Finite-size scaling analysis of the critical behavior of a general epidemic process in 2D

C. Argolo [b], Yan Quintino [b], Iram Gleria [a,*], M.L. Lyra [a]

[a] Instituto de Física, Universidade Federal de Alagoas, 57072-970 Maceió-AL, Brazil
[b] Instituto Federal de Ciência e Tecnologia do Estado de Alagoas, 57020-510 Maceió-AL, Brazil

ARTICLE INFO

ABSTRACT

We investigate the critical behavior of a stochastic lattice model describing a General Epidemic Process. By means of a Monte Carlo procedure, we simulate the model on a regular square lattice and follow the spreading of an epidemic process with immunization. A finite size scaling analysis is employed to determine the critical point as well as some critical exponents. We show that the usual scaling analysis of the order parameter moment ratio does not provide an accurate estimate of the critical point. Precise estimates of the critical quantities are obtained from data of the order parameter variation rate and its fluctuations. Our numerical results corroborate that this model belongs to the dynamic isotropic percolation universality class. We also check the validity of the hyperscaling relation and present data collapse curves which reinforce the accuracy of the estimated critical parameters.

© 2010 Elsevier B.V. Open access under the Elsevier OA license.

1. Introduction

The study of non-equilibrium phase transitions is of central importance in the analysis of complex phenomena. The contact process (CP) is one of the simplest models presenting a dynamic transition from an active state into an absorbing state [1–4]. The CP is the prototype model for the directed percolation (DP) universality class. Given a *D*-dimensional lattice, the CP considers each site being in either an active or inactive state. The inactive sites becomes active, with a given probability, when it is neighbor of an active neighbor. The active state has a finite lifetime and, for sufficiently short lifetimes, the whole system is driven to the absorbing state with only inactive sites. A stationary active state with a finite fraction of sites in the active state is obtained above a critical lifetime. The influence of particle diffusion in the critical behavior of absorbing state phase transitions has been a subject of growing interest, since analytical and numerical studies showed that diffusion is an important mechanism that can influence the critical behavior [5–14]. In particular, strong deviations from the directed percolation universality class have been recently reported for models with coupled diffusive and non-diffusive fields [15–18].

In real systems, however, epidemic spreading is a much more complex phenomenon than that described by the contact process. Individuals acquire immunization for their own protection and a realistic description of epidemic spreading should include this aspect in the model. In Ref. [19] the process of mutation was introduced in order to study its effect in the epidemic spreading with immunization. The case in which an individual acquires perfect immunization is known as a General Epidemic Process (GEP). An individual can be infected and later on it is immunized. Once immunized it remains in this state during the whole process and the disease can only spread to parts of the system that have not been infected

---

* Corresponding author.
*E-mail address:* iram@pq.cnpq.br (I. Gleria).

before. In Ref. [20] a trial to simulate systems with immunization has been developed on a basis of real data. The authors proposed a model that can serve as a basis for the development of other algorithms to simulate real epidemics. In the same work, the authors also considered the strategy of population vaccination.

In this paper, we study the spreading of an epidemic whose critical behavior places the model in the DIP (Dynamic Isotropic Percolation or GEP) universality class [21]. Our approach is the one based on stochastic spatially structured models. In the last years, a great number of works have shown the relevance of this kind of approach to describe biological population problems [22–40]. We focus on the stochastic lattice model for a susceptible–infected–immunized system introduced by Satulovsky and Tomé [24,25]. This model exhibits a phase diagram with an active phase where epidemic spreads indefinitely and an inactive phase where the immunization process predominates and the epidemic spreading stops after reaching a finite portion of the system. In order to find the critical properties of the model we perform Monte Carlo simulations and we use finite size scaling analysis, which is employed to determine the critical point as well as some critical exponents. We calculate the static critical exponents associated with the non-equilibrium phase transition from the active state into the absorbing state. It is worth mentioning that some of the dynamic critical exponents associated with the synchronous version of this model have already been obtained [35]. We determine a set of critical exponents which allows the classification of the model's universality class. We show that the cumulant technique does not provide an accurate procedure to directly locate the critical point but we precisely locate the critical point using the derivative of density of immunized individuals with respect to the control parameter.

The paper is structured as follows. In Section 2 we present the model and simulations. Section 3 presents our results and Section 4 concludes.

2. Model and simulations

Let us consider a model that mimics the spreading of an epidemic process in a population. We denote by X the susceptible individuals, that may get infected by contact with infected individuals Y. These can recover and become the immune population Z. The model comprehends the following set of reactions:

$$X + Y \overset{b}{\to} 2Y$$
$$Y \overset{c}{\to} Z$$

(1)
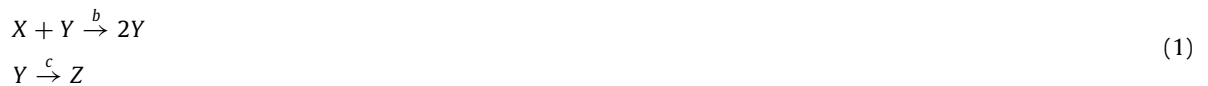
which describe the acyclic process $X \to Y \to Z$.

The present model is defined on a regular square lattice where each site can be in one of the states: occupied by a susceptible, an infected or an immune individual and they obey the stochastic rules that we will present. These rules are basic for the relevant reactions that characterize a simple susceptible–infected–immune system. They are taken into account here by considering the stochastic lattice model defined by an asynchronous global dynamics composed by the following set of local Markovian rules:

(a) Infection may occur if a susceptible individual (X), which occupies a site, has at least one site occupied by an infected individual (Y) in its neighborhood, reaction $X + Y \overset{b}{\to} 2Y$. The process occurs with probability $b/4$ times the number of infected individuals in the neighbor sites. The infection rate b is a parameter related to infected individual proliferation and it is divided by $z = 4$, where z is the lattice coordination number;

(b) In a site occupied by an infected individual (Y), an immune individual (Z) can be born (recovering process) with probability c (immunization rate) spontaneously, reaction $Y \overset{c}{\to} Z$. In this reaction, there occurs an instantaneous immunization process.

The condition $b + c = 1$ is obeyed, with b being the infection probability and c the immunization probability. This model may exhibit infinitely many absorbing states and presents a continuous phase transition. The critical behavior will be characterized by measuring a set of relevant static critical exponents obtained by the use of a finite size scaling analysis of the critical order parameter and its relative fluctuation. In what follows, we show results from simulations on finite lattices with $N = L^2$ sites (L is the linear size). Each lattice sweep is considered as the time unit or one Monte Carlo Step (MCS). The whole process is updated sequentially. We start from an initial condition with a single infected individual at the center of the square lattice covered by susceptible individuals. Once the system is placed in the initial condition we apply the local rules (a) and (b). An example of a configuration obtained by simulations is shown in Fig. 1. For high values (low c) of the infection probability b (upper panel), the epidemic spreads leaving a cluster of inactive sites composed of immune individuals and some groups of individuals that remain susceptible thereafter. Clusters grow with a front of infected individuals which remains the border. Later the cluster assumes a limiting shape and spreads to infinity with a nonzero probability. When c is increased (b is decreased) the threshold of the epidemic is reached (lower panel) and above this threshold, the epidemic will stop leaving a cluster with a few immune individuals and the rest of the lattice covered with susceptible individuals. The critical cluster presents an irregular shape of fractal nature. These clusters correspond to configurations where populations of susceptible, infected and immunized under condition of low densities, are grouped into small clusters of each species and they are isolated from each other. The system evolves in time and eventually reaches stationary states. The order parameter is the density of immune individuals:

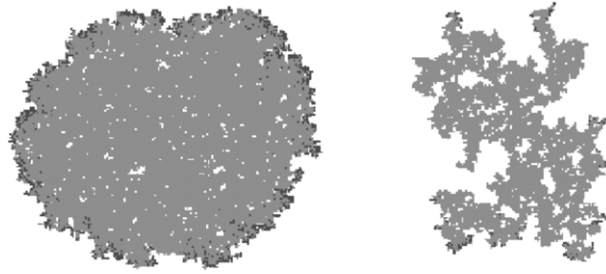$$\rho_0 = \frac{\langle N_z \rangle}{N}.$$

Fig. 1. Configurations at $c = 0.15$, $t = 250$ MCS (upper panel) and $c = 0.22$, $t = 400$ MCS (lower panel) for the spreading of the epidemic model on a square lattice with $N = 200^2$ sites. The figures were generated from a single infected individual located at the origin (center) of a lattice covered by susceptible individuals. Infected individuals are in black, susceptible individuals are in white and immune sites are in gray. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
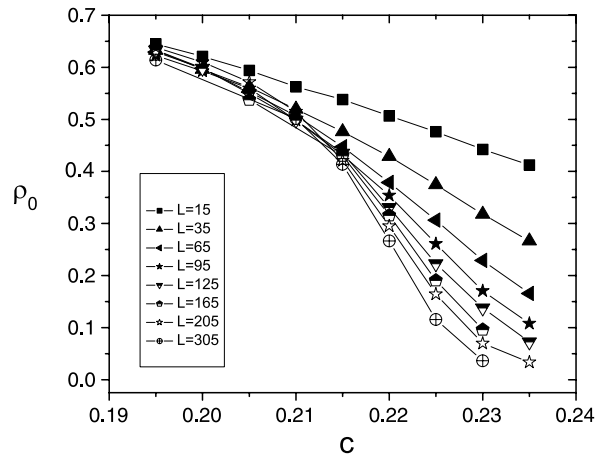

Fig. 2. Density of immune individuals ρ_0 in the active state versus c for distinct linear lattice sizes L.

3. Results

In Fig. 2 we show the density of immune individuals ρ_0, as a function of the spontaneous immunization probability c, as obtained from simulations on lattices of distinct sizes $L = 15, 35, 65, 95, 125, 165, 205, 305$. As $L \to \infty$ a transition from a state with nonzero density of immune individuals to the immunized absorbing takes place by increasing the values of c. The values used in our simulations for c ranged from 0.195 up to 0.235 with step 0.005 (b is calculated by the relation $b + c = 1$). A commonly used technique to locate the critical point explores the finite-size behavior of the ratio between the second moment and the squared first moment of the order parameter. For the general epidemic process, it corresponds to the ratio of moments of the average number of immunized individuals, defined as:

$$U_L(c) = \frac{\langle N_Z^2 \rangle}{\langle N_Z \rangle^2}. \tag{2}$$

In the limit of large L, such a ratio of order parameter moments usually becomes independent of the system size at the critical point, due to the fractal character of the active zone. However, this is not the case for the present model, as we discuss below. In Fig. 3(a), we plot $U_L(c)$ obtained from simulations performed in distinct lattice sizes. For the present class of absorbing state phase transition that takes place at a propagation front, the relative fluctuation becomes independent of the system size within the entire active phase, with finite size corrections at the critical point. This feature is directly related to the presence of permanently immunized sites. They reduce the available area of susceptible sites, thus leaving the activity restricted to occur at the fractal border of infected clusters even well within the active phase. Therefore, the cumulant technique does not provide an accurate procedure to directly locate the critical point for the present class of critical phenomena. Alternatively, we precisely locate the critical point by the use of the derivative of density of immunized individuals with respect to the control parameter c as shown in Fig. 3(b). Further, we refine the critical recovery rate c_c using the criteria of power-law size dependence of the order parameter at the critical point.

Once having located the critical point, finite size scaling relations were used to compute the critical exponents characterizing such a non-equilibrium phase transition. In particular, the order parameter obeys the power law $\rho_0(c_c, L) \propto L^{-\beta/\nu}$, while its logarithmic derivative scales as $\partial \ln \rho_0(c_c, L)/\partial c \propto L^{1/\nu}$. These scaling laws are depicted in Figs. 4 and 5 from which we estimate $\beta/\nu = 0.105(5)$ and $\nu = 1.32(2)$ for the square lattice.
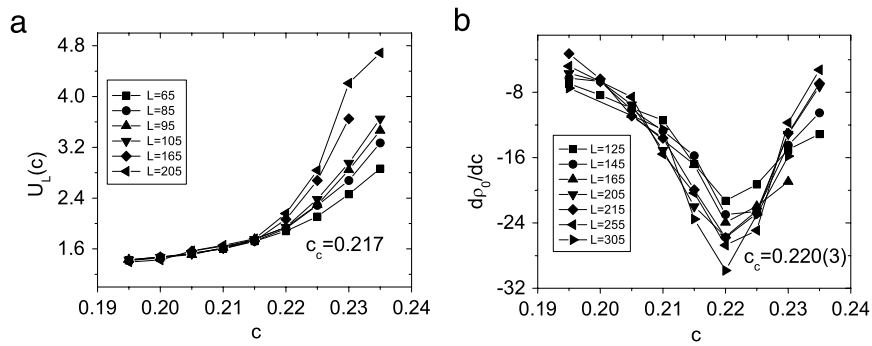
Fig. 3. (a) The moment ratio $U_L(c)$ as a function of the immunization rate c for distinct lattice sizes. From this figure, we (poorly) estimate the critical immunization rate $c_c \approx 0.217$. (b) Derivative of density of immune individuals in relation to c. The curves reach a minimum at $c_c = 0.220(3)$ signalizing the second order phase transition.
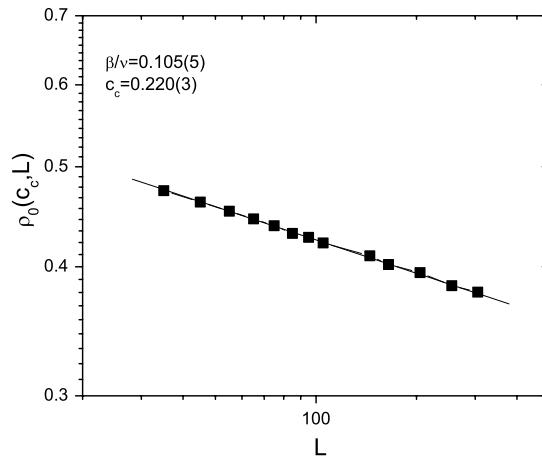

Fig. 4. Log–log plot of the order parameter versus the linear size L. From the best fit to a power-law we estimate the critical exponent ratio $\beta/\nu = 0.105(5)$ for the square lattice.
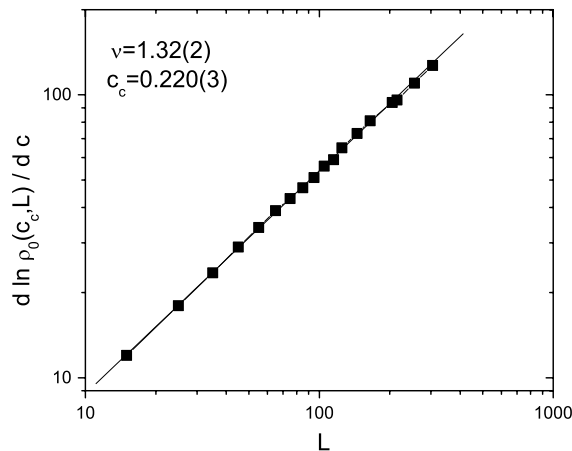

Fig. 5. Log–log plot of the logarithmic derivative of the critical order parameter versus L. From the best fit to a power-law we estimate the critical exponent $\nu = 1.32(2)$ for the square lattice.

In Fig. 6 we calculate the order parameter fluctuations

$$\Delta\rho = N(\langle N_Z^2 \rangle - \langle N_Z \rangle^2)$$

for the square lattice versus c for several lattice sizes. The data for the order parameter fluctuations at the critical point are used in Fig. 7 to obtain the critical exponent ratio γ'/ν since $\Delta\rho \propto L^{\gamma'/\nu}$ at the critical point $c_c = 0.220(3)$. In Fig. 8 we
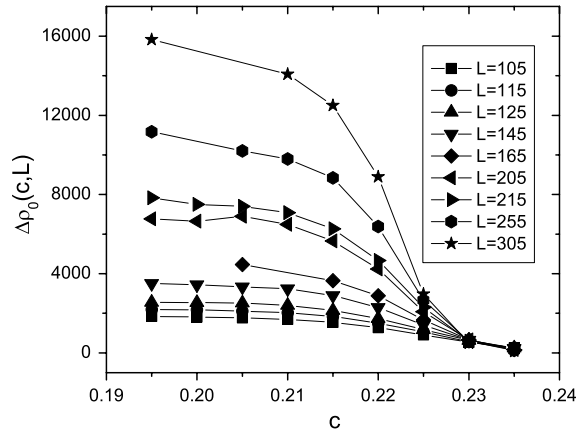
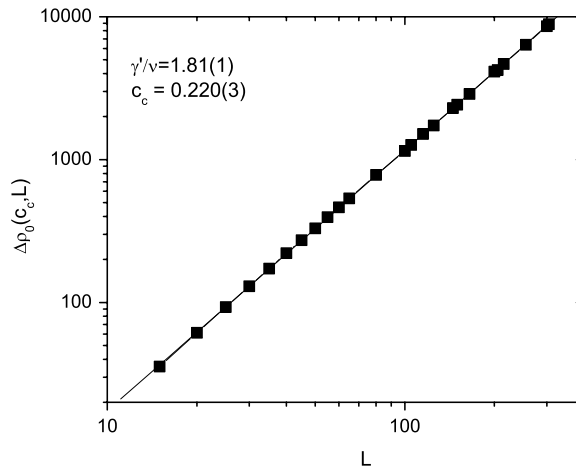**Fig. 6.** Order parameter fluctuations $\Delta\rho$ versus c for distinct linear lattice sizes L.

Fig. 7. Log–log plot of the order parameter fluctuations $\Delta\rho$ versus L at the critical point. The exponent ratio γ'/ν is estimated from the slope of the fitted straight line from which we obtained $\gamma'/\nu = 1.81(1)$.
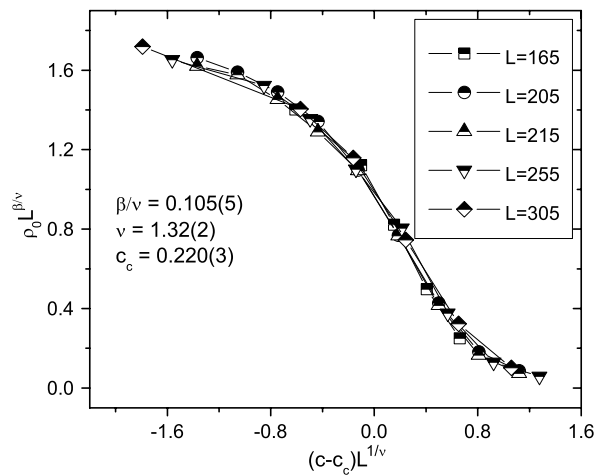

Fig. 8. Data collapse of the order parameter density computed from different linear lattice sizes L and using $c_c = 0.220(3)$, $\beta/\nu = 0.105(5)$ and $\nu = 1.32(2)$.

present data collapse of the order parameter density computed from different lattice sizes. Using $c_c = 0.220(3)$, the ratio of critical exponents $\beta/\nu = 0.105(5)$ and the critical exponent $\nu = 1.32(2)$ are confirmed. We also present in Fig. 9, the data

Table 1
Values of critical exponents β/ν, ν and γ'/ν for the square lattice. For comparison we show the corresponding values for the 2D GEP.

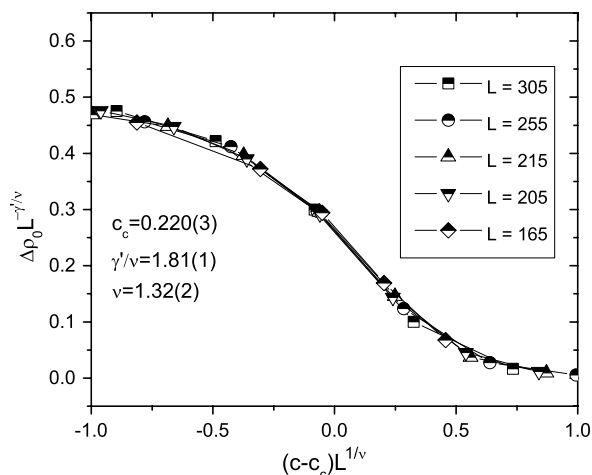|	β/ν	ν	γ'/ν
Our model	0.105(5)	1.32(2)	1.81(1)
GEP-2D	0.104	1.33	1.795

Fig. 9. Data collapse of the order parameter density fluctuations computed from different linear lattice sizes L and using $c_c = 0.220(3)$, $\gamma'/\nu = 1.81(1)$ and $\nu = 1.32(2)$.

collapse of the order parameter density fluctuation computed from different lattice sizes. These are results for the square lattice and using $c_c = 0.220(3)$. The exponents $\gamma'/\nu = 1.81(1)$ and $\nu = 1.32(2)$ are estimated considering this critical density. We point out that the uncertainty in the location of the critical point was taken into consideration in the estimate of the error bars on the critical exponents.

Finally in Table 1 we present the values of β/ν, ν and γ'/ν for the general epidemic process in the square lattice. The values for GEP in 2D are shown at the end of the table for comparison. In this case of the square lattice the results are similar to those of GEP. Our estimated value for γ'/ν is consistent with the hyperscaling relation $2\beta/\nu + \gamma'/\nu = 2$ for the square lattice.

4. Conclusions

We have investigated the critical behavior of a stochastic spatial structured model in which susceptible, infected and immunized individuals reside on the sites of a square lattice and are described by discrete stochastic variables. From numerical simulations of this irreversible model and using finite size scaling analysis, we computed some relevant critical exponents governing this non-equilibrium phase transition. We have found that the cumulant technique based on an order parameter moment ratio does not show the usual scale-invariance at the critical point observed in equilibrium phase transition. This feature is due to the presence of permanent immunized sites. Given that the absorbing state phase transition occurs at a fractal-like propagation front, the relative order parameter fluctuation becomes independent of the system size in the entire active phase. Therefore the cumulant technique does not provide an accurate procedure to directly locate the critical point. We provided an accurate estimate of the critical point using the derivative of density of immunized individuals with respect to the control parameter. The critical exponents were determined from the finite-size scaling behavior of the variation rate of the order parameter density at the critical point as well as from the scaling of the order parameter fluctuation. The results are consistent with the predicted Dynamic Isotropic Percolation universality class [21].

Acknowledgements

References

[1] J. Marro, R. Dickman, Nonequilibrium Phase Transitions in Lattice Models, Cambridge University Press, Cambridge, 1999.
[2] R. Dickman, in: V. Privman (Ed.), Nonequilibrium Statistical Mechanics in One Dimension, Cambridge University, Cambridge, 1996.
[3] H. Hinrichsen, Adv. Phys. 49 (2000) 815.
[4] T.E. Harris, Ann. Probab. 2 (1974) 969.
[5] J. Ramasco, M.A. Muñoz, C.A. da Silva Santos, Phys. Rev. E 69 (2004) 045105(R).

 [6] I. Dornic, H. Chaté, M.A. Muñoz, Phys. Rev. Lett. 94 (2005) 100601.
 [7] R. Kree, B. Schaub, B. Schmittmann, Phys. Rev. A 39 (1989) 2214.
 [8] F. van Wijland, K. Oerding, H.J. Hilhorst, Physica A 251 (1998) 179.
 [9] K. Oerding, F. van Wijland, J.P. Leroy, H.J. Hilhost, J. Stat. Phys. 99 (2000) 1365.
[10] R. Dickman, D.S. Maia, J. Phys. A: Math. Theor. 41 (2008) 405002.
[11] J.E. Freitas, L.S. Lucena, L.R. da Silva, H.J. Hilhost, Phys. Rev. E 61 (2000) 6330.
[12] D.S. Maia, R. Dickman, J. Phys.: Condens. Matter 19 (2007) 065143.
[13] H.K. Janssen, Phys. Rev. E 64 (2001) 058101.
[14] M.M. de Oliveira, R. Dickman, Phys. Rev. E 74 (2006) 011124.
[15] C. Argolo, Y. Quintino, Y. Siqueira, Iram Gleria, M.L. Lyra, Phys. Rev. E 80 (2009) 061127.
[16] U.L. Fulco, D.N. Messias, M.L. Lyra, Phys. Rev. E 63 (2001) 066118.
[17] E. Macnadbay, R. Bezerra, U.L. Fulco, M.L. Lyra, C. Argolo, Physica A 342 (2004) 249.
[18] N.V. da Costa, U.L. Fulco, M.L. Lyra, I.M. Gleria, Phys. Rev. E 75 (2007) 031112.
[19] S.M. Dammer, H. Hinrichsen, Phys. Rev. E 68 (2003) 016114.
[20] S. Hoya White, A. Martín del Rey, G. Rodrígues Sánches, Appl. Math. Comput. 186 (2007) 193.
[21] P. Grassberger, Math. Biosci. 63 (1983) 157.
[22] K. Tainaka, Phys. Rev. Lett. 63 (1989) 2688.
[23] R. Durrett, S. Levin, Theor. Popul. Biol. 46 (1994) 363.
[24] J. Satulovsky, T. Tomé, Phys. Rev. E 49 (1994) 5073.
[25] D.R. de Souza, T. Tomé, Physica A 389 (2010) 1142.
[26] N. Boccara, O. Roblin, M. Roger, Phys. Rev. E 50 (1994) 4531.
[27] A. Provata, G. Nicolis, F. Baras, J. Chem. Phys. 110 (1999) 8361.
[28] T. Antal, M. Droz, Phys. Rev. E 63 (2001) 056119.
[29] T. Antal, M. Droz, A. Lipowsky, G. Odor, Phys. Rev. E 64 (2001) 036118.
[30] O. Ovaskainen, K. Sato, J. Bascompe, I. Hanski, J. Theoret. Biol. 215 (2002) 95.
[31] M.A.M. de Aguiar, E.M. Rauch, Y. Bar-Yam, Phys. Rev. E 67 (2003) 047102.
[32] K.C. de Carvalho, T. Tomé, Modern Phys. Lett. B 18 (2004) 873.
[33] G. Szabó, G.A. Sznaider, Phys. Rev. E 69 (2004) 031911.
[34] M. Mobilia, I.T. Georgiev, U.C. Täuber, Phys. Rev. E 73 (2006) 040903.
[35] E. Arashiro, T. Tomé, J. Phys. A 40 (2007) 887.
[36] E. Arashiro, A.L. Rodrigues, M.J. de Oliveira, T. Tomé, Phys. Rev. E 77 (2008) 061909.
[37] A.L. Rodrigues, T. Tomé, Braz. J. Phys. 38 (2008) 87.
[38] T. Tomé, M.J. de Oliveira, Phys. Rev. E 72 (2005) 026130.
[39] D. Bertrand, Y. Siqueira, M.L. Lyra, Iram Gleria, C. Argolo, Physica A 386 (2007) 748.
[40] D.S. Maia, R. Dickman, J. Phys.: Condens. Matter 19 (2007) 065143.