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ON PERMUTATION CHARACTERS OF WREATH PRODUCTS

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It is known that the character rings of symmetric groups S_n and the character rings of hyperoctahedral groups $S_2 \sim S_n$ are generated by (transitive) permutation characters. These results of Young are generalized to wreath products $G \sim H$ (G a finite group, H a permutation group acting on a finite set). It is shown that the character ring of $G \sim H$ is generated by permutation characters if this holds for G, H and certain subgroups of H. This result can be sharpened for wreath products $G \sim S_n$; if the character ring of G has a basis of transitive permutation characters, then the same holds for the character ring of $G \sim S_n$.

0. Introduction

It is well known that the character rings of symmetric groups S_n and the character rings of hyperoctahedral groups $S_2 \sim S_n$ possess $\mathbb{Z}$-bases consisting of transitive permutation characters (cf. [4, 6]). In order to generalize these results we prove the following two theorems:

Theorem 1. *Let G denote a finite group and H a subgroup of S_n. If the characters of G and the characters of all intersections of H with Young-subgroups of S_n are $\mathbb{Z}$-linear combinations of permutation characters, then the same holds for the characters of $G \sim H$.*

Theorem 2. *If the character ring of G has a $\mathbb{Z}$-basis of transitive permutation characters then the same holds for the character rings of the monomial groups $G \sim S_n$.*

1. Some remarks on wreath products

Let G denote a finite group and H a subgroup of S_n, where S_n is the symmetric group on $\mathbf{N} := \{1, \ldots, n\}$. The set

$$G \sim H := \{(f, \pi) \mid f : \mathbf{N} \to G \text{ and } \pi \in H\}$$

together with the composition law

$$(f, \pi)(f', \pi') := (ff'_\pi, \pi\pi')$$

(where $\pi\pi'(i) := \pi(\pi'(i))$, $ff'_\pi(i) := f(i)f'_\pi(i) := f(i)f'(\pi^{-1}(i))$, for all $i \in \mathbf{N}$) constitutes a group, *the wreath product of G with H.* The group $G \sim S_n$ is called *the monomial group of degree n over G.*

Let F be a representation of G over the complex field $\mathbf{C}$ with representation space V. If m denotes a positive integer, then we obtain an ordinary representation of $G \sim S_m$ with representation space

$$\overset{m}{\otimes} V := V \otimes_{\mathbf{C}} \ldots \otimes_{\mathbf{C}} V \qquad (m \text{ factors})$$

by putting

$$(f, \pi)(v_1 \otimes \ldots \otimes v_m) := f(1)v_{\pi^{-1}(1)} \otimes \ldots \otimes f(m)v_{\pi^{-1}(m)},$$

for each $v_1, \ldots, v_m \in V$.

Following the notation of [2] we denote this representation by

$$\overset{\widetilde{m}}{\#} F,$$

since it extends the m-fold outer tensor product $\#^m F$ of F with itself which is a representation of the normal subgroup

$$G^* := \{(f, 1) \mid f : \mathbf{N} \to G\} \trianglelefteq G \sim S_m .$$

If furthermore D is a representation of $U \leq S_m$, then

$$D'(f, \pi) := D(\pi)$$

yields a representation D' of $G \sim U$.

It follows from [2] that the ordinary irreducible representations of $G \sim H$ are of the form

(1.1) $\qquad (R \otimes S') \uparrow G \sim H,$

where $\mathfrak{R}$ is the restriction of an outer tensor product representation $\#(\widetilde{\#}^{n_i} F_i)$ of $(G \sim S_{n_1}) \times \ldots \times (G \sim S_{n_r}) = G \sim (S_{n_1} \times \ldots \times S_{n_r}) \leqslant G \sim S_n$ to $G \sim (H \cap \mathsf{X}_i S_{n_i})$, F_i ordinary irreducible representations of G, and S is an ordinary irreducible representation of the intersection $H \cap \mathsf{X}_i S_{n_i}$ of H with the Young subgroup $\mathsf{X}_i S_{n_i} = S_{n_1} \times \ldots \times S_{n_r}$ of S_n.

We are now going to prove two important relations on the character of $\widetilde{\#}^m F$.

Let χ be a class function on G and $(f, \pi) \in G \sim S_m$. Let $g_1, \ldots, g_{c(\pi)}$ be the cycleproducts associated with the $c(\pi)$ cyclic factors of π with respect to f (cf. [2]) and put

$$(\chi; m)(f, \pi) := \prod_{i=1}^{c(\pi)} \chi(g_i) .$$

It is easy to see that $(\chi; m)$ is a class function of $G \sim S_m$ and that the following holds (cf. [3]):

Lemma 1.1. *If χ is the character of F, then $(\chi; m)$ is the character of $\widetilde{\#}^m F$.*

Let now sgn π denote the sign of the permutation π and define

$$(\chi; 1^m)(f, \pi) := \operatorname{sgn} \pi \cdot (\chi; m)(f, \pi) .$$

Putting $(\chi; 0) := (\chi; 1^0) :\equiv 1$, we can prove the following result which generalizes formula IV on page 290 in [5]:

Theorem 1.2.

(i) $$(\chi_1 + \chi_2; m) = \sum_{\nu=0}^{m} [(\chi_1; \nu)(\chi_2; m-\nu)] \uparrow G \sim S_m ,$$

(ii) $$(\chi_1 - \chi_3; m) = \sum_{\nu=0}^{m} (-1)^{m-\nu} [(\chi_1; \nu)(\chi_3; 1^{m-\nu})] \uparrow G \sim S_m .$$

The proof of Theorem 1.2 is based on Lemma 1.3 for which we first introduce the abbreviation

$$\chi^\nu := [(\chi_1; \nu)(\chi_2; m-\nu)] \uparrow G \sim S_m .$$

For $(f, \pi) \in G \sim S_m$ we denote by $\pi_1, ..., \pi_{c(\pi)}$ the $c(\pi)$ cyclic factors of π with corresponding cycleproducts $g_1, ..., g_{c(\pi)}$ with respect to f. Let n_i denote the length of π_i.

Lemma 1.3. *We have*

$$\chi^{\nu}(f, \pi) = \sum_{(I,J)} \prod_{i \in I} \chi_1(g_i) \prod_{j \in J} \chi_2(g_j) \ ,$$

where the sum is taken over all pairs (I, J) *of subsets of* $\{1, ..., c(\pi)\}$ *such that* $I \cup J = \{1, ..., c(\pi)\}$, $I \cap J = \emptyset$, $\Sigma_{i \in I} n_i = \nu$, $\Sigma_{j \in J} n_j = m - \nu$.

Proof. Let (ρ_k) be a system of left coset representatives of $S_\nu \times S_{m-\nu}$ in S_m. Hence $((e; \rho_k))$, where $e(i) = 1$ for all $i \in \{1, ..., m\}$, is a corresponding system for $G \sim (S_\nu \times S_{m-\nu})$ in $G \sim S_m$. The definition of induced characters yields

$$\chi^{\nu}(f, \pi) = \sum_k \ [\overline{(\chi_1; \nu)(\chi_2; m-\nu)}] \ ((e; \rho_k)^{-1} \ (f; \pi)(e, \rho_k)) \ ,$$

where the bar denotes that the value of the function is 0 if the argument is not in $G \sim (S_\nu \times S_{m-\nu})$.

Let now N_i be the set of symbols which are contained in the cyclic factor π_i of π. We then have

$$(e, \rho_k)^{-1}(f, \pi)(e, \rho_k) = (f_{\rho_k^{-1}}, \rho_k^{-1} \pi \rho_k) \in G \sim (S_\nu \times S_{m-\nu})$$

if and only if

$$\rho_k(N_i) \subseteq \{1, ..., \nu\} \text{ or } \{\nu + 1, ..., m\}, \quad \text{for all } i = 1, ..., c(\pi) .$$

Let us consider two representatives ρ_k and ρ_l and assume that the following holds:

$$\rho_k^{-1}(N_i) \text{ and } \rho_l^{-1}(N_i) \subseteq \{1, ..., \nu\}$$

or

$$\rho_k^{-1}(N_i) \text{ and } \rho_l^{-1}(N_i) \subseteq \{\nu + 1, ..., m\}$$

for all i. This together with

$$(\rho_l^{-1} \rho_k)(\rho_k^{-1}(N_i)) = \rho_l^{-1}((\rho_k \rho_k^{-1})(N_i)) = \rho_l^{-1}(N_i)$$

implies

$$\rho_l^{-1}\rho_k \in S_\nu \times S_{m-\nu}$$

i.e. $k = l$. Hence

$$\chi^\nu(f;\pi) = \sum_{(I,J)} (\chi_1;\nu)(f_I,\pi_I)\cdot(\chi_2;m-\nu)(f_J,\pi_J)\,,$$

where I and J are as above, (f_I,π_I) and (f_J,π_J) are the parts of the corresponding conjugate of (f,π) in $G\sim S_\nu$ and $G\sim S_{m-\nu}$.

It is easy to see that conjugation with elements of the form $(e;\rho)$ does not change the classes of the cycle products.
Hence we have

$$(\chi_1;\nu)(f_I,\pi_I) = \prod_{i\in I}\chi_1(g_i)\ ,$$

$$(\chi_2;m-\nu)(f_J,\pi_J) = \prod_{j\in J}\chi_2(g_j)\ .$$

This completes the proof of Lemma 1.3.

Proof of Theorem 1.2 (i). We have

$$(\chi_1+\chi_2;m)(f;\pi) = \prod_{i=1}^{c(\pi)}(\chi_1(g_i)+\chi_2(g_i))\ .$$

The multiplication yields all possible terms

$$\prod_{i\in I}\chi_1(g_i)\prod_{j\in J}\chi_2(g_j)\ ,$$

where $I\cap J=\emptyset$ and $I\cup J=\{1,\dots,c(\pi)\}$.

Lemma 1.3 has shown us that each of these terms occurs once and in exactly one χ^ν, namely the ν which satisfies $\nu=\Sigma_{i\in I}\,n_i$. This proves Theorem 1.2(i).

Proof of Theorem 1.2 (ii). Replacing χ_2 by $-\chi_3$ in χ^ν, we obtain from Theorem 1.2(i):

$$(\chi_1-\chi_3;m) = \sum_{\nu=0}^{m}\chi^\nu\ ,$$

where

$$\chi^{\nu}(f,\pi) = \sum_{(I,J)} \prod_{i\in I} \chi_1(g_i) \prod_{j\in J} (-\chi_3)(g_j)$$

$$= \sum_{(I,J)} \prod_{i\in I} \chi_1(g_i)\Big(\prod_{j\in J} \chi_3(g_j)\Big)(-1)^{|J|}.$$

On the other hand the proof of Lemma 1.3 yields

$$[(\chi_1;\nu)(\chi_3;1^{m-\nu})]\uparrow G\sim S_m(f,\pi) =$$

$$= \sum_{(I,J)} \prod_{i\in I} \chi_1(g_i)\Big(\prod_{j\in J} \chi_3(g_j)\Big)\operatorname{sgn}\pi_J,$$

and $\operatorname{sgn}\pi_J = \operatorname{sgn}(\Pi_{j\in J}\pi_j) = \Pi_{j\in J}\operatorname{sgn}\pi_j$.

As $\operatorname{sgn}\pi_j = (-1)^{n_j+1}$ and $\Sigma_{j\in J} n_j = m-\nu$, we obtain

$$\operatorname{sgn}\pi_J = (-1)^{m-\nu+|J|} = (-1)^{|J|}(-1)^{m-\nu}.$$

Hence

$$[(\chi_1;\nu)\cdot(\chi_3;1^{m-\nu})]\uparrow G\sim S_m(f,\pi) = (-1)^{m-\nu}\chi^{\nu}(f,\pi).$$

This completes the proof of Theorem 1.2.

2. Proof of Theorem 1

Let χ denote an ordinary character of G. Then by the assumption of Theorem 1 we have

$$\chi = \chi_i - \chi_j,$$

where χ_i and χ_j are permutation characters.

Theorem 1.2 implies that $(\chi;m)$ is a $\mathbb{Z}$-linear combination of the characters

$$[(\chi_i;\nu)\cdot(\chi_j;1^{m-\nu})]\uparrow G\sim S_m.$$

Let F_i and F_j be permutation representation of G with characters χ_i

and χ_j. Lemma 1.1 implies that $(\chi_i; \nu)$ is the character of $\widetilde{\#}^\nu F_i$ and that $(\chi_i; 1^{m-\nu})$ is the character of $\widetilde{\#}^{m-\nu} F_j \otimes [1^{m-\nu}]'$ (recall that as usual (cf. [2]) $[1^{m-\nu}]$ denotes the alternating representation of $S_{m-\nu}$).

It follows from the definition that $\widetilde{\#}^\nu F_i$ and $\widetilde{\#}^{m-\nu} F_j$ are permutation representations.

Furthermore we know that the character of $[1^{m-\nu}]$ is the difference of the character of the representation of $S_{m-\nu}$ which is induced by the identity representation of the alternating subgroup $A_{m-\nu} \leqslant S_{m-\nu}$ and the identity character:

$$\chi^{(1^{m-\nu})} = \chi^{IA_{m-\nu}} \uparrow S_{m-\nu} - \chi^{IS_{m-\nu}} .$$

As inner and outer tensor products of permutation representations and representations induced by permutation representations are again permutation representations, we obtain that $[(\chi_i; \nu) \cdot (\chi_j; 1^{m-\nu})] \uparrow G \sim S_m$ is a difference of two permutation characters.

As the representation R in (1.1) is a restriction of an outer tensor product of representations with characters of the form $(\chi; m)$, we obtain that the character of R is a $\mathbb{Z}$-linear combination of permutation characters.

S is a representation of the intersection of H and Young-subgroup of S_n. Hence, the character of $(R \otimes S') \uparrow G \sim H$ is a $\mathbb{Z}$-linear combination of permutation characters.

3. Proof of Theorem 2

According to the assumption of the theorem let $\{\psi_1, ..., \psi_h\}$ denote a $\mathbb{Z}$-basis of the character ring of G which consists of transitive permutation characters (so that h is the number of conjugacy classes of G). Let G_i be a subgroup of G, the identity representation IG_i of which induces a representation $IG_i \uparrow G$ with character ψ_i, $1 \leqslant i \leqslant h$.

The groups G_i are obviously pairwise non-conjugate. It follows from [2,3.7] that the number of the subgroups

$$(3.1) \qquad \mathop{\times}_{i=1}^{h} (G_i \sim S_{\alpha(i)}), \qquad \alpha(i) \text{ partition of } n_i , \qquad \sum_{i=1}^{h} n_i = n ,$$

where $S_{\alpha(i)}$ is a Young-subgroup of S_{n_i}, is equal to the number of conjugacy classes of $G \sim S_n$.

Hence Theorem 2 will be proved once we have shown that every irreducible character of $G \sim S_n$ is a $\mathbb{Z}$-linear combination of characters induced by identity characters of the groups (3.1).

Let us first consider representations of $G \sim S_n$ which are of the form

$$\overset{\sim}{\#}{}^{n} F \otimes [\beta]' ,$$

where $\beta = (\beta_1, ..., \beta_k)$ is a partition of n and $[\beta]$ the corresponding ordinary irreducible representation of S_n (cf. [2,4.6]).

The character χ^β of $[\beta]$ satisfies (cf. [2,4.41]):

$$(3.2) \qquad \chi^\beta = \det(\chi^{(\beta_i + j - i)}) = \sum_\rho \operatorname{sgn} \rho (I(\beta; \rho) \uparrow S_n) ,$$

where the sum is taken over all $\rho \in S_k$ such that all the $\beta_i + \rho(j) - 1$ are non-negative and $I(\beta; \rho)$ denotes the identity character of the Young-subgroup

$$\mathop{\times}_{i=1}^{k} S_{\beta_i + \rho(i) - i} .$$

This formula (3.2), Lemma 1.1, and [1, 38.5)(i)] yield that the character of $\overset{\sim}{\#}{}^n F \otimes [\beta]'$ is equal to

$$(3.3) \qquad \sum_\rho \operatorname{sgn} \rho \left(\prod_{i=1}^{k} (\chi; \beta_i + \rho(j) - i) \right) \uparrow G \sim S_n ,$$

where χ denotes the character of F.

χ is a $\mathbb{Z}$-linear combination of the characters ψ_i. Hence Theorem 1.2 implies that the characters $(\chi; m)$ are $\mathbb{Z}$-linear combinations of the characters which are induced by products of characters of the following form:

$$(3.4) \qquad (\psi_i; r) \quad \text{and} \quad (\psi_i; 1^r) .$$

Lemma 3.1. *We have that*

(i) $(\psi_i; r)$ *is induced by the identity character of* $G_i \sim S_r$,

(ii) $(\psi_i; 1^r)$ *is a* $\mathbb{Z}$*-linear combination of characters which are induced by identity characters of groups* $G_i \sim S_\alpha$ *with Young-subgroups* S_α *of* S_r.

Proof. The permutation representation which corresponds to $(\psi_i; r)$ acts transitively on

$$\left\{ \bigotimes_{j=1}^{r} (g_{h_j} \otimes_{\mathbb{C}G_i} 1_{\mathbb{C}}) \mid 1 \leq h_j \leq |G : G_i| \right\}$$

(where $\{g_1 = 1_G, g_2, ...\}$ is a system of left coset representatives of G_i in G) which is a basis of the corresponding representation module. Obviously the stabilizer of $\otimes^r (1_G \otimes_{\mathbb{C}G_i} 1_{\mathbb{C}})$ is $G_i \sim S_r$. This proves (i).

(ii) follows from (3.2) and (i).

Thus, (3.3)–(3.4) and Lemma 3.1 imply that the character of $\widetilde{\#}^n F \otimes [\beta]'$ is a linear combination of characters, induced by the identity characters of subgroups of $G \sim S_n$ which are conjugate to subgroups given in (3.1).

As every ordinary irreducible representation of $G \sim S_n$ is of the form (cf. [2])

$$\left(\left(\widetilde{\#}^{m_1} F_1 \otimes [\beta^1]'\right) \# \ldots \# \left(\widetilde{\#}^{m_k} F_k \otimes [\beta^k]'\right)\right) \uparrow G \sim S_n$$

with irreducible representations F_i of G, the assertion follows from the considerations above.

4. The characters of $S_m \sim S_n$

Let α be a partition of n and χ^α the corresponding irreducible character of S_n. Let $I(\alpha)$ and $A(\alpha)$ denote the characters of S_n which are induced by the 1-character and the alternating character of $S_\alpha = S_{\alpha_1} \times S_{\alpha_2} \times ... \times S_{\alpha_{h}}$.

The associated partition of α is denoted by α' (cf. [2,1.34]), the lexicographic order of partitions by $\subseteq$. It is well-known that $\{\chi^\alpha\}$, $\{I(\alpha)\}$ and $\{A(\alpha)\}$, where α runs through all partitions of n, are $\mathbb{Z}$-bases of the character ring of S_n and that the following holds:

Theorem 4.1. *Let*

$$\chi^{\alpha} = \sum a_{\beta} I(\beta) = \sum b_{\beta} A(\beta'), \quad I(\alpha) = \sum c_{\beta}\chi^{\beta}, \quad A(\alpha') = \sum d_{\beta}\chi^{\beta}.$$

Then, we have

(i) $a_\alpha = b_\alpha = c_\alpha = d_\alpha = 1$,
(ii) $\beta \subset \alpha$ *implies* $a_\beta = c_\beta = 0$,
(iii) $\alpha \subset \beta$ *implies* $b_\beta = d_\beta = 0$.

Now, we are going to generalize this result to wreath products of the form $S_m \sim S_n$. Let $\alpha^1, \alpha^2, ..., \alpha^k$ be the partitions of m and assume $\alpha^i \subseteq \alpha^j$ for all $i \geqslant j$. A complete system of ordinary irreducible representations of $S_m \sim S_n$ is given by:

$$\left[\left(\overset{\tilde{n}_1}{\#}[\alpha^1]\otimes[\beta^1]'\right) \# \left(\overset{\tilde{n}_2}{\#}[\alpha^2]\otimes[\beta^2]'\right)\# \right.$$
$$\left. ... \# \left(\overset{\tilde{n}_k}{\#}[\alpha^k]\otimes[\beta^k]'\right)\right] \uparrow S_m \sim S_n \;,$$

where $\Sigma n_i = n$ and β^i is a partition of n_i. The corresponding character is denoted by $\chi(\beta^1, \beta^2, ..., \beta^k)$. Let us define two other representations of $S_m \sim S_n$ which are associated with $(\beta^1, ..., \beta^k)$: $I(\beta^1, ..., \beta^k)$ denotes the character of $S_m \sim S_n$ which is induced by the 1-character of $\times_{i=1}^{k} S_{\alpha^i} \sim S_{\beta^i}$; hence,

$$I(\beta^1, ..., \beta^k) = \left(\overset{k}{\underset{i=1}{\#}}\left(\overset{\tilde{n}_i}{\#} I(\alpha^i)\otimes I(\beta^i)'\right)\right) \uparrow S_m \sim S_n \;,$$

and let

$$A(\beta^1, ..., \beta^k) = \left(\overset{k}{\underset{i=1}{\#}}\left(\overset{\tilde{n}_i}{\#} A(\alpha^i)\otimes A(\beta^i)'\right)\right) \uparrow S_m \sim S_n \;,$$

$$A'(\beta^1, ..., \beta^k) = \left(\overset{k}{\underset{i=1}{\#}}\left(\overset{\tilde{n}_i}{\#} A(\sigma^i)\otimes A(\tau^i)'\right)\right) \uparrow S_m \sim S_n \;,$$

where $\sigma^i = \alpha^{i'}$ and $\tau^i = \beta^{i'}$ for all i. Obviously, the mapping $A(\beta^1, ..., \beta^k) \mapsto A'(\beta^1, ..., \beta^k)$ is a permutation of $\{A(\beta^1, ..., \beta^k)\}$.

Let $\gamma^1, ..., \gamma^k$ be partitions of $l_1, ..., l_k$ and $\Sigma l_i = n$. We define:

$$(\beta^1, ..., \beta^k) \subset (\gamma^1, ..., \gamma^k)$$

if and only if either $n_j < l_j$ for an index j and $n_i = l_i$ for all $i < j$, or $n_i = l_i$ for all i and $\beta^j \subset \gamma^j$ for an index j and $\beta^i = \gamma^i$ for all $i < j$ (cf. [4]).

It follows from the considerations in Section 3 that $\{I(\beta^1, ..., \beta^k)\}$ is a $\mathbb{Z}$-basis of the character ring of $S_m \sim S_n$, and analogous calculations show that the same holds for $\{A(\beta^1, ..., \beta^k)\}$. These considerations and Theorem 4.1 also imply:

Theorem 4.2. *Let*

$$\chi(\beta^1, ..., \beta^k) = \sum a(\gamma^1, ..., \gamma^k) \cdot I(\gamma^1, ..., \gamma^k)$$
$$= \sum b(\gamma^1, ..., \gamma^k) \cdot A'(\gamma^1, ..., \gamma^k),$$
$$I(\beta^1, ..., \beta^k) = \sum c(\gamma^1, ..., \gamma^k) \cdot \chi(\gamma^1, ..., \gamma^k),$$
$$A'(\beta^1, ..., \beta^k) = \sum d(\gamma^1, ..., \gamma^k) \cdot \chi(\gamma^1, ..., \gamma^k).$$

Then we have

(i) $a(\beta^1, ..., \beta^k) = b(\beta^1, ..., \beta^k) = c(\beta^1, ..., \beta^k) = d(\beta^1, ..., \beta^k) = 1$,
(ii) $(\gamma^1, ..., \gamma^k) \subset (\beta^1, ..., \beta^k)$ *implies* $a(\gamma^1, ..., \gamma^k) = c(\gamma^1, ..., \gamma^k) = 0$,
(iii) $(\beta^1, ..., \beta^k) \subset (\gamma^1, ..., \gamma^k)$ *implies* $b(\gamma^1, ..., \gamma^k) = d(\gamma^1, ..., \gamma^k) = 0$.

Theorem 4.2 implies that $\chi(\beta^1, ..., \beta^k)$ is the only common constituent of $I(\beta^1, ..., \beta^k)$ and $A'(\beta^1, ..., \beta^k)$, and it occurs with multiplicity 1.

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