilon-1)\{k+\cosh(\lambda\xi)+\sinh(\lambda\xi)\}}\right\}, \quad (42)$$

where $k$ is an arbitrary constant.

Again, substituting the value of $(G'/G)$ into Eq. (16) and simplifying, we achieve the following multiple explicit solutions:

When $\Omega=\lambda^2-4\mu\upsilon+4\mu>0$ and $\lambda(\upsilon-1)\neq 0$ (or $\mu(\upsilon-1)\neq 0$),

$$u_{2_1}(x,t)=\mp(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(d^2(\upsilon-1)+\mu-\lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}^{-1}. \quad (43)$$

$$u_{2_2}(x,t)=\mp(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(d^2(\upsilon-1)+\mu-\lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}^{-1}. \quad (44)$$

$$u_{2_3}(x,t)=\mp(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(d^2(\upsilon-1)+\mu-\lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left[d-\frac{1}{2(\upsilon-1)}\left\{\lambda+\sqrt{\Omega}\left(\tanh\left(\sqrt{\Omega}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{\Omega}\xi\right)\right)\right\}\right]^{-1}. \quad (45)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu < 0$ and $\lambda\,(\upsilon-1)\neq 0$ (or $\mu(\upsilon-1)\neq 0$),

$$u_{2_{12}}(x,t)=\mp(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(d^2(\upsilon-1)+\mu-\lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d+\frac{1}{2(\upsilon-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}^{-1}. \quad (46)$$

$$u_{2_{13}}(x,t)=\mp(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(d^2(\upsilon-1)+\mu-\lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}^{-1}. \quad (47)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\mu = 0$ and $\lambda(\upsilon-1)\neq 0$,

$$u_{2_{23}}(x,t)=\mp(\lambda-2d(\upsilon-1))\sqrt{\frac{\alpha}{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\pm\frac{2(d^2(\upsilon-1)+\mu-\lambda d)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{d-\frac{\lambda k}{(\upsilon-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\right\}^{-1}. \quad (48)$$

where $k$ is an arbitrary constant.

The other families of exact solutions of Eq. (8) are omitted for convenience.

Again, substituting the value of $(G'/G)$ into Eq. (17) and simplifying, we achieve the following solutions:

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu > 0$ and $\lambda\,(\upsilon-1)\neq 0$ (or $\mu(\upsilon-1)\neq 0$),

$$u_{3_1}(x,t)=\pm\frac{(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}\pm\frac{\sqrt{\alpha(4\mu(\upsilon-1)-\lambda^2)}}{4(\upsilon-1)\sqrt{\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}^{-1}. \quad (49)$$

$$u_{3_2}(x,t)=\pm\frac{(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}\pm\frac{\sqrt{\alpha(4\mu(\upsilon-1)-\lambda^2)}}{4(\upsilon-1)\sqrt{\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}^{-1}. \quad (50)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu < 0$ and $\lambda\,(\upsilon-1)\neq 0$ (or $\mu(\upsilon-1)\neq 0$),

$$u_{3_{12}}(x,t)=\pm\frac{(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}+\frac{1}{2(\upsilon-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}\pm\frac{\sqrt{\alpha(4\mu(\upsilon-1)-\lambda^2)}}{4(\upsilon-1)\sqrt{\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}+\frac{1}{2(\upsilon-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}^{-1}. \quad (51)$$

$$u_{3_{13}}(x,t)=\pm\frac{(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}\pm\frac{\sqrt{\alpha(4\mu(\upsilon-1)-\lambda^2)}}{4(\upsilon-1)\sqrt{\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}--\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}^{-1}. \quad (52)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\mu = 0$ and $\lambda(\upsilon-1)\neq 0$,

$$u_{3_{23}}(x,t)=\pm\frac{(\upsilon-1)\sqrt{\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{\lambda k}{(\upsilon-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\right\}\pm\frac{\sqrt{\alpha(4\mu(\upsilon-1)-\lambda^2)}}{4(\upsilon-1)\sqrt{\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{\lambda k}{(\upsilon-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\right\}^{-1} \quad (53)$$

where $k$ is an arbitrary constant.

The other families of exact solutions of Eq. (8) are omitted for convenience

Finally, substituting the value of $(G'/G)$ into Eq. (18) and simplifying, we achieve the following multiple explicit solutions:

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu > 0$ and $\lambda\,(\upsilon-1)\neq 0$ (or $\mu(\upsilon-1)\neq 0$),

$$u_{4_1}(x,t)=\pm\frac{(\upsilon-1)\sqrt{-2\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}\mp\frac{1}{2(\upsilon-1)\sqrt{\frac{-2\beta}{4\mu\alpha(\upsilon-1)-\lambda^2\beta}}}\times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}^{-1}. \quad (54)$$

$$u_{4_2}(x,t) = \pm\frac{(\upsilon-1)\sqrt{-2\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}} \times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\cot h\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\} \mp\frac{1}{2(\upsilon-1)\sqrt{\frac{-2\beta}{4\mu\alpha(\upsilon-1)-\lambda^2\alpha}}} \times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{\Omega}\cot h\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right\}^{-1}. \quad (55)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu < 0$ and $\lambda(\upsilon-1)\neq 0$ (or $\mu(\upsilon-1)\neq 0$),

$$u_{4_{12}}(x,t) = \pm\frac{(\upsilon-1)\sqrt{-2\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}} \times\left\{\frac{\lambda}{2(\upsilon-1)}+\frac{1}{2(\upsilon-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\} \mp\frac{1}{2(\upsilon-1)\sqrt{\frac{-2\beta}{4\mu\alpha(\upsilon-1)-\lambda^2\alpha}}} \times\left\{\frac{\lambda}{2(\upsilon-1)}+\frac{1}{2(\upsilon-1)}\left(-\lambda+\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}^{-1}. \quad (56)$$

$$u_{4_{13}}(x,t) = \pm\frac{(\upsilon-1)\sqrt{-2\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}} \times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\} \mp\frac{1}{2(\upsilon-1)\sqrt{\frac{-2\beta}{4\mu\alpha(\upsilon-1)-\lambda^2\alpha}}} \times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\xi\right)\right)\right\}^{-1}. \quad (57)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\mu = 0$ and $\lambda(\upsilon-1)\neq 0$,

$$u_{4_{23}}(x,t) = \pm\frac{(\upsilon-1)\sqrt{-2\alpha}}{\sqrt{4\mu\beta(\upsilon-1)-\lambda^2\beta}} \times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{\lambda k}{(\upsilon-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\right\} \mp\frac{1}{2(\upsilon-1)\sqrt{\frac{-2\beta}{4\mu\alpha(\upsilon-1)-\lambda^2\alpha}}} \times\left\{\frac{\lambda}{2(\upsilon-1)}-\frac{\lambda k}{(\upsilon-1)\{k+\cosh(\lambda\xi)-\sinh(\lambda\xi)\}}\right\}^{-1}. \quad (58)$$

where $k$ is an arbitrary constant.

The other families of exact solutions of Eq. (8) are omitted for convenience.

The above determined solutions are very helpful to understand the wave propagation in dislocations within crystals, the Blochwall motion of magnetic crystals, the propagation of a splay wave along a lied membrane, the particle wave propagation for spinless particles in phi-theory, the wave propagation of magnetic flux on a Josephson line, etc.

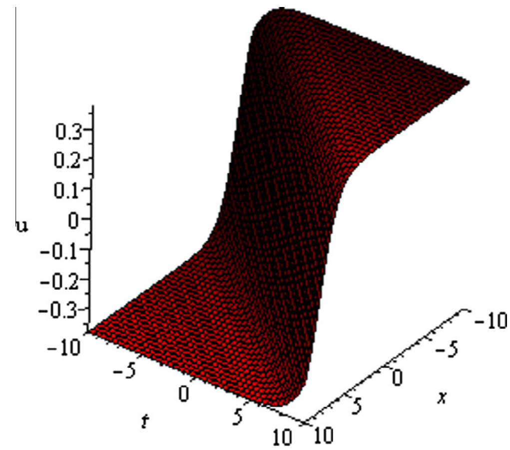


**Fig. 1.** Topological kink type solitary wave, Shape of (19) with $-10 \leqslant x, t \leqslant 10$.

## 4. Physical explanations

In this section we will discuss the physical explanations and graphical representation of the above determined four families of the solutions.

The introduction of dispersion without introducing nonlinearity destroys the solitary wave as different Fourier harmonics start propagating at different group velocities. On the other hand, introducing nonlinearity without dispersion also prevents the formation of solitary waves, because the pulse energy is frequently pumped into higher frequency modes. However, if both dispersion and nonlinearity are present, solitary waves can be sustained. Similarly to dispersion, dissipation can also give rise to solitary waves when combined with nonlinearity. Hence it is more interesting to point out that the delicate balance between the nonlinearity effect of $u^3$ and the dissipative effect of $u_{xx}$ gives rise to solitons solitary waves, that after a full interaction with others the solitons come back retaining their identities with the same speed and shape. The (1+1)-dimensional Klein–Gordon equation has solitary wave solutions that have exponentially decaying wings. If two solitons of the Klein–Gordon equation collide, the solitons just pass through each other and emerge unchanged. There are various types of traveling wave solutions that are of particular interest in solitary wave theory. The type of traveling waves depends on the variation of the physical parameters. If the exact solutions of the Klein–Gordon equation arise in a complex form according to the variations of the physical parameters, then the wave propagation for any varied instance is characterized by $|u(x, t)|$. For some special values of the

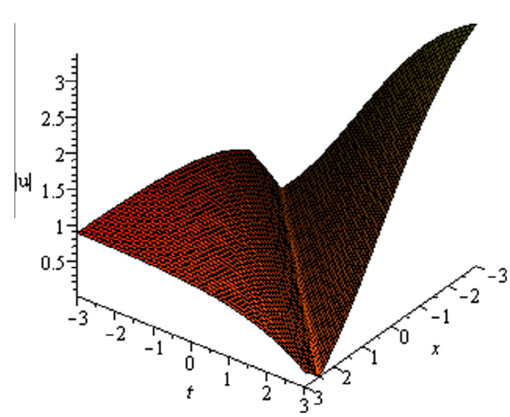


**Fig. 2.** Singular kink type solitary wave, Shape of (44) with $-10 \leqslant x, t \leqslant 10$.

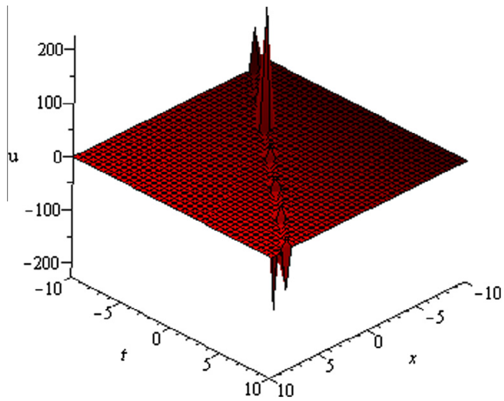


**Fig. 3.** Topological soliton solitary wave, Shape of (20) with $-10 \leqslant x, t \leqslant 10$.

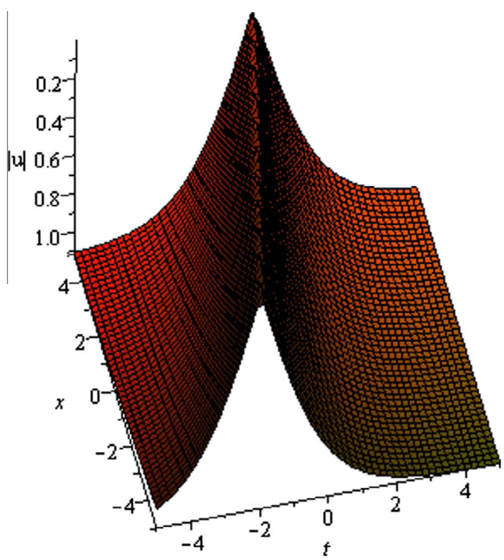


**Fig. 4.** Peakon type solitary wave, Shape of (26) with $-5 \leqslant x, t \leqslant 5$.

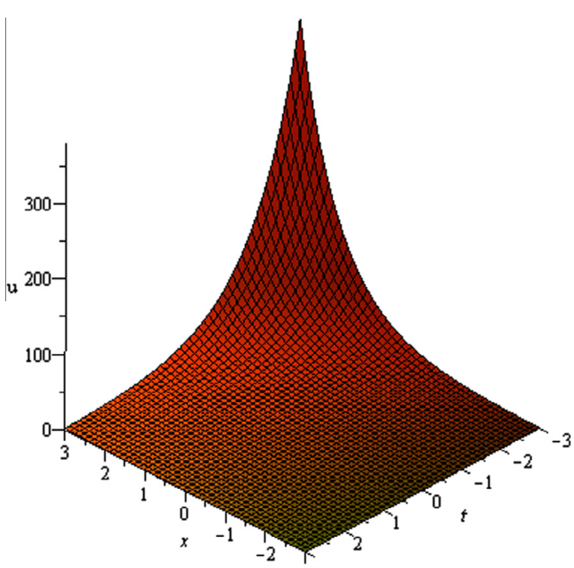


**Fig. 5.** 1-soliton solitary wave, Shape of (27) with $-3 \leqslant x, t \leqslant 3$.

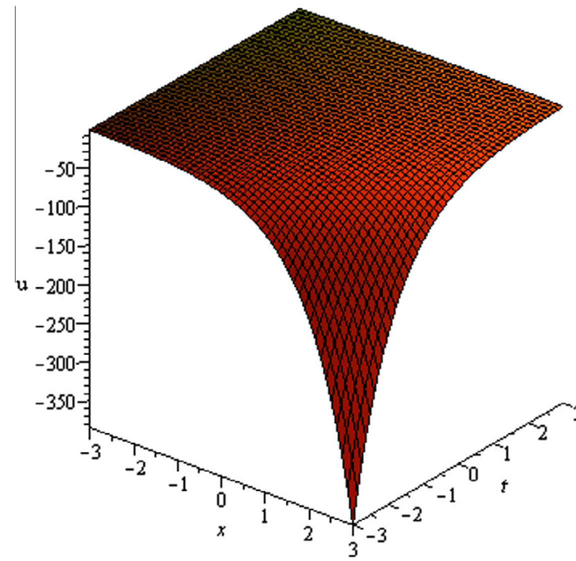


**Fig. 6.** Anti 1-soliton solitary wave, Shape of (26) with $-3 \leqslant x, t \leqslant 3$.

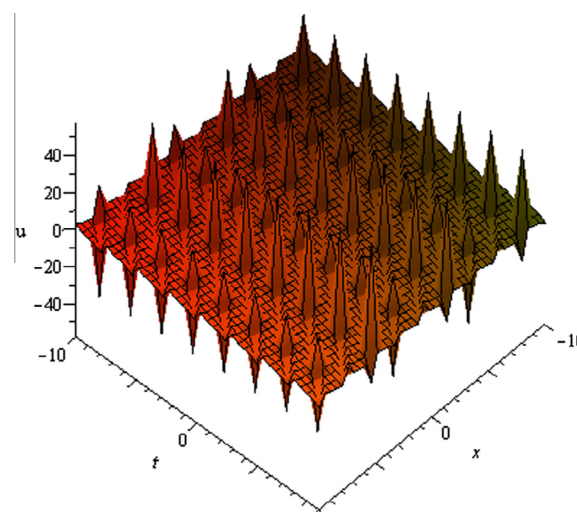


**Fig. 7.** Periodic wave solution, Shape of (30) with $-10 \leqslant x, t \leqslant 10$.

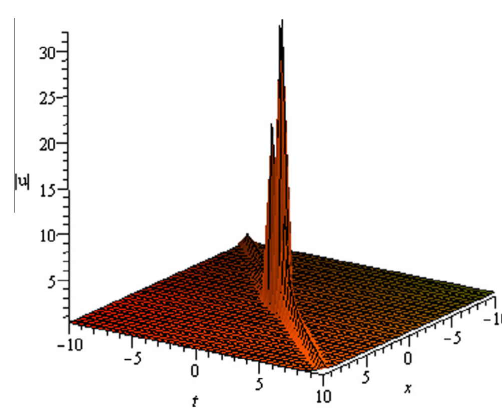


**Fig. 8.** Dark soliton solitary wave, Shape of (54) with $-10 \leqslant x, t \leqslant 10$.

physical parameters, the traveling wave solutions originated from the obtained exact explicit solutions as follows:

Solutions (19) and (43) corresponding to fixed values $\alpha = 1$, $\beta = 1$, $\mu = -1$, $\lambda = 1$, $\nu = -1$, $d = 1$ with $-10 \leqslant x, t \leqslant 10$ represented the exact solitary wave solutions of kink type. Again, according to the values $\alpha = 1$, $\beta = 1$, $\mu = 0$, $\lambda = 1$, $\nu = -1$, $d = 1$ and $-10 \leqslant x, t \leqslant 10$, solutions (41), (42), (48), (53) and (58) are also given the exact solitary wave solutions of kink type. The exact kink type solitary wave solution is shown in Fig. 1. For the fixed values $\alpha = 0.1$, $\beta = 0.2$, $\mu = 0.1$, $\lambda = 0.5$, $\nu = 2$, $d = 0$ and $-3 \leqslant x, t \leqslant 3$, the solution (44) represents the solitary wave solution of singular kink type. The solitary wave solution of singular kink type is shown in Fig. 2.

Solutions (20), (22)–(25), (44), (49), (50), (54), (55) and (58) represent exact solitary wave solutions of topological soliton type corresponding to the fixed values of the physical parameters $\alpha = 1$, $\beta = 1$, $\mu = -1$, $\lambda = 1$, $\nu = -1$, $d = 1$. The soliton solution is a specially localized solution, hence $u'(\xi), u''(\xi), u'''(\xi) \to 0$ as $\xi \to \pm\infty$, $\xi = x - c\,t$. The graphical representation of topological soliton type solitary wave is shown in Fig. 3. Fig. 4 shows the shape of exact solitary wave solution of peakon type; obtained from the solution (26) corresponding to the fixed values $\alpha = 0.5$, $\beta = 0.4$, $\mu = -0.1$, $\lambda = 0.5$, $\nu = 0.5$, $d = -0.5$, $-10 \leqslant x, t \leqslant 10$.

Solutions (26)–(29) have the exact anti 1-soliton and 1-soliton type solitary wave solutions corresponding to $\alpha = 1$, $\beta = 1$, $\mu = -1$, $\lambda = 1$, $\nu = -1$, $d = 1$. The exact solutions of anti 1-soliton and 1-soliton type solitary wave solutions are shown graphically in Fig. 5 and 6 respectively.

For the fixed values $\alpha = 1$, $\beta = 1$, $\mu = 3$, $\lambda = 2$, $\nu = 2$, $d = 1$, $-10 \leqslant x, t \leqslant 10$, solutions (30), (21), (32), (33), (35)–(38), (39), (45), (46), (51), (56) and (57) represent periodic wave solutions. Periodic solutions are traveling wave solutions that are periodic such as $\cos(x - t)$. The exact periodic traveling wave solution is presented graphically in Fig. 7. Solutions (31), (34), (44), (47), (52), (53), (56) and (57) represent dark soliton type solitary wave solutions according to the same fixed values. The solutions (54), (55) and (58) also represent dark soliton type solitary wave solutions corresponding to $\alpha = 1$, $\beta = 1$, $\mu = -1$, $\lambda = 1$, $\nu = -1$, $d = 1$. The dark soliton type solitary wave solution is shown in Fig. 8.

## 5. Conclusions

The novel $(G'/G)$-expansion method is successfully applied to establish traveling wave solutions to the famous Klein–Gordon equation. The performance of this method is reliable, convincing and can be used to other NLEEs in finding exact solutions. The method gives more general solutions which contain further arbitrary constants and the arbitrary constants imply that these solutions have rich local structures. It is important to notice that the basic $(G'/G)$-expansion method, the improved $(G'/G)$-expansion and the generalized and improved $(G'/G)$-expansion method are only special case of the novel $(G'/G)$-expansion method. By means of this scheme, we found some fresh traveling wave solutions of the above mentioned equation. Although the method is applied to the celebrated Klein–Gordon equation it can be applied to many other NLEEs, and this is our task in the future. The obtained solutions can be utilized to further analyze by the physicists on varied instance.

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