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Research on characteristics of noise-perturbed M–J sets based on equipotential point algorithm

Yuanyuan Sun*, Kaining Hou, Xudong Zhao, Zhixing Lu

School of Computer Science and Technology, Dalian University of Technology, Dalian, Liaoning, 116024, China

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ABSTRACT

As the classical ones among the fractal sets, Julia set (abbreviated as J set) and Mandelbrot set (abbreviated as M set) have been explored widely in recent years. In this study, J set and M set under additive noise perturbation and multiplicative noise perturbation are created by equipotential point algorithm. Changes of the J set and M set under random noise perturbation as well as the close correlation between them are studied. Experimental results show that either additive noise perturbation or multiplicative noise perturbation may cause dramatic changes on J set. On the other hand, when the M set is perturbed by additive noise, it almost changes nothing but its position; when the M set is perturbed by multiplicative noise, its inner structures change with the stabilized areas shrinking, but it keeps the symmetry with respect to *X* axis. In addition, the J set and the M set still share the same stabilized periodic point in spite of noise perturbation.

1. Introduction

Euclidean geometry shows its limitations when facing the realistic complicated world, combining with mystery and irregularity. In the mid of 1970 s, fractal geometry theory is established, which provided a new method to describe and explore the rules and structures of these irregular situations [16]. The irregular structures in traditional mathematics studied under fractal geometry theory display new rules and are of wonderful beauty [13].

M set and J set are famous fractal sets generated from the nonlinear complex mapping $f : z \leftarrow z^2 + c$. These years a great number of achievements have been made on the study of escape time algorithm [11,12,19,24–29]. Jovanovic discovered properties of M set by visual analysis on its calculation path [18] and thus escape time algorithm have been further improved [14,21,30]. There are also much research on the generation of the M–J set with noise perturbation [1–10,15,17,20].

In this paper, the J set and the M set under random noise perturbation are constructed using equipotential point algorithm [22,23] and their characteristic features are explored. In addition, the relationship between these two sets is identified by calculating the stabilized periodic point coordinates.

2. Equipotential point algorithm

The advantages of equipotential point algorithm over classical escape time algorithm include the clearer image structure, more visible iterative process and easier observation of M–J sets' macro features. The steps of the algorithm are as follows.

* Corresponding author. Tel.: +86 041184706003.
E-mail address: syuan@dlut.edu.cn (Y. Sun).

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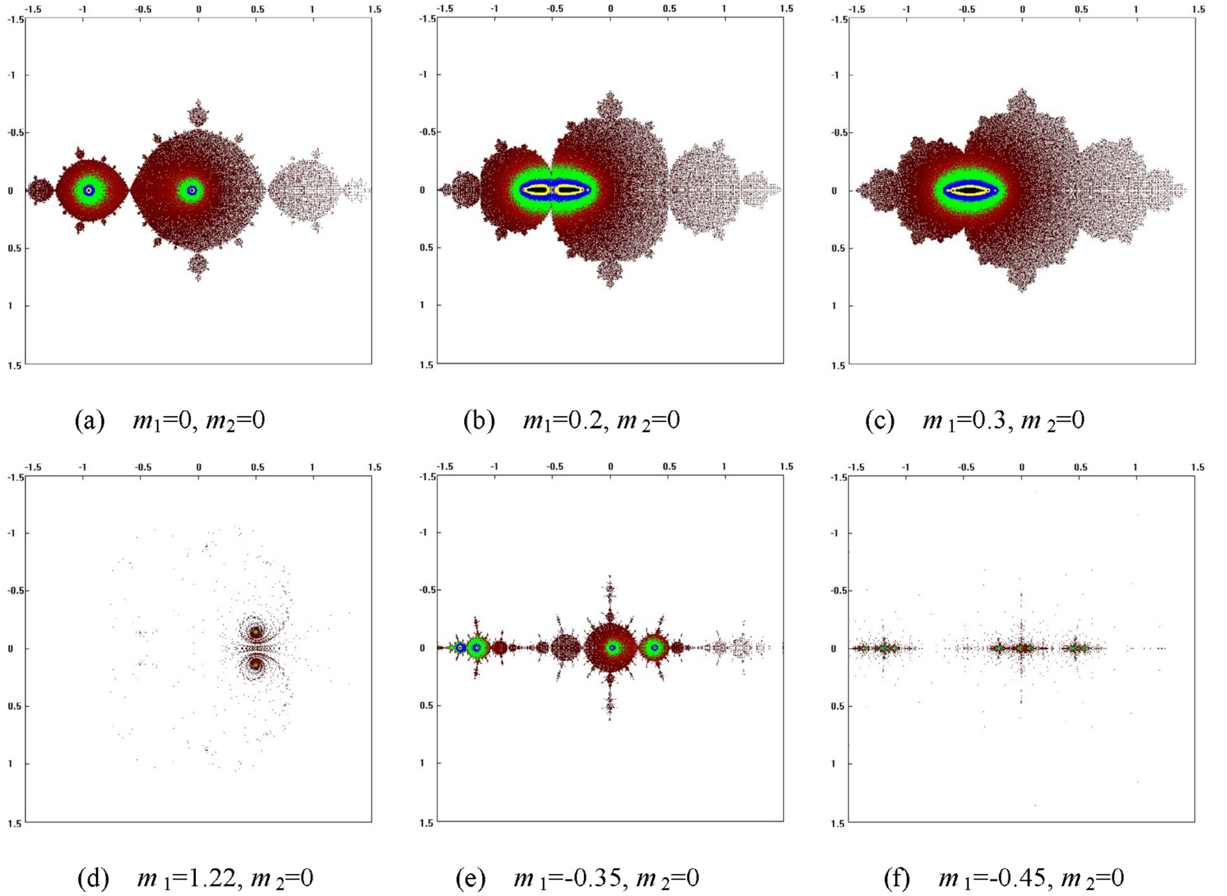


Fig. 1. J sets of $c(-0.95, 0)$ under additive noise perturbations, noise m_1 changes and $m_2 = 0$.

(1) For the complex mapping $Z_{n+1} = Z_n^2 + C$, designate $C(C = p + \mathbf{i}q,\ p, q \in \mathbf{R})$ as the complex constant, L as the escape radius and N as the escape time limit. The point z_0 is from the two-dimensional window and the mapping region is $M \times M$. Denote $counter[M][M]$ as the two-dimensional array with the initial value 0.
(2) For z_0, its coordinate on the screen is $(i_0, j_0), i_0, j_0 \in \mathbf{N}, 0 \le i_0 \le M, 0 \le j_0 \le M$.
(3) If $|f^k(z_0)| < L, k \in \mathbf{N}, 1 \le k \le N$ and $f^k(z_0) \subset W$, or if $|f^k(z_0)| < L$, $|f^l(z_0)| \ge L$, $1 \le k < l, l \le N$ and $f^l(z_0) \subset W$, then $counter[i][j]++$ (an incremental increase of the counter); and the coordinate of $f^k(z_0)$ or $f^l(z_0)$ in the corresponding mapping area on the screen is (i, j).
(4) Repeat steps (2) and (3) until all points in the screen are covered.
(5) To determine whether the point is internal or external is according to the escape radius R. Since the internal points can reflect the fractal features of the M–J sets, only the internal points are discussed in this paper.
(6) The color of the point (i_0, j_0) is defined by the value of $counter[i_0][j_0]$.

A defined c may create a J set. If c is added by a parameter, then different J sets can be obtained by different parameters. The iterative process of the J set under additive noise perturbation is as follows [7].

$$x_{n+1} = x_n^2 - y_n^2 + p + m_1,$$

$$y_{n+1} = 2x_n y_n + q + m_2,$$

where $x_i, y_i \in \mathbf{R}, i \in \mathbf{N}$, m_1 and m_2 are noise intensities, $m_1, m_2 \in \mathbf{R}$.

Similar to additive noise perturbation, the only difference of multiplicative noise perturbation is that there is a parameter multiplied in front of the variable. The iterative process is as follows.

$$x_{n+1} = (1 + k_1)x_n^2 - (1 + k_1)y_n^2 + p,$$

$$y_{n+1} = (2 + k_2)x_n y_n + q,$$

where k_1, k_2 are noise intensities, and $k_1, k_2 \in \mathbf{R}$.

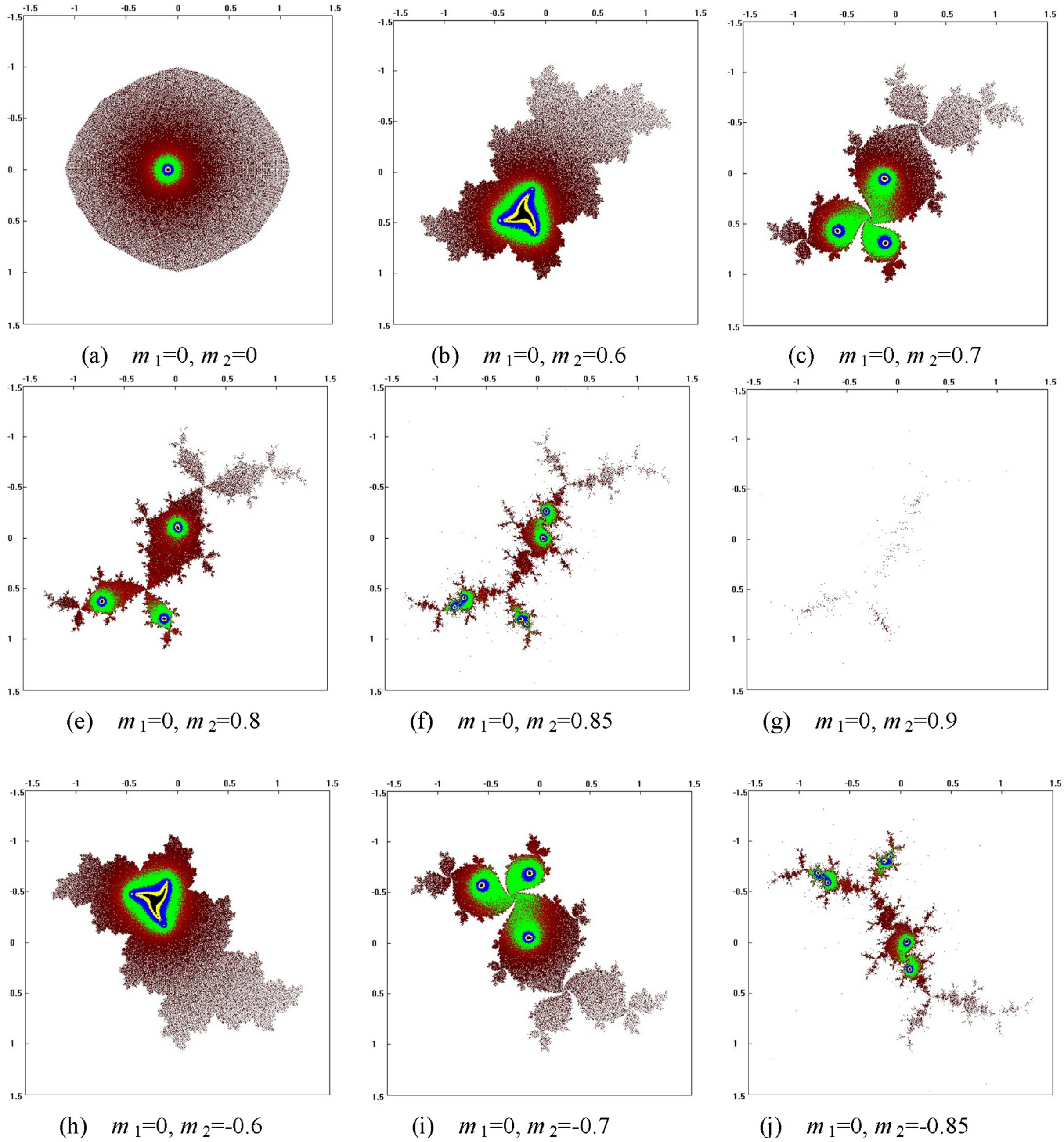


Fig. 2. J sets of $c(-0.1, 0)$ under additive noise perturbations, noise m_1 =0 and m_2 changes.

3. Additive noise perturbation

There are many affinities between J set and M set because both of them are generated from $f : z \leftarrow z^2 + c$. J set, for which the value of point c is invariable, is iterated with z as the initial points; point c in J set corresponds to a point in M set. That is to say, M set is a collection of all possible values of point c and each point stands for a different J set. If point c is inside of M set, then the corresponding J set should be interconnected and vice versa. Aiming at such feature, the additive noise perturbation is performed on both J set and M set. This paper also analyzes the relationship between the movement of point c in M set and the corresponding J set.

3.1. J set under additive noise perturbation

In the experiment, additive noise perturbations are performed on two J sets. Their two corresponding points, c_1 $(-0.95, 0)$ and c_2 $(-0.1, 0)$, are chosen from X axis area of the M set. The experimental results are observed by changing noise intensity.

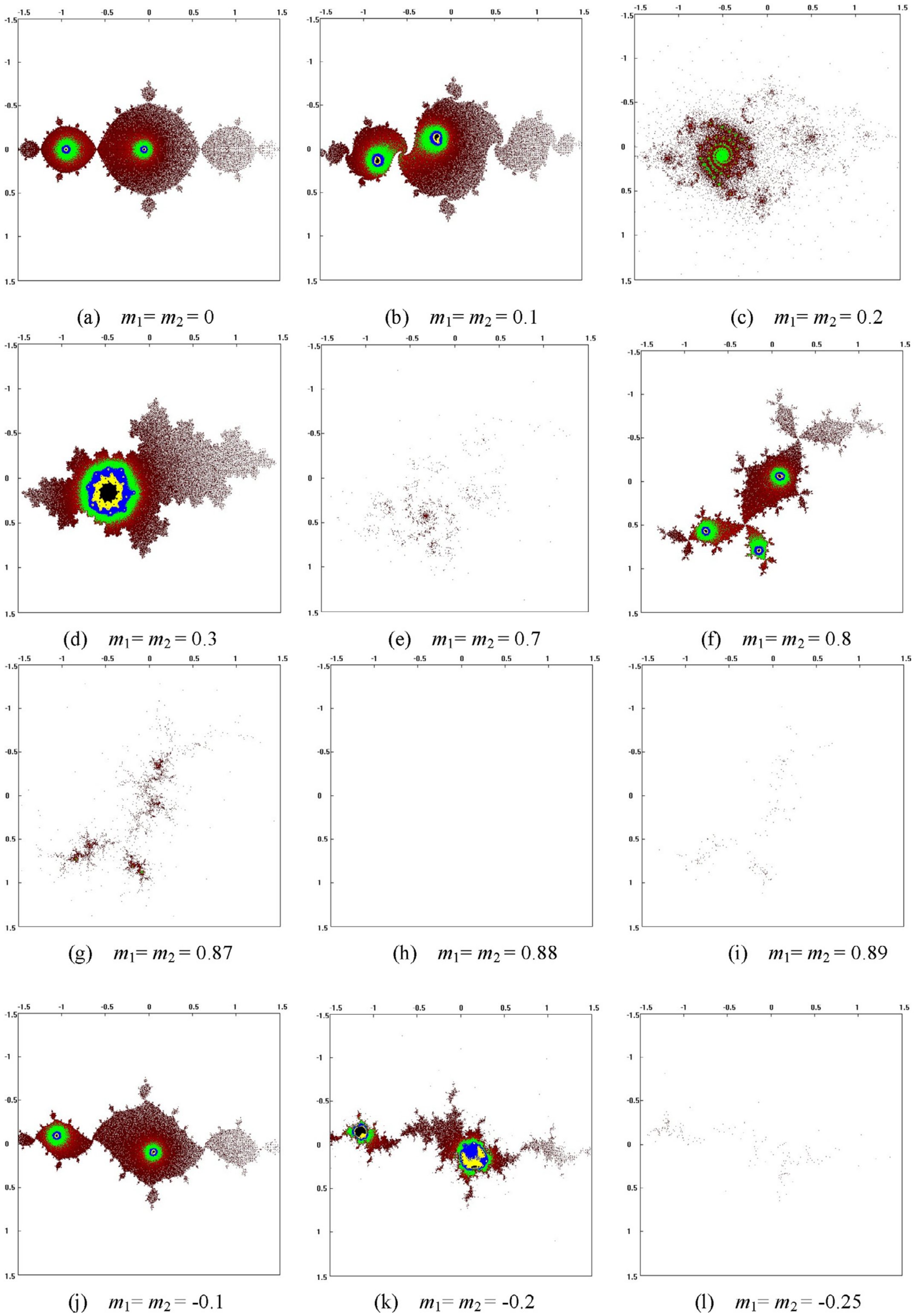
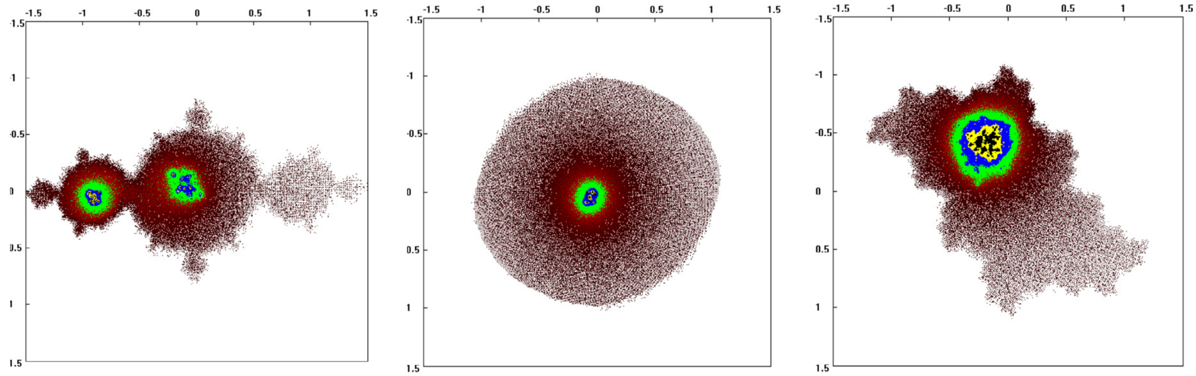
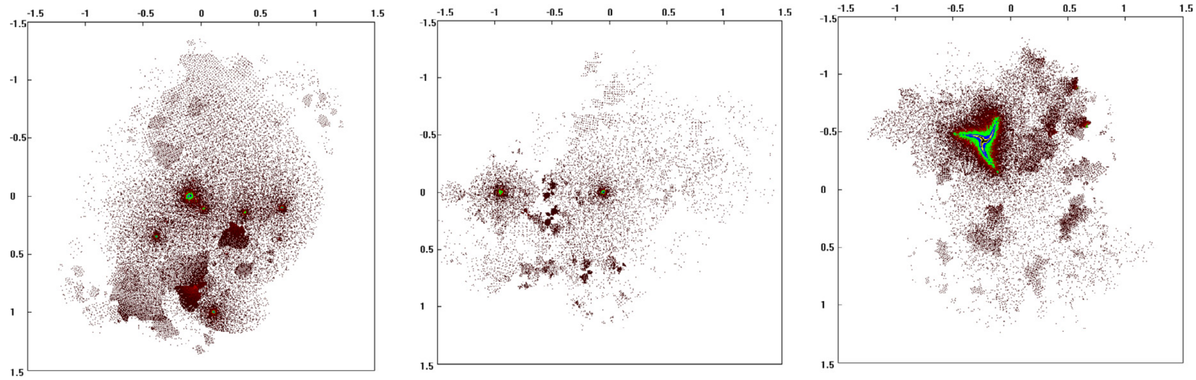


Fig. 3. J sets of $c(-0.95, 0)$ under additive noise perturbations, noise m_1 and m_2 change at the same time.



(a) c=(-0.95, 0) (b) c=(-0.1, 0) (c) c=(-0.11, -0.62)

Fig. 4. J sets under random additive noise perturbations, noise intensity keeping around 0.01.



(a) c=(-0.95, 0) (b) c=(-0.1, 0) (c) c=(-0.11, -0.62)

Fig. 5. J sets under random additive noise perturbations, noise intensity keeping around 0.1.

Fig. 1 shows the variation of the J set with the value of point c $(-0.95, 0)$ under conditions that m_1 changes and m_2 keeps zero. Fig. 1(a) shows a 2-cycle J set with zero noise intensity. According to the experimental results, if m_1 is an increasing positive number, the J set will gradually transform from 2-cycle to 1-cycle (see Fig. 1(b) and (c)). If $m_1 = 1.22$, the J set tends to be scattered; if m_1 continues to increase, the J set will be totally escaped and scattered (see Fig. 1(d)). Otherwise if m_1 is a decreasing negative number, the J set will change from 2-cycle to 4-cycle (see Fig. 1(e)). The image will be scattered and eventually escape as m_1 decreasing (see Fig. 1(f)).

Fig. 2 shows the variation of the J set with the value of point c $(-0.1, 0)$ under conditions that m_1 keeps zero and m_2 changes. Fig. 2(a) shows a 1-cycle J set with zero noise intensity. According to the experimental results, if $m_2 \in [0, 0.9]$ and is increasing, the J set will gradually transform from 1-cycle to 3-cycle and then to 6-cycle (see Fig. 2(b)–(f)). Meanwhile, its structure evolves from being interconnected to disconnected until Cantor dust appear and finally completely escape (see Fig. 2(g)). If $m_2 \in [-0.9, 0]$ and is decreasing, it shares the similar evolution with the J sets when $m_2 \in [0, 0.9]$ and is increasing (see Fig. 2(h)–(j)). There is also some interesting phenomenon in these J sets. Comparing the images in Fig. 2(c)(f) and Fig. 2(i)(j), we find that if the values of m_2 are opposite numbers the images will be symmetry around the horizontal axis.

Fig. 3 shows the variation of J set with the value of point $c(-0.95, 0)$ under the condition that m_1 and m_2 changes equivalently at the same time. According to experimental results, if m_1 and m_2 are increasing simultaneously in the scope of [0, 0.89] with the same value, inner structure of the J set changes according to the following rules. The initial connected 2-cycle image become disconnected and even totally escape (see Fig. 3(a)–(c)); as the value increasing, it becomes to be a 1-cycle J set again and tends to be disconnected and totally escape (see Fig. 3(d) and (e)); it finally changes to be 3-cycle and then becomes disconnected, eventually escaping (see Fig. 3(f)–(i)). In this process, a detail should be mentioned that when $m_1 = m_2 = 0.88$, the J set totally escapes and shows nearly no image; when m_1 and m_2 increases to 0.89, Cantor dust appears. However, as m_1 and m_2 increases further more, no Cantor dust appears again. Inner structure of the J set changes simply when m_1 and m_2 decreases simultaneously with the same value in the range of $[-0.25, 0]$ (see Fig. 3(j)–(l)). The initial connected 2-cycle image just changes gradually to be disconnected and eventually completely escape.

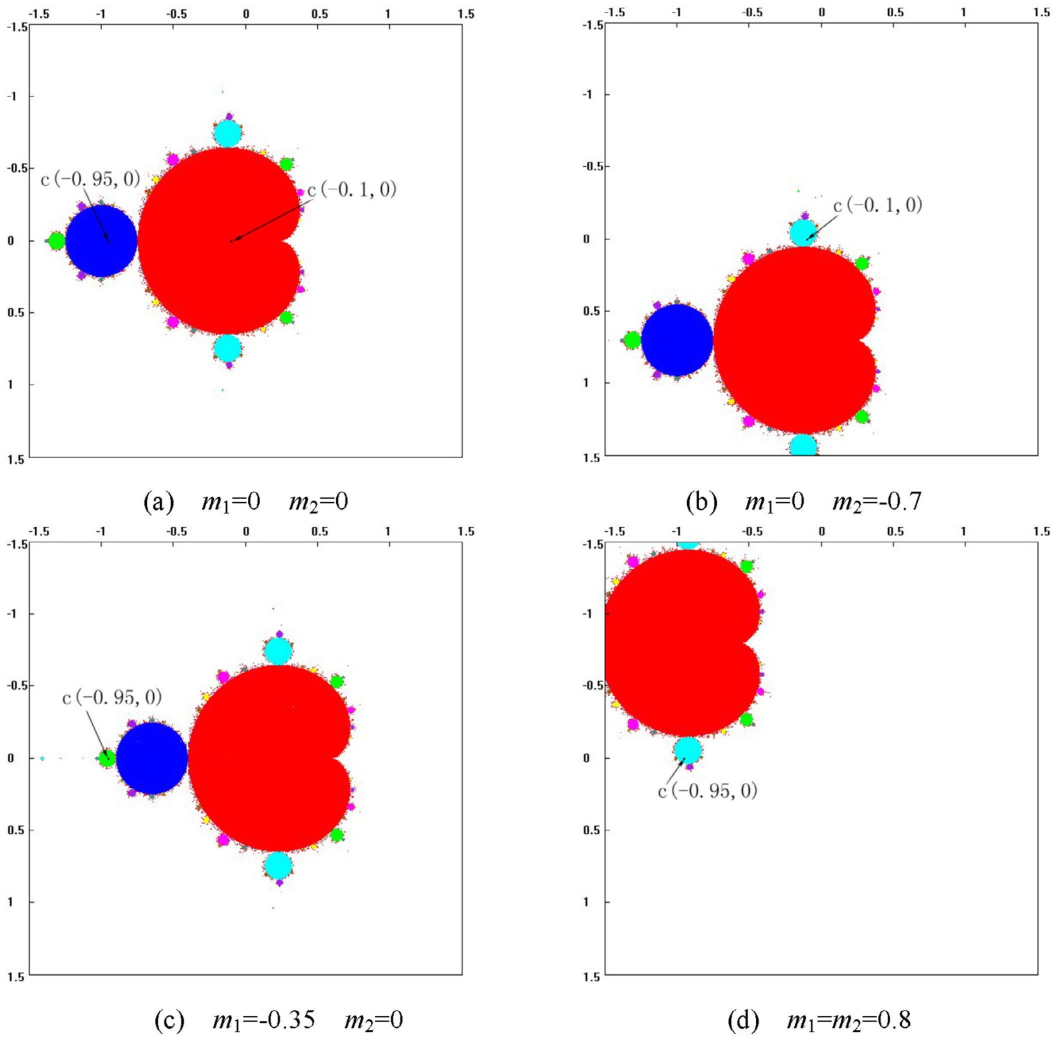


Fig. 6. M sets under additive noise perturbations.

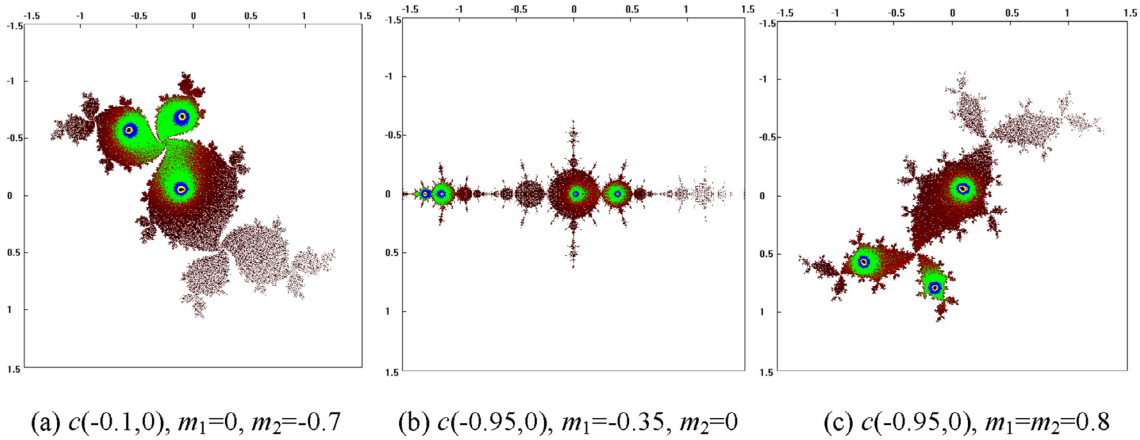


Fig. 7. J sets corresponding to the points of M sets in Fig. 6(b)–(d).

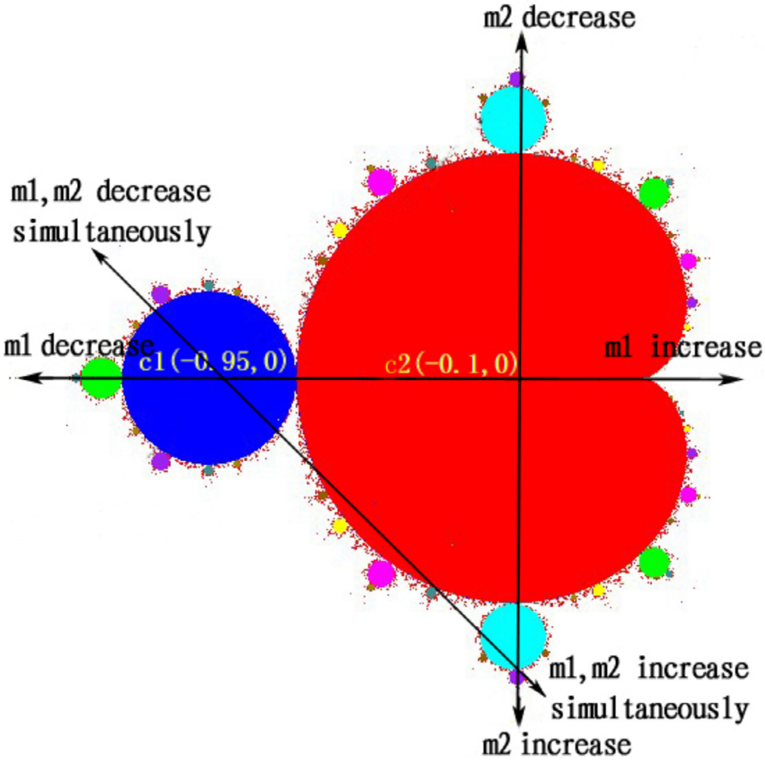


Fig. 8. Movement of point c in the M set.

Table 1
Periodic points of J set and M set under the same additive noise perturbation.

	$m_1 = -0.35, m_2 = 0$	$m_1 = 0, m_2 = -0.7$	$m_1 = m_2 = 0.8$
J set	(0.019430; 0) (−1.299620; 0) (0.389019; 0) (−1.148660; 0)	(−0.108717; −0.046786) (−0.090370; −0.689827) (−0.567695; −0.575321)	(0.090153; −0.058381) (−0.145281; 0.789474) (−0.752162; 0.570609)
M set	(0.019430; 0) (−1.299620; 0) (0.389019; 0) (−1.148660; 0)	(−0.108717; −0.046786) (−0.090370; −0.689827) (−0.567695; −0.575321)	(0.090153; −0.058381) (−0.145281; 0.789474) (−0.752162; 0.570609)

Fig. 4 and 5 show the J sets with random noise perturbations, noise intensity around 0.01 and 0.1, respectively. Comparing the non-perturbed J sets with the ones in Fig. 4, there almost no difference between them with the noise intensity keeping around 0.01. However, the noise can destroy the stability of J sets if the random noise intensity increases to 0.1 level, as shown in Fig. 5. The pictures in Fig. 5 are drawn by only 10 iterations and the majority of J sets will escape within 20 iterations.

3.2. M set under multiplicative noise perturbation

The M sets under additive noise perturbations are shown in Fig. 6. Two points $c_1(-0.95,0)$ and $c_2(-0.1,0)$, which are the parameters of the J sets as discussed in Section 3.1, are marked in Fig. 6.

Fig. 6 shows the changes of the M sets under the same additive noise perturbations with the J sets in Section 3.1. It is noted that additive noise perturbation only causes the movement of M set. That is to say, m_2 results in the movement of the image along the longitudinal axis (see Fig. 6(b)); m_1 causes the image moving towards the horizontal direction (see Fig. 6(c)); if m_1 and m_2 changes simultaneously with the same intensity, the image moves in the direction with the longitudinal axis of a 45°angle (see Fig. 6(d)). Fig. 6 illustrates the same points are in different positions of different noise perturbed M sets. To make it clear, we draw J sets in Fig. 7 with c corresponding to the points of M sets in Fig. 6.

It has been known that the center position of the M set 1-cycle region is $C = 0$ and the boundary of the 1-cycle region, the large cardioid-shaped region of the M set, is

$$C = 1/4 + \sin^2\frac{\theta}{2}e^{i\theta}(0 \le \theta \le 2\pi).$$

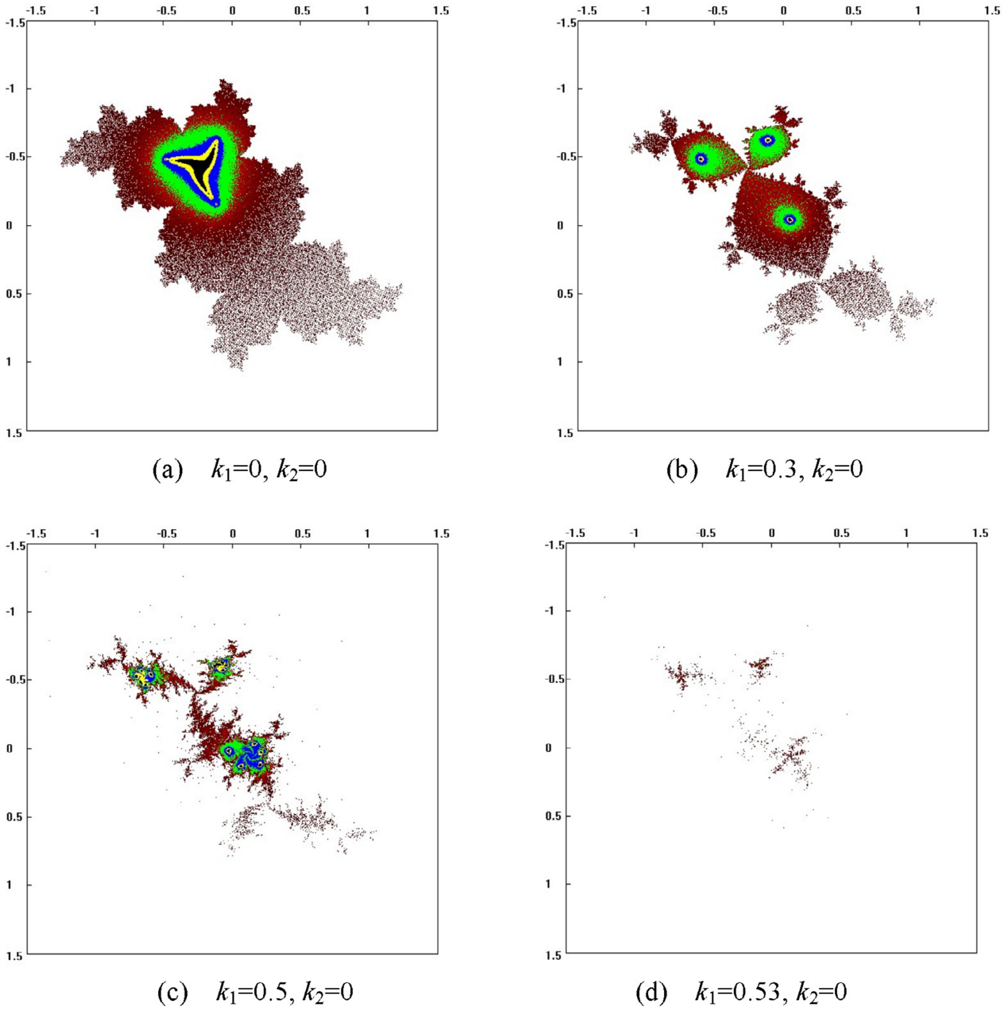


Fig. 9. J sets of $c(-0.11, -0.62)$ under multiplicative noise perturbations, noise k_1 changes and $k_2 = 0$.

The center position of 2-cycle region is $C = -1$ and the boundary of the region, the circular area, is

$$C = -1 + \frac{1}{4}e^{i\theta}(0 \le \theta \le 2\pi).$$

When $m_1 m_2 \neq 0$, the M–J sets are perturbed by additive noise. The impact of the noise on the iteration function is mainly reflected in the value of C. Let $C' = C + m_1 + im_2$, then the J set corresponding to C' is the same as the one corresponding to C.

At the same time, the additive noise perturbations only change the positions of the M sets instead of the structures. The center position of 1-cycle region for the perturbed M set is $C = -m_1 - im_2$. The boundary of the 1-cycle region changes to

$$C = 1/4 - \sqrt{m_1^2 + m_2^2}e^{ia} + \sin^2\frac{\theta}{2}e^{i\theta}, \quad \left(0 \le \theta \le 2\pi, a = \arctan\frac{m_2}{m_1}\right).$$

The 2-cycle region is the circular area with the center $C = -1 - m_1 - im_2$ and radius 1/4. For 3 or larger cycle stable regions, only position movements occur with the shape of their boundaries unchanged.

In order to explain how additive noise influences the J set from another point of view, we mark the movement trajectory of the points c_1 and c_2 in the M set as m_1 and m_2 changes. Fig. 8 shows the movement of c_1 and c_2 when their corresponding J sets are perturbed by additive noise. Fig. 8 shows obviously, as the change of noise intensity, point c will move along the direction of the arrow, appearing in different cycles and eventually leap over the M set. Therefore, the corresponding J sets become scattered and disconnected and eventually disappear.

The transverse line in Fig. 8 indicates the process J set changes in Fig. 1. The change of the J set when m_1 increases and m_2 keeps zero corresponds to the movement of point c in positive direction of the transverse line. Thus the change of the corresponding J set should be 2-cycle→1-cycle→scatter and escape (see Fig. 1 (a)–(d)). The change of the J set when m_1 decreases and m_2

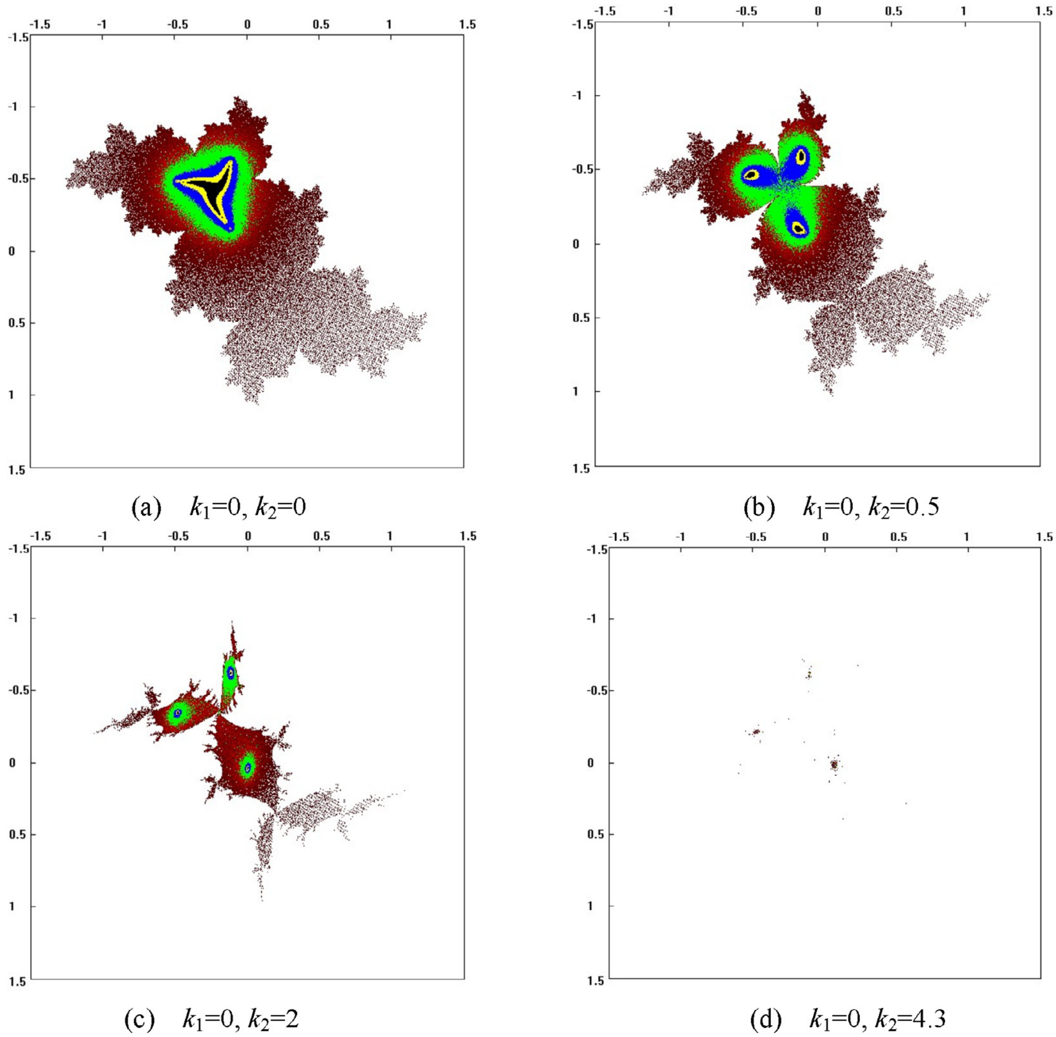


Fig. 10. J sets of $c(-0.11, -0.62)$ under multiplicative noise perturbations, noise $k_1 = 0$ and k_2 changes.

keeps zero corresponds to the movement of point c in negative direction of the transverse line. The change of the corresponding J set is 2-cycle→4-cycle→scatter and escape (see Fig. 1 (e)–(f)). The change of the J set when m_1 keeps zero and m_2 increases corresponds to the movement of point c in positive direction of the vertical line. Thus the change of the corresponding J set should be 1-cycle→3-cycle→6-cycle→scatter and escape (see Fig. 2 (a)–(g)). The negative direction of the diagonal line in Fig. 8 shows the change of the J set in Fig. 3 under the condition that m_1 equals to m_2 as well as m_1 and m_2 increase simultaneously with the same value. The change of the J set is 2-cycle→scatter and escape→1-cycle→scatter and escape→3-cycle→scatter and escape (see Fig. 3(a)–(i)).

3.3. Coordinate of the stabilized periodic point

It is well known that M set is the collection of the parameters of the J set. So, whether there are some affinities between these two sets when perturbed by the same noise intensity? This study tries to prove it by searching a stabilized periodic point coordinate. Cycles and inner structure of the J set change when perturbed by additive noise. Iterating sufficiently, it will be stabilized to several steady points. These stabilized points are called the stabilized periodic points. However, what the M set shows under the same parameter? Experimental results (see Table 1) show that both J set and M set share the same stabilized periodic points if they iterate sufficiently.

4. Multiplicative noise perturbation

In order to study the properties of J set and M set under the same multiplicative noise perturbation and their correlations, a point $c(-0.11,-0.62)$ is selected inside cycle 1 and near the edge of cycle 3 of the M set. The corresponding J set and M set are perturbed by the same multiplicative noise and the experimental results are analyzed.

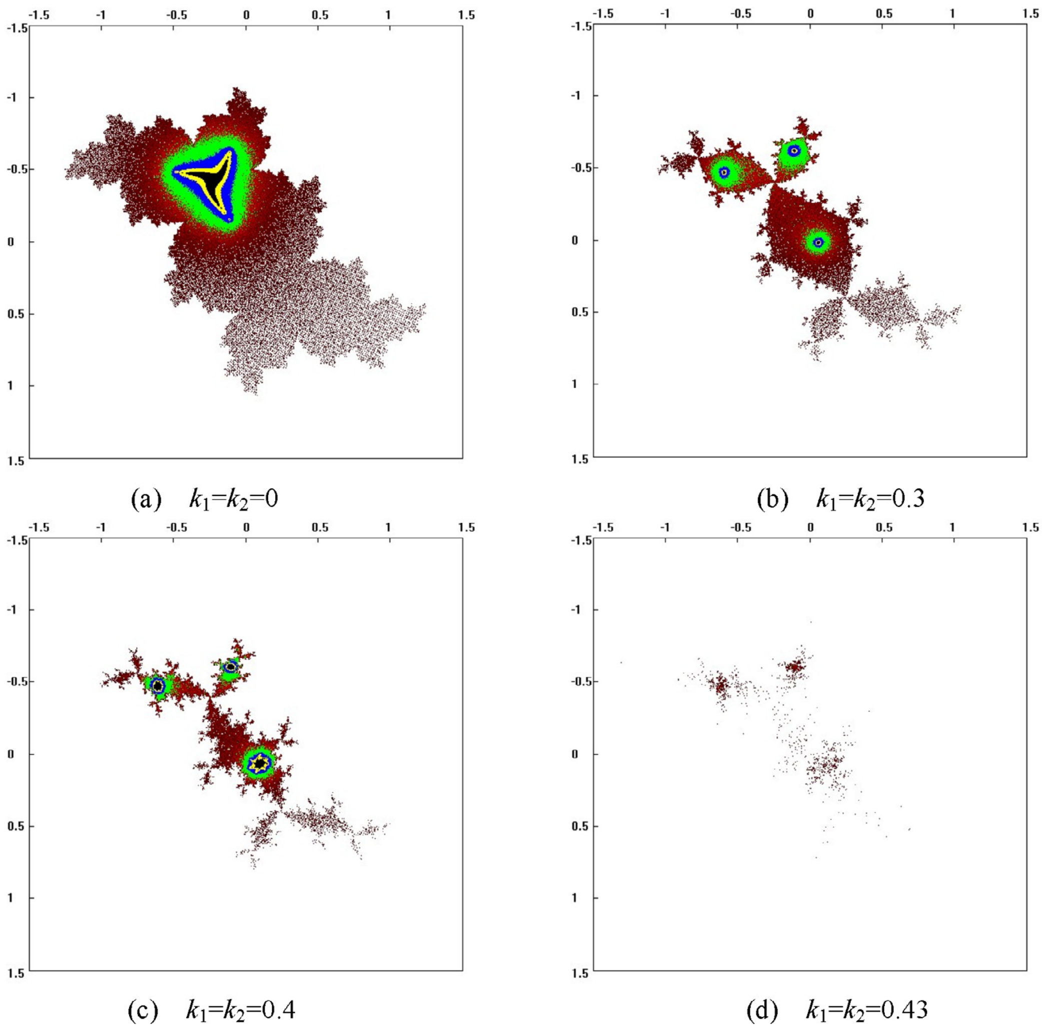


Fig. 11. J sets of $c(-0.11, -0.62)$ under multiplicative noise perturbations, noise k_1 and k_2 change at the same time.

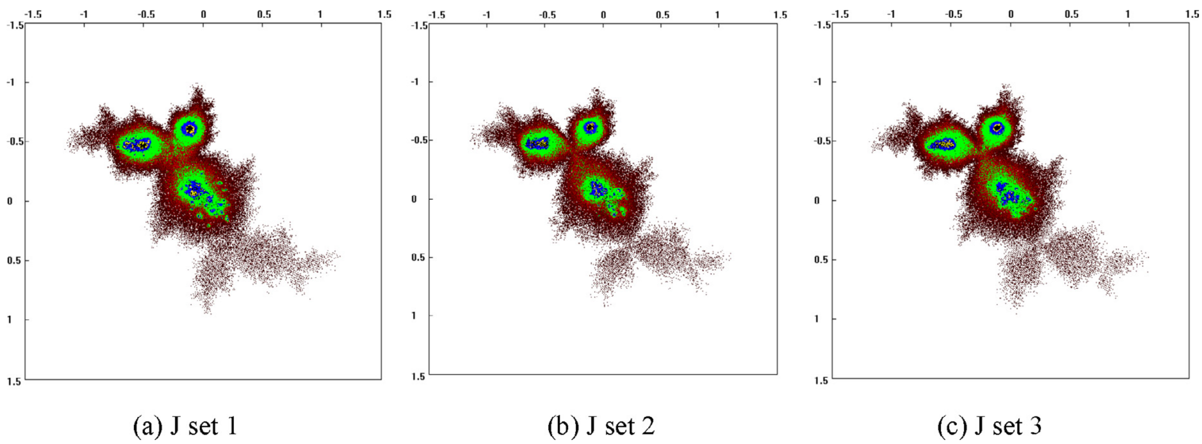


Fig. 12. J sets of $c(-0.11, -0.62)$ under three series of random multiplicative noise perturbations, noise intensity keeping in $(0, 0.5)$.

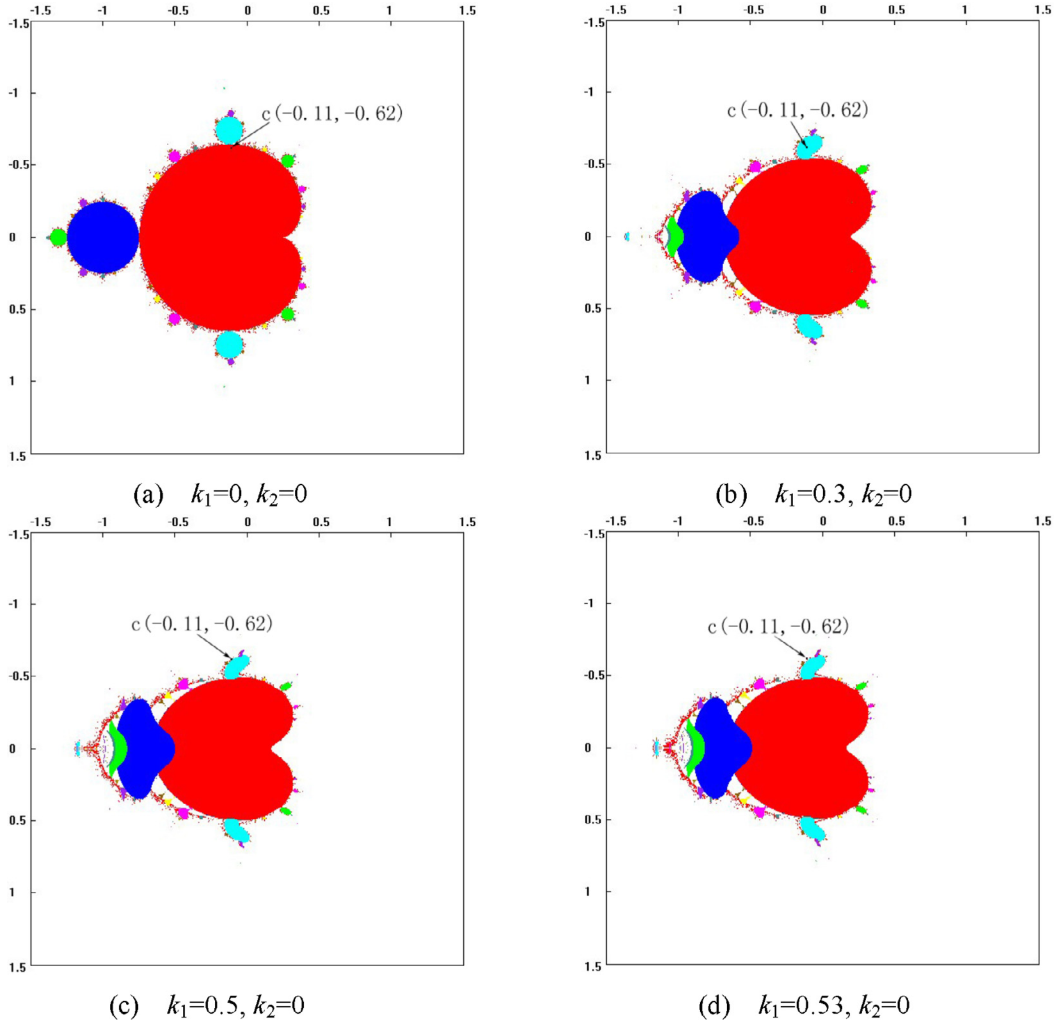


Fig. 13. M sets under multiplicative noise perturbations, noise k_1 changes and $k_2 = 0$.

Table 2
The periodic point for M–J set under the same multiplicative noise perturbation.

	$k_1 = k_2 = 0$	$k_1 = k_2 = 0.3$	$k_1 = k_2 = 0.4$
J set	(−0.233804, −0.422456)	(−0.591193, −0.469351) (0.057984,0.018197) (−0.106060, −0.617573)	(0.096398; 0.065664) (−0.103027; −0.604808) (−0.607250; −0.470453)
M set	(−0.233804, −0.422456)	(−0.591193, −0.469351) (0.057984,0.018197) (−0.106060,−0.617573)	(0.096398; 0.065664) (−0.103027; −0.604808) (−0.607250; −0.470453)

4.1. J set under multiplicative noise perturbation

Fig. 9 shows iteration process of the J set with c (−0.11, −0.62) under conditions k_1 changes and k_2 keeps zero. According to Fig. 9, when k_1 increases from zero, J set evolves from 1-cycle to 3-cycle initially, and its inner parts tend to contract. But when $k_1 = 0.5$, J set shows to be 5-cycle and its inner part is disconnected. When c keeps increasing to 0.53, the Cantor dust appears and then the J set completely escapes.

In Fig. 10, as k_2 increases the J set changes from 1-cycle to 3-cycle firstly and then the inner part contracts. The value of k_2 changes in a relatively larger scope than that of k_1. When $k_2 = 4$, the J set shows an obscure periodic feature and tends to scatter. If k_2 increase to 4.3, the Cantor dust appears, and eventually all points escape.

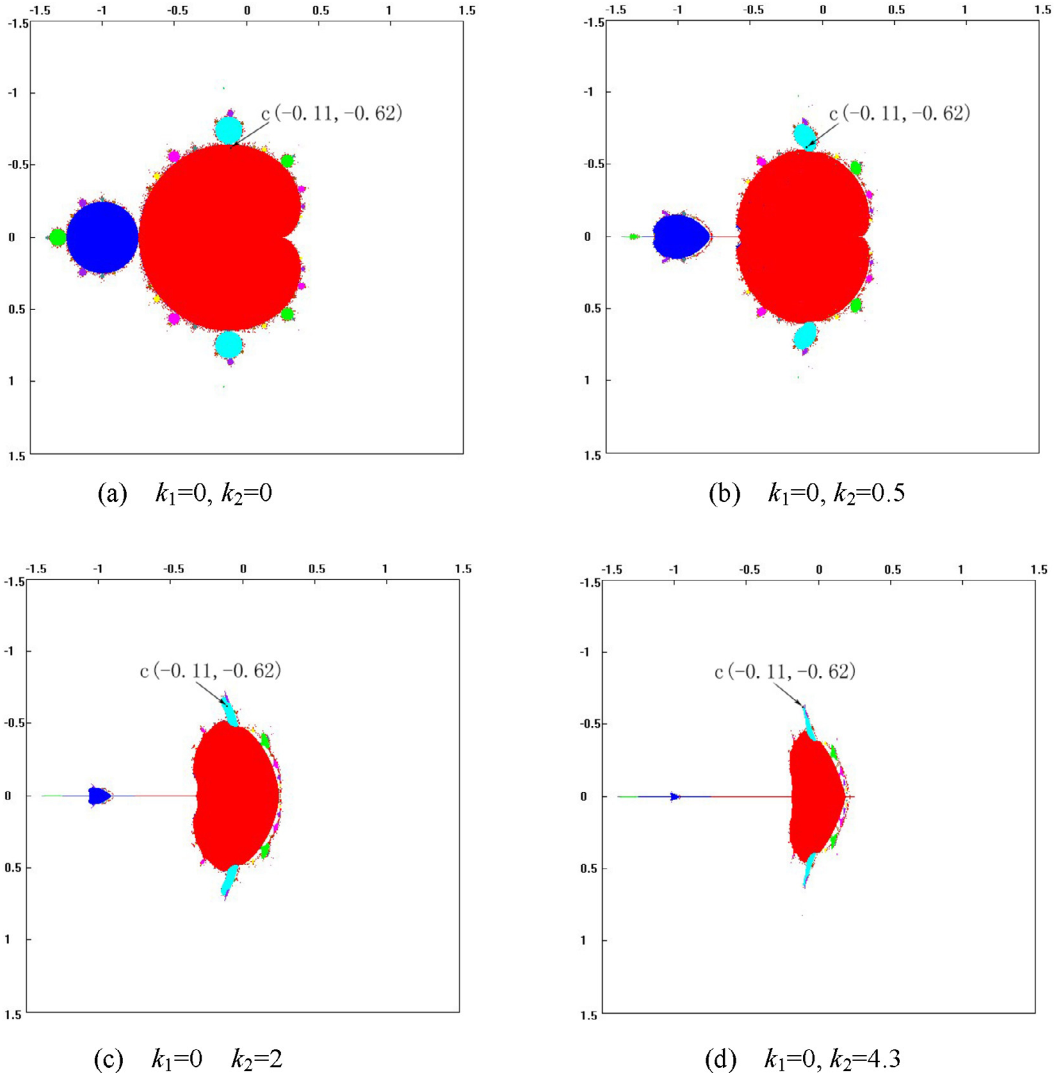


Fig. 14. M sets under multiplicative noise perturbations, noise $k_1 = 0$ and k_2 changes.

Fig. 11 shows the experimental results under the condition that k_1 and k_2 changes equivalently. Effect of noise perturbation is obvious even if c changes slightly. The process of contract is further shortened. When $k_1 = k_2 = 0.43$, Cantor dust appears. And it finally escapes as the value of k_1 and k_2 increases.

Fig. 12 shows the J sets under three series of random multiplicative noise perturbations, noise intensity keeping in the range of (0, 0.5). The pictures in Fig. 12 not only have similarities with each other, but also with the constant noise perturbed J set in Fig. 11(b), keeping 3-cycle stability. It is found that J sets will escape quickly if random noise increases above 0.5.

Plenty of experimental results indicate the range of noise intensity which keeps the structures of the J sets relatively stable. The J sets perturbed by the random noise within this range almost have the same periodicity and stable region with that of constant noise perturbed ones.

4.2. M set under multiplicative noise perturbation

When studying the M set under multiplicative noise perturbation, this paper lays emphasis on the study of the movement of point c in M set when perturbed. In Fig. 13, when k_2 keeps zero and k_1 increases at the range (0, 0.53), M set changes significantly because of being perturbed by multiplicative noise: the stabilized areas of M set are still inside the escaped area, but shrinks; its inner structure seems to be stretched in horizontal direction; the external escaped area changes little. Correspondingly, due to the effect of noise perturbation, the selected point c at the edge of cycle 1 in the M set moves to cycle 3, then to the edge of cycle 3 and eventually escapes out of the M set.

Fig. 14 shows the change of M set under the condition $k_1 = 0$ and k_2 changes, which seems to have some similarity to the M set in Fig. 13. Point c moves from the edge of cycle 1 to the inner side of cycle 1, then to the edge of cycle 3 and eventually to the

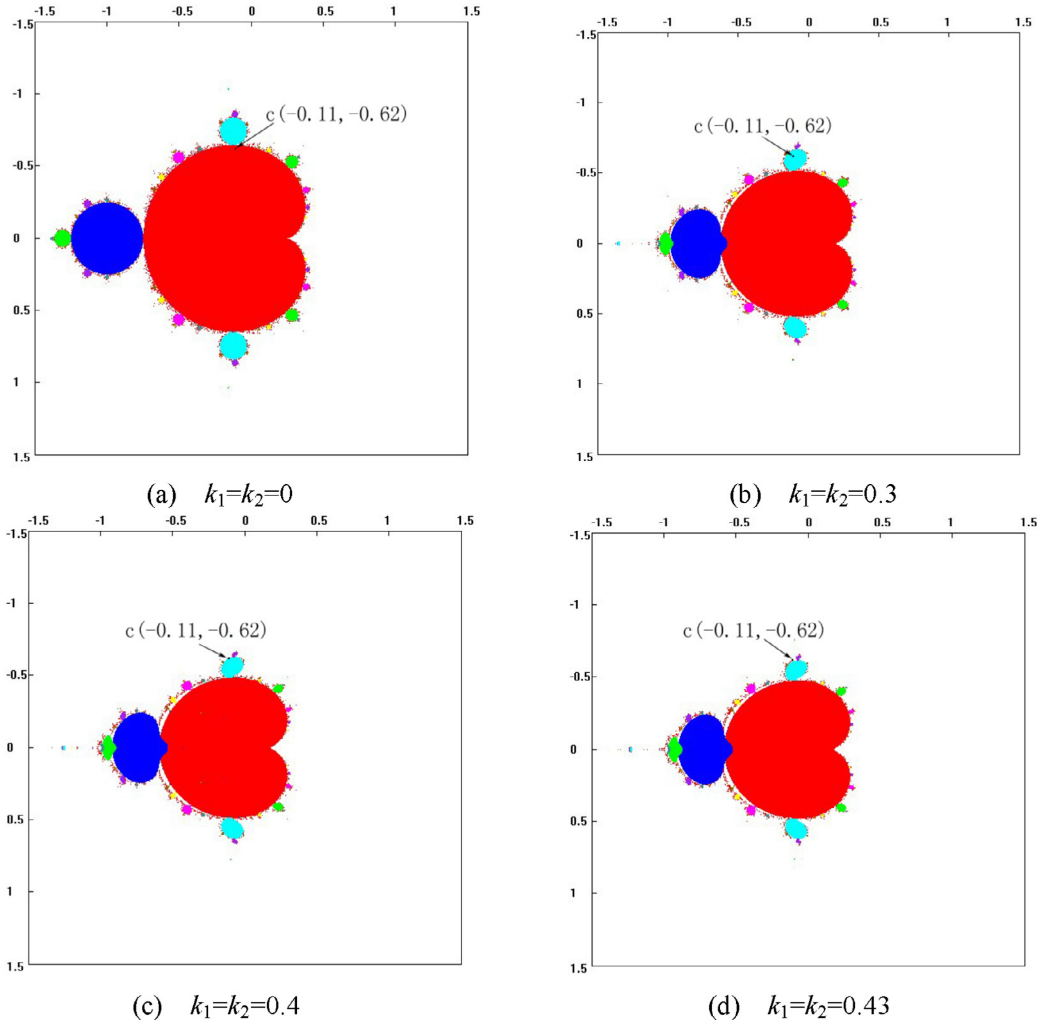


Fig. 15. M sets under multiplicative noise perturbations, noise k_1 and k_2 change at the same time.

outside of M set. One of the differences is that k_2 changes in a relatively larger range than k_1. This means that the change of k_1 leads to a greater change of M set than that caused by k_2. At the same time the outer part of M set alters as the inner stabilized part being extruded horizontally.

When M set is perturbed simultaneously by k_1 and k_2 with same values, point c moves through cycle 1 and cycle 3 and finally to the outside part of M set (Fig. 15). During this process, variations of k_1 and k_2 are in a smaller scope. And the areas of the stabilized part of the M set contract without obvious change in the inner structure.

It can be seen from Fig. 13 to Fig. 15 that whatever the noise strength is the perturbed M sets keep symmetrical with respect to X axis. As defined in Section 2, the iteration of M set under multiplicative noise perturbation is

$$x_{n+1} = (1 + k_1)x_n^2 - (1 + k_1)y_n^2 + p,$$

$$y_{n+1} = (2 + k_2)x_n y_n + q,$$

which can be converted to

$$Z_{n+1} = 1 + \frac{k_1 + k_2}{2} Z_n^2 + \frac{k_1 - k_2}{2} \overline{Z}_n^2 + C.$$

Now we give the following theorem.

Theorem The M set under multiplicative noise perturbation are symmetrical around X axis.

Proof. Let $C = ce^{i\theta}$, now we prove $f^k(C) = \overline{f^k(\overline{C})}$ $(k = 1, 2, \cdots, N)$ when $\theta \in [-\pi, \pi)$ by mathematical induction.

When $k = 1$

$$f^1(C) = \left(1 + \frac{k_1 + k_2}{2}\right)c^2 e^{2i\theta} + \frac{k_1 - k_2}{2}c^2 e^{-2i\theta} + ce^{i\theta},$$

then we have

$$f^1(\bar{C}) = \left(1 + \frac{k_1 + k_2}{2}\right)c^2 e^{-2i\theta} + \frac{k_1 - k_2}{2}c^2 e^{2i\theta} + ce^{-i\theta}$$

so $f^1(C) = \overline{f^1(\bar{C})}$.

Assume that $f^k(C) = \overline{f^k(\bar{C})}$ is true, then we have

$$f^{k+1}(C) = f^k(f^1(C)) = \overline{f^k(\overline{f^1(C)})} = \overline{f^k(f^1(\bar{C}))} = \overline{f^{k+1}(\bar{C})}.$$

The proof is completed. □

4.3. The stabilized periodic point

Like additive noise perturbation, we also discuss the periodic point for the M–J set under the same multiplicative noise perturbation. Experimental results show that the M–J set shares the same periodic point if perturbed by multiplicative noise, which is shown in Table 2.

5. Conclusions

In this study, M–J sets under additive noise perturbation and multiplicative noise perturbation are created, and changing characteristics of the J set are demonstrated and explained by analyzing topology change of the M set which is perturbed by noise. We draw the following conclusions from the experimental results:

(1) It means the value of c changes if the J set is perturbed by additive noise. Therefore, every pair of additive noise perturbation parameters corresponds to a different J set.
(2) When the M set is perturbed by additive noise, the structures of stabilized areas in the M set do not change. Different perturbation parameters only cause the movement of the M set.
(3) If the J set is perturbed by multiplicative noise, structures and cycles of the J set will change differently as the change of noise perturbation parameters.
(4) When the M set is perturbed by multiplicative noise, its inner structure changes with the stabilized areas contracting. Meanwhile, parameter k_1 is more sensitive than k_2 to the M set. And the M set keeps the symmetry with respect to X axis.
(5) If perturbed by the same noise and given the same value to c, features of the J set are decided by the position of the point c in the M set. And both sets share the same stabilized periodic point if they iterate sufficiently.
(6) The J sets perturbed by the random noise within certain range almost have the same periodicity and stable region with that of constant noise perturbed ones.

Future work will focus on the further exploration of the fractal characteristics of the generalized M–J sets with noise perturbations, structural influences of the M–J sets caused by various kinds of noise perturbations, and the inherent relationship between the M sets and the J sets, and so on.

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