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Spectral flow and Dixmier traces

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Abstract

We obtain general theorems which enable the calculation of the Dixmier trace in terms of the asymptotics of the zeta function and of the heat operator in a general semi-finite von Neumann algebra. Our results have several applications. We deduce a formula for the Chern character of an odd $\mathcal{L}^{(1,\infty)}$-summable Breuer–Fredholm module in terms of a Hochschild 1-cycle. We explain how to derive a Wodzicki residue for pseudo-differential operators along the orbits of an ergodic $\mathbf{R}^n$ action on a compact space X. Finally, we give a short proof of an index theorem of Lesch for generalised Toeplitz operators.

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1. Introduction

There is a generalisation of the usual setting of noncommutative geometry [Co1,Co2,Co3] where one replaces spectral triples by Breuer–Fredholm modules. In this situation one is given a Hilbert space $\mathcal{H}$, a C^*-algebra $\mathcal{A}$ represented in a semifinite von Neumann algebra $\mathcal{N}$ which acts on $\mathcal{H}$ and a self-adjoint unbounded operator D_0 affiliated to $\mathcal{N}$ and such that the commutator $[a, D_0]$ is bounded for a

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dense set of $a \in \mathscr{A}$ [CPS]. This situation arises for example in the twisted L^2-index theorem of Gromov [Gr]. There are also other interesting invariants of operators affiliated to $\mathscr{N}$ such as L^2 spectral flow studied in [CP1,CP2]. We became interested in the Dixmier trace and its relation to the zeta function partly as a result of the local index formula of Connes and Moscovici [CM]. In [CM] a formula for spectral flow in an $\mathscr{L}^{(p,\infty)}$-summable Fredholm module (the notation for these symmetric ideals is explained below) is given. It is natural to try to relate this formula and those for spectral flow in [CP1,CP2].

In the course of this investigation, we became aware of the subtleties in the zeta function approach to the Dixmier trace especially in the general semifinite case that we were interested in. Specifically for $T \in \mathscr{L}^{(p,\infty)}$, we asked the question of when the functional $A \to \mathrm{tr}(AT^s)$ on $\mathscr{N}$ may be used to calculate the Dixmier trace $\mathrm{tr}_\omega(AT^p)$. The strongest known result of which we were aware is contained in [Co4, Proposition 4, p. 306] for compact operators $T \geqslant 0$ whose singular values $\mu_n(T)$ satisfy $\sum_{n=0}^{N-1} \mu_n(T) = O(\log(N))$, when either $\lim_{s\to 1}(s-1)\,\mathrm{tr}(T^s)$ or $\lim_{N\to\infty}(\log N)^{-1}\sum_{n=0}^{N-1}\mu_n(T)$ exists they both do and are equal. While the somewhat nontrivial proof is not given there, it does follow as Connes states from the Hardy–Littlewood Tauberian Theorem ([H, Theorem 98] is a good reference). In the Ph.D. Thesis of Prinzis [P] an extension of this latter result was claimed in type II setting however, the proof was flawed. Additional interesting information is contained in [Co4, p. 563] where the Dixmier trace is expressed in terms of the asymptotics of the trace of the 'heat operator' $e^{\lambda^{-2/p}T^{-2}}$ as $\lambda \to \infty$. Subsequently, a proof for this result due to Connes was published for $p > 1$ in [GVF].

Our aim in this paper is to prove the strongest possible theorem relating the zeta function, the asymptotics of the trace of the heat operator and the Dixmier trace in the type I and type II setting of $\mathscr{L}^{(p,\infty)}$ summable (Breuer-)Fredholm modules $(1 \leqslant p < \infty)$. We obtain the most general results possible in the most general semifinite case without assuming that any of the above limits exist. To do this we need a rather novel approach to the Dixmier trace which we explain in the Section 1. The essence of our approach is contained in Theorem 1.5 where we observe that there are really two Dixmier traces, one which might naturally be regarded as being constructed from an invariant mean on $L^\infty(\mathbf{R})$ (with the additive group structure on $\mathbf{R}$) and the other an invariant mean on $L^\infty(\mathbf{R}_+^*)$ with the multiplicative group structure on $\mathbf{R}_+^*$. The former trace is natural from the viewpoint of the zeta function while the latter is that encountered in [Co4]. Our key observation in Section 3, where we prove the main Theorems 3.1 and 3.8, is that in order to calculate the Dixmier trace using the zeta function these traces have to be chosen in pairs related one to the other via the isomorphism from $\mathbf{R}$ to $\mathbf{R}_+^*$ given by the exponential function.

Choose a faithful, normal, semi-finite trace τ on $\mathscr{N}$ (τ will be fixed throughout). Let D_0 have resolvent in the ideal of "compact operators" in $\mathscr{N}$. An odd $\mathscr{L}^{(1,\infty)}$ summable unbounded (Breuer-)Fredholm module for a Banach *-algebra, $\mathscr{A}$ is a triple $(\mathscr{N}, \mathscr{A}, D_0)$ where $\mathscr{A} \subset \mathscr{N}$ is such that $[a, D_0]$ is bounded for all a in a dense subalgebra of $\mathscr{A}$ and $(1 + D_0^2)^{-1/2} \in \mathscr{L}^{(1,\infty)}$. Our main results (in Section 3) concern the asymptotics of $\tau(A(1+D_0^2)^{-s})$ as $s\to 1/2$ for $A\in\mathcal{N}$ and how this relates to the Dixmier trace $\tau_\omega(A(1+D_0^2)^{-1/2})$. Then in Section 4 we consider the asymptotics of the trace of the heat semigroup of D_0^2 deriving in particular the formula of [Co4, p. 563] for the Dixmier trace. Section 5 generalises all of the previous formulae to the case where $(1+D_0^2)^{-1/2}\in\mathcal{L}^{(p,\infty)}$ with $p>1$.

In Theorem 6.2 we apply our results on the zeta function approach to the Dixmier trace, using [CP1,CP2], to derive a general formula for the Chern character of an $\mathcal{L}^{(1,\infty)}$ summable Breuer–Fredholm module $(\mathcal{N},\mathcal{A},D_0)$.

In Section 7 we give a brief overview of the results in [P] on a Wodzicki residue formula for the Dixmier trace of pseudo-differential operators tangential to a minimal ergodic action of $\mathbf{R}^n$ on a compact space. Our aim here is to show how the results of the earlier sections may be used to overcome a technical difficulty in Prinzis' approach.

Section 8 contains our short proof of the theorem of Lesch giving the index of a generalised Toeplitz operator associated with an action of $\mathbf{R}$ on a C^*-algebra equipped with an invariant trace. The argument depends in an essential way on our results in Section 3 on the type II Dixmier trace and zeta function and shows that the index theory of Toeplitz operators with noncommutative symbol is a corollary of results in noncommutative geometry.

1.1. Generalities on singular traces

We have two groups, the additive group $\mathbf{R}$ and the multiplicative group $\mathbf{R}_+^*$ of positive reals. The exponential map and the log are mutually inverse isomorphisms between these groups. Notice that exp takes translation by $a\in\mathbf{R}$ to dilation by $\exp(a)\in\mathbf{R}_+^*$ and dilation by $b\in\mathbf{R}_+^*$ to the transformation $x\mapsto x^b$ on $\mathbf{R}_+^*$. Let G_1 and G_2 be given by taking the semidirect product of the group $\mathbf{R}$ and dilations and the semidirect product of the group of powers with $\mathbf{R}_+^*$, respectively. That is, G_1 is the set $\mathbf{R}\times\mathbf{R}_+^*$ with multiplication:

$$(a,s)(b,t)=(a+sb,st).$$

While, G_2 is the set $\mathbf{R}_+^*\times\mathbf{R}_+^*$ with multiplication:

$$(s,t)(x,y)=(sx^t,ty).$$

Then, exp and log induce mutually inverse isomorphisms of G_1 and G_2. For example, the isomorphism $G_1\to G_2$ is given by

$$(a,s)\mapsto(\exp(a),s):G_1\to G_2.$$

Definition 1.1. We define the isomorphism $L:L^\infty(\mathbf{R})\to L^\infty(\mathbf{R}_+^*)$ by $L(f)=f\circ\log$. We also define the Hardy and Cesaro means (transforms) on $L^\infty(\mathbf{R})$ and $L^\infty(\mathbf{R}_+^*)$, respectively, by

$$H(f)(u) = \frac{1}{u}\int_0^u f(v)\,dv \quad \text{for } f \in L^\infty(\mathbf{R}), \quad u \in \mathbf{R}$$

and

$$M(g)(t) = \frac{1}{\log t}\int_1^t g(s)\frac{ds}{s} \quad \text{for} \quad g \in L^\infty(\mathbf{R}_+^*), \quad t > 0.$$

We refer to H as the mean for the additive group $\mathbf{R}$.

Then a brief calculation yields for $g \in L^\infty(\mathbf{R}_+^*)$,

$$LHL^{-1}(g)(r) = \frac{1}{\log r}\int_0^{\log r} g(e^u)\,du = \frac{1}{\log r}\int_1^r g(v)\frac{dv}{v} = M(g)(r).$$

So indeed L intertwines the two means.

Definition 1.2. We also define the following families of self-maps on these L^∞ spaces: let T_b denote translation by $b \in \mathbf{R}$, D_a denote dilation by $a \in \mathbf{R}_+^*$ and let P^a denote exponentiation by $a \in \mathbf{R}_+^*$. That is,

$$T_b(f)(x) = f(x+b) \quad \text{for} \quad f \in L^\infty(\mathbf{R}),$$

$$D_a(f)(x) = f(ax) \quad \text{for} \quad f \in L^\infty(\mathbf{R})$$

and

$$P^a(f)(x) = f(x^a) \quad \text{for} \quad f \in L^\infty(\mathbf{R}_+^*).$$

Some of the basic relations between these L^∞ spaces and their self-maps are provided for easy access by the following proposition.

Proposition 1.3. *$L^\infty(\mathbf{R})$ together with the self-maps, D_a, T_b, and H $(a>0, b \in \mathbf{R})$ is related to $L^\infty(\mathbf{R}_+^*)$ together with the self-maps, P^a, D_a, and M $(a>0)$ via the isomorphism*

$$L : L^\infty(\mathbf{R}) \to L^\infty(\mathbf{R}_+^*)$$

and the following identities:

(1) $LD_aL^{-1} = P^a$ *for* $a>0$,
(2) $LT_bL^{-1} = D_{\exp(b)}$ *for* $b \in \mathbf{R}$ (*or* $LT_{\log(a)}L^{-1} = D_a$ *for* $a>0$),
(3) $LHL^{-1} = M$,

(4) $D_aH = HD_a$ *and* $P^aM = MP^a$ *for* $a>0$,
(5) $\lim_{t\to\infty}(HT_b - T_bH)f(t) = 0$ *for* $f\in L^\infty(\mathbf{R})$ *and* $b\in\mathbf{R}$,
(6) $\lim_{t\to\infty}(MD_a - D_aM)f(t) = 0$ *for* $f\in L^\infty(\mathbf{R}_+^*)$ *and* $a>0$.

Proof. We have already shown (3). The calculations for (1), (2), and (4) are equally straightforward. To see (5), take $b\in\mathbf{R}$ and $f\in L^\infty(\mathbf{R})$, then

$$\begin{aligned}(HT_b - T_bH)f(t) &= \frac{1}{t}\int_0^t f(x+b)\,dx - \frac{1}{t+b}\int_0^{t+b} f(x)\,dx\\ &= \frac{1}{t}\int_b^{t+b} f(x)\,dx - \frac{1}{t+b}\int_0^{t+b} f(x)\,dx\\ &= \left(\frac{1}{t} - \frac{1}{t+b}\right)\int_b^{t+b} f(x)\,dx - \frac{1}{t+b}\int_0^b f(x)\,dx\\ &= \frac{b}{t(t+b)}\int_b^{t+b} f(x)\,dx - \frac{1}{t+b}\int_0^b f(x)\,dx.\end{aligned}$$

In absolute value this is less than or equal to

$$\frac{||f||\cdot|b|}{|t+b|} + \frac{||f||\cdot|b|}{|t+b|} = \frac{2||f||\cdot|b|}{|t+b|},$$

which vanishes as $t\to\infty$.

The proof of (6) is similar. □

We give G_1 and G_2 the discrete topology to simplify the discussion and note that they are amenable being extensions of one abelian group by a second. They act as groups of homeomorphisms of $\mathbf{R}$ and $\mathbf{R}_+^*$, respectively, via $\tilde{\alpha}_{a,s}(y) = a + sy$ for $(a,s)\in G_1$, $y\in\mathbf{R}$ and $\alpha_{s,t}(x) = sx^t$ for $(s,t)\in G_2$, $x\in\mathbf{R}_+^*$. Furthermore, there are actions of the groups G_1 and G_2 on $L^\infty(\mathbf{R})$ and $L^\infty(\mathbf{R}_+^*)$. These actions are generated by $\{T_b, D_a \mid b\in\mathbf{R}, a\in\mathbf{R}_+^*\}$ in the case of G_1 and $\{D_a, P^c | a, c\in\mathbf{R}_+^*\}$ in the case of G_2 and L intertwines these actions. Thus, we have actions

$$G_1\times L^\infty(\mathbf{R})^*\to L^\infty(\mathbf{R})^* \quad\text{and}\quad G_2\times L^\infty(\mathbf{R}_+^*)^*\to L^\infty(\mathbf{R}_+^*)^*$$

given, respectively, by

$$[(a,s),\tilde{\omega}]\mapsto\tilde{\alpha}^*_{a,s}(\tilde{\omega}) \quad\text{where } \tilde{\alpha}^*_{a,s}(\tilde{\omega})(f) = \tilde{\omega}(f\circ\tilde{\alpha}^{-1}_{a,s})$$
$$\text{for } (a,s)\in G_1, \tilde{\omega}\in L^\infty(\mathbf{R})^*, f\in L^\infty(\mathbf{R})$$

and

$$[(s,t),\omega]\mapsto\alpha^*_{s,t}(\omega) \quad\text{where } \alpha^*_{s,t}(\omega)(f) = \omega(f\circ\alpha^{-1}_{s,t})$$
$$\text{for } (s,t)\in G_2, \omega\in L^\infty(\mathbf{R}_+^*)^*, f\in L^\infty(\mathbf{R}_+^*).$$

These are weak*-continuous actions because, for example, if $\omega_\beta \to \omega$ is a net in $L^\infty(\mathbf{R}_+^*)^*$ then

$$|\alpha_{s,t}^*(\omega_\beta)(f) - \alpha_{s,t}^*(\omega)(f)| = |\omega_\beta(f \circ \alpha_{s,t}^{-1}) - \omega(f \circ \alpha_{s,t}^{-1})| \to 0$$

as $f \circ \alpha_{s,t}^{-1} \in L^\infty(\mathbf{R}_+^*)$.

If we use these remarks together with the previous proposition we obtain the following.

Proposition 1.4. *Given any continuous functional $\tilde{\omega}$ on $L^\infty(\mathbf{R})$ which is invariant under H and G_1* then *$\tilde{\omega} \circ L^{-1}$ is a continuous functional on $L^\infty(\mathbf{R}_+^*)$ invariant under M and G_2. Conversely, composition with L converts an M and G_2 invariant continuous functional on $L^\infty(\mathbf{R}_+^*)$ into an H and G_1 invariant continuous functional on $L^\infty(\mathbf{R})$.*

1.2. Existence of invariant singular traces

We denote by $C_0(\mathbf{R}_+^*)$ the continuous functions on $\mathbf{R}_+^*$ vanishing at infinity. Our aim in this subsection is to prove the following result.

Theorem 1.5. *There exists a state ω on $L^\infty(\mathbf{R}_+^*)$ satisfying the following conditions:*

(1) $\omega(C_0(\mathbf{R}_+^*)) \equiv 0$.
(2) *If f is real-valued in $L^\infty(\mathbf{R}_+^*)$ then*

$$ess \liminf_{t\to\infty} f(t) \leqslant \omega(f) \leqslant ess \limsup_{t\to\infty} f(t).$$

(3) *If the essential support of f is compact then $\omega(f) = 0$.*
(4) *For all $c \in \mathbf{R}_+^*$, $\omega(D_c f) = \omega(f)$ for all $f \in L^\infty(\mathbf{R}_+^*)$.*
(5) *For all $a \in \mathbf{R}_+^*$ and all $f \in L^\infty(\mathbf{R}_+^*)$ $\omega(P^a f) = \omega(f)$.*
(6) *For all $f \in L^\infty(\mathbf{R}_+^*)$, $\omega(Mf) = \omega(f)$.*

Using the preceding proposition we obtain the following:

Corollary 1.6. *There exists a state $\tilde{\omega}$ on $L^\infty(\mathbf{R})$ satisfying the following conditions:*

(1) $\tilde{\omega}(C_0(\mathbf{R})) \equiv 0$.
(2) *If f is real-valued in $L^\infty(\mathbf{R})$ then*

$$ess \liminf_{t\to\infty} f(t) \leqslant \tilde{\omega}(f) \leqslant ess \limsup_{t\to\infty} f(t).$$

(3) *If the essential support of f is compact then $\tilde{\omega}(f) = 0$.*
(4) *For all $c \in \mathbf{R}$, $\tilde{\omega}(T_c f) = \tilde{\omega}(f)$ for all $f \in L^\infty(\mathbf{R})$.*

(5) *For all* $a \in \mathbf{R}_+^*$ *and all* $f \in L^\infty(\mathbf{R})$ $\tilde{\omega}(D_a f) = \tilde{\omega}(f)$.
(6) *For all* $f \in L^\infty(\mathbf{R})$, $\tilde{\omega}(Hf) = \tilde{\omega}(f)$.

Notice that L sends $C_0(\mathbf{R})$ into $C_0(\mathbf{R}_+^*)$. Also, we observe that condition (2) of the corollary is equivalent to the statement that if $f \in L^\infty(\mathbf{R})$ is continuous and $\lim_{|t| \to \infty} f(t)$ exists then $\tilde{\omega}(f) = \lim_{|t| \to \infty} f(t)$. The rest of this subsection will be devoted to the proof of the theorem. Introduce the set S consisting of all positive functionals $\omega \in L^\infty(\mathbf{R}_+^*)^*$ normalised so that $\omega(1) = 1$ and such that condition (1) of the theorem holds.

Clearly S is a convex and weak* closed subset of the unit ball. Moreover, S is nonempty as we can define $\omega \in (C[0, \infty])^*$ by $\omega(f) = f(\infty)$ then ω is positive and $\omega(1) = ||\omega|| = 1$. So extending ω to $\tilde{\omega} \in L^\infty(\mathbf{R}_+^*)^*$ by Hahn–Banach yields a nontrivial element of S (note that positivity of the extension is well known, for example see [KR, Theorem 4.3.2]).

It is straightforward to verify G_2 acts affinely (i.e. preserving convex combinations) on S by restriction of the dual action on $L^\infty(\mathbf{R}_+^*)^*$. As we have remarked earlier the action is weak* continuous and G_2 is amenable since it is the extension of an abelian group by an abelian group (and so too is G_1). Hence by Rickert's Theorem [G] there is a fixed point ω_0 for this action. This fixed point satisfies conditions (1), (4) and (5) of the theorem. Condition (3) holds because if $f \geqslant 0$ and has compact support then there is a continuous function $g \geqslant f$ a.e. with $g(\infty) = 0$ and so,

$$0 \leqslant \omega_0(f) \leqslant \omega_0(g) = 0.$$

To see that ω_0 satisfies condition (2), let f be real valued and let C denote the *ess* $\limsup_{t \to \infty} f(t)$. Then for each $\varepsilon > 0$ there exists a function g with the support of g compact and $(f - g) \leqslant C + \varepsilon$ a.e. Then $\omega_0(f) \leqslant C + \varepsilon$. Similarly $\omega_0(f)$ is bounded below by the essential $\liminf_{t \to \infty} f(t)$.

Let M^* denote the linear map on $L^\infty(\mathbf{R}_+^*)^*$ given by $M^*\omega(f) = \omega(Mf)$. Finally, to prove (6) we note first that M leaves $C_0(\mathbf{R}_+^*)$ and the constant functions invariant and hence M^* leaves S invariant. By Proposition 1.3 (part (6) and the second half of part (4)), we see that the action of M^* (*on* S!) commutes with the dual actions of the generators, D_a and P^a of G_2 (*on* S!). It follows then that for any fixed point ω_0 of the G_2 action, $\omega_0 \circ M^*$ is another fixed point of the G_2 action on S. In other words, M^* leaves the set of G_2 fixed points of S invariant. Thus M^* leaves the set of functionals in S satisfying conditions (1)–(5) invariant. The collection of fixed points for G_2 is clearly a weak-* compact convex set invariant under the (affine) action of M^*. It follows from the Kakutani–Markov Theorem [E] that M^* itself has a fixed point in this subset which is therefore a functional satisfying conditions (1)–(6) of the theorem completing the proof.

Remark. The spirit of the approach of this section goes back to Dixmier [Dix1]. The approach of Connes [Co4] is different in a slightly subtle way which we will not go into fully here. Suffice to say that [Dix1] uses dilation invariant functionals from the start while [Co4] uses the Cesaro mean to obtain a dilation invariant functional (that is, starting from a state ω on $L^\infty(\mathbf{R}_+^*)^*$ one observes that $\omega \circ M^*$ is dilation invariant). This difference is important to us in Sections 5 and 6.

1.3. Notation

We are interested in certain ideals of operators in the von Neumann algebra $\mathcal{N}$ defined using our faithful, normal, semifinite trace τ.

Definition 1.7. If $S \in \mathcal{N}$ the *t-th generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{||SE|| \mid E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-E) \leqslant t\}.$$

We will mostly explain the results we need about these singular values later in the text although a full exposition is contained in [F,FK]. We write $T_1 \prec\prec T_2$ to mean that $\int_0^t \mu_s(T_1)\, ds \leqslant \int_0^t \mu_s(T_2)\, ds$ for all $t > 0$.

Definition 1.8. If $\mathcal{I}$ is a $*$-ideal in $\mathcal{N}$ which is complete in a norm $||\cdot||_{\mathcal{I}}$ then we will call $\mathcal{I}$ an *invariant operator ideal* if

(1) $||S||_{\mathcal{I}} \geqslant ||S||$ for all $S \in \mathcal{I}$,
(2) $||S^*||_{\mathcal{I}} = ||S||_{\mathcal{I}}$ for all $S \in \mathcal{I}$,
(3) $||ASB||_{\mathcal{I}} \leqslant ||A||\, ||S||_{\mathcal{I}} ||B||$ for all $S \in \mathcal{I}$, $A, B \in \mathcal{N}$.

Since $\mathcal{I}$ is an ideal in a von Neumann algebra, it follows from 1.1.6, Proposition 10 of [Dix] that if $0 \leqslant S \leqslant T$ and $T \in \mathcal{I}$, then $S \in \mathcal{I}$ and $||S||_{\mathcal{I}} \leqslant ||T||_{\mathcal{I}}$. Much more is true, especially in type I case but we shall not need it here, see [GK].

The main examples of such ideals that we consider in this paper are the spaces

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \mid ||T||_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds < \infty \right\}$$

and with $p > 1$,

$$\psi_p(t) = \begin{cases} t & \text{for } 0 \leqslant t \leqslant 1, \\ t^{1-\frac{1}{p}} & \text{for } 1 \leqslant t, \end{cases}$$

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \mid ||T||_{\mathcal{L}^{(p,\infty)}} := \sup_{t>0} \frac{1}{\psi_p(t)} \int_0^t \mu_s(T)\, ds < \infty \right\}.$$

There is also the equivalent definition

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \mid \sup_{t>0} \frac{t}{\psi_p(t)} \mu_t(T) < \infty \right\}.$$

It is well-known (see e.g. [Co4,GK]) that for $T_1 \in \mathcal{N}$, $T_2 \in \mathcal{L}^{(p,\infty)}(\mathcal{N})$, $p \in [1, \infty)$, the condition $T_1 \prec\prec T_2$ implies that $T_1 \in \mathcal{L}^{(p,\infty)}(\mathcal{N})$.

As we will not change $\mathcal{N}$ throughout the paper we will suppress the $(\mathcal{N})$ to lighten the notation. On this point, however, the reader should note that $\mathcal{L}^{(p,\infty)}$ is often taken to mean an ideal in the algebra $\tilde{\mathcal{N}}$ of measurable operators affiliated to $\mathcal{N}$. Our notation is, however, consistent with that of [Co4] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$.

For most of the paper T is a positive operator in $\mathcal{L}^{(1,\infty)}$. There is a map from the positive operators in $\mathcal{L}^{(1,\infty)}$ to $L^\infty[0, \infty)$ given by $T \to f_T$ where $f_T(t) = \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds$. We may extend f_T to all of $\mathbf{R}$ by defining it to be zero on the negative reals. Depending on the circumstances we can thus regard f_T as either an element of $L^\infty(\mathbf{R})$ or $L^\infty(\mathbf{R}_+^*)$.

Henceforth, we use the notation $\tau_\omega(T)$ for $\omega(f_T)$ where $\omega \in L^\infty(\mathbf{R}_+^*)^*$ satisfies the conditions of Theorem 1.5. We also write

$$\tau_\omega(T) = \omega - \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds.$$

It follows from [Co4, IV.2.β], (see also [DPSS, Example 2.5]) that $\tau_\omega(\cdot)$ is additive and positively homogeneous on the positive part of $\mathcal{L}^{(1,\infty)}$ and hence extends to a positive linear functional on $\mathcal{L}^{(1,\infty)}$ (again denoted by τ_ω). It is in fact an example of a singular trace on $\mathcal{N}$ (cf. the discussion in [Co4,DPSS]).

2. Preliminary results

It is useful to have an estimate on the singular values of the operators in $\mathcal{L}^{(1,\infty)}$.

Lemma 2.1. *For $T \in \mathcal{L}^{(1,\infty)}$ positive there is a constant $K > 0$ such that for each $p \geqslant 1$,*

$$\int_0^t \mu_s(T)^p\, ds \leqslant K^p \int_0^t \frac{1}{(s+1)^p}\, ds.$$

Proof. By Fack and Kosaki [FK], Lemma 2.5(iv), for all $0 \leqslant T \in \mathcal{N}$ and all continuous increasing functions f on $[0, \infty)$ with $f(0) \geqslant 0$, we have $\mu_s(f(T)) = f(\mu_s(T))$ for all $s > 0$. Combining this fact with the well-known result of Hardy–Littlewood–Pólya (see e.g. [F, Lemma 4.1]), we see that $T_1 \prec\prec T_2$, $0 \leqslant T_1, T_2 \in \mathcal{N}$

implies $T_1^p \prec\prec T_2^p$ for all $p \in (1, \infty)$. Now, by definition of $\mathscr{L}^{(1,\infty)}$ the singular values of T satisfy $\int_0^t \mu_s(T)\,ds = O(\log t)$ so that for some $K > 0$,

$$\int_0^t \mu_s(T)\,ds \leqslant K \int_0^t \frac{1}{(s+1)}\,ds \quad \forall t > 0.$$

In other words $\mu_s(T) \prec\prec K/(1+s)$ and the assertion of lemma follows immediately. □

Theorem 2.2 (Weak*-Karamata theorem). *Let $\tilde{\omega} \in L^\infty(\mathbf{R})^*$ be a dilation invariant state and let β be a real valued, increasing, right continuous function on $\mathbf{R}_+$ which is zero at zero and such that the integral $h(r) = \int_0^\infty e^{-\frac{t}{r}}\,d\beta(t)$ converges for all $r > 0$ and $C = \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} h(r)$ exists. Then*

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} h(r) = \tilde{\omega} - \lim_{t\to\infty} \frac{\beta(t)}{t}.$$

Remark. The classical Karamata theorem states, in the notation of the theorem, that if the ordinary limit $\lim_{r\to\infty} \frac{1}{r} h(r) = C$ exists then $C = \lim_{t\to\infty} \frac{\beta(t)}{t}$. The proof of this classical result is obtained by replacing, in the proof of Theorem 2.2, $\tilde{\omega} - \lim$ throughout by the ordinary limit.

Proof. Let

$$g(x) = \begin{cases} x^{-1} & \text{for } e^{-1} \leqslant x \leqslant 1, \\ 0 & \text{for } 0 \leqslant x < e^{-1}, \end{cases}$$

so that g is right continuous at e^{-1}. Then for $r > 0$, $t \to e^{-t/r} g(e^{-t/r})$ is left continuous at $t = r$. Thus, the Riemann–Stieltjes integral $\int_0^\infty e^{-t/r} g(e^{-t/r})\,d\beta(t)$ exists for each $r > 0$. We claim that for any polynomial p

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} \int_0^\infty e^{-t/r} p(e^{-t/r})\,d\beta(t) = C \int_0^\infty e^{-t} p(e^{-t})\,dt.$$

To see this first compute for $p(x) = x^n$,

$$\frac{1}{r} \int_0^\infty e^{-t/r} e^{-nt/r}\,d\beta(t) = \frac{1}{r} \int_0^\infty e^{-(n+1)t/r}\,d\beta(t).$$

Therefore

$$\frac{1}{n+1} \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r/(n+1)} \int_0^\infty e^{-(n+1)t/r}\,d\beta(t) = \frac{C}{n+1}$$

by dilation invariance of $\tilde{\omega}$. Thus

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\int_0^\infty e^{-t/r}e^{-nt/r}\,d\beta(t) = C\int_0^\infty e^{-t}(e^{-t})^n\,dt.$$

Since $\tilde{\omega}$ is linear the claim follows for all p.

Choose sequences of polynomials $\{p_n\}$, $\{P_n\}$ such that for all $x\in[0,1]$

$$-1\leqslant p_n(x)\leqslant g(x)\leqslant P_n(x)\leqslant 3$$

and such that p_n and P_n converge a.e. to $g(x)$. Then since $\tilde{\omega}$ is positive it preserves order:

$$\begin{aligned} C\int_0^\infty e^{-t}p_n(e^{-t})\,dt &= \tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\int_0^\infty e^{-t/r}p_n(e^{-t/r})\,d\beta(t) \\ &\leqslant \tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\int_0^\infty e^{-t/r}g(e^{-t/r})\,d\beta(t) \\ &\leqslant \cdots \leqslant C\int_0^\infty e^{-t}P_n(e^{-t})\,dt. \end{aligned}$$

By the Lebesgue Dominated Covergence Theorem both $\int_0^\infty e^{-t}p_n(e^{-t})dt$ and $\int_0^\infty e^{-t}P_n(e^{-t})dt$ converge to $\int_0^\infty e^{-t}g(e^{-t})dt$ as $n\to\infty$. But a direct calculation yields $\int_0^\infty e^{-t}g(e^{-t})\,dt = 1$ and

$$\int_0^\infty e^{-t/r}g(e^{-t/r})\,d\beta(t) = \beta(r).$$

Hence

$$C = \tilde{\omega} - \lim_{r\to\omega}\frac{1}{r}\int_0^\infty e^{-t/r}g(e^{-t/r})\,d\beta(t) = \tilde{\omega} - \lim_{r\to\infty}\frac{\beta(r)}{r}. \qquad \square$$

Recall that for any τ-measurable operator T, the distribution function of T is defined by

$$\lambda_t(T) := \tau(\chi_{(t,\infty)}(|T|)), \quad t>0,$$

where $\chi_{(t,\infty)}(|T|)$ is the spectral projection of $|T|$ corresponding to the interval (t,∞) (see [FK]). By Proposition 2.2 of [FK],

$$\mu_s(T) = \inf\{t\geqslant 0\colon\ \lambda_t(T)\leqslant s\},$$

we infer that for any τ-measurable operator T, the distribution function $\lambda_{(\cdot)}(T)$ coincides with the (classical) distribution function of $\mu_{(\cdot)}(T)$. From this formula and the fact that λ is right-continuous, we can easily see that for $t>0$, $s>0$

$$s\geqslant\lambda_t \Leftrightarrow \mu_s\leqslant t.$$

Or equivalently,

$$s < \lambda_t \Leftrightarrow \mu_s > t.$$

Using Remark 3.3 of [FK] this implies that:

$$\int_0^{\lambda_t} \mu_s(T)\, ds = \int_{[0,\lambda_t)} \mu_s(T)\, ds = \tau(|T|\chi_{(t,\infty)}(|T|)), \quad t > 0. \qquad (*)$$

Lemma 2.3. *For $T \in \mathscr{L}^{(1,\infty)}$ and $C > ||T||_{\mathscr{L}^{(1,\infty)}}$* we have eventually

$$\lambda_{\frac{1}{t}}(T) \leqslant Ct \log t.$$

Proof. Suppose not and there exists $t_n \uparrow \infty$ such that $\lambda_{\frac{1}{t_n}}(T) > Ct_n \log t_n$ and so for $s \leqslant Ct_n \log t_n$ we have $\mu_s(T) \geqslant \mu_{Ct_n \log t_n}(T) > \frac{1}{t_n}$. Then for sufficiently large n

$$\int_0^{Ct_n \log t_n} \mu_s(T)\, ds > \frac{1}{t_n} Ct_n \log t_n = C \log t_n.$$

Choose $\delta > 0$ with $C - \delta > ||T||_{\mathscr{L}^{(1,\infty)}}$. Then for sufficiently large n

$$\begin{aligned} C \log t_n &= (C-\delta) \log t_n + \delta \log t_n > ||T||_{\mathscr{L}^{(1,\infty)}} \log(Ct_n) + ||T||_{\mathscr{L}^{(1,\infty)}} \log(\log(t_n+1)) \\ &= ||T||_{\mathscr{L}^{(1,\infty)}} \log(Ct_n \log(t_n+1)). \end{aligned}$$

This is a contradiction with the inequality $\int_0^t \mu_s(T)\, ds \leqslant ||T||_{\mathscr{L}^{(1,\infty)}} \log(t+1)$, which holds for any $t > 0$ due to the definition of the norm in $\mathscr{L}^{(1,\infty)}$. $\square$

An assertion somewhat similar to Proposition 2.4 was formulated in [P] and supplied with an incorrect proof. We use a different approach.

Proposition 2.4. *For $T \in \mathscr{L}^{(1,\infty)}$ positive let ω be a G_2 invariant state on $L^\infty(\mathbf{R}_+^*)$. For every $C > 0$*

$$\begin{aligned} \tau_\omega(T) &= \omega - \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds \\ &= \omega - \lim_{t\to\infty} \frac{1}{\log(1+t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T)) \\ &= \omega - \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^{Ct \log t} \mu_s(T)\, ds \end{aligned}$$

and if one of the ω-limits is a true limit then so are the others.

Proof. We first note that

$$\int_0^t \mu_s(T)\,ds \leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)\,ds + 1, \quad t>0.$$

Indeed, the inequality above holds trivially if $t \leqslant \lambda_{\frac{1}{t}}(T)$. If $t > \lambda_{\frac{1}{t}}(T)$, then

$$\int_0^t \mu_s(T)\,ds = \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)\,ds + \int_{\lambda_{\frac{1}{t}}(T)}^t \mu_s(T)\,ds.$$

Now $s > \lambda_{\frac{1}{t}}(T)$ implies that $\mu_s(T) \leqslant \frac{1}{t}$ so we have

$$\int_0^t \mu_s(T)\,ds \leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)\,ds + \frac{1}{t}(t - \lambda_{\frac{1}{t}}(T)) \leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)\,ds + 1.$$

Using this observation and the lemma above we see that for $C > \|T\|_{\mathscr{L}^{(1,\infty)}}$ and any fixed $\alpha > 1$ eventually

$$\begin{aligned}\int_0^t \mu_s(T)\,ds &\leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)\,ds + 1 \leqslant \int_0^{Ct\log t} \mu_s(T)\,ds + 1\\ &\leqslant \int_0^{t^\alpha} \mu_s(T)\,ds + 1\end{aligned}$$

and so eventually

$$\begin{aligned}\frac{1}{\log(1+t)}\int_0^t \mu_s(T)\,ds &\leqslant \frac{1}{\log(1+t)}\left(\int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)\,ds + 1\right)\\ &\leqslant \frac{1}{\log(1+t)}\left(\int_0^{Ct\log t} \mu_s(T)\,ds + 1\right)\\ &\leqslant \frac{\log(1+t^\alpha)}{\log(1+t)\log(1+t^\alpha)}\left(\int_0^{t^\alpha} \mu_s(T)\,ds + 1\right).\end{aligned}$$

Taking the ω-limit we get

$$\begin{aligned}\tau_\omega(T) &\leqslant \omega - \lim_{t\to\infty} \frac{1}{\log(1+t)}\int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)\,ds\\ &\leqslant \omega - \lim_{t\to\infty} \frac{1}{\log(1+t)}\int_0^{Ct\log t} \mu_s(T)\,ds\\ &\leqslant \omega - \lim_{t\to\infty} \frac{\alpha}{\log(1+t^\alpha)}\int_0^{t^\alpha} \mu_s(T)\,ds = \alpha\tau_\omega(T),\end{aligned}$$

where the last line uses G_2 invariance. Since this holds for all $\alpha > 1$ and using $(*)$ we get the conclusion for ω-limits and $C > \|T\|_{\mathscr{L}^{(1,\infty)}}$. The assertion for an arbitrary

$0 < C \leqslant ||T||_{\mathscr{L}^{(1,\infty)}}$ follows immediately by noting that for $C' > ||T||_{\mathscr{L}^{(1,\infty)}}$ one has eventually

$$\int_0^t \mu_s(T)\,ds \leqslant \int_0^{Ct\log t} \mu_s(T)\,ds \leqslant \int_0^{C't\log t} \mu_s(T)\,ds.$$

To see the last assertion of the proposition suppose that $\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds = A$ then by the above argument we get

$$A \leqslant \liminf_{t\to\infty} \frac{1}{\log(1+t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T)) \leqslant \limsup_{t\to\infty} \frac{1}{\log(1+t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T)) \leqslant \alpha A$$

for all $\alpha > 1$ and hence $\lim_{t\to\infty} \frac{1}{\log(1+t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T)) = A$. On the other hand if the limit $\lim_{t\to\infty} \frac{1}{\log(1+t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T))$ exists and equals B say then

$$\limsup_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds \leqslant B \leqslant \alpha \liminf_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds$$

for all $\alpha > 1$ and so

$$\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds = B$$

as well. The remaining claims follow similarly. $\square$

3. The zeta function and the Dixmier trace

The zeta function of positive $T \in \mathscr{L}^{(1,\infty)}$ is given by

$$\zeta(s) = \tau(T^s)$$

and for $A \in \mathscr{N}$ we set

$$\zeta_A(s) = \tau(AT^s).$$

We are interested in the asymptotic behaviour of $\zeta(s)$ and $\zeta_A(s)$ as $s \to 1$.

Now it is elementary to see that the discussion of singular traces is relevant because by Lemma 2.1 we have for some $K > 0$ and all $s > 1$

$$\tau(T^s) = \int_0^\infty \mu_r(T^s)\,dr = \int_0^\infty \mu_r(T)^s\,dr$$
$$\leqslant \int_0^\infty \frac{K^s}{(1+r)^s}\,dr = \frac{K^s}{s-1}.$$

From this it follows that $\{(s-1)\tau(T^s) \mid s>1\}$ is bounded. Now for A bounded $|(s-1)\tau(AT^s)| \leqslant ||A||(s-1)\tau(T^s)$ so that $(s-1)\tau(AT^s)$ is also bounded and hence for any $\tilde{\omega} \in L^\infty(\mathbf{R})^*$ satisfying conditions (1)–(3) of Corollary 1.6

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(AT^{1+\frac{1}{r}}) \tag{3.1}$$

exists.

Here, we think of $r \to \frac{1}{r}\tau(AT^{1+\frac{1}{r}})$ as a function on all of $\mathbf{R}$ by extending it to be identically zero for $r<1$. For notational convenience one might like to think of (3.1) as $\tilde{\omega} - \lim_{s\to 1}(s-1)\tau(AT^s)$ but this of course does not (strictly speaking) make sense whereas if $\lim_{s\to 1}(s-1)\tau(AT^s)$ exists then it is $\lim_{r\to\infty}\frac{1}{r}\tau(AT^{1+\frac{1}{r}})$.

In the following theorem, we will take $T \in \mathscr{L}^{(1,\infty)}$ positive, $||T|| \leqslant 1$ with spectral resolution $T = \int \lambda\, dE(\lambda)$. We would like to integrate with respect to $d\tau(E(\lambda))$; unfortunately, these scalars $\tau(E(\lambda))$ are, in general, all infinite. To remedy this situation, we instead must integrate with respect to the increasing (negative) real-valued function $N_T(\lambda) = \tau(E(\lambda) - 1)$ for $\lambda > 0$. Away from 0, the increments $\tau(\Delta E(\lambda))$ and $\Delta N_T(\lambda)$ are, of course, identical.

In a recent email, Alain Connes has sent us a proof of the more difficult implication of Proposition 4 of [Co4, p. 306]. This is the essential point in the proof of the second statement of the theorem below for $\mathscr{N} = \mathscr{B}(\mathscr{H})$. While his argument is admittedly simpler it is similar in spirit to the proof below as it uses Karamata's approach to the classical Hardy–Littlewood Tauberian Theorem [H, Theorem 98], as suggested by Connes in [Co4].

Theorem 3.1. *For $T \in \mathscr{L}^{(1,\infty)}$ positive, $||T|| \leqslant 1$ and $\tilde{\omega} \in L^\infty(\mathbf{R})^*$ satisfying all the conditions of Corollary* 1.6, *let $\tilde{\omega} = \omega \circ L$ where L is given in Section* 1.1, *then we have*

$$\tau_\omega(T) = \tilde{\omega} - \lim_{r^\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}}).$$

If $\lim_{r\to\infty}\frac{1}{r}\tau(T^{1+\frac{1}{r}})$ *exists then*

$$\tau_\omega(T) = \lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}})$$

for an arbitrary dilation invariant functional $\omega \in L^\infty(\mathbf{R}^*_+)^*$.

Proof. By (3.1) we can apply the weak*-Karamata theorem to $\frac{1}{r}\tau(T^{1+\frac{1}{r}})$. First write $\tau(T^{1+\frac{1}{r}}) = \int_{0^+}^1 \lambda^{1+\frac{1}{r}}\, dN_T(\lambda)$. Thus setting $\lambda = e^{-u}$

$$\tau(T^{1+\frac{1}{r}}) = \int_0^\infty e^{-\frac{u}{r}}\, d\beta(u),$$

where $\beta(u) = \int_u^0 e^{-v} dN_T(e^{-v}) = -\int_0^u e^{-v} dN_T(e^{-v})$. Since the change of variable $\lambda = e^{-u}$ is strictly decreasing, β is, in fact, nonnegative and increasing. By the weak*-Karamata theorem applied to $\tilde{\omega} \in L^\infty(\mathbf{R})^*$

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}}) = \tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u}.$$

Next with the substitution $\rho = e^{-v}$ we get

$$\tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u} = \tilde{\omega} - \lim_{u\to\infty} \frac{1}{u}\int_{e^{-u}}^1 \rho \, dN_T(\rho). \tag{3.2}$$

Set $f(u) = \frac{\beta(u)}{u}$. We want to make the change of variable $u = \log t$ or in other words to consider $f \circ \log = Lf$. We use the discussion in Section 1.1 which tells us that if we start with a G_2 and M invariant functional $\omega \in L^\infty(\mathbf{R}_+^*)^*$ then the functional $\tilde{\omega} = \omega \circ L$ is G_1 and H invariant as required by the theorem. Then we have

$$\begin{aligned}\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}}) &= \tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u} = \tilde{\omega} - \lim_{u\to\infty} f(u)\\ &= \omega - \lim_{t\to\infty} Lf(t) = \omega - \lim_{t\to\infty} \frac{1}{\log t}\int_{1/t}^1 \lambda \, dN_T(\lambda).\end{aligned}$$

Now, by Proposition 2.4

$$\omega - \lim_{t\to\infty} \frac{1}{\log t}\int_{1/t}^1 \lambda \, dN_T(\lambda) = \omega - \lim_{t\to\infty} \frac{1}{\log t}\tau(\chi_{(\frac{1}{t},1]}(T)T) = \tau_\omega(T).$$

This completes the proof of the first part of the theorem.

The proof of the second part is similar. Using the classical Karamata theorem (see the remark following the statement of Theorem 2.2) we obtain the following analogue of (3.2):

$$\lim_{r\to\infty} \frac{1}{r}\tau(T^{1+r}) = \lim \frac{\beta(u)}{u} = \lim_{u\to\infty} \frac{1}{u}\int_{e^{-u}}^1 \rho \, dN_T(\rho).$$

Making the substitution $u = \log t$ on the right-hand side we have

$$\lim_{u\to\infty} \frac{1}{u}\int_{e^{-u}}^1 \rho \, dN_T(\rho) = \lim_{t\to\infty} \frac{1}{\log t}\int_{\frac{1}{t}}^1 \lambda \, dN_T(\lambda) = \tau_\omega(T),$$

where in the last equality we need only dilation invariance of the state $\omega \in L^\infty(\mathbf{R}_+^*)^*$ and not the full list of conditions of Corollary 1.6. $\square$

The map on positive $T \in \mathcal{L}^{(1,\infty)}$ to $\mathbf{R}$ given by $T \to \tau_\omega(T)$ can be extended by linearity to a $\mathbf{C}$ valued functional on all of $\mathcal{L}^{(1,\infty)}$. Then the functional

$$A \mapsto \tau_\omega(AT) \tag{$**$}$$

for $A \in \mathcal{N}$ and fixed $T \in \mathcal{L}^{(1,\infty)}$ is well defined. We intend to study the properties of $(**)$. Part of the interest in this functional stems from the following result as well as the use of the Dixmier trace in noncommutative geometry [Co4].

Lemma 3.2. *Let $T \in \mathcal{L}^{(1,\infty)}$, then*

(i) *For $A \in \mathcal{N}$ we have*

$$\tau_\omega(AT) = \tau_\omega(TA).$$

(ii) *Assume that D_0 is an unbounded self adjoint operator affiliated with $\mathcal{N}$ such that $T = (1 + D_0^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$. If $[A_j, |D_0|]$ is a bounded operator for $A_j \in \mathcal{N}, j = 1, 2$ then*

$$\tau_\omega(A_1 A_2 T) = \tau_\omega(A_2 A_1 T).$$

Proof. (i) This is Proposition A.2 of [CM]. The proof is elementary, first show that $\tau_\omega(UTU^*) = \tau_\omega(T)$ then use linearity to extend to arbitrary $T \in \mathcal{L}^{(1,\infty)}$. Replace T by TU then use linearity again.

(ii) We remark that $[A_j, |D_0|]$ defining a bounded operator means that the A_j leave $dom(|D_0|) = dom(D_0)$ invariant and that $[A_j, |D_0|]$ is bounded on this domain (see [BR, 3.2.55], and its proof for equivalent but seemingly weaker conditions). As $|D_0| - (1 + D_0^2)^{1/2}$ is bounded, $[A_j, (1 + D_0^2)^{1/2}]$ defines a bounded operator whenever $[A_j, |D_0|]$ does. As $T^{-1} = (1 + D_0^2)^{1/2}$ and $T : \mathcal{H} \to dom(T^{-1})$, we see that the formal calculation:

$$[A_j, T] = A_j T - T A_j = T(T^{-1} A_j - A_j T^{-1})T = T[T^{-1}, A_j]T$$

makes sense as an everywhere-defined operator on $\mathcal{H}$. That is,

$$[A_j, T] = T[(1 + D_0^2)^{1/2}, A_j]T \in (\mathcal{L}^{(1,\infty)})^2 \subseteq \mathcal{L}^1.$$

Then we have, using part (i),

$$\tau_\omega(A_1 A_2 T) = \tau_\omega(A_2 A_1 T) - \tau_\omega([A_1, T]A_2).$$

Since the operator in the last term is trace class we are done. $\square$

As a corollary of this lemma we see that (**) can be used to define a trace on certain subalgebras of $\mathcal{N}$. We aim to give several formulas for it. The first involves the zeta function. We begin with some preliminary lemmas.

Lemma 3.3. *Let $T \geqslant 0$, $b \geqslant 0$ be bounded operators*

(i) *If $||b|| \leqslant M$ then for any $1 \leqslant s < 2$*

$$(b^{1/2}Tb^{1/2})^s \leqslant M^{s-1}b^{1/2}T^s b^{1/2}.$$

(ii) *If $m > 0, \mathbf{1}$ denotes the identify operator and $b \geqslant m\mathbf{1}$ then for any $1 \leqslant s < 2$*

$$(b^{1/2}Tb^{1/2})^s \geqslant m^{s-1}b^{1/2}T^s b^{1/2}.$$

Proof. One can prove a weaker version of part (i) using singular values as a special case of [FK, Lemma 4.5]. However, we feel that the stronger version has some independent interest. Now (i) is equivalent to

$$\left(\left(\frac{b}{M}\right)^{1/2} T \left(\frac{b}{M}\right)^{1/2}\right)^s \leqslant \left(\frac{b}{M}\right)^{1/2} T^s \left(\frac{b}{M}\right)^{1/2}.$$

So we can assume that $M = 1$ and therefore $b \leqslant \mathbf{1}$. Letting $A = b^{1/2}$ we have $0 \leqslant A \leqslant \mathbf{1}$ and we want

$$(ATA)^s \leqslant AT^s A.$$

Equivalently, we want

$$(ATA)(ATA)^{s-1} \leqslant ATT^{s-1}A$$

or, letting $r = s - 1$ we want

$$(ATA)(ATA)^r \leqslant ATT^r A$$

for $0 \leqslant r < 1$. Using the integral formula for the rth power of a positive operator, we want

$$\int_0^\infty t^{-r}(ATA)(1 + tATA)^{-1}ATA\, dt \leqslant \int_0^\infty t^{-r}AT(1 + tT)^{-1}TA\, dt,$$

which would follow from

$$\int_0^\infty t^{-r}[AT(A(1 + tATA)^{-1}A - (1 + tT)^{-1})TA]\, dt \leqslant 0.$$

So, it would be enough to see that

$$A(1+tATA)^{-1}A \leqslant (1+tT)^{-1}.$$

Since the left-hand side of this inequality is a norm-continuous function of A, we can approximate A by a sequence $\{A_n\}$ with $0 < A_n \leqslant 1$. Then it suffices to prove that

$$A_n(1+tA_nTA_n)^{-1}A_n \leqslant (1+tT)^{-1}.$$

Or

$$(1+tA_nTA_n) \geqslant A_n(1+tT)A_n$$

or

$$\mathbf{1} \geqslant A_n^2.$$

So, (i) holds.

The argument for (ii) is very similar but easier. As in the proof of (i) we can assume $m=1$ and letting $A=b^{1/2}$ we have $A \geqslant \mathbf{1}$ and we want

$$(ATA)^s \geqslant AT^sA$$

for $1 \leqslant s < 2$. We argue as above with all of the inequalities reversed. Since $A \geqslant \mathbf{1}$ it is invertible and we need no approximations. Our final line for the argument then becomes $\mathbf{1} \leqslant A^2$ and so (ii) is done. $\square$

Lemma 3.4. *For $T \geqslant 0$ in $\mathscr{L}^{(1,\infty)}$ and any b in $\mathscr{N}$ with $b \geqslant m\mathbf{1} > 0$,*

$$\lim_{s\to 1^+} [(s-1)\tau(bT^s) - (s-1)\tau((b^{1/2}Tb^{1/2})^s)] = 0.$$

Proof. Let $M = ||b||$ then by Lemma 3.3

$$(M^{s-1}-1)\tau(b^{1/2}T^sb^{1/2}) \geqslant \tau[(b^{1/2}Tb^{1/2})^s - b^{1/2}T^sb^{1/2}] \geqslant (m^{s-1}-1)\tau[b^{1/2}T^sb^{1/2}].$$

Hence

$$\begin{aligned}&(M^{s-1}-1)(s-1)\tau(b^{1/2}T^sb^{1/2})\\ &\quad\geqslant (s-1)\tau[(b^{1/2}Tb^{1/2})^s] - (s-1)\tau[b^{1/2}T^sb^{1/2}]\\ &\quad\geqslant (m^{s-1}-1)(s-1)\tau(b^{1/2}T^sb^{1/2}).\end{aligned}$$

Now let $s \to 1^+$:

$$0 \geqslant \limsup_{s \to 1^+} \left((s-1)\tau[(b^{1/2}Tb^{1/2})^s] - (s-1)\tau[bT^s]\right)$$
$$\geqslant \liminf_{s \to 1^+} \left((s-1)\tau[(b^{1/2}Tb^{1/2})^s] - (s-1)\tau[bT^s]\right) \geqslant 0. \qquad \square$$

Lemma 3.5. *If* $b \geqslant 0$, $T \geqslant 0$, $T \in \mathcal{L}^{(1,\infty)}$ *and* $b \in \mathcal{N}$ *then there is a constant* $C > 0$ *depending on* b, T *such that for all* $0 < \varepsilon < 1$.

$$\limsup_{s \to 1^+} |(s-1)\tau[(b^{1/2}Tb^{1/2})^s] - (s-1)\tau[((b+\varepsilon)^{1/2}T(b+\varepsilon)^{1/2})^s]| \leqslant C\varepsilon^{1/4}.$$

Proof. To shorten the notation let $A = b^{1/2}Tb^{1/2}$ and $B = (b+\varepsilon)^{1/2}T(b+\varepsilon)^{1/2})$ so that there is an $M > 0$ such that $||A||_s \leqslant M||T||_s$ and $||B||_s \leqslant M||T||_s$ for all $0 < \varepsilon < 1$ and $1 < s < 2$, where $|| \cdot ||_s$ is the Schatten class norm. Then

$$|\tau[(b^{1/2}Tb^{1/2})^s] - \tau[((b+\varepsilon)^{1/2}T(b+\varepsilon)^{1/2})^s]| \leqslant ||A^s - B^s||_1$$

and

$$||A^s - B^s||_1 \leqslant ||A^{s/2}(A^{s/2} - B^{s/2})||_1 + ||(A^{s/2} - B^{s/2})B^{s/2}||_1.$$

Apply the [BKS] inequality to the RHS of the previous line (for a discussion of this inequality for operator ideals in semifinite von Neumann algebras see the references in [CPS]) using $1 > s/2$ to obtain

$$\begin{aligned}
||A^s - B^s||_1 &\leqslant ||A^{s/2}||_2||A^{s/2} - B^{s/2}||_2 + ||A^{s/2} - B^{s/2}||_2||B^{s/2}||_2 \\
&\leqslant ||A^{s/2}||_2|| \, |A - B|^{s/2}||_2 + || \, |A - B|^{s/2}||_2||B^{s/2}||_2 \\
&= ||A||_s^{s/2}||A - B||_s^{s/2} + ||A - B||_s^{s/2}||B||_s^{s/2} \\
&\leqslant 2M^{s/2}||T||_s^{s/2}||A - B||_s^{s/2} \\
&= 2M^{s/2}(\tau(T^s))^{1/2}||A - B||_s^{s/2}.
\end{aligned}$$

Hence

$$|(s-1)\tau(b^{1/2}Tb^{1/2})^s - (s-1)\tau[((b+\varepsilon)^{1/2}T(b+\varepsilon)^{1/2})^s]|$$
$$\leqslant 2M^{s/2}((s-1)\tau(T^s))^{1/2}[(s-1)||A - B||_s^s]^{1/2}.$$

Now using the argument at the beginning of this section there is a $K > 0$ depending only b, T such that

$$\limsup_{s \to 1^+} 2M^{s/2}((s-1)\tau(T^s))^{1/2} \leqslant K.$$

On the other hand,

$$\begin{aligned}||A-B||_s &\leqslant ||b^{1/2}T(b^{1/2}-(b+\varepsilon)^{1/2})||_s+||((b+\varepsilon)^{1/2}-b^{1/2})T(b+\varepsilon)^{1/2})||_s\\ &\leqslant ||b^{1/2}||\,||T||_s||b^{1/2}-(b+\varepsilon)^{1/2}||+||(b+\varepsilon)^{1/2}-b^{1/2}||\,||T||_s||(b+\varepsilon)^{1/2}||\\ &\leqslant K_2\sqrt{\varepsilon}||T||_s=K_2\sqrt{\varepsilon}(\tau(T^s))^{1/s}\end{aligned}$$

for some constant $K_2>0$. Thus

$$\limsup_{s\to 1^+}\ [(s-1)||A-B||_s^s]^{1/2}\leqslant \limsup_{s\to 1^+}\ [(s-1)\tau(T^s)]^{1/2}(K_2\sqrt{\varepsilon})^{s/2}\leqslant(\text{const})\varepsilon^{1/4}$$

as required. □

Proposition 3.6. *If $b\geqslant 0, T\geqslant 0,\ T\in\mathscr{L}^{(1,\infty)}$ and $b\in\mathcal{N}$ then $\lim_{s\to 1^+}(s-1)\tau(bT^s)$ exists if and only if $\lim_{s\to 1^+}(s-1)\tau((b^{1/2}Tb^{1/2})^s)$ exists and in this case they are equal. Moreover*, *in any case for any $\tilde{\omega}\in L^\infty(\mathbf{R})^*$ satisfying conditions* (1)–(4) *of Corollary* 1.6.

$$\tilde{\omega}-\lim_{r\to\infty}\frac{1}{r}\tau(bT^{1+\frac{1}{r}})=\tilde{\omega}-\lim_{r\to\infty}\frac{1}{r}\tau((b^{1/2}Tb^{1/2})^{1+\frac{1}{r}}).$$

Proof. It suffices to prove

$$\limsup_{r\to\infty}\left|\frac{1}{r}\tau(bT^{1+\frac{1}{r}})-\frac{1}{r}\tau((b^{1/2}Tb^{1/2})^{1+\frac{1}{r}})\right|=0.$$

Now,

$$\begin{aligned}&\limsup_{r\to\infty}\frac{1}{r}|\tau(bT^{1+\frac{1}{r}})-\tau((b^{1/2}Tb^{1/2})^{1+\frac{1}{r}})|\\ &\quad\leqslant\limsup_{r\to\infty}\frac{1}{r}|\tau(bT^{1+\frac{1}{r}})-\tau((b+\varepsilon)T^{1+\frac{1}{r}})|\\ &\qquad+\ \limsup_{r\to\infty}\frac{1}{r}|\tau((b+\varepsilon)^{1/2}T^{1+\frac{1}{r}}(b+\varepsilon)^{1/2})-\tau[((b+\varepsilon)^{1/2}T(b+\varepsilon)^{1/2})^{1+\frac{1}{r}}]|\\ &\qquad+\ \limsup_{r\to\infty}\frac{1}{r}|\tau[((b+\varepsilon)^{1/2}T(b+\varepsilon)^{1/2})^{1+\frac{1}{r}}]-\tau((b^{1/2}Tb^{1/2})^{1+\frac{1}{r}})|\\ &\quad\leqslant\limsup_{r\to\infty}\frac{1}{r}\tau(T^{1+\frac{1}{r}})\varepsilon+0+C\varepsilon^{1/4}\end{aligned}$$

by Lemmas 3.4 and 3.5. As this holds for all $\varepsilon>0$ we are done. □

Corollary 3.7. *If $b \geqslant 0, T \geqslant 0$, $T \in \mathcal{L}^{(1,\infty)}$ and $b \in \mathcal{N}$ then if any one of the following limits exist they all do and if ω is chosen to satisfy the conditions of Theorem* 1.5 *they are all equal to $\tau_\omega(bT)$*

(1) $\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(b^{1/2}Tb^{1/2})\, ds$;
(2) $\lim_{r\to\infty} \frac{1}{r}\tau(bT^{1+\frac{1}{r}})$;
(3) $\lim_{r\to\infty} \frac{1}{r}\tau((b^{1/2}Tb^{1/2})^{1+\frac{1}{r}})$.

Proof. The simultaneous existence and equality of (2) and (3) follows from Proposition 3.6. If (3) exists then (1) exists and is equal to (3) by the second part of Theorem 3.1.

Conversely, if (1) exists then it equals $\tau_\omega(b^{1/2}Tb^{1/2})$ by definition. Then applying Lemma 3.2(i), we have (1) equal to $\tau_\omega(bT)$ and so for all $\varepsilon > 0$ there is an $M > 0$ such that for $t \geqslant M$

$$\tau_\omega(bT) - \varepsilon \leqslant \frac{1}{\log(1+t)} \int_0^t \mu_s(b^{1/2}Tb^{1/2})\, ds \leqslant \tau_\omega(bT) + \varepsilon.$$

Hence for $t \geqslant M$

$$(\tau_\omega(bT) - \varepsilon) \int_0^t \frac{1}{1+s}\, ds \leqslant \int_0^t \mu_s(b^{1/2}Tb^{1/2})\, ds \leqslant (\tau_\omega(bT) + \varepsilon) \int_0^t \frac{1}{1+s}\, ds.$$

Following [P] introduce three functions

$$g_2(t) = \begin{cases} \frac{1}{M}\int_0^M \mu_s(b^{1/2}Tb^{1/2})\, ds & \text{if } t < M, \\ \mu_t(b^{1/2}Tb^{1/2}) & \text{if } t \geqslant M, \end{cases}$$

$$g_1(t) = \begin{cases} (\tau_\omega(bT) - \varepsilon)\frac{1}{M}\int_0^M \frac{1}{1+s}\, ds & \text{if } t < M, \\ (\tau_\omega(bT) - \varepsilon)\frac{1}{1+t} & \text{if } t \geqslant M, \end{cases}$$

$$g_3(t) = \begin{cases} (\tau_\omega(bT) + \varepsilon)\frac{1}{M}\int_0^M \frac{1}{1+s}\, ds & \text{if } t < M, \\ (\tau_\omega(bT) + \varepsilon)\frac{1}{1+t} & \text{if } t \geqslant M. \end{cases}$$

Then $g_1 \prec\prec g_2 \prec\prec g_3$ and thus $g_1^{1+\frac{1}{r}} \prec\prec g_2^{1+\frac{1}{r}} \prec\prec g_3^{1+\frac{1}{r}}$. So we have for $t \geqslant M$

$$(\tau_\omega(bT) - \varepsilon)^{1+\frac{1}{r}} \left[M\left(\frac{1}{M}\int_0^M \frac{1}{1+s}ds\right)^{1+\frac{1}{r}} + \int_M^t \left(\frac{1}{1+s}\right)^{1+\frac{1}{r}} ds\right]$$
$$\leqslant M\left(\frac{1}{M}\int_0^M \mu_s(b^{1/2}Tb^{1/2})ds\right)^{1+\frac{1}{r}} + \int_M^t \mu_s(b^{1/2}Tb^{1/2})^{1+\frac{1}{r}} ds$$
$$\leqslant (\tau_\omega(bT) + \varepsilon)^{1+\frac{1}{r}} \left[M\left(\frac{1}{M}\int_0^M \frac{1}{1+s}ds\right)^{1+\frac{1}{r}} + \int_M^t \left(\frac{1}{1+s}\right)^{1+\frac{1}{r}} ds\right].$$

Let $t \to \infty$ so that

$$(\tau_\omega(bT) - \varepsilon)^{1+\frac{1}{r}} \left[M\left(\frac{1}{M}\int_0^M \frac{1}{1+s}ds\right)^{1+\frac{1}{r}} + r\left(\frac{1}{1+M}\right)^{\frac{1}{r}}\right]$$
$$\leqslant M\left(\frac{1}{M}\int_0^M \mu_s(b^{1/2}Tb^{1/2})\,ds\right)^{1+\frac{1}{r}} + \tau((b^{1/2}Tb^{1/2})^{1+\frac{1}{r}}) - \int_0^M \mu_s(b^{1/2}Tb^{1/2})^{1+\frac{1}{r}} ds$$
$$\leqslant (\tau_\omega(bT) + \varepsilon)^{1+\frac{1}{r}} \left[M\left(\frac{1}{M}\int_0^M \frac{1}{1+s}ds\right)^{1+\frac{1}{r}} + r\left(\frac{1}{1+M}\right)^{\frac{1}{r}}\right].$$

Multiply by $\frac{1}{r}$ and let $r \to \infty$,

$$\tau_\omega(bT) - \varepsilon \leqslant \lim_{r\to\infty} \frac{1}{r}\tau((b^{1/2}Tb^{1/2})^{1+\frac{1}{r}}) \leqslant (\tau_\omega(bT) + \varepsilon).$$

Hence the result. □

Theorem 3.8. *Let $A \in \mathcal{N}$, $T \geqslant 0$, $T \in \mathcal{L}^{(1,\infty)}$.*

(i) *If $\lim_{s\to 1^+} (s-1)\tau(AT^s)$ exists then it is equal to $\tau_\omega(AT)$ where we choose ω as in the proof of Theorem* 3.1.

(ii) *More generally, if we choose functionals ω and $\tilde{\omega}$ as in the proof of Theorem* 3.1 *then*

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(AT^{1+\frac{1}{r}}) = \tau_\omega(AT).$$

Proof. For part (i) we first assume that A is self-adjoint. Write $A = a^+ - a^-$ where $a^\pm$ are positive. Choose $\tilde{\omega}$ as in the proof of Theorem 3.1, then

$$\begin{aligned}\lim_{s\to 1^+} (s-1)\tau(AT^s) &= \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(AT^{1+\frac{1}{r}}) \\ &= \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(a^+T^{1+\frac{1}{r}}) - \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(a^-T^{1+\frac{1}{r}}) \\ &= \tau_\omega(a^+T) - \tau_\omega(a^-T) \\ &= \tau_\omega(AT).\end{aligned}$$

Here, the third equality uses first Proposition 3.6 and then Theorem 3.1. The reduction from the general case to the self-adjoint case now follows in a similar way.

For part (ii), we assume that A is positive. By Lemma 3.2(i), Theorem 3.1, and Proposition 3.6 we have

$$\begin{aligned}\tau_\omega(AT) = \tau_\omega(A^{1/2}TA^{1/2}) &= \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau((A^{1/2}TA^{1/2})^{1+\frac{1}{r}}) \\ &= \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(AT^{1+\frac{1}{r}}).\end{aligned}$$

For general A we reduce to the case A positive as in the proof of part (i). □

4. The heat semigroup formula

Throughout this section $T \geqslant 0$. We define $e^{-T^{-2}}$ as the operator that is zero on $\ker T$ and on $\ker\, T^\perp$ is defined in the usual way by the functional calculus. We remark that if $T \geqslant 0$, $T \in \mathcal{L}^{(p,\infty)}$ for some $p \geqslant 1$ then $e^{-tT^{-2}}$ is trace class for all $t > 0$.

Our aim in this section is to prove the following:

Theorem 4.1. *If* $A \in \mathcal{N}$, $T \geqslant 0$, $T \in \mathcal{L}^{(1,\infty)}$ *then,*

$$\omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(Ae^{-\lambda^{-2}T^{-2}}) = \Gamma(3/2)\tau_\omega(AT)$$

for $\omega \in L^\infty(\mathbf{R}_+^*)^*$ *satisfying the conditions of Theorem* 1.5.

Let $\zeta_A(p+\frac{1}{r}) = \tau(AT^{p+\frac{1}{r}})$. Notice that $\frac{1}{2}\Gamma(\frac{p}{2})\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta_A(p+\frac{1}{r})$ always exists. Hence we can reduce the hard part of the proof of Theorem 4.1 to the following preliminary result.

Proposition 4.2. *If* $A \in \mathcal{N}$, $T \geqslant 0$, $T \in \mathcal{L}^{(p,\infty)}$, $1 \leqslant p < \infty$ *then, choosing* ω *and* $\tilde{\omega}$ *as in the proof of Theorem* 3.1, *we have*

$$\omega - \lim \frac{1}{\lambda}\tau(Ae^{-T^{-2}\lambda^{-2/p}}) = \frac{1}{2}\Gamma\left(\frac{p}{2}\right)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta_A\left(p + \frac{1}{r}\right).$$

Proof. We have, using the Laplace transform,

$$T^s = \frac{1}{\Gamma(s/2)}\int_0^\infty t^{s/2-1}e^{-tT^{-2}}\, dt.$$

Then

$$\zeta_A(s) = \tau(AT^s) = \frac{1}{\Gamma(s/2)}\int_0^\infty t^{s/2-1}\tau(Ae^{-tT^{-2}})\, dt.$$

Make the change of variable $t = 1/\lambda^{2/p}$ so that the preceding formula becomes

$$\frac{p}{2}\Gamma(s/2)\zeta_A(s) = \int_0^\infty \lambda^{-\frac{s}{p}-1}\tau(Ae^{-\lambda^{-2/p}T^{-2}})\, d\lambda.$$

We split this integral into two parts, $\int_0^1$ and $\int_1^\infty$ and call the first integral $R(r)$ where $s = p + \frac{1}{r}$. Then

$$R(r) = \int_0^1 \lambda^{-\frac{1}{pr}-2}\tau(Ae^{-\lambda^{-2/p}T^{-2}})\, d\lambda = \int_1^\infty t^{\frac{p}{2}+\frac{1}{2r}-1}\tau(Ae^{-tT^{-2}})\, dt.$$

The integrand decays exponentially in t as $t\to\infty$ because $T^{-2} \geqslant ||T^2||^{-1}\mathbf{1}$ so that for $A \geqslant 0$

$$\tau(Ae^{-tT^{-2}}) \leqslant \tau(Ae^{-T^{-2}}e^{-\frac{t-1}{||T^2||}}).$$

Then we can conclude that $R(r)$ is bounded independently of r and so $\lim_{r\to\infty}\frac{1}{r}R(r) = 0$. For the other integral the change of variable $\lambda = e^\mu$ gives

$$\int_1^\infty \lambda^{-\frac{1}{pr}-2}\tau(Ae^{-\lambda^{-2/p}T^{-2}})\, d\lambda = \int_0^\infty e^{-\frac{\mu}{pr}}\, d\beta(\mu),$$

where $\beta(\mu) = \int_0^\mu e^{-v}\tau(Ae^{-e^{-\frac{2}{v}p}T^{-2}})\, dv$. Hence we can now write

$$\frac{p}{2}\Gamma\left(\left(p+\frac{1}{r}\right)\Big/2\right)\zeta_A\left(p+\frac{1}{r}\right) = \int_0^\infty e^{-\frac{\mu}{pr}}\, d\beta(\mu) + R(r).$$

Now consider

$$\frac{p}{2}\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\Gamma\left(\frac{p}{2}+\frac{p}{2r}\right)\zeta_A\left(p+\frac{1}{r}\right) = \frac{p}{2}\Gamma(p/2)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta_A\left(p+\frac{1}{r}\right).$$

Then

$$\frac{p}{2}\Gamma(p/2)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta_A\left(p+\frac{1}{r}\right) = p\tilde{\omega} - \lim_{r\to\infty} \frac{1}{pr}\int_0^\infty e^{-\mu/pr}\, d\beta(\mu)$$

(remembering that the term $\frac{1}{r}R(r)$ has limit zero as $r\to\infty$). By dilation invariance and Theorem 2.2 we then have

$$\frac{p}{2}\Gamma(p/2)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta_A\left(p+\frac{1}{r}\right) = p\tilde{\omega} - \lim_{\mu\to\infty} \frac{\beta(\mu)}{\mu}. \tag{4.0}$$

Making the change of variable $\lambda = e^v$ in the expression for $\beta(\mu)$ we get

$$\frac{\beta(\mu)}{\mu} = \frac{1}{\mu}\int_1^{e^\mu} \lambda^{-2}\tau(Ae^{-T^{-2}\lambda^{-2/p}})\, d\lambda.$$

Make the substitution $\mu = \log t$ so the RHS becomes

$$\frac{1}{\log t}\int_1^t \lambda^{-2}\tau(Ae^{-T^{-2}\lambda^{-2/p}})\, d\lambda = g_1(t).$$

This is the Cesaro mean of

$$g_2(\lambda) = \frac{1}{\lambda}\tau(Ae^{-T^{-2}\lambda^{-2/p}}).$$

So as we chose $\omega\in L^\infty(\mathbf{R}_+^*)^*$ to be G_2 and M invariant we have $\omega(g_1) = \omega(g_2)$. Recalling that we choose $\tilde{\omega}$ to be related to ω as in Theorem 3.1 and so using (4.0) we obtain

$$\omega - \lim \frac{1}{\lambda}\tau(Ae^{-T^{-2}\lambda^{-2/p}}) = \frac{1}{2}\Gamma\left(\frac{p}{2}\right)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta_A\left(p+\frac{1}{r}\right). \qquad \square$$

To prove the theorem consider first the case where A is bounded, $A\geqslant 0$ and use the Proposition 4.2 and Theorem 3.8 to assert that

$$\Gamma(3/2)\tau_\omega(AT) = \Gamma(3/2)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(AT^{1+\frac{1}{r}}) = \omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(Ae^{-\lambda^{-2}T^{-2}}).$$

Then for self-adjoint A write $A = a^+ - a^-$ where $a^\pm$ are positive so that

$$\begin{aligned}\Gamma(3/2)\tau_\omega(AT) &= \Gamma(3/2)(\tau_\omega(a^+T) - \tau_\omega(a^-T))\\ &= \omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(a^+e^{-\lambda^{-2}T^{-2}}) - \omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(a^-e^{-\lambda^{-2}T^{-2}})\\ &= \omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(Ae^{-\lambda^{-2}T^{-2}}).\end{aligned}$$

We can extend to general bounded A by a similar argument.

4.1. The 'smaller' ideal

The curious feature of our proof of this heat kernel formula of Connes for the Dixmier trace is that we need to go via the zeta function and hence need the pair of functionals $\tilde{\omega}$ and ω as in Theorem 3.1. There is a special case of the previous result for which we can avoid the introduction of these functionals and hence avoid using the full strength of the assumptions in Theorem 1.5.

The operators $T \in \mathcal{L}^{(1,\infty)}$ satisfying $\mu_s(T) \leqslant C/s$ for some $C > 0$ form an ideal as well. For this 'smaller ideal', which is the one that usually arises in geometric applications, there is a direct proof of a special case of the heat kernel formula which does not use the zeta function.

For simplicity we restrict to $A = 1$. This direct proof uses the Laplace transform: $T = \frac{1}{\Gamma(1/2)}\int_0^\infty u^{-1/2}e^{-uT^{-2}}du$ (with our usual convention that $e^{-T^{-2}}$ is defined to be zero on *ker* T). Thus we have

$$\frac{\Gamma(3/2)}{\log(1+t)}\int_0^t \mu_s(T)\,ds = \frac{1}{2\log(1+t)}\int_0^t \int_0^\infty u^{-1/2}e^{-u/\mu_s(T^2)}\,du\,ds. \tag{4.1}$$

Using the basic fact that if f is increasing $\mu_s(f(T)) = f(\mu_s(T))$ [FK] we have

$$\frac{1}{\log(1+t)}\int_0^t \lambda^{-2}\tau(e^{-\lambda^{-2}T^{-2}})\,d\lambda = \frac{1}{\log(1+t)}\int_0^t \int_0^\infty \lambda^{-2}e^{-\lambda^{-2}/\mu_s(T^2)}ds\,d\lambda$$

and we have to show that this has the same ω limit as (4.1). Change variable in this integral by $u = \lambda^{-2}$ then

$$\begin{aligned}&\frac{1}{\log(1+t)}\int_0^t \lambda^{-2}\tau(e^{-\lambda^{-2}T^{-2}})\,d\lambda\\ &\quad= \frac{1}{2\log(1+t)}\int_{1/t^2}^\infty \int_0^\infty u^{-1/2}e^{-u/\mu_s(T^2)}\,ds\,du.\end{aligned} \tag{4.2}$$

Subtract (4.1) from (4.2) and rewrite the difference as

$$\frac{1}{2\log(1+t)}\int_0^{\infty}\int_0^{\infty}(-\chi_{[0,t]}(s)\chi_{[0,1/t^2]}(u) + \chi_{[1/t^2,\infty)}(u)\chi_{[t,\infty)}(s))u^{-1/2}e^{-u/\mu_s(T^2)}\,du\,ds. \tag{4.3}$$

To prove equality of the ω-limits of (4.1) and (4.2) we have to estimate the two integrals in (4.3). The first of these is

$$\frac{1}{2\log(1+t)}\int_0^{1/t^2}\int_0^{t} u^{-1/2}e^{-u/\mu_s(T^2)}\,ds\,du.$$

As we can (and do) assume that $\mu_s(T^2)\leqslant 1$ for all s, $e^{-u/\mu_s(T^2)}\leqslant e^{-u}$. Thus the integral is bounded by

$$\frac{1}{2\log(1+t)}\int_0^{t}\left(\int_0^{1/t^2} u^{-1/2}e^{-u}\,du\right)ds = \frac{1}{2\log(1+t)}t\int_0^{1/t^2} u^{-1/2}e^{-u}\,du.$$

Now

$$\gamma\left(\frac{1}{2},\frac{1}{t^2}\right) = \int_0^{1/t^2} u^{-1/2}e^{-u}\,du$$

is the incomplete Γ function which has an expansion of the form (see [AS])

$$\gamma\left(\frac{1}{2},\frac{1}{t^2}\right) = \frac{1}{t}\sum_0^{\infty}\frac{(-1)^n}{n!}\frac{1}{t^{2n}(\frac{1}{2}+n)}.$$

So we conclude that

$$t\int_0^{1/t^2} u^{-1/2}e^{-u}\,du = \sum_0^{\infty}\frac{(-1)^n}{n!}\frac{1}{t^{2n}(\frac{1}{2}+n)},$$

which is bounded as $t\to\infty$. Thus as $t\to\infty$

$$\frac{1}{2\log(1+t)}\int_0^{1/t^2}\int_0^{t} u^{-1/2}e^{-u/\mu_s(T^2)}\,ds\,du\to 0.$$

For the second integral in (4.3) we first make a number of preliminary observations. We make some changes of variable in letting $r=s/t$ and $v=ut^2$. Then we find that

$$\int_{1/t^2}^{\infty}\int_t^{\infty} u^{-1/2}e^{-u/\mu_s(T^2)}\,ds\,du = \int_1^{\infty}\int_1^{\infty} v^{-1/2}e^{-v/t^2\mu_{rt}(T^2)}\,dr\,dv.$$

Now we exploit the assumption that $\mu_s(T) = O(1/s)$ and use $v^{-1/2} < 1$. Thus $\mu_{rt}(T^2) \leqslant C/(rt)^2$ for some constant C and

$$\int_1^\infty \int_1^\infty v^{-1/2} e^{-v/t^2 \mu_{rt}(T^2)}\, dr\, dv \leqslant \int_1^\infty \int_1^\infty e^{-vr^2/C}\, dv\, dr$$
$$= \int_1^\infty C\frac{1}{r^2} e^{-r^2/C}\, dr < \infty.$$

Dividing by $\log(1+t)$ and taking $t \to \infty$ shows that the second integral in (4.3) gives a function of t which vanishes at infinity.

Now choose $\omega \in L^\infty(\mathbf{R}_+^*)^*$ satisfying conditions (1)–(3), (6) of Theorem 1.5. Taking the ω-limit on (4.3) gives zero. Writing $\tau_\omega(T) = \omega - \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds$ we obtain, using the same reasoning as at the end of Proposition 4.2, the result that

$$\omega - \lim_{\lambda\to\infty} \lambda^{-1} \tau(e^{-\lambda^{-2}T^{-2}}) = \Gamma(3/2)\tau_\omega(T).$$

5. The $\mathscr{L}^{(p,\infty)}$-summable case

If $T \in \mathscr{L}^{(p,\infty)}$ for $p > 1$, $T \geqslant 0$ then $\mu_s(T) = O(\frac{1}{s^{1/p}})$. Moreover, $\tau(T^{p+\frac{1}{r}}) = \int_0^1 \lambda^{p+1/r} dN_T(\lambda)$ where $N_T(\lambda) = \tau(E(\lambda) - 1)$ for $\lambda > 0$ where $T = \int \lambda\, dE(\lambda)$ is the spectral resolution for T.

We now establish some $\mathscr{L}^{(p,\infty)}$ versions of our previous results.

Lemma 5.1. *For $T \in \mathscr{L}^{(p,\infty)}$ and ω and $\tilde{\omega}$ as in the proof of Theorem 3.1 we have*

$$p\tau_\omega(T^p) = \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} \tau(T^{p+\frac{1}{r}}).$$

Proof. Set $\lambda = e^{-u/p}$ so that

$$\frac{1}{r}\tau(T^{p+\frac{1}{r}}) = p\frac{1}{pr}\int_0^\infty e^{-u/rp}\, d\beta(u),$$

where $\beta(u) = \int_0^u e^{-v} dN_T(e^{-v/p})$. So using dilation invariance:

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(T^{p+\frac{1}{r}}) = p\tilde{\omega} - \lim_{r\to\infty} \frac{1}{pr}\int_0^1 e^{-u/pr}\, d\beta(u) = p\tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u}$$

by the weak*-Karamata theorem. Reasoning as in the proof of Theorem 3.1 and substituting $\lambda = e^{-v/p}$ and $u = \log t$ we have

$$\begin{aligned}\tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u} &= \omega - \lim_{t\to\infty} \frac{1}{\log t}\int_{t^{-1/p}}^{1} \lambda^p dN_T(\lambda)\\ &= \omega - \lim_{t\to\infty} \frac{1}{\log t}\tau(\chi_{(\frac{1}{t},\infty)}(T^p)T^p) = \tau_\omega(T^p). \qquad \square\end{aligned}$$

Corollary 5.2. *Let* $T \geqslant 0$, $T \in \mathscr{L}^{(p,\infty)}$ *then*

$$\omega - \lim \frac{1}{\lambda}\tau(e^{-T^{-2}\lambda^{-2/p}}) = \Gamma\left(1+\frac{p}{2}\right)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(T^{p+\frac{1}{r}})$$

with the usual convention that $e^{-T^{-2}}$ *is zero on* $\ker T$.

Proof. Combine Proposition 4.2 and Lemma 5.1. $\square$

Our aim is now to prove the $\mathscr{L}^{(p,\infty)}$ version of Theorem 3.8 and the following result of Connes'.

Theorem 5.3. *If A is bounded,* $T \geqslant 0$, $T \in \mathscr{L}^{(p,\infty)}$ *for* $p \geqslant 1$

$$\omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(Ae^{-\lambda^{-2/p}T^{-2}}) = \Gamma(1+p/2)\tau_\omega(AT^p),$$

where $e^{-T^{-2}}$ *is defined to be zero on* $\ker T$.

To this end let us consider the steps in the proof of Theorem 3.8. The key results are Proposition 3.6 and Corollary 3.7. Proposition 3.6 rests on the preceding lemmas. These lemmas have analogues in the case of $\mathscr{L}^{(p,\infty)}$. The first nonobvious extension is Lemma 3.3 which we replace by

Lemma 5.4. *Let* $0 \leqslant T \in \mathscr{L}^{(1,\infty)}$ *and let* $0 \leqslant b \in \mathscr{N}$, $||b|| \leqslant M$.

(i) *For any* $s \geqslant 1$

$$\mu_t(b^{1/2}Tb^{1/2})^s \leqslant M^{s-1}\mu_t(b^{1/2}T^sb^{1/2}), \quad t > 0.$$

(ii) *If* $b \geqslant m1$, *then* $s \geqslant 1$

$$\mu_t(b^{1/2}Tb^{1/2})^s \geqslant m^{s-1}\mu_t(b^{1/2}T^sb^{1/2}), \quad t > 0.$$

Proof. The first result is a special case of [FK] Lemma 4.5. To obtain the second result, we shall (without loss of generality) assume that $||T|| \leqslant 1$. Let $T = \int_0^1 \lambda dE^T(\lambda)$

be the spectral decomposition of T. Note that it follows from the assumption $T \in \mathscr{L}^{(1,\infty)}$ that $\tau(E^T(1/n, 1]) < \infty$ for all $n \in \mathbb{N}$. We set for brevity

$$p_n := E^T(1/n, 1], \quad q_n := l(b^{-1/2}p_n) = r(p_n b^{-1/2}), \quad \mathcal{N}_n := q_n \mathcal{N} q_n, \ n \in \mathbb{N},$$

where $l(\cdot)$ and $r(\cdot)$ are left and right support projections, respectively. Note that $\mathcal{N}_n =$ is a finite von Neumann algebra and that restriction of the trace τ on $\mathcal{N}_n$ is semifinite for every $n \in \mathbb{N}$. From assertion (i) we have

$$\mu_t(b^{-1/2}(p_nTp_n)^{-1}b^{-1/2})^s \leqslant m^{-(s-1)}\mu_t(b^{-1/2}(p_nTp_n)^{-s}b^{-1/2}), \quad n \in \mathbb{N}. \qquad (5.1)$$

Note that $b^{-1/2} \geqslant \frac{1}{M}$ and therefore $b^{-1/2}(p_nTp_n)^{-1}b^{-1/2}$, $b^{-1/2}(p_nTp_n)^{-s}b^{-1/2}$ are invertible elements in $\mathcal{N}_n$ for all $n \geqslant 1$.

Now we need a following simple observation: if $(\mathcal{M}, \tau)$ is a finite von Neumann algebra and $0 \leqslant x$ is an invertible τ-measurable operator affiliated with $\mathcal{M}$, then the elements x^{-1} and $\mu_{(\cdot)}(x)^{-1}$ are equimeasurable, or equivalently, $\mu_{(\cdot)}(x^{-1})$ is the decreasing rearrangement of the function $\mu_{(\cdot)}(x)^{-1}$. To see the validity of this observation, set for brevity $f(\lambda) := \frac{1}{\lambda}$, $x = \int_0^\infty \lambda \, dE^x(\lambda)$, $y = f(x) = \int_0^\infty f(\lambda) dE^x(\lambda) = \int_0^\infty \lambda \, dE^y(\lambda)$ and note that $E^y(\Delta) = E^x(f^{-1}(\Delta))$ for every Borel subset $\Delta \subseteq [0, \infty)$. In particular,

$$E^y(s, \infty) = E^x(f^{-1}(s, \infty)) = E^x\left(0, \frac{1}{s}\right) = 1 - E^x\left[\frac{1}{s}, \infty\right), \quad s > 0,$$

whence

$$\lambda_s(y) = \tau(1) - \lambda_{\frac{1}{s}-0}(x), \quad s > 0.$$

If instead of the algebra $(\mathcal{M}, \tau)$ and the element x we consider the von Neumann algebra $L_\infty(0, \tau(1))$ and the element $\mu_{(\cdot)}(x)$, then the preceding equality becomes

$$\lambda_s((\mu_{(\cdot)}(x))^{-1}) = \tau(1) - \lambda_{\frac{1}{s}-0}(\mu_{(\cdot)}(x)), \quad s > 0$$

(where we use the notation $\lambda_{(\cdot)}$ for the classical distribution function of the elements $(\mu_{(\cdot)}(x))^{-1}$ and $\mu_{(\cdot)}(x)$). Our observation now follows from comparison of the two preceding equalities, taking into account a crucial fact, namely that $\lambda_{\frac{1}{s}-0}(x) = \lambda_{\frac{1}{s}-0}(\mu_{(\cdot)}(x))$ for all $s > 0$. This latter fact easily follows from the equality $\lambda_s(x) = \lambda_s(\mu_{(\cdot)}(x))$ and the assumption that $\mathcal{M}$ is finite.

Now we can continue the proof of the lemma. From inequality (5.1) taking the inverses we get

$$\mu_t^{-1}(b^{-1/2}(p_nTp_n)^{-1}b^{-1/2})^s \geqslant m^{(s-1)}\mu_t^{-1}(b^{-1/2}(p_nTp_n)^{-s}b^{-1/2}),$$
$$t>0,\ n\in\mathbb{N},\ s\geqslant 1.$$

Since $0\leqslant x\leqslant y$ implies $\mu_{(\cdot)}(x)\leqslant\mu_{(\cdot)}(y)$ we immediately infer from the preceding inequality

$$\mu_{(\cdot)}(\mu_t^{-1}(b^{-1/2}(p_nTp_n)^{-1}b^{-1/2})^s)\geqslant m^{(s-1)}\mu_{(\cdot)}(\mu_t^{-1}(b^{-1/2}(p_nTp_n)^{-s}b^{-1/2})),\quad n\in\mathbb{N}.$$

The elements $(b^{-1/2}(p_nTp_n)^{-1}b^{-1/2})^s$ and $b^{-1/2}(p_nTp_n)^{-s}b^{-1/2}$ are invertible positive elements from $\mathcal{N}_n$, and by the preceding observation we know that the elements $\mu_{(\cdot)}^{-1}(b^{-1/2}(p_nTp_n)^{-1}b^{-1/2})^s$ and $(b^{1/2}(p_nTp_n)b^{1/2})^s$ (respectively, $\mu_{(\cdot)}^{-1}(b^{-1/2}(p_nTp_n)^{-s}b^{-1/2})$ and $b^{1/2}(p_nTp_n)^sb^{1/2})$ are equimeasurable, thus the preceding inequality may be equivalently rewritten as

$$\mu_{(\cdot)}((b^{1/2}(p_nTp_n)b^{1/2})^s)\geqslant m^{(s-1)}\mu_{(\cdot)}(b^{1/2}(p_nTp_n)^sb^{1/2}),\quad n\in\mathbb{N}.$$

To complete the proof of the lemma it is sufficient to show that

$$\mu_{(\cdot)}((b^{1/2}(p_nTp_n)b^{1/2})^s)\to\mu_{(\cdot)}((b^{1/2}Tb^{1/2})^s) \tag{5.2}$$

and

$$\mu_{(\cdot)}(b^{1/2}(p_nTp_n)^sb^{1/2})\to\mu_{(\cdot)}(b^{1/2}T^sb^{1/2}) \tag{5.3}$$

in measure. Since $\mu_{(\cdot)}(x^s)=\mu_{(\cdot)}^s(x)$ for all $x\in\mathcal{N}$ and all $s>0$, to establish the first convergence, it is sufficient to show that

$$\mu_{(\cdot)}(b^{1/2}(p_nTp_n)b^{1/2})\to\mu_{(\cdot)}(b^{1/2}Tb^{1/2}).$$

To this end we shall need the following result [CS, Corollary 2.3].

If $E(\mathcal{N})$ is a symmetric operator space associated with a separable symmetric operator space $E(0,\infty)$, then $||xe_n||_{E(\mathcal{N})}\to 0$ and $||e_nx||_{E(\mathcal{N})}\to 0$ for every $x\in E(\mathcal{N})$ and every sequence $\{e_n\}$ of orthogonal projections in $\mathcal{N}$ decreasing to 0.

Consider the symmetric function space $L_1+L_\infty(0,\infty)$ and let E be its closed separable symmetric subspace obtained by taking the norm closure of $L_1\cap L_\infty(0,\infty)$. It is easy to see that E is a separable symmetric function space (in a sense it is an analogue of the space c_0 of all bounded sequences converging to 0). It is clear from the cited result from [CS] and the definition of p_n that

$||T - Tp_n||_{E(\mathcal{N})} \to 0$ and $||T - p_n T||_{E(\mathcal{N})} \to 0$, whence $||T - p_n T p_n||_{E(\mathcal{N})} \to 0$ and also

$$||b^{1/2}(p_n T p_n) b^{1/2} - b^{1/2} T b^{1/2}||_{E(\mathcal{N})} \to 0.$$

Using the continuity of embedding of any $E(\mathcal{N})$ into the space $\tilde{\mathcal{N}}$ of all τ-measurable operators affiliated with $\mathcal{N}$ we get from the preceding convergence that

$$b^{1/2}(p_n T p_n) b^{1/2} - b^{1/2} T b^{1/2} \to 0$$

in measure. Now using [FK], Lemma 3.4 (ii) and the fact

$$\lim_{t \to \infty} \mu_t(b^{1/2}(p_n T p_n) b^{1/2}) = \lim_{t \to \infty} \mu_t(b^{1/2} T b^{1/2}) = 0$$

we get

$$\mu_{(\cdot)}(b^{1/2}(p_n T p_n) b^{1/2}) - \mu_{(\cdot)}(b^{1/2} T b^{1/2}) \to 0$$

in measure, i.e. (5.2) is established. The proof of (5.3) is very similar, after we note that $(p_n T p_n)^s = (p_n T^s p_n)$ and therefore we omit the details. $\square$

Next, some remarks are needed for Lemma 3.5. For the $\mathcal{L}^{(p,\infty)}$ case the statement reads if $b \geqslant 0$, $T \geqslant 0$, $T \in \mathcal{L}^{(p,\infty)}$ with b bounded then there is a constant $C > 0$ depending on b, T such that for all $0 < \varepsilon < 1$.

$$\limsup_{s \to p^+} \{(s-p)\tau(b^{1/2} T b^{1/2})^s - (s-p)\tau[((b+\varepsilon)^{1/2} T (b+\varepsilon)^{1/2})^s]\} \leqslant C\varepsilon^{\frac{1}{4}}.$$

For the proof we use the same argument for all $1 < p < 2$ but for $p \geqslant 2$ we use Cauchy–Schwartz in place of the BKS inequality so that in fact the proof is more elementary. The proofs of Proposition 3.6 and Corollary 3.7 also generalise to give us the following:

Proposition 5.5. *If* $b \geqslant 0$, $T \geqslant 0$, $T \in \mathcal{L}^{(p,\infty)}$ *with b bounded then* $\lim_{s \to p^+} (s-p)\tau(bT^s)$ *exists if and only if* $\lim_{s \to p^+} (s-p)\tau((b^{1/2} T b^{1/2})^s)$ *exists and in this case they are equal. Moreover, in any case for any* $\omega \in L^\infty(\mathbf{R}_+)^*$ *chosen to satisfy the conditions of Theorem* 1.5

$$\omega - \lim_{r \to \infty} \frac{1}{r} \tau(b T^{p+\frac{1}{r}}) = \omega - \lim_{r \to \infty} \frac{1}{r} \tau((b^{1/2} T b^{1/2})^{p+\frac{1}{r}}).$$

Now the proof of Theorem 3.8(i) generalises to give the

Theorem 5.6. *If A is bounded, $T \geqslant 0$, $T \in \mathcal{L}^{(p,\infty)}$ and*

$$\lim_{s \to p^+} (s-p)\tau(AT^s)$$

exists then it is equal to $p\tau_\omega(AT^p)$.

Finally, it is now straightforward to extend the arguments we used in the proof of Theorem 4.1 to prove Theorem 5.3.

6. Application to spectral flow and index formulae

We fix an unbounded self-adjoint operator D_0 on H affiliated with $\mathcal{N}$. Recalling the discussion in the introduction we have:

Definition. We say that $(\mathcal{N}, D_0)$ is an *odd bounded $\mathcal{L}^{(1,\infty)}$-summable Breuer–Fredholm module* for a Banach $*$-algebra $\mathcal{A}$ if $\mathcal{A}$ is represented in $\mathcal{N}$ and if $(1+D_0^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$ and $[D_0, a]$ is bounded for all a in a dense $*$-subalgebra of $\mathcal{A}$.

Recall that these assumptions imply that a leaves the domain of D_0 invariant. In this section, we apply our results to $\mathcal{L}^{(1,\infty)}$ summable Breuer–Fredholm modules in order to establish a relationship between the formula for spectral flow in [CP2] and the formula in [CM]. In [CM] assumptions are made about the discreteness of the spectrum of D_0 which are clearly unrealistic when $\mathcal{N}$ is not type I.

We now summarise some well-known notions (cf. [PR]). Let $\mathcal{K}_\mathcal{N}$ be the τ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$) and $\pi : \mathcal{N} \to \mathcal{N}/\mathcal{K}_\mathcal{N}$ the canonical mapping. A Breuer–Fredholm operator is one that maps to an invertible operator under π. For a unitary $u \in \mathcal{A}$ the path

$$D_t^u := (1-t)D_0 + tuD_0u^*$$

of unbounded self-adjoint Breuer–Fredholm operators is continuous in the sense that

$$F_t^u := D_t^u(1+(D_t^u)^2)^{-\frac{1}{2}}$$

is a continuous path of self-adjoint Breuer–Fredholm operators in $\mathcal{N}$. Recall that the Breuer–Fredholm index of a Breuer–Fredholm operator F is defined by

$$ind(F) = \tau(Q_{\ker F}) - \tau(Q_{\operatorname{coker} F}),$$

where $Q_{\ker F}$ and $Q_{\operatorname{coker} F}$ are the projections onto the kernel and cokernel of F.

Definition. If $\{F_t\}$ is a continuous path of self-adjoint Breuer–Fredholm operators in $\mathcal{N}$, then the definition of *spectral flow* of the path, $sf(\{F_t\})$ is based on the following sequence of observations in [P1]:

1. The map $t \mapsto sign(F_t)$ is usually discontinuous as is the projection-valued mapping $t \mapsto P_t = \frac{1}{2}(sign(F_t) + 1)$.
2. However, $t \mapsto \pi(P_t)$ is continuous.
3. If P and Q are projections in $\mathcal{N}$ and $||\pi(P) - \pi(Q)|| < 1$ then
$$PQ : rng(Q) \to rng(P)$$
is a Breuer–Fredholm operator and so $ind(PQ) \in \mathbf{R}$ is well defined.
4. If we partition the parameter interval of $\{F_t\}$ so that the $\pi(P_t)$ do not vary much in norm on each subinterval of the partition then
$$sf(\{F_t\}) := \sum_{i=1}^{n} ind(P_{t_{i-1}} P_{t_i})$$
is a well defined and (path-) homotopy-invariant number which agrees with the usual notion of spectral flow in the type I_∞ case.

We denote by $sf(D_0, uD_0u^*) = sf(\{F_t\})$ the *spectral flow* of this path [P1,P2] which is an integer in the $\mathcal{N} = \mathcal{B}(\mathcal{H})$ case and a real number in the general semifinite case. This real number $sf(D_0, uD_0u^*)$ recovers the pairing of the K-homology class $[D_0]$ of $\mathcal{A}$ with the $K^1(\mathcal{A})$ class $[u]$.

Let P denote the projection onto the nonnegative spectral subspace of D_0. It is also well known that spectral flow along $\{D_t^u\}$ is equal to the Breuer–Fredholm index of the operator PuP acting on $P\mathcal{H}$. When $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and the spectrum of D_0 is discrete [CM] show that

$$ind(PuP) = \tfrac{1}{2}\tau_\omega(u^*[D_0, u]|D_0|^{-1}).$$

We aim to generalise this formula to the situation where $\mathcal{N}$ is a general semifinite von Neumann algebra and link this formula with the expression for spectral flow.

Lemma 6.1. *Let D_0 be an unbounded self-adjoint operator affiliated with $\mathcal{N}$ so that $(1 + D_0^2)^{-1/2}$ is in $\mathcal{L}^{(1,\infty)}$. Let A_t and B be in $\mathcal{N}$ for $t \in [0,1]$ with A_t self-adjoint and $t \mapsto A_t$ continuous. Let $D_t = D_0 + A_t$ and let p be a real number with $1 < p < 4/3$. Then, the quantity*

$$\tau(B(1 + D_0^2)^{-p/2} - B(1 + D_t^2)^{-p/2})$$

is uniformly bounded independent of $t \in [0,1]$ and $p \in (1, \frac{4}{3})$.

Proof. We estimate

$$\begin{aligned}|\tau(B(1+D_0^2)^{-p/2}-B(1+D_t^2)^{-p/2})| &\leqslant ||B(1+D_0^2)^{-p/2}-B(1+D_t^2)^{-p/2}||_1\\ &\leqslant ||B||\cdot||(1+D_0^2)^{-p/2}-(1+D_t^2)^{-p/2}||_1\\ &\leqslant ||B||\cdot||\,|(1+D_0^2)^{-1}-(1+D_t^2)^{-1}|^{p/2}||_1\\ &=||B||\cdot||(1+D_0^2)^{-1}-(1+D_t^2)^{-1}||_{p/2}^{p/2}.\end{aligned}$$

Where the last *inequality* follows from the BKS inequality, see [BKS], or the discussion and references in [CPS].

Now, by Lemma 2.9 of [CP1] we have

$$(1+D_0^2)^{-1}-(1+D_t^2)^{-1}=W_t+Z_t,$$

where

$$W_t=D_0(1+D_0^2)^{-1}A_t(1+D_t^2)^{-1}$$

and

$$Z_t=(1+D_0^2)^{-1}A_tD_t(1+D_t^2)^{-1}.$$

Now, since $p/2$ is less than 1, $||\cdot||_{p/2}$ is not a norm: however, by either 4.9 (iii) or 4.7 (i) of [FK] we have

$$||W_t+Z_t||_{p/2}^{p/2}\leqslant||W_t||_{p/2}^{p/2}+||Z_t||_{p/2}^{p/2}.$$

Thus, it suffices to see that $||W_t||_{p/2}^{p/2}$ and $||Z_t||_{p/2}^{p/2}$ are bounded independent of p and t.

Now, $(1+D_0^2)^{-1/2}$ and $(1+D_t^2)^{-1/2}$ are both in $\mathscr{L}^{(1,\infty)}$ by Lemma 6 of [CP1] and therefore in $\mathscr{L}^q$ for any $q>1$. In particular, $(1+D_0^2)^{-1}$ and $(1+D_t^2)^{-1}$ are both in $\mathscr{L}^{2/3}$ and $\mathscr{L}^{3/4}$.

Also, $p<4/3$ implies $4-3p>0$ and since we also have $p>1$, we get $r_p:=\frac{2p}{4-3p}>3/2$ and we easily calculate:

$$\frac{1}{2/3}+\frac{1}{r_p}=\frac{1}{p/2}.$$

So, by Hölder's inequality [FK, Theorem 4.2] , we get

$$\begin{aligned}||W_t||_{p/2}^{p/2} &= ||D_0(1+D_0^2)^{-1}A_t(1+D_t^2)^{-1}||_{p/2}^{p/2}\\ &\leqslant \{||D_0(1+D_0^2)^{-1}||_{r_p}||A_t||\cdot||(1+D_t^2)^{-1}||_{2/3}\}^{p/2}\\ &= \{[\tau(|D_0(1+D_0^2)|^{-r_p})]^{1/r_p}||A_t||\,[\tau(1+D_t^2)^{-2/3}]^{3/2}\}^{p/2}\\ &\leqslant \{[\tau(1+D_0^2)^{-r_p/2})]^{1/r_p}||A_t||f(||A_t||)[\tau((1+D_0^2)^{-2/3}]^{3/2}\}^{p/2},\end{aligned}$$

where $f(t) = 1 + \frac{1}{2}(t^2 + t\sqrt{4+t^2})$ by Lemma 6 of [CP1]. Since $r_p > 3/2$ we have

$$[(1+D_0^2)^{-1/2}]^{3/2} \geqslant [(1+D_0^2)^{-1/2}]^{r_p}.$$

Thus, we obtain our final inequality for $||W_t||_{p/2}^{p/2}$:

$$||W_t||_{p/2}^{p/2} \leqslant \{[\tau((1+D_0^2)^{-3/4})]^{1/r_p}||A_t||f(||A_t||)\cdot||(1+D_0^2)^{-1}||_{2/3}\}^{p/2}.$$

This last quantity is clearly a continuous function of t and p for $t\in[0,1]$ and $p\in(1,\frac{4}{3})$. As $p\to 1$ (and so $r_p\to 2$) we see that the estimate for $||W_t||_{p/2}^{p/2}$ converges to a continuous function of $t\in[0,1]$ and so remains bounded at this end of $(1,\frac{4}{3})$. On the other hand, as $p\to\frac{4}{3}$ (and so $r_p\to\infty$) we again see that the estimate for $||W_t||_{p/2}^{p/2}$ converges to a continuous function of $t\in[0,1]$ and so remains bounded at the right-hand side of $(1,\frac{4}{3})$. That is, the estimate for $||W_t||_{p/2}^{p/2}$ is bounded independent of $t\in[0,1]$ and $p\in(1,\frac{4}{3})$.

A slightly different calculation for $||Z_t||_{p/2}^{p/2}$, yields the inequality

$$||Z_t||_{p/2}^{p/2} \leqslant \{||(1+D_0^2)^{-1}||_{2/3}||A_t||\cdot||f(A_t)||^{1/2}[||(1+D_0^2)^{-1}||_{3/4}^{3/4}]^{1/r_p}\}^{p/2}.$$

Similar considerations to those above show that $||Z_t||_{p/2}^{p/2}$ is also bounded independent of $t\in[0,1]$ and $p\in(1,\frac{4}{3})$. This completes the proof. □

In [CP2, Corollary 9.4] we proved the following. Let $\mathcal{N}$ be a factor and $(\mathcal{N}, D_0)$ be a $\mathcal{L}^{(1,\infty)}$-summable Breuer–Fredholm module for the unital Banach $*$-algebra, $\mathcal{A}$, and let $u\in\mathcal{A}$ be a unitary such that $[D_0,u]$ is bounded. Let P be the projection on the nonnegative spectral subspace of D_0. Then for each $p>1$

$$ind(PuP) = sf(D_0, uD_0u^*) = \frac{1}{\tilde{C}_{p/2}}\int_0^1 \tau(u[D_0,u^*](1+(D_t^u)^2)^{-p/2})\,dt,$$

where

$$D_t^u = D_0 + tu[D_0, u^*] = D_0 + A_t \quad \text{for } A_t = tu[D_0, u^*] \;\; t \in [0, 1]$$

and $\tilde{C}_{\frac{p}{2}} = \int_{-\infty}^{\infty} (1 + x^2)^{-\frac{p}{2}} dx$. (Note that a similar formula appears in Theorem 2.17 of [CP1] except that there the exponent $p > \frac{3}{2}$. The improvement in the lower bound on the exponent uses the theory of theta summable Fredholm modules in [CP2].) The removal of the assumption that $\mathcal{N}$ be a factor is not hard (see for example the discussion in the appendix to [PR]). The main point to note is that when $\mathcal{N}$ is a general semi-finite von Neumann algebra then the map $u \to ind(PuP)$ is clearly dependent on the choice of trace τ, there being no canonical choice. However, this is not important for our discussion in this paper.

Theorem 6.2. *Let $(\mathcal{N}, D_0)$ be a $\mathcal{L}^{(1,\infty)}$-summable Breuer–Fredholm module for the unital Banach $*$-algebra, $\mathcal{A}$, and let $u \in \mathcal{A}$ be a unitary such that $[D_0, u]$ is bounded. Let P be the projection on the nonnegative spectral subspace of D_0. Then with ω chosen as in Theorem* 1.5,

$$\begin{aligned} ind(PuP) = sf(D_0, uD_0u^*) &= \lim_{p \to 1^+} \tfrac{1}{2}(p-1)\tau(u[D_0, u^*](1 + D_0^2)^{-p/2}) \\ &= \tfrac{1}{2}\tau_\omega(u[D_0, u^*](1 + D_0^2)^{-1/2}) \\ &= \tfrac{1}{2}\tau_\omega(u[D_0, u^*]|D_0|^{-1}), \end{aligned}$$

where the last equality only holds if D_0 has a bounded inverse.

Remark. (1) The equality

$$ind(PuP) = \tfrac{1}{2}\tau_\omega(u[D_0, u^*]|D_0|^{-1}) \tag{6.1}$$

proved above should be compared with Theorem IV.2.8 of [Co4]. In the case where $\mathcal{N} = \mathcal{B}(\mathcal{H})$ the RHS of (6.1) is a Hochschild 1-cocycle on $\mathcal{A}$ which is known to equal the Chern character of the $\mathcal{L}^{(1,\infty)}$-summable Fredholm module $(\mathcal{A}, D_0, \mathcal{H})$.

(2) Since any 1-summable module is clearly a $\mathcal{L}^{(1,\infty)}$-summable module, the theorem implies that any unbounded 1-summable module must have a trivial pairing with $K_1(\mathcal{A})$ and is therefore uninteresting from the homological point of view.

Proof. By the extension of Corollary 9.4 of [CP2] to the case where $\mathcal{N}$ is a general semifinite von Neumann algebra, we have for each $p > 1$, that

$$ind(PuP) = \frac{1}{\tilde{C}_{p/2}} \int_0^1 \tau(u[D_0, u^*](1 + (D_t^u)^2)^{-p/2})\, dt,$$

where the notation is described in the paragraph preceding the theorem. Now, by Lemma 6.1, we have that

$$|\tau(u[D_0, u^*][(1+(D_t^u)^2)^{-p/2} - (1+D_0^2)^{-p/2}])|$$

is uniformly bounded independent of t and p for $1<p<\frac{4}{3}$. Since, $\tilde{C}_{p/2} \to \infty$ as $p \to 1^+$, we see that:

$$\begin{aligned}
&\left| ind(PuP) - \frac{1}{\tilde{C}_{p/2}} \tau(u[D_0, u^*](1+D_0^2)^{-p/2}) \right| \\
&= \left| \frac{1}{\tilde{C}_{p/2}} \int_0^1 \tau(u[D_0, u^*](1+(D_t^u)^2)^{-p/2})\, dt - \frac{1}{\tilde{C}_{p/2}} \int_0^1 \tau(u[D_0, u^*](1+D_0^2)^{-p/2})\, dt \right| \\
&\leqslant \frac{1}{\tilde{C}_{p/2}} \int_0^1 |\tau(u[D_0, u^*][(1+(D_t^u)^2)^{-p/2} - (1+D_0^2)^{-p/2}])|\, dt \\
&\leqslant \frac{\text{Constant}}{\tilde{C}_{p/2}} \to 0.
\end{aligned}$$

Now, it is elementary that as $p \to 1^+$

$$\frac{2}{p-1} = \int_{|x| \geqslant 1} \left(\frac{1}{|x|}\right)^p dx \sim \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{1+x^2}}\right)^p dx = \tilde{C}_{p/2}.$$

This ends the proof of the first equality.

The second equality follows from Theorem 3.8(i).

The third equality follows from the fact that $(\sqrt{1+D_0^2})^{-1} - |D_0|^{-1}$ is very trace-class:

$$(\sqrt{1+D_0^2})^{-1} - |D_0|^{-1} = (\sqrt{1+D_0^2})^{-1}|D_0|^{-1}(\sqrt{1+D_0^2} + |D_0|)^{-1}. \qquad \square$$

7. Non-smooth foliations and pseudo-differential operators

The main aim of Prinzis' thesis [P] is to establish a Wodzicki residue formula for the Dixmier trace of certain pseudo-differential operators associated to non-smooth actions of $\mathbf{R}^n$ on a compact space X. We will not reproduce the full details of [P], indeed the subject deserves a far more complete analysis than we have space for here.

The set-up is the group-measure space construction of Murray–von Neumann. Thus X is a compact space equipped with a probability measure v and a continuous free minimal ergodic action α of $\mathbf{R}^n$ on X leaving v invariant. We write the action as

$x \to t.x$ for $x \in X$ and $t \in \mathbf{R}^n$. Then the crossed product $L^\infty(X, v) \times_\alpha \mathbf{R}^n$ is a type II factor contained in the bounded operators on $L^2(\mathbf{R}^n, L^2(X, v))$. We describe the construction. For a function $f \in L^1(\mathbf{R}^n, L^\infty(X, v)) \subset L^\infty(X, v) \times_\alpha \mathbf{R}^n$ the action of f on a vector ξ in $L^2(\mathbf{R}^n, L^2(X, v))$ is defined by twisted left convolution as follows:

$$(\tilde{\pi}(f)\xi)(s) = \int_{\mathbf{R}^n} \alpha_s^{-1}(f(t))\xi(s-t)dt.$$

Here, $f(t)$ is a function on X acting as a multiplication operator on $L^2(X, v)$. The twisted convolution algebra

$$L^1(\mathbf{R}^n, L^\infty(X, v)) \cap L^2(\mathbf{R}^n, L^2(X, v))$$

is a dense subspace of $L^2(\mathbf{R}^n, L^2(X, v))$ and there is a canonical faithful, normal, semifinite trace, Tr, on the von Neumann algebra that it generates. This von Neumann algebra is

$$\mathcal{N} = (\tilde{\pi}(L^\infty(X, v) \times_\alpha \mathbf{R}^n))''.$$

For functions $f, g : \mathbf{R}^n \to L^\infty(X)$ which are in $L^2(\mathbf{R}^n, L^2(X, v))$ and whose twisted left convolutions $\tilde{\pi}(f)$, $\tilde{\pi}(g)$ define bounded operators on $L^2(\mathbf{R}^n, L^2(X, v))$, this trace is given by

$$Tr(\tilde{\pi}(f)^* \tilde{\pi}(g)) = \int_{\mathbf{R}^n} \int_X f(t, x)g(t, x)^* \, dv(x)dt,$$

where we think of f, g as functions on $\mathbf{R}^n \times X$.

Identify $L^2(\mathbf{R}^n)$ with $L^2(\mathbf{R}^n) \otimes 1 \subset L^2(\mathbf{R}^n, L^2(X, v))$ then any scalar-valued function f on $\mathbf{R}^n$ which is the Fourier transform $f = \hat{g}$ of a bounded L^2 function, g will satisfy $f \in L^2(\mathbf{R}^n, L^2(X, v))$ and $\tilde{\pi}(f)$ will be a bounded operator.

Pseudo-differential operators are defined in terms of their symbols. A smooth symbol of order m is a function $a : X \times \mathbf{R}^n \to \mathbf{C}$ such that for each $x \in X$ a_x, defined by $a_x(t, \xi) = a(t.x, \xi)$, satisfies:

(1) $\sup\{|\partial_\xi^\beta \partial_t^\gamma a_x(t, \xi)(1 + |\xi|)^{-m+|\beta|}| (t, \xi) \in \mathbf{R}^n \times \mathbf{R}^n, \beta, \gamma \in \mathbf{N}^n, |\beta| + |\gamma| \leqslant M\} < \infty$ for all $M \in \mathbf{N}$;

(2) $\xi \to a_x(0, \xi)$ is a smooth function on $\mathbf{R}^n$ into the space $\mathscr{C}^\infty(X)$, the set of continuous functions f on X such that $t \to (x \to f(t.x))$ is smooth on $\mathbf{R}^n$.

Each symbol a defines a pseudo-differential operator $Op(a)$ on $C(X) \otimes C_c^\infty(\mathbf{R}^n)$ by

$$Op(a)f(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{it\xi} a(t.x, \xi) \hat{f}(x, \xi) \, d\xi, \quad f \in C(X) \otimes C_c^\infty(\mathbf{R}^n).$$

The principal symbol of a pseudo-differential operator A on X is the limit

$$\sigma_m(A)(x,\xi) = \lim_{\lambda\to\infty} \frac{a(x,\lambda\xi)}{\lambda^m}(x,\xi)\in(X\times\mathbf{R}^n\backslash\{0\})$$

if it exists. We say A is elliptic if its symbol a is such that a_x is elliptic for all $x\in X$. Prinzis studies invertible positive elliptic pseudo-differential operators A with a principal symbol. Henceforth, we will only consider such operators. The zeta function of such an operator is $\zeta(z)=\tau(A^z)$ and this exists because A^z is in the trace class in $\mathcal{N}$, [P] for $\Re z< -n/m$. Prinzis shows that

$$\lim_{x\to -\frac{n}{m}^-}\left(x+\frac{n}{m}\right)\zeta(x) = -\frac{1}{(2\pi)^n m}\int_{X\times S^{n-1}} \sigma_m(A)(x,\xi)^{-\frac{n}{m}}\,dv(x)\,d\xi \tag{7.1}$$

and that $A^{-\frac{n}{m}}\in\mathcal{L}^{(1,\infty)}$.

Our contribution to this situation is to note that (7.1) combined with Theorem 5.6 implies that we have the relation

$$\tau_\omega(A^{-\frac{n}{m}}) = \frac{1}{(2\pi)^n n}\int_{X\times S^{n-1}} \sigma_m(A)(x,\xi)^{-\frac{n}{m}}\,dv(x)\,d\xi.$$

In other words, we have a type II Wodzicki residue for evaluating the Dixmier trace of these pseudo-differential operators.

8. Lesch's index theorem

Here, we consider a unital C^*-algebra $\mathcal{A}$ with a faithful finite trace, τ satisfying $\tau(\mathbf{1})=1$ and a continuous action α of $\mathbf{R}$ on $\mathcal{A}$ leaving τ invariant. In this section, we deduce the index theorem of M. Lesch as a corollary of our zeta function approach to the Dixmier Trace formula for the index of generalised Toeplitz operators in this situation. See [L,PR].

We let H_τ denote the Hilbert space completion of $\mathcal{A}$ in the inner product $(a|b)=\tau(b^*a)$. Then $\mathcal{A}$ is a Hilbert Algebra and the left regular representation of $\mathcal{A}$ on itself extends by continuity to a representation, $a\mapsto\pi_\tau(a)$ of $\mathcal{A}$ on H_τ [Dix]. In what follows, we will drop the notation π_τ and just denote the action of $\mathcal{A}$ on H_τ by juxtaposition.

We now look at the induced representation, $\tilde{\pi}$, of the crossed product C^*-algebra $\mathcal{A}\times_\alpha\mathbf{R}$ on $L^2(\mathbf{R},H_\tau)$. That is, $\tilde{\pi}$ is the representation $\pi\times\lambda$ obtained from the covariant pair, (π,λ) of representations of the system $(\mathcal{A},\mathbf{R},\alpha)$ defined for $a\in\mathcal{A}$, $t,s\in\mathbf{R}$ and $\xi\in L^2(\mathbf{R},H_\tau)$ by

$$(\pi(a)\xi)(s) = \alpha_s^{-1}(a)\xi(s)$$

and

$$\lambda_t(\xi)(s) = \xi(s-t).$$

Then, for a function $x \in L^1(\mathbf{R}, \mathscr{A}) \subset \mathscr{A} \times_\alpha \mathbf{R}$ the action of $\tilde{\pi}(x)$ on a vector ξ in $L^2(\mathbf{R}, H_\tau)$ is defined as follows:

$$(\tilde{\pi}(x)\xi)(s) = \int_{-\infty}^{\infty} \alpha_s^{-1}(x(t))\xi(s-t)\,dt.$$

Now the twisted convolution algebra $L^1(\mathbf{R}, \mathscr{A}) \cap L^2(\mathbf{R}, H_\tau)$ is a dense subspace of $L^2(\mathbf{R}, H_\tau)$ and also a Hilbert Algebra in the given inner product. As such, there is a canonical faithful, normal, semifinite trace, Tr, on the von Neumann algebra that it generates. Of course, this von Neumann algebra is identical with

$$\mathscr{N} = (\tilde{\pi}(\mathscr{A} \times_\alpha \mathbf{R}))''.$$

For functions $x, y : \mathbf{R} \to \mathscr{A} \subset H_\tau$ which are in $L^2(\mathbf{R}, H_\tau)$ and whose twisted left convolutions $\tilde{\pi}(x)$, $\tilde{\pi}(y)$ define bounded operators on $L^2(\mathbf{R}, H_\tau)$, this trace is given by

$$Tr(\tilde{\pi}(y)^*\tilde{\pi}(x)) = \langle x|y \rangle = \int_{-\infty}^{\infty} \tau(x(t)y(t)^*)dt.$$

In particular, if we identify $L^2(\mathbf{R}) = L^2(\mathbf{R}) \otimes 1_{\mathscr{A}} \subset L^2(\mathbf{R}, H_\tau)$ then any scalar-valued function x on $\mathbf{R}$ which is the Fourier transform $x = \hat{f}$ of a bounded L^2 function, f will have the properties that $x \in L^2(\mathbf{R}, H_\tau)$ and $\tilde{\pi}(x)$ is a bounded operator. For such scalar functions x, the operator $\tilde{\pi}(x)$ is just the usual convolution by the function x and is usually denoted by $\lambda(x)$ since it is just the integrated form of λ. The next lemma follows easily from these considerations.

Lemma 8.1. *With the hypotheses and notation discussed above*:

(i) *if* $h \in L^2(\mathbf{R})$ *with* $\lambda(h)$ *bounded and* $a \in \mathscr{A}$, *then defining* $f : \mathbf{R} \to H_\tau$ *via* $f(t) = ah(t)$ *we see that* $f \in L^2(\mathbf{R}, H_\tau)$ *and* $\tilde{\pi}(f) = \pi(a)\lambda(h)$ *is bounded*,

(ii) *if* $g \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ *and* $a \in \mathscr{A}$ *then* $\pi(a)\lambda(\hat{g})$ *is trace-class in* $\mathscr{N}$ *and*

$$Tr(\pi(a)\lambda(\hat{g})) = \tau(a)\int_{-\infty}^{\infty} g(t)\,dt.$$

Proof. To see part (i), let $\xi \in C_c(\mathbf{R}, H_\tau) \subseteq L^2(\mathbf{R}, H_\tau)$. Then

$$(\tilde{\pi}(f)\xi)(s) = \int_{-\infty}^{\infty} \alpha_s^{-1}(f(t))\xi(s-t)\,dt$$
$$= \int_{-\infty}^{\infty} \alpha_s^{-1}(a)h(t)\xi(s-t)\,dt$$

$$
\begin{aligned}
&= \alpha_s^{-1}(a) \int_{-\infty}^{\infty} h(t)\xi(s-t)\, dt \\
&= \alpha_s^{-1}(a)(\lambda(h)\xi)(s) \\
&= (\pi(a)\lambda(h)\xi)(s).
\end{aligned}
$$

To see part (ii) we can (and do) assume that g is nonnegative and a is self-adjoint. Then let $g = g^{1/2}g^{1/2}$ so that $g^{1/2} \in L^2 \cap L^\infty$ and so $\lambda(\widehat{g^{1/2}})$ is bounded. Now,

$$\pi(a)\lambda(\hat{g}) = \pi(a)\lambda(\widehat{g^{1/2}})\pi(1_{\mathscr{A}})\lambda(\widehat{g^{1/2}}).$$

Then, $\pi(a)\lambda(\widehat{g^{1/2}}) = \tilde{\pi}(x)$ where $x(t) = a\widehat{g^{1/2}}(t)$ and $\pi(1_{\mathscr{A}})\lambda(\widehat{g^{1/2}}) = \tilde{\pi}(y)$ where $y(t) = 1_{\mathscr{A}}\widehat{g^{1/2}}(t)$. So, $\tilde{\pi}(x)$ and $\tilde{\pi}(y)$ are in $\mathscr{N}_{sa}$ and $\pi(a)\lambda(\hat{g}) = \tilde{\pi}(x)\tilde{\pi}(y)$.

Hence,

$$
\begin{aligned}
Tr(\pi(a)\lambda(\hat{g})) &= Tr(\tilde{\pi}(x)\tilde{\pi}(y)) \\
&= \int_{-\infty}^{\infty} \tau(x(t)y(t))\, dt \\
&= \tau(a) \int_{-\infty}^{\infty} \left|\widehat{g^{1/2}}(t)\right|^2 dt \\
&= \tau(a) \int_{-\infty}^{\infty} g(s)\, ds. \qquad \square
\end{aligned}
$$

Now, $\mathscr{N}$ is a semifinite von Neumann algebra with faithful, normal, semifinite trace, Tr, and a faithful representation $\pi : \mathscr{A} \to \mathscr{N}$ [Dix]. For each $t \in \mathbf{R}$, λ_t is a unitary in $U(\mathscr{N})$. In fact the one-parameter unitary group $\{\lambda_t \mid t \in \mathbf{R}\}$ can be written $\lambda_t = e^{itD}$ where D is the unbounded self-adjoint operator

$$D = \frac{1}{2\pi i}\frac{d}{ds}$$

which is affiliated with $\mathscr{N}$. In the Fourier Transform picture (i.e., the spectral picture for D) of the previous proposition, D becomes multiplication by the independent variable and so $f(D)$ becomes pointwise multiplication by the function f. That is,

$$\tilde{\pi}(\hat{f}) = \lambda(\hat{f}) = f(D).$$

And, hence, if f is a bounded L^1 function, then:

$$Tr(f(D)) = \int_{-\infty}^{\infty} f(t)\, dt.$$

By this discussion and the previous lemma, we have the following result.

Lemma 8.2. *If $f \in L^{-1}(\mathbf{R}) \cap L^{\infty}(\mathbf{R})$ and $a \in \mathscr{A}$ then $\pi(a)f(D)$ is trace-class in $\mathscr{N}$ and*

$$Tr(\pi(a)f(D)) = \tau(a)\int_{-\infty}^{\infty} f(t)\,dt.$$

We let δ be the densely defined (unbounded) $*$-derivation on $\mathscr{A}$ which is the infinitesimal generator of the representation $\alpha: \mathbf{R} \to Aut(\mathscr{A})$ and let $\hat{\delta}$ be the unbounded $*$-derivation on $\mathscr{N}$ which is the infinitesimal generator of the representation $Ad \circ \lambda: \mathbf{R} \to Aut(\mathscr{N})$ (here $Ad(\lambda_t)$ denotes conjugation by λ_t). Now if $a \in dom(\delta)$ then clearly $\pi(a) \in dom(\hat{\delta})$ and $\pi(\delta(a)) = \hat{\delta}(\pi(a))$. By Bratteli and Robinson [BR, Proposition 3.2.55] (and its proof) we have that $\pi(\delta(a))$ leaves the domain of D invariant and

$$\pi(\delta(a)) = 2\pi i[D, \pi(a)].$$

We are now in a position to state and prove Lesch's index theorem.

Theorem 8.3. *Let τ be a faithful finite trace on the unital C^*-algebra, $\mathscr{A}$, which is invariant for an action α of $\mathbf{R}$. Let $\mathscr{N}$ be the semifinite von Neumann algebra $(\tilde{\pi}(\mathscr{A} \times_{\alpha} \mathbf{R}))''$, and let D be the infinitesimal generator of the canonical representation λ of $\mathbf{R}$ in $U(\mathscr{N})$. Then, the representation $\pi: \mathscr{A} \to \mathscr{N}$ defines a $\mathscr{L}^{(1,\infty)}$ summable Breuer–Fredholm module $(\mathscr{N}, D)$ for $\mathscr{A}$. Moreover, if P is the nonnegative spectral projection for D and $u \in U(\mathscr{A})$ is also in the domain of δ, then $T_u := P\pi(u)P$ is Breuer–Fredholm in $P\mathscr{N}P$ and*

$$ind(T_u) = \frac{1}{2\pi i}\tau(u\delta(u^*)).$$

Proof. It is easy to see that D satisfies $(1 + D^2)^{-1/2} \in \mathscr{L}^{(1,\infty)}$. By the previous discussion, for any $a \in dom(\delta)$ we have $\pi(\delta(a)) = 2\pi i[D, \pi(a)]$. Since the domain of δ is dense in $\mathscr{A}$ we see that π defines a $\mathscr{L}^{(1,\infty)}$ summable Breuer–Fredholm module for $\mathscr{A}$.

Now, by Theorem 6.2 and Lemma 8.2

$$\begin{aligned}
ind(T_u) &= \lim_{p \to 1^+} \frac{1}{2}(p-1)\, Tr(\pi(u)[D, \pi(u^*)](1 + D^2)^{-p/2}) \\
&= \lim_{p \to 1^+} \frac{1}{2}(p-1)\frac{1}{2\pi i}\, Tr(\pi(u\delta(u^*))(1 + D^2)^{-p/2}) \\
&= \lim_{p \to 1^+} \frac{1}{2}(p-1)\frac{1}{2\pi i}\tau(u\delta(u^*))\int_{-\infty}^{\infty} (1 + t^2)^{-p/2}\,dt \\
&= \lim_{p \to 1^+} \frac{1}{2\pi i}\tau(u\delta(u^*))\frac{1}{2}(p-1)\tilde{C}_{p/2} \\
&= \frac{1}{2\pi i}\tau(u\delta(u^*)). \qquad \square
\end{aligned}$$

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