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Quartic functional equations

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Received 4 November 2002

Available online 17 March 2005

Submitted by William F. Ames

Abstract

In this paper, we solve a new functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

and prove the stability of this equation.

1. Introduction

In 1940, S.M. Ulam [11] gave the following question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.12.062

D.H. Hyers [3] has excellently answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. In 1978, Th.M. Rassias [9] proved a generalized version of the theorem of Hyers for approximately liner mappings. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [4–8].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

clearly has $f(x)=x^2$ as a solution when f is a real valued function of a real variable. So, it is natural that Eq. (1) is called a quadratic functional equation, and every solution of the quadratic functional equation (1) is said to be a quadratic function.

A Hyers–Ulam stability of the quadratic functional equation (1) was proved by F. Skof [10] for function $f: E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space. In [2], S. Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1).

Now, we consider the following new functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{2}$$

It is easy to see that the function $f(x)=x^4$ ($x \in \mathbb{R}$) satisfies above functional equation (2). Hence, it is natural that Eq. (2) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

In this paper, we solve above quartic functional equation and prove the stability of a quartic functional equation (2).

2. A solution of quartic functional equations

It is well known [1] that a function $f: X \to Y$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all $x \in X$. The biadditive function B is given by

$$B(x,y)=\frac{1}{4}\big(f(x+y)-f(x-y)\big). \tag{3}$$

In this section, we prove that a function $f: X \to Y$ between real vector spaces satisfies the functional equation (2) if and only if there exists a symmetric biquadratic function F such that $f(x)=F(x,x)$ for all $x \in X$. Throughout this section X and Y will be real vector spaces.

Theorem 2.1. *A function $f: X \to Y$ satisfies the functional equation* (2) *if and only if there exists a symmetric biquadratic function $F: X \times X \to Y$ such that $f(x)=F(x,x)$ for all $x \in X$.*

Proof. Assume that f satisfies the functional equation (2). Putting $x=y=0$ in (2), we have $f(0)=0$. Putting $x=0$ in (2), we have $f(y)=f(-y)$ for all $y \in X$. Putting $y=0$

and $y = x$ in (2), we obtain that $f(2x) = 16f(x)$ and $f(3x) = 81f(x)$ for all $x \in X$, respectively. Actually, we can lead to $f(nx) = n^4 f(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Replacing x and y by $x + y$ and $x - y$ in (2), respectively, we get

$$\begin{aligned} f(3x+y) + f(x+3y) &= 4f(2x) + 4f(2y) + 24f(x+y) - 6f(x-y) \\ &= 64f(x) + 64f(y) + 24f(x+y) - 6f(x-y) \end{aligned} \tag{4}$$

for all $x, y \in X$. Replacing x and y by $x + y$ and $2y$ in (2), respectively, we get

$$4f(x+2y) + 4f(x) = f(x+3y) + f(x-y) + 6f(x+y) - 24f(y) \tag{5}$$

for all $x, y \in X$. Interchange x and y in (5) to get the relation

$$4f(y+2x) + 4f(y) = f(y+3x) + f(y-x) + 6f(y+x) - 24f(x) \tag{6}$$

for all $x, y \in X$. Adding (5) to (6) and using (4), we lead to

$$f(x+2y) + f(2x+y) = 9f(x) + 9f(y) + 9f(x+y) - f(x-y) \tag{7}$$

for all $x, y \in X$. Using (2), we have

$$\begin{aligned} &9f(2x+z) + 9f(2x-z) + 9f(2y+z) + 9f(2y-z) \\ &\quad = 36f(x+z) + 36f(x-z) + 216f(x) - 54f(z) \\ &\qquad + 36f(y+z) + 36f(y-z) + 216f(y) - 54f(z) \end{aligned} \tag{8}$$

for all $x, y, z \in X$.

On the other hand, replacing x and y by $2x + z$ and $2y + z$ in (7), respectively, we get

$$\begin{aligned} &9f(2x+z) + 9f(2y+z) \\ &\quad = f(2x+4y+3z) + f(4x+2y+3z) - 9f(2x+2y+2z) + f(2x-2y) \end{aligned} \tag{9}$$

for all $x, y, z \in X$. Replacing x and y by $2x - z$ and $2y - z$ in (7), respectively, we get

$$\begin{aligned} &9f(2x-z) + 9f(2y-z) \\ &\quad = f(2x+4y-3z) + f(4x+2y-3z) - 9f(2x+2y-2z) + f(2x-2y) \end{aligned} \tag{10}$$

for all $x, y, z \in X$. Adding (9) to (10) and using (2), we get

$$\begin{aligned} &9f(2x+z) + 9f(2y+z) + 9f(2x-z) + 9f(2y-z) \\ &\quad = 4f(x+2y+3z) + 4f(x+2y-3z) + 24f(x+2y) - 6f(3z) \\ &\qquad + 4f(2x+y+3z) + 4f(2x+y-3z) + 24f(2x+y) - 6f(3z) \\ &\qquad - 144f(x+y+z) - 144f(x+y-z) + 32f(x-y) \end{aligned} \tag{11}$$

for all $x, y, z \in X$. By (8) and (11), we get

$$\begin{aligned} &36f(x+z) + 36f(x-z) + 216f(x) - 54f(z) \\ &\qquad + 36f(y+z) + 36f(y-z) + 216f(y) - 54f(z) \\ &\quad = 4f(x+2y+3z) + 4f(x+2y-3z) + 24f(x+2y) - 6f(3z) \\ &\qquad + 4f(2x+y+3z) + 4f(2x+y-3z) + 24f(2x+y) - 6f(3z) \\ &\qquad - 144f(x+y+z) - 144f(x+y-z) + 32f(x-y) \end{aligned} \tag{12}$$

for all $x, y, z \in X$. Referring to the process of (9)–(11), when $x = 2x + z$, $y = 2y - z$ in (7) and $x = 2x - z$, $y = 2y + z$ in (7), we get

$$\begin{aligned}
&9f(2x+z) + 9f(2y-z) + 9f(2x-z) + 9f(2y+z) \\
&\quad = f(2x+4y-z) + f(4x+2y+z) - 9f(2x+2y) + f(2x-2y+2z) \\
&\qquad + f(2x+4y+z) + f(4x+2y-z) - 9f(2x+2y) + f(2x-2y-2z) \\
&\quad = 4f(x+2y+z) + 4f(x+2y-z) + 24f(x+2y) - 6f(z) \\
&\qquad + 4f(2x+y+z) + 4f(2x+y-z) + 24f(2x+y) - 6f(z) \\
&\qquad - 288f(x+y) + 16f(x-y+z) + 16f(x-y-z)
\end{aligned} \tag{13}$$

for all $x, y, z \in X$. Replacing z by $3z$ in (13) and then using (12), we have

$$\begin{aligned}
&9f(2x+3z) + 9f(2y-3z) + 9f(2x-3z) + 9f(2y+3z) \\
&\quad = 4f(x+2y+3z) + 4f(x+2y-3z) + 24f(x+2y) - 6f(3z) \\
&\qquad + 4f(2x+y+3z) + 4f(2x+y-3z) + 24f(2x+y) - 6f(3z) \\
&\qquad - 288f(x+y) + 16f(x-y+3z) + 16f(x-y-3z) \\
&\quad = 36f(x+z) + 36f(x-z) + 216f(x) - 54f(z) \\
&\qquad + 36f(y+z) + 36f(y-z) + 216f(y) - 54f(z) \\
&\qquad + 144f(x+y+z) + 144f(x+y-z) - 32f(x-y) \\
&\qquad - 288f(x+y) + 16f(x-y+3z) + 16(x-y-3z)
\end{aligned} \tag{14}$$

for all $x, y, z \in X$. On the other hand, putting $x = x - y + 3z$ and $y = x - y - 3z$ in (7), we have

$$\begin{aligned}
&9f(x-y+3z) + 9f(x-y-3z) \\
&\quad = f(3x-3y+3z) + f(3x-3y-3z) - 9f(2x-2y) + f(6z) \\
&\quad = 81f(x-y+z) + 81f(x-y-z) - 144f(x-y) + 1296f(z)
\end{aligned} \tag{15}$$

for all $x, y, z \in X$. Multiplying both sides of (15) by 16/9, we obtain

$$\begin{aligned}
&16f(x-y+3z) + 16f(x-y-3z) \\
&\quad = 144f(x-y+z) + 144f(x-y-z) - 256f(x-y) + 2304f(z)
\end{aligned} \tag{16}$$

for all $x, y, z \in X$. Applying (16) to (14), we have

$$\begin{aligned}
&9f(2x+3z) + 9f(2y-3z) + 9f(2x-3z) + 9f(2y+3z) \\
&\quad = 36f(x+z) + 36f(x-z) + 216f(x) - 54f(z) \\
&\qquad + 36f(y+z) + 36f(y-z) + 216f(y) - 54f(z) \\
&\qquad + 144f(x+y+z) + 144f(x+y-z) - 32f(x-y) - 288f(x+y) \\
&\qquad + 144f(x-y+z) + 144f(x-y-z) - 256f(x-y) + 2304f(z)
\end{aligned} \tag{17}$$

for all $x, y, z \in X$.

Referring to the process of (9)–(11), when $x = 2x + 3z$, $y = 2x - 3z$ in (7) and $x = 2y - 3z$, $y = 2y + 3z$ in (7), and using (2), we get

$$\begin{aligned}
&9f(2x+3z)+9f(2x-3z)+9f(2y-3z)+9f(2y+3z)\\
&\quad= f(6x+3z)+f(6x-3z)-9f(4x)+f(6z)\\
&\qquad+f(6y+3z)+f(6y-3z)-9f(4y)+f(6z)\\
&\quad=81f(2x+z)+81f(2x-z)-2304f(x)+1296f(z)\\
&\qquad+81f(2y+z)+81f(2y-z)-2304f(y)+1296f(z)\\
&\quad=81\big(4f(x+z)+4f(x-z)+24f(x)-6f(z)\big)\\
&\qquad+81\big(4f(y+z)+4f(y-z)+24f(y)-6f(z)\big)\\
&\qquad-2304f(x)-2304f(y)+2592f(z)\\
&\quad=324f(x+z)+324f(x-z)+1944f(x)-486f(z)\\
&\qquad+324f(y+z)+324f(y-z)+1944f(y)-486f(z)\\
&\qquad-2304f(x)-2304f(y)+2592f(z)
\end{aligned}\tag{18}$$

for all $x, y, z \in X$. By (17) and (18), we get

$$\begin{aligned}
&f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)\\
&\quad=2\big[f(x+y)+f(x-y)+f(y+z)+f(y-z)+f(z+x)+f(z-x)\big]\\
&\qquad-4\big[f(x)+f(y)+f(z)\big]
\end{aligned}\tag{19}$$

for all $x, y, z \in X$.

Define a function $F : X \times X \to Y$ by

$$F(x,y)=\frac{1}{12}\big(f(x+y)+f(x-y)-2f(x)-2f(y)\big)$$

for all $x, y \in X$. Then we have $F(x,x) = f(x)$ for all $x \in X$ and F is symmetric since f is even.

Now, we claim that $Q = F(\cdot, y) : X \to Y$ defined by $Q(x) = F(x, y)$ is quadratic for each fixed $y \in X$. Using (19) and evenness of f, we get

$$\begin{aligned}
&12\big[F(x+z,y)+F(x-z,y)-2F(x,y)-2F(z,y)\big]\\
&\quad=f(x+z+y)+f(x+z-y)-2f(x+z)-2f(y)\\
&\qquad+f(x-z+y)+f(x-z-y)-2f(x-z)-2f(y)\\
&\qquad-2f(x+y)-2f(x-y)+4f(x)+4f(y)\\
&\qquad-2f(z+y)-2f(z-y)+4f(z)+4f(y)=0
\end{aligned}$$

for all $x, z \in X$. This shows that $Q = F(\cdot, y)$ is quadratic. Since F is symmetric, $Q' = F(x, \cdot) : X \to Y$ defined by $Q'(y) = F(x, y)$ is quadratic for fixed $x \in X$.

Conversely, assume that a function $F : X \times X \to Y$ is symmetric biquadratic such that $f(x) = F(x, x)$ for all $x \in X$. Then

$$
\begin{aligned}
&f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\\
&\quad = F(2x+y,2x+y)+F(2x-y,2x-y)\\
&\qquad -4F(x+y,x+y)-4F(x-y,x-y)-24F(x,x)+6F(y,y)\\
&\quad = 2F(x,2x+y)+2F(x+y,2x+y)-F(y,2x+y)\\
&\qquad +2F(x,2x-y)+2F(x-y,2x-y)-F(y,2x-y)\\
&\qquad -4F(x+y,x+y)-4F(x-y,x-y)-24F(x,x)+6F(y,y)\\
&\quad = 2\big(2F(x,x)+2F(x,x+y)-F(x,y)\big)\\
&\qquad +2\big(2F(x+y,x)+2F(x+y,x+y)-F(x+y,y)\big)\\
&\qquad -\big(2F(y,x)+2F(y,x+y)-F(y,y)\big)\\
&\qquad +2\big(2F(x,x)+2F(x,x-y)-F(x,y)\big)\\
&\qquad +2\big(2F(x-y,x)+2F(x-y,x-y)-F(x-y,y)\big)\\
&\qquad -\big(2F(y,x)+2F(y,x-y)-F(y,y)\big)\\
&\qquad -4F(x+y,x+y)-4F(x-y,x-y)-24F(x,x)+6F(y,y)\\
&\quad = 8\big(F(x,x+y)+F(x,x-y)\big)-4\big(F(y,x+y)+F(y,x-y)\big)\\
&\qquad -16F(x,x)-8F(x,y)+8F(y,y)\\
&\quad = 8\big(2F(x,x)+2F(x,y)\big)-4\big(2F(y,x)+2F(y,y)\big)\\
&\qquad -16F(x,x)-8F(x,y)+8F(y,y)\\
&\quad = 0
\end{aligned}
$$

for all $x, y \in X$. Hence, f satisfies the functional equation (2). □

3. Stability of a quartic functional equation

Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$Df(x,y) := f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)$$

for all $x, y \in X$.

Theorem 3.1. *Let a function $f : X \to Y$ satisfy*

$$\big\|Df(x,y)\big\| \leqslant \delta \tag{20}$$

for all $x, y \in X$, where $\delta \geqslant 0$. Then there exists a unique quartic function $F : X \to Y$ such that

$$\big\|f(x)-F(x)\big\| \leqslant \frac{1}{5}\big\|f(0)\big\| + \frac{\delta}{30}. \tag{21}$$

Proof. Putting $y=0$ in (20) and dividing both sides of its result by 32, we have

$$\left\| f(x)-\frac{1}{16}f(2x)\right\| \leqslant \frac{3}{16}\|f(0)\|+\frac{\delta}{32} \tag{22}$$

for all $x\in X$. We claim that

$$\left\| f(x)-\frac{1}{16^n}f(2^n x)\right\| \leqslant \left(1-\frac{1}{16^n}\right)\frac{1}{5}\|f(0)\|+\left(1-\frac{1}{16^n}\right)\frac{\delta}{30} \tag{23}$$

for all $x\in X$ and all $n\in\mathbb{N}$. We prove the assertion by induction on n. Inequality (22) yields the validity of (23) for $n=1$. Assume now that (23) holds for some n. Replacing x by $2x$ in (23) and then dividing both sides of its result by 16 we obtain

$$\begin{aligned}&\left\|\frac{1}{16}f(2x)-\frac{1}{16^{n+1}}f(2^{n+1}x)\right\|\\&\quad\leqslant \left(1-\frac{1}{16^n}\right)\frac{1}{16\cdot 5}\|f(0)\|+\frac{1}{16}\left(1-\frac{1}{16^n}\right)\frac{\delta}{30}\end{aligned}$$

for all $x\in X$. Hence, we have

$$\begin{aligned}&\left\| f(x)-\frac{1}{16^{n+1}}f(2^{n+1}x)\right\|\\&\quad\leqslant \left\| f(x)-\frac{1}{16}f(2x)\right\|+\left\|\frac{1}{16}f(2x)-\frac{1}{16^{n+1}}f(2^{n+1}x)\right\|\\&\quad\leqslant \left(1-\frac{1}{16^{n+1}}\right)\frac{1}{5}\|f(0)\|+\left(1+\frac{1}{16^{n+1}}\right)\frac{\delta}{30}\end{aligned}$$

for all $x\in X$. Therefore, the inequality (23) is true for all $x\in X$ and all $n\in\mathbb{N}$. Define

$$F_n(x)=\frac{1}{16^n}f(2^n x) \tag{24}$$

for all $n\in\mathbb{N}$ and all $x\in X$. For $m>n>0$, we have

$$\begin{aligned}\|F_m(x)-F_n(x)\| &= \left\|\frac{1}{16^m}f(2^m x)-\frac{1}{16^n}f(2^n x)\right\|\\&=\frac{1}{16^n}\left\|\frac{1}{16^{m-n}}f(2^{m-n}2^n x)-f(2^n x)\right\|\\&\leqslant \left(\frac{1}{16^n}-\frac{1}{16^m}\right)\frac{1}{5}\|f(0)\|+\left(\frac{1}{16^n}-\frac{1}{16^m}\right)\frac{\delta}{30}\end{aligned}$$

for all $x\in X$. Since the right-hand side of the above inequality tends to zero as $n\to\infty$, the sequence $\{F_n(x)\}$ is a Cauchy sequence for each $x\in X$. Since Y is complete, there exists a limit function $F(x)=\lim_{n\to\infty}F_n(x)$. By letting $n\to\infty$ in (23), we get the formula (21).

To show that F satisfies Eq. (2), replace x and y by $2^n x$ and $2^n y$ in (20), respectively, and divide by 16^n, then it follows that

$$\begin{aligned}&\frac{1}{16^n}\big\| f\big(2^n(2x+y)\big)+f\big(2^n(2x-y)\big)-4f\big(2^n(x+y)\big)\\&\quad-4f\big(2^n(x-y)\big)-24f(2^n x)+6f(2^n y)\big\| \leqslant \frac{\delta}{16^n}\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ we see that F satisfies (2) for all $x, y \in X$.

To prove the uniqueness of the function F, assume that there exists a function $F': X \to Y$ which satisfies (2) and (21). Then we have

$$\begin{aligned} \|F(x) - F'(x)\| &= \frac{1}{16^n}\|F(2^n x) - F'(2^n x)\| \\ &\leqslant \frac{1}{16^n}\left(\|F(2^n x) - f(2^n x)\| + \|f(2^n x) - F'(2^n x)\|\right) \\ &\leqslant \frac{1}{16^n}\left(\frac{2}{5}\|f(0)\| + \frac{\delta}{15}\right) \end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ we get $F(x) = F'(x)$ for all $x \in X$. This completes the proof of the theorem. □

References

[1] J. Aczél, J. Dhombres, Functional Equations in Several Variables, Cambridge Univ. Press, 1989.
[2] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg 62 (1992) 59–64.
[3] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941) 222–224.
[4] D.H. Hyers, Th.M. Rassias, Approximate homomorphisms, Aequationes Math. 44 (1992) 125–153.
[5] K.W. Jun, Y.H. Lee, On the Hyers–Ulam–Rassias stability of a pexiderized quadratic inequality, Math. Inequalities Appl. 4 (2001) 93–118.
[6] S.M. Jung, On the Hyers–Ulam stability of the functional equations that have the quadratic property, J. Math. Anal. Appl. 222 (1998) 126–137.
[7] S.M. Jung, On the Hyers–Ulam–Rassias stability of a quadratic functional equation, J. Math. Anal. Appl. 232 (1999) 384–393.
[8] Pl. Kannappan, Quadratic functional equation and inner product spaces, Results Math. 27 (1995) 368–372.
[9] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[10] F. Skof, Proprietà locali e approssimazione di operatori, Rend. Sem. Mat. Fis. Milano 53 (1983) 113–129.
[11] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1964.