Some applications of a generalized Martin's axiom

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Abstract

A generalization to higher cardinals of a variant of Martin's axiom is considered. Numerous applications are given in set theory and in set-theoretic topology.

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0. Introduction

This paper is dedicated to the memory of Boris Šapirovskiǐ. During the 1970's, we worked on similar problems and felt a close connection but never managed to meet. I invited him to the SETOP Conference in 1980, but the Soviet authorities did not permit him to come. Only at the end of his life, at the conference in honor of Mary Ellen Rudin in 1991 did we finally get together. His mind was as sharp as ever and I found him to be a warm human being I wanted to know better, but his body failed him. I was moved to be asked to contribute a paper in his memory; the choice requires some explanation. My recent work mainly involves forcing and large cardinals; I wanted to contribute something closer to his interests. The present paper was first written around 1977 and was supposed to appear in Transactions, but I never got around to correcting errors and making the changes required by the referee and subsequent developments. I still get requests for the preprint and most of it never appeared elsewhere. More to the point, the work

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refers to Šapirovskii in a number of places so I thought it might be appropriate for this volume. The editors agreed and so I have created an updated version of the original. I overlap a bit with the survey [67].

In the early 1970's, a number of set-theorists considered the problem of generalizing Martin's axiom [39] to higher cardinals. Their aims were to prove the consistency of a generalized Souslin's hypothesis, as well as to generalize the "combinatorial consequences" [35] of Martin's axiom. The first aim led to some surprising results of Laver, Shelah and Stanley, which we discuss in our concluding section. The second aim was accomplished independently by Laver [36], Baumgartner [2], and Shelah [49] in that order of priority. Baumgartner's version—although not the strongest—is the easiest to state and fits well into the classification of iteration axioms introduced in [35]. For that reason I shall work with it. I shall derive from Baumgartner's axiom some nontrivial topological and set-theoretic consequences. By "nontrivial" I mean that the Martin's axiom analogues either do not exist or are uninteresting.

Recall that Martin's axiom says that for each countable chain condition, partial order and collection of $< 2^\kappa$ dense subsets of it, there is a generic set meeting all of them. Also recall that the countable chain condition is the simplest nontrivial restriction on a partial order that ensures that forcing with that partial order preserves cardinals. Furthermore, the simplest forcing extension, namely the one that adjoins Cohen subsets of $\omega$, satisfies the countable chain condition. A generalization of Martin's axiom should at the minimum say that, given certain requirements on a partial order that ensure the preservation of cardinals, there is a generic set meeting all of $< 2^\kappa$ dense sets. One would expect that the partial order which adjoins subsets of $\omega_1$ using countable conditions would satisfy these requirements. One's first try therefore would be to require that the partial order be countably closed (every descending sequence has a lower bound) and satisfy the $\kappa_2$-chain condition (every collection of mutually incompatible elements has cardinality $< \kappa_2$). These requirements placed upon one partial order do suffice to preserve cardinals. However when one attempts to imitate the proof of the consistency of Martin's axiom with $2^{\kappa_0} > \kappa_1$ [52] to get the consistency of such a generalization of Martin's axiom with $2^{\kappa_0} > \kappa_2$, difficulties appear. Laver, Baumgartner, and Shelah each solved the problem by adding the technical condition that any two compatible elements have an inf (call such a partial order well-met) and by variously strengthening the chain condition. In particular, Baumgartner imposed the requirement that the partial order be $\kappa_1$-linked, i.e., be the union of $\kappa_1$-wise compatible subsets. Summing up, we state

**Baumgartner's axiom.** Let $\mathcal{P} = \langle P, \leq \rangle$ be a partial order such that

1. $\mathcal{P}$ is countably closed,
2. $\mathcal{P}$ is well-met,
3. $\mathcal{P}$ is $\kappa_1$-linked,

then if $\kappa < 2^{\kappa}$ and $\{D_{\alpha} < \kappa \}$ are dense subsets of $P$, then there is $G \subseteq P$ such that

4. $p \geq q \in G \implies p \in G$,
(5) if \( p, q \in G \), there is an \( r \in G \), \( r \leq p \) and \( r \leq q \),
(6) \( G \cap D_\alpha \neq \emptyset \) for all \( \alpha < \kappa \).

**Theorem 0.1** (Baumgartner). \( BA \) is consistent with \( 2^{\aleph_0} = \aleph_1 \) and \( 2^{\aleph_1} = \kappa \), where \( \kappa \geq \aleph_2 \) is regular.

Clearly, \( 2^{\aleph_1} - \aleph_2 \) implies \( BA \). In many applications one needs that \( \lambda < \kappa \) implies \( \lambda^{\aleph_0} < \kappa \) but Baumgartner [3] notes that one can in fact get the axiom to hold even if there is a \( \lambda < \kappa \) for which \( \lambda^{\aleph_0} = \kappa \).

Surprisingly, Shelah [50] has shown that the “well-met” condition cannot be removed (see Section 9). The requirement looks less artificial when subsumed in the following equivalent version of \( BA \) due jointly to Weiss and myself.

**Definition 0.2.** A subset \( S \) of a partial order is *centred* if each finite subset of \( S \) has a lower bound (not necessarily in \( S \)). A partial order is *countably compact* if each countable centred subcollection has a lower bound.

**Theorem 0.3.** In the statement of Baumgartner’s axiom the conditions “countably closed” and “well-met” may together be replaced by “countably compact”.

**Proof.** Certainly every countably closed well-met partial order is countably compact, for if \( \{p_n\} \in \omega \) is centred, then \( \{p_0, p_0 \land p_1, (p_0 \land p_1) \land p_2, \ldots \} \) is descending and its lower bound is \( \leq \) each \( p_n \). Conversely, let \( \mathcal{P} \) be a countably compact partial order. According to [20, Theorem 26, p. 52] any partial order can be isomorphically embedded in a lattice. Consider \( \mathcal{P} \) as a sub-partial order of \( \mathcal{L} \) and let \( \mathcal{T} \) be \( \mathcal{P} \) together with all nonzero meets of finite subsets of \( \mathcal{P} \), with the order inherited from \( \mathcal{L} \). \( \mathcal{T} \) is well-met by construction. Suppose \( \mathcal{P} \) is countably compact and \( \{q_n\} \in \omega \) is a descending sequence in \( \mathcal{P} \), say \( q_n = \land F_n \), \( F_n \) a finite subset of \( P \). Without loss of generality assume the \( F_n \) are increasing. Then \( \cup \{F_n; n \in \omega \} \) is centred, so by countable compactness it has a lower bound \( p \). Since \( \mathcal{P} \) is dense in \( \mathcal{T} \), \( \mathcal{T} \) is \( \aleph_1 \)-linked if \( \mathcal{P} \) is. \( \mathcal{T} \) is thus a candidate for the application of \( BA \). If \( G \) is a collection of dense subsets of \( \mathcal{P} \) and \( G \) is \( \mathcal{D} \)-generic for \( \mathcal{T} \), then \( G \cap P \) is \( \mathcal{D} \)-generic for \( \mathcal{P} \). □

The organization of this paper is as follows. In Section 1 we derive a useful consequence “\( P_1 \)” of \( BA \) in terms of \( \sigma \)-ideals and Lusin sets. In Section 2 we use \( P_1 \) plus \( CH \) to construct an \( L \)-space with large weight. In Section 3 we obtain conditions under which caliber \( \mathfrak{m} \) implies separability, employing the method of Lusin sets. In Section 4 we compute from \( BA \) plus \( CH \) (\( BACH \)) the expected Baire category kind of results. In Section 5 we use \( BACII \) to obtain normality of various spaces. Section 6 applies \( BACH \) to topological products and introduces various combinatorial principles involving stationary sets. In Section 7 we exploit the particular properties of Baumgartner’s model for \( BACH \) plus \( 2^{\aleph_1} > \aleph_2 \) to obtain results—such as (generalizations of) \( \Diamond \)—not obviously following from the axiom.
itself. In Section 8 we observe that BACH plus \( 2^{\aleph_1} > \aleph_2 \) implies Kurepa's hypothesis. In Section 9 we discuss results of Laver, Shelah, and Stanley and make some general concluding remarks about iteration axioms.

I should like to thank Jim Baumgartner not only for inventing his brand of generalized Martin's axiom but for correcting a number of errors in an earlier version of this paper and for giving permission for the inclusion of several of his results. I am also grateful to the referees and to Ken Kunen for pointing out various errors in various versions of the manuscript, and to Lee Stanley for several illuminating remarks.

1. \( \sigma \)-ideals and Lusin sets

The most useful "combinatorial" consequence of Martin's axiom is now known as "\( p = c \)" but we shall call it \( P \).

\( P \): Suppose \( \{ A_\alpha \}_{\alpha < \kappa}, \kappa < 2^{\aleph_0} \) are infinite subsets of \( \omega \) with each finite intersection infinite. Then there is an infinite \( A \subseteq \omega \) such that for every \( \alpha, A - A_\alpha \) is finite.

Similarly, the most useful consequence of BACH is the obvious analogue of \( P \):

\( P_1 \): Suppose \( \{ A_\alpha \}_{\alpha < \kappa}, \kappa < 2^{\aleph_1} \) are subsets of \( \omega_1 \) with each countable intersection uncountable. Then there is an uncountable \( A \subseteq \omega_1 \) such that for every \( \alpha, A - A_\alpha \) is countable.

A forcing proof of the consistency of \( P_1 \) is in [33]. A similar argument proves

**Theorem 1.1.** BACH implies \( P_1 \).

The proof is a straightforward generalization of the proof that MA implies \( P \). We sketch it. \( \langle h, H \rangle \), where \( h \) is a countable subset of \( \omega_1 \) and \( H \) is a countable subset of \( \kappa \), is a condition. \( \langle h', H' \rangle \preceq \langle h, H \rangle \) if \( h' \supseteq h, H' \supseteq H \), and for each \( \alpha \in H, h' - h \subseteq A_\alpha \). The partial order is clearly countably closed and well-met. Any two conditions with the same first coordinate are compatible, so under CH, the partial order is \( \aleph_1 \)-linked. For each \( \alpha < \kappa \),

\[ D_\alpha = \{ \langle h, H \rangle : \alpha \in H \} \]

is dense. For each \( \beta < \omega_1 \),

\[ E_\beta = \{ \langle h, H \rangle : (\exists \gamma > \beta) [\gamma \in h] \} \]

is dense. Let \( G \) be generic for the \( D_\alpha \) and \( E_\beta \). Then \( \bigcup \{ h : (\exists H) [\langle h, H \rangle \in G] \} \) is the desired subset of \( \omega_1 \).

In practice I have found an equivalent formulation of \( P_1 \) to be very useful. Before proving the equivalence, I will temporarily refer to it as \( P_1' \).
Let $Y$ be a set of size $\aleph_1$. Let $\mathcal{F}$ be a nontrivial $\sigma$-ideal on $Y$, i.e., a collection of subsets of $Y$ such that

1. if $y \in Y$ then $\{y\} \in \mathcal{F}$,
2. if $A \subset B \in \mathcal{F}$, then $A \in \mathcal{F}$,
3. for any countable $\mathcal{F} \subseteq \mathcal{F}$, $Y \in \mathcal{F}$,
4. there is a $\mathcal{F} \subseteq \mathcal{F}$, $|\mathcal{F}| < 2^{\aleph_1}$, such that each $I \in \mathcal{F}$ is included in a countable union of members of $\mathcal{F}$.

Then there is an $L \subseteq Y$ which is $\mathcal{F}$-Lusin, i.e., $L$ is uncountable and for each $I \in \mathcal{F}$, $L \cap I$ is countable.

To see that $P_1$ implies $P_1'$, apply $P_1$ to the family of complements of members of $\mathcal{F}$. Conversely, let $\mathcal{F}$ be the ideal generated by the complements of the $A_i$.

Note that if in what we now will call $P_1$ we require $|\mathcal{F}| < \aleph_1$, we get a true statement, proved by a standard diagonal argument. With $|\mathcal{F}| \leq 2^{\aleph_0}$ we get a typical consequence of CH.

Particular interesting cases are when $\mathcal{F}$ is the ideal of first-category subsets of the real line, and when $\mathcal{F}$ is the ideal of sets of measure zero. In the former case $\mathcal{F}$ is the collection of $F_\sigma$ first-category sets; in the latter $\mathcal{F}$ is the collection of $G_\delta$-sets of measure 0. In the former case $L$ is a Lusin set, explaining our terminology. The reader can easily construct many other examples. The importance of $P_1$ then is that from the strengthening BACH of CH, one can diagonalize with weaker hypotheses. As we shall see, this enables us to take constructions usually performed with CH and build in extra pathology. Before getting to that kind of application however, let's see two easy examples of the utility of thinking in terms of $\sigma$-ideals.

**Theorem 1.2.** $P_1$ implies any maximal almost disjoint family of subsets of $\omega_1$ has cardinality $2^{\aleph_1}$.

**Proof.** Let $\mathcal{A}$ be an almost disjoint (i.e., pairwise intersections countable) family of subsets of $\omega_1$. Countable $\mathcal{A}$ are not maximal so we may assume $\mathcal{A}$ is uncountable. Then $\mathcal{A} \cup \{\omega_1 - \bigcup \mathcal{A}\}$ generates a nontrivial $\sigma$-ideal $\mathcal{F}$ on $\omega_1$. An $\mathcal{F}$-Lusin set would be almost disjoint from each member of $\mathcal{A}$.

**Theorem 1.3.** $P_1$ implies the closed unbounded filter on $\omega_1$ cannot be generated by fewer than $2^{\aleph_1}$ sets.

**Proof.** Working with the dual nonstationary allegedly so generated $\sigma$-ideal, get a Lusin set. Every uncountable subset of $\omega_1$—in particular the Lusin set—includes an uncountable nonstationary set. This latter set won’t be in the ideal.
be the family $\mathcal{F}$ of cocountable subsets of $\omega_2$. For $\alpha \in \omega_2$, let $\mathcal{F}_\alpha = \{ S \subseteq \omega_2 : S \text{ is cocountable and } \alpha \in S \}$. Then $\{ \mathcal{F}_\alpha : \alpha < \omega_2 \}$ generates a proper $\sigma$-ideal on $\mathcal{F}$. If $\mathcal{L}$ were Lusin for the ideal, so would be any $\mathcal{M} \subseteq \mathcal{L}$ of power $\mathfrak{K}_1$. But any $\mathfrak{K}_1$ cocountable subsets of $\omega_2$ have a point $\alpha$ in common. Hence $\mathcal{M}$ would have uncountable intersection with $\mathcal{F}_\alpha$.

$P_1$ plus CH in fact captures almost the entire flavour of BACH according to a result of Weiss. Bell [6] has shown that $P$ is equivalent to Martin’s axiom for partial orders which are $\sigma$-centred, i.e., the union of countably many centred subsets. A partial order is $\mathfrak{K}_1$-centred if it’s the union of $\mathfrak{K}_1$-centred subsets. Weiss [66] generalized Bell’s result to prove

**Theorem 1.4.** Assume $P_1$ plus CH plus for each $\kappa < 2^{\mathfrak{K}_1}$, $\kappa^{\mathfrak{K}_0} < 2^{\mathfrak{K}_1}$. Then generalized Martin’s axiom holds for countably compact $\mathfrak{K}_1$-centred partial orders.

It is not clear whether the extra cardinality hypothesis can be dropped in the case when $\kappa^{\mathfrak{K}_0} \geq 2^{\mathfrak{K}_1}$. It is not needed if one is content to meet $< \mathfrak{K}_0$ dense sets.

Incidentally, Steprans [53] proved that the $\mathfrak{K}_1$-centred version of generalized Martin’s axiom is strictly weaker than BA although such weakness hasn’t shown up in topological applications. It’s perhaps worth mentioning that

**Theorem 1.5.** BA implies every $\mathfrak{K}_1$-linked countably closed well-met partial order of cardinality $< 2^{\mathfrak{K}_1}$ is $\mathfrak{K}_1$-centred.

**Proof.** I find the topological method of proof more intuitive; readers who disagree may easily recover a partial order version. By standard techniques (see e.g. [35] or [67]) then, assume $X$ is a compact Hausdorff space with a $\pi$-base $\mathcal{B}$ which when ordered by inclusion is isomorphic to the given partial order. Consider the space $X^{\omega_1}$ with the product topology. The basic open sets in the product generated from elements of the $\pi$-base for $X$ form a $\pi$-base $\mathcal{B}$ for $X^{\omega_1}$. It is routine to verify that this $\pi$-base is also $\mathfrak{K}_1$-linked, countably closed, and well-met. $X^{\omega_1}$ is compact Hausdorff, so the usual genericity argument establishes that in $X^{\omega_1}$ the intersection of $< 2^{\mathfrak{K}_1}$ dense open sets is dense. For each $U \in \mathcal{B}$, let

$$D_U = \{ B \in \mathcal{B} : \text{for some } \alpha < \omega_1, \pi_\alpha(B) = U \}.$$

Then $D_U$ is dense open in $X^{\omega_1}$. Let $p \in \bigcap\{ D_U : U \in \mathcal{B} \}$. $\{ \pi_\alpha(p) : \alpha < \omega_1 \}$ is dense in $X$, so $\mathcal{B}$ is $\mathfrak{K}_1$-centred. \hfill $\square$

2. $L$-spaces with large weight

We first define our terms.

**Definition 2.1.** An $L$-space is a hereditarily Lindelöf regular nonseparable space. A Lusin space is an uncountable CCC (every collection of disjoint open sets is countable) space in which every first-category set is countable.
We assume the reader is familiar with the cardinal functions character, π-character, weight, ω-weight, and density (see [27] or [24]); we introduce another one:

**Definition 2.2.** \( G_δ \)-density \( d^1(X) = \kappa_0 \cdot \min \{|Y| : Y \text{ meets every nonempty } G_δ \} \).

We are going to illustrate the suggestion that BACH enables us to build in extra pathology when doing CH constructions by producing an \( L \)-space with large weight. One can construct \( L \)-spaces using CH, but if in addition one wants their weight to be large, say \( > \kappa_2 \), CH does not suffice, since the weight of an \( L \)-space is \( \leq 2^{2^{\kappa_0}} \) (see e.g. [27]) which, if say CH and \( 2^{\kappa_0} = \kappa_2 \) are assumed, is just \( \kappa_2 \).

Hajnal and Juhász constructed \( L \)-spaces with high weight using forcing [21]; we shall give a simple argument to show

**Theorem 2.3.** \( P_1 \) plus CH entails the existence of \( L \)-spaces of weight any \( \kappa \) such that \( \kappa_2 < \kappa_0 < 2^{\kappa_1} \).

(\( \kappa_1 \) (and \( \kappa \)) of course can be arbitrarily large.

In [62] it is observed that Lusin spaces are hereditarily Lindelöf and that nonseparable ones can frequently be found, assuming CH. More particularly,

**Theorem 2.4** [62]. CH implies that if \( X \) is uncountable, regular, CCC, has no notated points, is Baire, and \( \pi(X) < 2^{\kappa_0} \), then \( X \) has a dense Lusin subspace.

(\( X \) is Baire if no nonempty open set is first-category.)

A dense Lusin subspace of a nonseparable space is then an \( L \)-space. The simplest example of an \( X \) satisfying the conditions in Theorem 2.4 is the subspace of \( 2^{\omega_1} \) (i.e., the product of \( \kappa_1 \) copies of the two-point discrete space) consisting of all functions with countable support. Kunen [32] has proved that MA plus not CH implies there are no Lusin spaces.

We prove Theorem 2.3 by constructing a dense Lusin subspace of \( 2^\kappa \). However we can state a more general result:

**Theorem 2.5.** Assume \( P_1 \). Let \( X \) be regular CCC Baire without isolated points, \( \pi(X)^{\kappa_0} < 2^{\kappa_1} \) and \( d^1(X) = \kappa_1 \). Then \( X \) has a dense Lusin subspace.

**Proof.** In [62] it is shown that the first-category ideal in a CCC regular space \( C \) is generated by \( \pi(X)^{\kappa_0} \) many sets. We will look in fact at the trace of this ideal on a \( G_δ \)-dense \( Y \subseteq X, |Y| = \kappa_1 \). Points of \( Y \) are nowhere dense in \( X \) and hence in \( Y \).

Since \( Y \) is \( G_δ \)-dense and \( X \) is Baire, \( \{F \cap Y : F \text{ is first-category in } X \} \) is in fact a proper \( \sigma \)-ideal on \( Y \). Therefore by \( P_1 \) there is a Lusin subspace \( L \) of \( Y \). Any Lusin subspace is dense in some open set. Using the fact that \( Y \) as a dense subspace of a CCC space is CCC, one can get a Lusin subspace of \( Y \) which is dense in \( Y \); get a Lusin subspace of each of a maximal disjoint collection of open subsets of \( Y \) and take the union. We thus get a Lusin space dense in \( Y \) and hence in \( X \). \( \square \)
Baumgartner [4] points out that if only $\pi(x) < 2^\kappa$ is assumed, it is unclear whether the space can be obtained from just $P_1$ plus CH, but that weak restriction consistently suffices if one is careful in one's Lowenheim–Skolem arguments.

**Proof of Theorem 2.3.** It is well known that $2^\kappa$ is CCC for all $\kappa$, and that it is compact—hence Baire—and has no isolated points. For $\kappa > 2^{\aleph_0}$ it is not separable; for $\kappa < 2^{\aleph_0}$ its density is $\leq \kappa$. The $\pi$-weight of $2^\kappa$ is $\kappa$. It remains to check that $d'(2^\kappa) = \kappa$. In the particular case of product spaces, the calculation of $d'$ is known, but the following lemma of Ginsburg (included with his kind permission) is of independent interest.

**Lemma 2.6.** If $X$ is regular and countably compact, then $d'(x) \leq d(X)^{\aleph_0}$.

**Proof.** Let $D$ be dense in $X$, $|D| - d(X)$. For each countably infinite $A \subseteq D$, choose a limit point $x_A$ of $A$. Let $D_1 = \{x_A: A$ is a countably infinite subset of $D\}$. Clearly $|D_1| < |D|^{\aleph_0}$. We claim $D_1$ meets every nonempty $G_\delta$. For let $H = \bigcap\{U_n: n < \omega\}$, $U_n$ open, $H \neq \emptyset$. By regularity find open sets $V_n$, $n < \omega$ such that $\bigcap_{n=1}^{\infty} V_n \subseteq V_n$, all $n$. Choose $x_A \in (U_n - \overline{U}_{n+1}) \cap D$, and let $A = \{x_n: n < \omega\}$. Then $x_A \in \bigcap\{V_n: n < \omega\} \subseteq H$ so $D_1 \cap H \neq \emptyset$. □

Finally, to see that the weight of the Lusin space $L$ is $\kappa$, observe that $\kappa = \pi(2^\kappa) = \pi(L) \leq w(L) \leq w(2^\kappa) = \kappa$. Juhász has raised the general question of various cardinal functions skipping values. In this connection it is interesting to observe that every subspace of $L$ has weight either $\kappa$ or $< \kappa$. To see this, note that uncountable subspaces are somewhere dense so the above calculation works, while countable subspaces by CH have weight $\leq \aleph_1$. Similarly, $\pi$-weight omits all cardinals between $\aleph_0$ and $\kappa$. This has found application in some work of Juhász and Weiss [31].

I do not have as simple an argument as the Lusin set one for producing hereditarily separable regular spaces of cardinality $> 2^{\aleph_0}$. (Such spaces are not Lindelöf.) However as we note in Section 8, BACH plus $2^{\aleph_1} > \aleph_2$ implies the combinatorial principle $W(\kappa)$ for all infinite $\kappa < 2^{\aleph_1}$. As noted in [28], $W(\kappa)$ implies the existence of a hereditarily separable regular space of cardinality $\kappa$, as well as CH. It also yields the existence of a hereditarily Lindelöf regular space of weight $\kappa$, but our Lusin set construction is considerably easier.

**3. Caliber $\aleph_1$ versus separability**

First we introduce some additional cardinal functions that will prove useful.

**Definition 3.1.** $\kappa$-tensity: $d_0(X) = \aleph_0 \cdot \min\{|D|: \text{for each } x \in X \text{ there is } E \subseteq D, |E| \leq \kappa, x \in \overline{E}\}$.

**Pretightness:** $\theta(X) = \sup\{\min\{\lambda: \text{there is } Y \subseteq X - \{x\}, x \in \overline{Y}, |Y| = \lambda; \text{or there isn't and } \lambda = 0\}: x \in X\}$. 

Removing the sup's, we get the various functions at a point \( x \).

A space has \textit{caliber} \( \mathbb{K}_1 \) if every point-countable collection of open sets is countable. I have investigated the question of which spaces with caliber \( \mathbb{K}_1 \) are separable in \([57, 59]\). Also see the papers of Šapirovskii listed in the References below, which contain many fine related results. An unexpected application of \( P_1 \) is

\textbf{Theorem 3.2.} Assume \( P_1 \). If \( |X| < 2^{\mathbb{K}_1} \) and \( \pi(X) \leq \mathbb{K}_1 \) and \( X \) has caliber \( \mathbb{K}_1 \), then \( X \) is separable.

Analysis of the proof will lead us to new results on cardinal invariants not requiring Baumgartner’s axiom. If \( \mathcal{W} \) is a collection of open subsets of a space \( X \) and \( x \in X \), define \( st(x, \mathcal{W}) = \{W \in \mathcal{W}: x \in W\} \). Let \( \mathcal{Y} \) be a \( \pi \)-base of the space \( X \) of the theorem. If \( X \) is not separable, \( \{st(x, \mathcal{Y}): x \in X\} \) generates a proper \( \sigma \)-ideal on \( \mathcal{Y} \). By \( P_1 \) there is an uncountable \( \mathcal{Y} \subseteq \mathcal{Y} \) such that \( \mathcal{Y} \cap st(x, \mathcal{Y}) \) is countable for all \( x \). But then \( \mathcal{Y} \) is uncountable point-countable, a contradiction.

As one might expect, if \( |X| \leq \mathbb{K}_1 \), \( P_1 \) is not required. Indeed neither is \( "\pi(X) \leq \mathbb{K}_1" \). See \([57]\). The stipulation that \( |X| < 2^{\mathbb{K}_1} \) cannot be dropped: let \( \mathcal{F} \) be a \( \sigma \)-ideal on \( \omega_1 \), let \( \Sigma(\mathcal{F}) = \{f \in 2^{\omega_1}: \{\alpha: f(\alpha) = 1\} \in \mathcal{F}\} \) inherit the subspace topology from \( 2^{\omega_1} \); in \([8]\) it is shown that if \( \mathcal{F} \) is the ideal of nonstationary sets or the ideal generated by a maximal almost disjoint collection of sets, then \( \Sigma(\mathcal{F}) \) is a counterexample. Interestingly, under \( P_1 \) such \( \Sigma(\mathcal{F}) \) do not have caliber \( \mathbb{K}_1 \) if \( \mathcal{F} \) has \( < 2^{\mathbb{K}_1} \) generators.

The \( \pi \)-weight did not play an integral role in the proof of the theorem; isolating the key concept we make the following

\textbf{Definition 3.3.} A collection \( \mathcal{Y} \) of open sets is \textit{countably generated} if there is a countable set \( D \) such that \( \mathcal{Y} = \bigcup \{st(d, \mathcal{Y}): d \in D\} \). A space is \( \kappa \)-c.g. if every collection of \( \kappa \) open sets is countably generated.

Clearly a space is separable iff it is \( \kappa \)-c.g. for all \( \kappa \) iff some \( \pi \)-base is countably generated.

\textbf{Lemma 3.4.} \textit{Separable implies} \( \mathbb{K}_1 \)-c.g. \textit{implies caliber} \( \mathbb{K}_1 \). \textit{No implication can be reversed.}

\textbf{Proof.} The \( \Sigma(\mathcal{F}) \) discussed above show that the latter implication cannot be reversed. That the former cannot will follow from the next result, which is that \( \mathbb{K}_1 \)-c.g. is arbitrarily productive while, as is well known, a product of nontrivial separable spaces is separable iff no more than \( 2^{\mathbb{K}_0} \) factors are involved. \( \square \)

\textbf{Theorem 3.5.} Let \( \{X_\alpha: \alpha \in A\} \) each be \( \mathbb{K}_1 \)-c.g. Then so is their product.
Proof. Without loss of generality we may assume we are dealing with basic open sets. Since these involve only finitely many coordinates, we may assume \(|A| \leq \aleph\). Let \(\{U_\beta : \beta < \omega_1\}\) be basic open in the product. Generate a new topology on each \(X_\alpha\) by taking \(\emptyset, X_\alpha,\) and the \(\alpha\)-projections of the \(U_\beta\) as subbase. These new topologies are separable, therefore so is their product. But the countable dense subset of the product generates the \(U_\beta\). □

Restating Theorem 3.2 we have

**Theorem 3.6.** Assume \(P_1\). If \(|X| < 2^{\aleph_1}\) and \(X\) has caliber \(\mathfrak{K}_1\), then \(X\) is \(\mathfrak{K}_1\)-c.g.

It is interesting to compare this with Šapirovskii's

**Theorem 3.7** [45]. Assume\(MA plus not CH. Then every compact CCC Hausdorff space is \(\kappa\)-c.g. for all \(\kappa < 2^{\aleph_0}\).

We can ensure point-countability in the proof of Theorem 3.2 or 3.6 without obviously bounding the cardinality of \(X\):

**Theorem 3.8.** Assume \(P_1\). If \(d_{\mathfrak{K}_0}(X) < 2^{\aleph_1}\) and \(X\) has caliber \(\mathfrak{K}_1\), then \(X\) is \(\mathfrak{K}_1\)-c.g.

The point is that each \(st(x, \mathcal{A})\) is included in a countable union of \(st(d, \mathcal{A})\)'s, \(d \in D\). It is easy to see that

**Lemma 3.9.** If \(\min(\pi X(X), t(X)) < \kappa, d_{\mathfrak{K}_0}(X) < d(X) \cdot \min(\pi X(X), t(X))\).

Examining the tightness version first, we have

**Corollary 3.10.** Assume \(P_1\). If \(d(X) < 2^{\aleph_1}\) and \(t(X) \leq \mathfrak{K}_0\) and \(X\) has caliber \(\mathfrak{K}_1\), then \(X\) is \(\mathfrak{K}_1\)-c.g.

Unfortunately \(t(X) \leq \mathfrak{K}_0\) plus \(d(X) \leq \mathfrak{K}_1\) plus caliber \(\mathfrak{K}_1\) were already known to imply separability [57, 3.21], so we don't have much new here. The case of \(\pi\)-character is more interesting.

**Corollary 3.11.** (1) Assume \(P_1\). If \(d(X) < 2^{\aleph_1}\) and \(\pi X(X) \leq \mathfrak{K}_0\) and \(X\) has caliber \(\mathfrak{K}_1\), then \(X\) is \(\mathfrak{K}_1\)-c.g.

(2) If \(\pi(X) \leq \mathfrak{K}_1\) and \(\pi X(X) \leq \mathfrak{K}_0\) and \(X\) has caliber \(\mathfrak{K}_1\), then \(X\) is separable.

(3) Assume \(CH\). If \(\pi X(X) \leq \mathfrak{K}_0\) and \(X\) is regular and has caliber \(\mathfrak{K}_1\), then \(X\) is separable.

**Proof.** (1) follows immediately from Theorem 3.8 and Lemma 3.9. (2) follows from the proof of Theorem 3.8 and the fact that Lusin sets exist for ideals with \(\mathfrak{K}_1\) generators. For (3), we use the fact that for regular CCC \(X, \pi(X) \leq (\pi X(X))^{\mathfrak{K}_0}\) [46]. (3) should be contrasted with the result of Efimov [14] that \(CH\) implies
first-countable Hausdorff spaces with caliber \( k_1 \) are separable. Assuming MA plus not CH, there is a normal one that isn’t [59]. □

If we apply the proof of Theorem 3.2 locally rather than globally, we obtain

**Theorem 3.12.** Assume \( P_1 \). If \( |X| < 2^{\aleph_1} \), \( X \) has caliber \( k_1 \), \( \{x\} \subseteq X \) is closed, and \( \pi_X(x) < k_1 \), then either \( \pi(x) = k_0 \) or \( x \) is isolated.

**Proof.** Let \( D \) countably generate the \( \pi \)-base \( \mathcal{Z} \) at \( x \). Then \( x \in D - \{x\} \), unless \( x \) is isolated. □

4. Baire category analogues

There are well-known techniques for translating variants of Martin’s axiom into topology. See e.g. [35]. One can similarly translate Baumgartner’s axiom, but because the well-met and linked requirements do not translate felicitously, it is better to give consequences rather than a translation. See [66] or [67] for a topological translation of \( P_1 \).

**Theorem 4.1.** Assume Baumgartner’s axiom. Let \( X \) be a compact Hausdorff space with \( d(X) < \aleph_1 \) in which every nonempty \( G_\delta \) has nonempty interior. Then \( X \) is not the union of \( 2^{\aleph_1} \) nowhere dense sets.

**Proof.** The nonempty closed \( G_\delta \)-subsets of \( X \) ordered by inclusion form a countably closed \( \aleph_1 \)-linked well-met partial order. If \( F \) is a nowhere dense subset of \( X \), the collection of all closed \( G_\delta \) disjoint from \( F \) is a dense subset of the partial order, since by regularity, every nonempty open set includes a nonempty closed \( G_\delta \). The reader can finish the proof via genericity and compactness. □

**Remark.** The reader will observe that if \( X \) has a countably closed \( \aleph_1 \)-linked well-met \( \pi \)-base then “compact Hausdorff” can be weakened to “Lindelöf regular”, provided BA gives us a countably compact generic set. If \( (\forall \kappa < \aleph_0) [k_\kappa < \aleph_1] \), it does. See the discussion after Corollary 7.18 below.

The most interesting space satisfying the hypotheses of the theorem is \( \beta N - N \) (assuming CH). For the elementary topology of \( \beta N - N \), see [64]. It is well known and easy to prove that if \( X \) is compact Hausdorff and any nonempty \( G_\delta \) has nonempty interior, then \( X \) is not the union of \( \leq \aleph_1 \) nowhere dense sets. Assuming Martin’s axiom, \( \beta N - N \) is not the union of \( \leq 2^{\aleph_0} \) nowhere dense sets [56]. Hechler [22] has constructed a model in which \( \aleph_2 < 2^{\aleph_0} \) and \( \beta N - N \) is the union of \( \aleph_2 \) nowhere dense sets.

Another well-known space to which the theorem applies is the completion of an \( \eta_1 \)-set. See [18] for the definition and properties of \( \eta_1 \)-sets.
There are still other spaces satisfying the hypotheses of Theorem 4.1. In particular they too can be found as growths, i.e., as $\beta X - X$ for some space $X$. If $X$ is locally compact, $\beta X - X$ is compact. If $X$ is locally compact and realcompact, every nonempty $G_\delta$ of $\beta X - X$ has nonempty interior [64, 4.21]. I do not know sharp conditions that ensure $d(\beta X - X) \leq 2^{\aleph_0}$, but it certainly suffices to have $w(\beta X) \leq 2^{\aleph_0}$. This will happen if $X$ is CCC and $\pi_1(X) \leq 2^{\aleph_0}$, since $\beta X$ will also have these properties and hence by [44,46] will have weight $\leq 2^{\aleph_0}$. The CCC condition is too strong since one can prove directly [10; 64, 5.12] that the density of the growth of the discrete space of cardinality $\aleph_1$ is $2^{\aleph_0}$. Hence BACH implies it too is not the union of $< 2^{\aleph_1}$ nowhere dense sets.

Shelah [49] observed that there is another way of generalizing the notion of first category in the context of generalization of Martin's axiom. Consider the topology on the Cartesian product of $\aleph_1$ copies of the two-point discrete space generated by boxes fixing countably many coordinates. Shelah proves that under his version of generalized Martin's axiom, which replaces "$\aleph_1$-linked" by the weaker condition that there is a closed unbounded $C \subseteq \omega_2$ and a regressive $f : \omega_2 \to \omega_2$, such that if $\alpha, \beta \in C$ and $\text{cf}(\alpha), \text{cf}(\beta) > \aleph_0$ and $f(\alpha) = f(\beta)$, then $p_\alpha$ and $p_\beta$ are compatible, the union of $< 2^{\aleph_1}$ sets, each of which is the union of $< \aleph_1$ nowhere dense sets in this topology, also has this property. The same proof works for BACH.

By Theorem 1.5 the condition that $d(X) \leq \aleph_1$ in Theorem 4.1 is no great weakening of BACH.

As one might expect, the density and $G_\delta$-requirements are necessary in calculating numbers of nowhere dense sets. A standard counterexample is the compact Aronszajn line [41] which is the union of $\aleph_1$ nowhere dense sets and has density $\aleph_1$. Weiss [65] constructs a compact line in which nonempty $G_\delta$ have nonempty interiors but which is the union of $\aleph_2$ nowhere dense sets.

5. Baumgartner’s axiom and normality

There is a chapter in [42] entitled “Martin's axiom and normality”. It is not obvious how to generalize the proofs given there or in the unifying paper [1] to higher cardinals. However the later formulation in [30] does generalize straightforwardly to enable us to prove the normality of various spaces from BACH and in particular to prove

**Theorem 5.1.** There is a model of set theory in which every normal space of character $< 2^{\aleph_1}$ is $\aleph_1$-collectionwise Hausdorff, but there is a normal space of character $\aleph_1$ which is not $\aleph_2$-collectionwise Hausdorff.

Recall that a space is $\kappa$-collectionwise Hausdorff if there exist pairwise disjoint open sets about the points in any closed discrete subspace of cardinality $\leq \kappa$. Let us first prove a generalization of a version of the main lemma in [30].
**Lemma 5.2.** Assume BA. Suppose \( H, K \) are subsets of a space \( X \) such that \( H \cap K = K \cap H = \emptyset \), and \( |H \cup K| < 2^{\aleph_1} \). Let \( \mathcal{A} \) be a family of closed subsets of \( X \) which is closed under countable unions and such that each point in \( H \cup K \) has a neighbourhood basis included in \( \mathcal{A} \). Let \( \mathcal{U} \) be a collection of \( \leq \aleph_1 \) subsets of \( X \) such that for every disjoint \( A, B \in \mathcal{U} \), there is a \( U \in \mathcal{U} \) with \( A \subseteq U \subseteq X - B \). Then there exist disjoint open sets about \( H \) and \( K \).

**Proof.** Let \( P = \{ (A, B) : A, B \in \mathcal{A}, A \cap \overline{K} = B \cap \overline{H} = \emptyset \} \). Define \( (A, B) \leq (A', B') \) if \( A \supseteq A' \) and \( B \supseteq B' \). Let \( \mathcal{P} = (P, \leq) \). For each \( x \in H \), let \( D_x = \{ (A, B) \in P : x \in \text{int } A \} \). For \( x \in K \), let \( E_x = \{ (A, B) \in P : x \in \text{int } B \} \). Then each \( D_x \) and \( E_x \) is a dense subset of \( P \), for given e.g. \( x \in H \) and \( (A, B) \in P \), pick \( A' \in \mathcal{A} \) such that \( x \in \text{int } A' \) and \( A' \cap (K \cap B) = \emptyset \). Then \( (A \cup A', B) \leq (A, B) \). \( P \) is clearly countably closed and well-met. To see that \( P \) is \( \mathfrak{K}_1 \)-linked, for \( U \in \mathcal{U} \) let \( P_U = \{ (A, B) : A \subseteq X - B \} \). Then \( P = \bigcup \{ P_U : U \in \mathcal{U} \} \) and each \( P_U \) is linked. Finally, let \( G \) be \( (\{ D_x : x \in H \} \cup \{ E_x : x \in K \}) \)-generic. Then \( \bigcup \{ \text{int } A : \text{for some } B, (A, B) \in G \} \) and \( \bigcup \{ \text{int } B : \text{for some } A, (A, B) \in G \} \) are the desired disjoint open sets about \( H \) and \( K \) respectively. \( \square \)

One of the standard first-countable normal (under MA plus not CH) non-\( \mathfrak{K}_1 \)-collectionwise Hausdorff spaces is obtained by taking a subset \( X \) of the real line of cardinality \( \mathfrak{K}_1 \), isolating a countable dense subset \( Q \) of \( X \) (without loss of generality assume \( Q \) is the set of rationals), and assigning as neighbourhoods to each irrational point, tails of a fixed sequence of rationals converging to it. We mimic this construction by taking a subset \( Y \) of the completion of an \( \eta_1 \)-set which has cardinality \( \mathfrak{K}_2 \) and includes the \( \eta_1 \)-set. Assume CH so that the \( \eta_1 \)-set has cardinality \( \mathfrak{K}_1 \). The points of the \( \eta_1 \)-sets are declared open in \( Y \). For each other point \( y \) of \( Y \), an \( \omega_1 \)-sequence from the \( \eta_1 \)-set converging to it (in the order topology) is chosen. Neighbourhoods of \( y \) are then tails (all but countably many terms) of the chosen sequence. Then it is easy to see that \( Y \) is not \( \mathfrak{K}_2 \)-collectionwise Hausdorff but is \( \mathfrak{K}_1 \)-collectionwise Hausdorff (since it is regular and \( G_\delta \)'s are open). \( Y \) is clearly locally Lindelöf and has character \( \mathfrak{K}_1 \). Let \( \mathcal{A} \) be the collection of the countable unions of basic clopen sets about the “irrational” points of \( Y \). By CH let \( \mathcal{U} \) be a basis of power \( \mathfrak{K}_1 \), closed under countable unions, for the topology on the set \( Y \) inherited from the completion of the \( \eta_1 \)-set. Then each member of \( \mathcal{U} \) is open in the stronger topology on \( Y \). In the weaker topology (as well as in the stronger) \( G_\delta \)'s are open and the members of \( \mathcal{A} \) are Lindelöf. It follows that \( \mathcal{U} \) satisfies the conditions of the lemma and hence that BACH plus \( 2^{\aleph_1} > \mathfrak{K}_2 \) implies \( Y \) is normal.

We thus have obtained from BACH plus \( 2^{\aleph_1} > \mathfrak{K}_2 \) a normal space of character \( \mathfrak{K}_1 \) which is \( \mathfrak{K}_1 \)-collectionwise Hausdorff, but not \( \mathfrak{K}_2 \)-collectionwise Hausdorff. It is not known whether these hypotheses imply every normal space of character \( \mathfrak{K}_1 \) is \( \mathfrak{K}_1 \)-collectionwise Hausdorff, but in the particular model of Baumgartner that conclusion holds. For more on the properties of his model and an indication of a proof of Theorem 5.1, see Section 7.
Definition 5.3. A space is \((k_1)\)-para-Lindelöf if every open cover (of size \(\leq k_1\)) has a locally countable refinement.

In [55] I proved every regular para-Lindelöf space is collectionwise Hausdorff. Fleissner has proved that \(V = L\) implies every regular \(k_1\)-para-Lindelöf space is collectionwise Hausdorff [16]. It is of interest therefore to note that \(Y\) is \(k_1\)-para-Lindelöf. The proof is a straightforward generalization of the situation of the MA case where we note that perfect normality implies countable paracompactness. We leave to the reader to prove that

Lemma 5.4. A normal space in which \(G_\delta\)'s are open and in which every closed set is the intersection of \(\leq k_1\) open sets is \(k_1\)-para-Lindelöf.

It remains only to show that every closed set is such an intersection. We need only consider sets of nonisolated points. But by normality, if \(F\) is such a set, there is an open \(U \supseteq F\) such that \(U - F\) consists of isolated points. There are only \(k_1\) of these and points are closed so we are done.

Another standard example of a first-countable normal (under MA plus not CH) non-\(k_1\)-collectionwise Hausdorff space is the Cantor tree [42]. The obvious generalization of this space using the binary tree on \(\omega_1\) and \(< 2^{\omega_1}\) of its nodes on the top level can be shown normal by Lemma 5.2. This answers a question of L. Sennott and R. Levy. Alternatively, for this special case one can generalize the methods of [55, Chapter III] which obtain normality by corresponding subsets of the dense set to subsets of the closed discrete set using almost disjoint coding. Shelah [49] gives the necessary coding lemma (from his axiom, but BACH will do):

Lemma 5.5. Assume BACH. Suppose \(\mathcal{P} \subseteq \{\eta \mid \alpha < \omega_1; \eta \in 2^{\omega_1}\}\), the intersection of any two members of \(\mathcal{P}\) is countable, \(|\mathcal{P}| < 2^{k_1}\), and \(\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2\), where \(\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset\). Then there is an \(S\) such that

\[
A \in \mathcal{P}_1 \implies A - S \text{ is countable},
A \in \mathcal{P}_2 \implies A \cup S \text{ is countable}.
\]

One can prove in a variety of ways (e.g. from the above examples or from Lemma 5.5) that BACH plus \(2^{k_1} > \kappa \geq k_1\) implies \(2^\kappa = 2^{k_1}\). Indeed—not unexpectedly—

Theorem 5.6. \(P_1\) plus CH plus \(2^{k_1} > \kappa \geq k_1\) implies \(2^\kappa = 2^{k_1}\).

I am grateful to Eric van Douwen for providing the following proof and suggesting I use it. It appropriately generalizes the usual Q-set results.

Theorem 5.7. \(P_1\) plus CH implies that if \(X\) is a space in which countable sets are closed, which has weight \(k_1\), and cardinality \(< 2^{k_1}\), then every subset of \(X\) is the intersection of \(\leq k_1\) open sets.
Theorem 5.6 easily follows from Theorem 5.7 by counting—take for example a subspace of cardinality \( \kappa \) of the Cartesian product of \( \mathbb{R} \) copies of the two-point discrete space with the topology generated by countable boxes.

To prove Theorem 5.7, let \( \mathcal{B} \) be a base of \( X \) of cardinality \( \mathfrak{c} \), closed under countable unions. Enumerate \( \mathcal{B} = \langle B_\alpha : \alpha < \omega_1 \rangle \) so that each element of the base appears cofinally often. For \( S, T \subseteq X \), let

\[
I_{S,T} = \{ \alpha < \omega_1 : S \subseteq B_\alpha \subseteq X - T \}.
\]

Then \( |I_{S,T}| = \mathfrak{c} \) for any two disjoint countable \( S, T \). Also, \( I_{S,T} = \bigcap_{x \in S} \bigcap_{x \in T} I_{\{x\}, \{y\}} \). Given \( Y \subseteq X \), it follows from \( P_1 \) that there is an uncountable \( I \subseteq \omega_1 \) such that for every \( y \in Y \) and every \( x \in X - Y \), \( |I - I_{\{y\}, \{x\}}| \leq \mathfrak{c} \). Then \( Y = \bigcap_{\alpha < \omega_1} \bigcup\{B_\beta : \alpha < \beta \in I \} \).

6. Products, stationary sets, and \( \Diamond \)

There are a number of results which are of the following form: a certain topological property concerning sequences is considered. It is shown to be preserved by countable products, and under MA, preserved under products of size \( < 2^{\mathfrak{c}} \). See e.g. [7,58]. As one might expect, BACH also produces theorems of this sort. The proofs of the natural generalizations are straightforward. However when one follows the set-theoretic rule of thumb that

\[
\text{infinite} : \omega = \text{stationary} : \omega_1
\]

some surprising results are obtained, in particular a generalization of \( \Diamond \). But first some background.

Let us recall my solution of a problem of Wilansky in [58]. I showed that if \( \{X_n\}_{n<\omega} \) were spaces such that there was in each a countable dense set such that each point was a sequential limit of it, then the product had this property (sequential separability). Also, if MA is assumed, sequential separability is preserved by products with \( < 2^{\mathfrak{c}} \) factors. The natural generalization is

**Theorem 6.1.** Assume BACH. For each \( \alpha < \kappa \), where \( \kappa < 2^{\mathfrak{c}} \), let \( X_\alpha \) be a space and \( D_\alpha \) be a subset of \( X_\alpha \) of cardinality \( \mathfrak{c} \), such that each point in \( X_\alpha \) is the limit of a net from \( D_\alpha \) indexed by \( \omega_1 \). Then there is \( D \subseteq X = \prod\{X_\alpha : \alpha < \kappa \} \), \( |D| = \mathfrak{c} \), such that each point in \( X \) is the limit of a net from \( D \) indexed by \( \omega_1 \).

**Proof.** Refer to [58] or see Section 7. The obvious modifications suffice; however CH is needed even when \( \kappa = \mathfrak{c} \), because we need to have a \( G_\delta \)-dense subset of the product of cardinality \( \mathfrak{c} \). \( \square \)

It is not difficult to see that Theorem 6.1 may be stated equivalently in terms of an \( \omega_1 \)-sequence such that each point is the limit of a subnet. Either way it is not very exciting. But let us introduce stationary sets into the situation.
Definition 6.2 [19]. An $\omega_1$-sequence $\{x_\alpha : \alpha \in \omega_1\}$ in a space $X$ is universal if for every $x \in X$ there is a stationary $S \subseteq \omega_1$ such that the net $\{x_\alpha : \alpha \in S\}$ converges to $x$.

Lemma 6.3 [19]. $\Diamond$ iff there is a universal sequence in $2^{\omega_1}$ iff every space of weight $\leq 2^{\aleph_0}$ has a universal sequence.

One naturally wonders whether the property of having a universal sequence is $< 2^{\aleph_1}$-productive under BACH. It trivially isn't since the GCH does not imply $\Diamond$, but of course the real question concerns BACH plus $2^{\aleph_1} > \aleph_2$. It is not known whether this is enough to yield $\Diamond$, but in Baumgartner's iteration model $\Diamond$ does hold; indeed one almost has that the property of having a universal sequence is $< 2^{\aleph_1}$-productive:

Theorem 6.4. There is a model of set theory in which BACH plus $2^{\aleph_1} > \aleph_2$ holds, and in which the product of $< 2^{\omega_1}$ spaces, each of weight $< 2^{\omega_1}$ and each having a universal sequence, also has a universal sequence.

We shall prove this in the next section, as well as give Baumgartner's example which shows that the weight restriction is necessary. It is interesting that, by the proof of Theorem 6.4, $\Diamond$ holds in Baumgartner's model regardless of whether it holds in the ground model. This also follows from more general considerations, namely that the partial order is countably closed and adjoins a new subset of $\omega_1$.

Theorem 6.4 yields an interesting generalization of $\Diamond$. Intuitively, the function form of $\Diamond$ says that there are $\aleph_1$ functions which trap all functions from $\omega_1$ to $\omega_1$ on initial segments, i.e., on countable sets. The proposition $\Diamond_{\aleph_1}(\kappa)$ defined below says that there are $\aleph_1$ functions which trap all functions from $\kappa$ to $\omega_1$ on countable sets.

Definition 6.5. $\Diamond_{\aleph_1}(\kappa)$ is the assertion that there exist $\{d_\xi : \xi < \omega_1\}$, each $d_\xi \in {}^\kappa \omega_1$, such that for every $f \in {}^\kappa \omega_1$ there is a stationary $S_f \subseteq \omega_1$ such that for every countable $T \subseteq \kappa$, for all but countably many $\xi \in S_f$, $d_\xi \upharpoonright T = f \upharpoonright T$.

It is easy to see that $\Diamond$ implies $\Diamond_{\aleph_1}(\omega_1)$. To prove the converse, let $\{d_\xi : \xi < \omega_1\}$ be as above. Let $f_\xi = d_\xi \upharpoonright \xi$. Claim $\{f_\xi : \xi < \omega_1\}$ satisfies $\Diamond$. Following the similar proof in [19] that a universal sequence in $2^{\omega_1}$ satisfies $\Diamond$, let $R = \{\alpha \in S_f : f_\alpha \upharpoonright \alpha \neq f \upharpoonright \alpha\}$. Claim $R$ is nonstationary. For $\alpha \in R$, let $x_\alpha$ be the least such that $f_\alpha$ and $f \upharpoonright \alpha$ differ on $x_\alpha$. Then the function mapping $\alpha$ to $x_\alpha$ is regressive so there is a stationary $R' \subseteq R$ and a $\beta \in \omega_1$, such that $x_\alpha = \beta$ for all $\alpha \in R'$. Thus $f_\alpha(\beta) \neq f(\beta)$ for all $\alpha \in R'$. But $R'$ is an uncountable subset of $S_f$ and $f_\alpha(\beta) = d_\alpha(\beta)$, contradicting $\Diamond_{\aleph_1}(\omega_1)$.

Theorem 6.6. There is a model of set theory in which BACH, $2^{\aleph_1} > \aleph_2$, and $(\forall \kappa < 2^{\aleph_1})(\Diamond_{\aleph_1}(\kappa))$ hold.
Proof. The discrete space of power $\mathfrak{K}$ has a universal sequence. Hence the product of $\kappa$ copies does. But then observe that if every open set about a point includes a tail of an $\mathfrak{K}$-sequence, so does every $G_\delta$ about it. □

Given the usefulness of $P_\kappa$, it is natural to look for a stationary analogue. First we define the appropriate strengthening of the concept of $\sigma$-ideal. Refer e.g. to [5].

Definition 6.7. If $f: \omega_1 \to \mathcal{P}(\omega_1)$ (the power set of $\omega_1$), then the diagonal union of $f$, denoted by $\nabla f$, is $\{p \in \omega_1: (\exists \alpha < p)(p \ni f(\alpha))\}$. Similarly, the diagonal intersection of $f$, $\Delta f$, is $\{p \in \omega_1: (\forall \alpha < p)(p \ni f(\alpha))\}$. An ideal $\mathcal{I}$ on $\omega_1$ is normal if for any $f: \omega_1 \to \mathcal{I}$, $\nabla f \in \mathcal{I}$.

It is well known that the nonstationary ideal is normal and that it is included in any normal ideal. Let $\mathcal{N}$ be the nonstationary ideal. Consider the Boolean algebra $\mathcal{P}(\omega_1)/\mathcal{N}$. Despite the apparent dependence of the definition of diagonal intersection on the choice of $f$, by [5, 2.91] diagonal intersection is well defined as an operation on subsets of $\mathcal{P}(\omega_1)/\mathcal{N}$. Indeed, according to an unpublished result of C.D. Herink, it operates as the inf for $\omega_1$-chains. We now state

$P'_{\kappa}$. Let $\mathcal{I}$ be a (proper) normal ideal on $\omega_1$. Suppose there is a $\mathcal{I} \subseteq \mathcal{I}$, $|\mathcal{I}| < 2^{\kappa_1}$, such that each $\mathcal{I} \in \mathcal{I}$ is included, except for a nonstationary set, in a diagonal union of members of $\mathcal{I}$. Then there is a stationary set having nonstationary intersection with each member of $\mathcal{I}$.

Equivalently,

$P'_{\kappa}$. Suppose $(S_\alpha)_{\alpha < \kappa}$, $\kappa < 2^{\kappa_1}$, are stationary subsets of $\omega_1$ with each diagonal intersection stationary. Then there is a stationary $S \subseteq \omega_1$ such that for every $\alpha$, $S - S_\alpha$ is nonstationary.

The proof of equivalence is routine except for the following lemma, which appears in [63, p. 85]. The lemma also follows from Herink's result.

Lemma 6.8. Let $\mathcal{A}$ be a family of stationary subsets of $\omega_1$. If every diagonal intersection of members of $\mathcal{A}$ is stationary, so is every diagonal intersection of diagonal intersections of members of $\mathcal{A}$.

Apparently stronger versions of $P'_{\kappa}$ may be obtained by replacing in either version the last occurrence of “nonstationary” by “countable”. As observed by Alan Taylor, the countable version of $P'_{\kappa}$ for ideals is obviously false—take $\mathcal{I}$ to be the nonstationary ideal, $\mathcal{I}$ to be the countable sets. Every nonstationary set is a diagonal union of countable sets, but every uncountable set includes a nonstationary set. I do not know whether the countable version of $P'_{\kappa}$ for intersections is actually stronger than $P'_{\kappa}$, but we shall prove it consistent. We state it formally as
\( P_5 \). Suppose \( \{ S_\alpha : \alpha < \kappa \} \) are stationary subsets of \( \omega_1 \) with each diagonal intersection stationary. Then there is a stationary \( S \subseteq \omega_1 \) such that for every \( \alpha \), \( S - S_\alpha \) is countable.

Originally I had formulated \( P_5 \) with "countable" replacing "diagonal". I am grateful to K. Kunen for pointing out the error and indicating how to modify my original incorrect consistency proof. Indeed, the diagonal intersection of even a decreasing chain of stationary sets need not be stationary. I am grateful to L. Harrington for the following example, and L. Temes for communicating it to me. Take an \( \omega \times \omega_1 \) Ulam matrix \( \{ A_\alpha \} \) of subsets of \( \omega_1 \) such that the sets in any row are disjoint and the union of the sets in any column is cocountable. Indeed, for each \( \beta \in \omega_1 \), let \( f_\beta : \omega \rightarrow \beta \) be onto and let \( A_\alpha = \{ \beta : f_\beta(n) = \alpha \} \). It is well known and easy to prove that for some \( n \), uncountably many \( A_\alpha \) are stationary. Let \( X_\beta = \bigcup \{ A_\alpha : \alpha < \beta \} \). Then \( \beta < \gamma \) implies \( X_\beta \supseteq X_\gamma \), and each \( X_\beta \) is stationary. But

\[ \forall \{ X_\beta : \beta < \omega_1 \} = \{ \gamma : (\forall \beta < \gamma) [g_\beta(n) \neq \gamma] \} = \emptyset. \]

Call a family of stationary sets almost disjoint if pairwise intersections are nonstationary, and strongly almost disjoint if pairwise intersections are countable. Assuming large cardinals, there consistently need not exist a (strongly) almost disjoint family of \( \kappa_2 \) stationary sets [17]. It is easy to see that \( \diamond \) entails the existence of a strongly almost disjoint family of power \( 2^{\kappa_1} \). By [26], if \( 2^{\kappa_0} < \kappa_{\omega_1} \) and \( 2^{\kappa_0} < 2^{\kappa_1} \) and \( 2^{\kappa_1} > \kappa_2 \), there is a strongly almost disjoint family of power \( \kappa_2 \).

**Theorem 6.9.** \( P_5 \) (\( P_5 \)) implies every (strongly) almost disjoint family of \( \geq \kappa_2 \) stationary sets can be extended to a maximal (strongly) almost disjoint family of power \( 2^{\kappa_1} \).

**Proof.** Given a (strongly) almost disjoint family of size \( \kappa_1 < \kappa < 2^{\kappa_1} \), the diagonal union of any subfamily of cardinality \( \kappa_1 \) has stationary complement. Working with the complements, the results follows. \( \square \)

It is interesting to note [4] that if \( 2^{\kappa_1} \) is blown up with Cohen subsets of \( \omega_1 \), \( \diamond \) holds but there is a maximal almost disjoint family of \( \kappa_2 \) stationary subsets of \( \omega_1 \).

The "\( \kappa_2 \)" in Theorem 6.9 cannot be replaced by "\( \kappa_1 \)". J. Baumgartner and A. Taylor pointed out to me that if \( \{ A_\alpha : 0 < \alpha < \omega_1 \} \) is a partition of \( \omega_1 \) into disjoint stationary sets, then letting \( B_0 = \bigcup \{ A_\alpha \cap (\alpha + 1) : 0 < \alpha < \omega_1 \} \) and \( B_\alpha = A_\alpha - (\alpha + 1), 0 < \alpha < \omega_1 \), \( \{ B_\alpha : \alpha < \omega_1 \} \) is also such a partition, having the additional property that any stationary set has stationary intersection with some \( B_\alpha \). For if \( X \cap B_\alpha \) is nonstationary for all \( \alpha \), then \( X = \bigcup \{ X \cap B_\alpha : \alpha < \omega_1 \} = \bigcap \{ X \cap B_\alpha : \alpha < \omega_1 \} \), which is nonstationary.

In the next section, we shall show that Baumgartner's model for BACH plus \( 2^{\kappa_1} > \kappa_2 \) is a model for \( P_5 \), so by the aforementioned [26] result, there are indeed maximal strongly almost disjoint families of cardinality \( 2^{\kappa_1} \) in that model. This also follows from the fact that \( \diamond \) holds in that model.
A referee points out that

**Theorem 6.10.** \( P_1 \) implies \( P_5 \) is equivalent to \( P_S \).

**Proof.** Let \( \{S_\alpha\}_{\alpha < \kappa}, \kappa < 2^\aleph_1 \), and \( S \) be stationary subsets of \( \omega_1 \) with \( S - S_\alpha \) nonstationary for all \( \alpha \). By \( P_1 \) there is an uncountable \( A \) such that \( A \cap (S - S_\alpha) \) is countable for all \( \alpha \). Let \( C_\alpha = \omega_1 - (S - S_\alpha) \). \( C_\alpha \) is closed unbounded. \( \bar{A} \cap C_\alpha \). The first term is countable and the second is a subset of \( C_\alpha \), so \( \bar{A} \cap S \) is thus the desired stationary set. \( \square \)

Matet [40] noted that

**Theorem 6.11.** \( P_S \) plus \( \diamond \) implies \( \diamond_{\kappa}(\lambda) \) for all \( \lambda \geq \aleph_1 \) such that \( \lambda^{< \lambda} < 2^{\aleph_1} \).

7. **Baumgartner’s model**

To get the strong results concerning stationary sets discussed in the previous section, as well as the normality assertion, Theorem 5.1, we seem to need to work in a particular model of BACH, rather than with the axiom itself. Fortunately we need to know very little about the model—call it \( \mathcal{B} \)—just that it’s constructed via a well-behaved iteration sequence from a model of CH. Say e.g. the sequence is countably closed and preserves cardinals at each stage and for initial and final segments. Assume the sequence has the usual nice splitting properties for iterated forcing. Assume it has length \( (2^{\aleph_1})^\mathcal{B} \) and that every countably closed \( \mathfrak{U}_1 \)-linked well-met partial order of cardinality \( < (2^{\aleph_1})^\mathcal{B} \) appears cofinally often. For special purposes one might want to assume the ground model satisfies GCH or \( V = L \), or specify the length of the iteration sequence. For the details of the construction, see [3].

With these preliminaries, let us prove that \( P_S \) holds in \( \mathcal{B} \). The following well-known result is needed.

**Lemma 7.1.** Let \( \mathcal{M}[G] \) be obtained via a countably closed notion of forcing. Let \( S \subseteq \mathcal{M} \) be a subset of \( \omega_1 \). Then \( S \) is stationary in \( \mathcal{M} \) iff it is stationary in \( \mathcal{M}[G] \).

To prove the consistency of \( P_S \), let \( \{A_\alpha\}_{\alpha < \kappa}, \kappa < 2^{\aleph_1} \) be stationary sets in \( \mathcal{B} \) with every diagonal intersection stationary. There is an intermediate model \( \mathcal{B}_0 \) in which the family of \( A_\alpha \)'s appears. By the lemma, all diagonal intersections of \( A_\alpha \)'s which appear in \( \mathcal{B}_0 \) are stationary there. The partial order \( \mathcal{P} \) that one would use for establishing \( P_1 \) for the family of \( A_\alpha \)'s is definable from the family and so appears in \( \mathcal{B}_0 \). This particular order (with different dense sets) will yield \( P_S \) for the \( A_\alpha \)'s. We may as well assume that \( \mathcal{B}_0 \) is our ground model and that the “next” generic set \( G \) is \( \mathcal{P} \)-generic over \( \mathcal{B}_0 \). As before,

\[
D_\alpha = \{ \langle h, H \rangle : \alpha \in H \}
\]
is dense. Let 
\[ S = \bigcup \{ h : \text{for some } H, \langle h, H \rangle \in G \}. \]
Claim for each closed unbounded \( C \in \mathcal{B}_{\omega}[G] \),
\[ E_C = \{ \langle h, H \rangle : \langle h, H \rangle \models \delta \cap C \neq 0 \} \]
is dense. To see this, as usual get a continuous strictly increasing sequence of countable ordinals \( \{ \gamma_\beta \}_{\beta < \omega_1} \) and a descending sequence of conditions \( \langle h_\beta, H_\beta \rangle \) below an \( \langle h, H \rangle \) forcing \( C \) closed unbounded, such that \( \langle h_\beta, H_\beta \rangle \models \gamma_\beta \in C \). Furthermore, for limit \( \sigma \) take \( \langle h_\sigma, H_\sigma \rangle = \langle \bigcup_{\beta < \sigma} h_\beta, \bigcup_{\beta < \sigma} H_\beta \rangle \). For each \( \beta < \omega_1 \), let \( B_\beta = \bigcap \{ A_\alpha : \alpha \in H_\beta \} \). Since any diagonal intersection of the \( A_\alpha \) is stationary, it follows that the \( B_\beta \) and indeed all diagonal intersections of them—are stationary. \( \beta : \beta = \gamma_\beta \) is closed unbounded, so there is a limit \( \sigma = \gamma_\sigma \), \( \sigma \in \bigcap \{ B_\beta : \beta < \sigma \} \). \( \sigma \) is then in \( \bigcap \{ A_\alpha : \eta \in H_\alpha \} \) since \( H_\sigma = \bigcup \{ H_\beta : \beta < \sigma \} \). It follows that \( \langle h_\sigma \cup \{ \sigma \}, H_\sigma \rangle \subseteq \langle h_\sigma, H_\sigma \rangle \). But then \( \langle h_\sigma \cup \{ \sigma \}, H_\sigma \rangle \in E_C \). This proves \( S \) is stationary.
The usual argument proves \( S \) is almost included in each \( A_\alpha \) By the lemma and absoluteness, these statements hold in \( \mathcal{B} \), completing the proof.

After seeing the proof for \( P_\delta \) and other propositions that required consideration of the model rather than the axiom, Baumgartner suggested that it would be desirable to formulate a stronger axiom holding in the model which would suffice to imply these results. He eventually did so and obtained almost all our results. The difficult task was the conceptualization of the axiom, rather than the proofs, which are mainly cleaned-up versions of earlier forcing proofs. Our work in the previous version of this section has largely been superseded by Baumgartner’s ideas and so we are grateful for his consent to include his work here.

As Baumgartner noted, his strengthening can be applied to any of the various generalized Martin’s axioms. For the sake of definiteness, we stick with BA and define:

**Strong BA.** Suppose \( \mathcal{P} \) is a countably compact \( \kappa \)-linked partial order. Let \( \kappa < 2^{\kappa_1} \) and suppose that for each \( s \in \bigcup \{ \kappa : \alpha < \omega_1 \} \) a set \( P_s \subseteq P \) is specified such that for each \( f \in \kappa_1^\omega \kappa \) and for each decreasing sequence \( \langle p_\beta : \alpha < \omega_1 \rangle, \{ \alpha < \omega_1 : (\exists q \in P_{f_\beta}) \beta < \alpha \} \subseteq q < p_\beta \rangle \) is stationary. Then there is a directed \( G \subseteq P \) such that for each \( f \in \omega_1^\omega \kappa, \{ \alpha : G \cap P_{f_\alpha} \neq 0 \} \) is stationary.

Following Baumgartner, we denote Strong BA plus CH by “\( \Sigma \)” and note that \( \Sigma \) implies for each \( \kappa < 2^{\kappa_1} \) that \( \kappa^{\omega_1} < 2^{\kappa_1} \). The following proof is due to a referee and replaces an incorrect proof in an earlier version. Consider the usual order \( \mathcal{P} \) for adding a Cohen subset of \( \omega_1 \) with countable conditions. For each \( g : \omega_1 \rightarrow 2, D_g = \{ p \in P : g \text{ does not extend } p \} \) is dense. Furthermore, \( g \upharpoonright h \) implies \( D_g \neq D_h \). There is no directed \( G \) which meets all \( D_g \) and the \( E_\alpha = \{ p \in P : \alpha \in \text{dom } p \} \), \( \alpha < \omega_1 \). Thus there is a collection of \( 2^{\kappa_1} \) dense sets which cannot be simultaneously met. Yet we can meet \( \kappa^{\omega_1} \) dense sets simultaneously by simply taking distinct dense \( P_s \) for each \( s \in \omega^\kappa \) and letting \( P_s = P_{s+1} \) for other \( s \)’s.
Given the consistency proof for BACH and the one for $P_S$, it is not difficult to prove

**Theorem 7.2.** There is a model of set theory in which $\Sigma$ holds and in which $2^{\aleph_1} = \kappa$, where $\kappa$ is a regular cardinal such that for every $\lambda < \kappa$, $\lambda^{\omega_0} < \kappa$.

See [3]. It is easy to see that

**Theorem 7.3.** $\Sigma$ implies BACH.

**Proof.** Let $\{D_\alpha\}_{\alpha < \kappa}$, $\kappa < 2^{\aleph_1}$, be dense subsets of an $\aleph_1$-linked countably compact $\mathcal{P}$. For $s \in \bigcup\{\kappa: \alpha < \omega_1\}$, let $P_s = D_{s(0)}$. \(\square\)

We leave as an exercise for the reader (once she has seen the other $\Sigma$ proofs) to prove

**Theorem 7.4.** $\Sigma$ implies $P_S$.

Instead we shall prove

**Theorem 7.5.** $\Sigma$ implies the product of $< 2^{\aleph_1}$ spaces, each of weight $< 2^{\aleph_1}$ and each having a universal sequence, also has one.

The intuitive idea of a consistency proof for the conclusion is to combine the forcing proof for $P_S$ with the Martin’s axiom technique for amalgamating sequences from factors into a sequence in a product, as done in [58]. My original “proof” did not correctly realize this correct idea, with the result that I missed the necessity of the weight restriction. Baumgartner produced a counterexample (see below) and a more elegant (and correct!) proof from $\Sigma$, which we present with his permission.

Let $X_\alpha$, $\alpha < \kappa$, be spaces of weight $< \aleph_1$, each with a universal sequence. By CH, $X = \Pi X_\alpha: \alpha < \kappa$ has a $G_\delta$-dense subset $D$ of cardinality $\aleph_1$. We claim some enumeration of $D$ is universal. Given $f$ mapping $\omega_1$ onto $D$, for each basic open $U$ in $X$, let $S(f, U) = \{\alpha: f(\alpha) \in U\}$. It suffices to prove the following

**Lemma 7.6.** $\Sigma$ implies there is an $f$ mapping $\omega_1$ onto $D$ such that for every $x \in X$, $\{S(f, U): x \in U\}$ is a family of stationary sets with each diagonal intersection stationary.

For assume the lemma. Let $x \in X$. $x$ is in $< 2^{\aleph_1}$ basic open sets since $2^{\aleph_1}$ is regular and the factors have weight $< 2^{\aleph_1}$. Thus by $P_S$ there is a stationary $S$ such that $S - S(f, U)$ is countable for each basic open $U$ containing $x$. Consider the net $f \upharpoonright S$. Any $U$ about $x$ contains a tail of $f \upharpoonright S$, so we are done.

To prove the lemma, let $\{U_\xi: \xi < \kappa\}$, $\kappa < 2^{\aleph_1}$, enumerate the basic open subsets of $X$. Let $\mathcal{P}$ be the set of all functions from countable ordinals into $D$, ordered by
extension. If \( s \in \kappa \) and \( \bigcap \{ U_{\beta}: \beta < \alpha \} \neq \emptyset \), let \( P_s = \{ p \in P: p(\alpha) \in \bigcap \{ U_{\beta}: \beta < \alpha \} \} \). If \( \bigcap \{ U_{\beta}: \beta < \alpha \} = \emptyset \), let \( P_s = P \). Let \( \langle p_{\alpha}: \alpha < \omega_1 \rangle \) be a decreasing sequence of conditions and let \( f \in \omega^{\omega} \). If for some \( \alpha \), \( \bigcap \{ U_{\beta}: \beta < \alpha \} = \emptyset \), then \( \langle \alpha < \omega_1: \exists q \in P_{f(\alpha)}(\forall \beta < \alpha)(q \leq p_{\beta}) \rangle \) is countable, so assume otherwise. Let \( C = \{ \alpha < \omega_1: (\forall \beta < \alpha)(\text{dom } p_{\beta} \subseteq \alpha) \} \). \( C \) is closed unbounded and if \( \alpha \in C \) then certainly there is a \( q \in P_{f(\alpha)} \) such that \( q \leq \text{each } p_{\beta}, \beta < \alpha \). So in either case the required set is stationary and hence \( \Sigma \) can be applied to give us a \( G \). Claim \( \cup G \) is the function called for in the lemma. Let \( x \in X \) and let \( \{ V_{\beta}: \beta < \omega_1 \} \) be the set of basic open neighbourhoods of \( x \). Let \( S_\beta = S(\bigcup \{ V_{\beta} \} \) and \( S = \{ \alpha: (\forall \beta < \alpha)[\alpha \in S_{\beta}] \}. \) If we can prove \( S \) is stationary, the lemma is established. Take \( h \in \omega^{\omega} \) such that for all \( \alpha, U_{h(\alpha)} = V_\alpha \). Then \( \langle \alpha: G \cap p_{h(\alpha)} \neq \emptyset \rangle \supseteq S \) and so by \( \Sigma \), \( S \) is stationary.

Baumgartner notes that \( \Sigma \) suffices to obtain various other propositions holding in the iteration model, for example that normal spaces of character \( < 2^{\omega_1} \) are \( \kappa \)-collectionwise Hausdorff. This is done by a \( \Sigma \) modification of a forcing version of Fleissner’s method in [15]. \( \Sigma \) also implies Fleissner’s \( \diamond \) for stationary systems, in fact \( \diamond_{\omega_1}(\kappa) \) versions. This collectionwise Hausdorff result does not seem to follow from \( \diamond_{\omega_1}(\kappa) \) for stationary systems since functions from \( \omega_0 \) to \( \kappa \) rather than vice versa are at stake. Plausible attempts to get \( \diamond \)-like principles from \( \omega_1 \) to \( \kappa \) are refutable: suppose e.g. that \( \{ f_\alpha: \alpha \in \omega_1, f_\alpha \in {}^\omega \omega \} \) trapped all \( f \in {}^\omega \omega \). Let \( \beta = \sup \{ \text{range } f_\alpha: \alpha \in \omega_1 \} \). Define \( f \in {}^{\omega_1} \omega \) by \( f(\gamma) = \beta + \gamma \). Then no \( f_\alpha \) traps \( f \). We shall obtain stronger versions of the collectionwise Hausdorff result at the end of this section by extending the methods of [55].

The following example of Baumgartner was produced in response to the “proof” in a previous version of this paper of Theorem 6.4 without the restriction on the factor spaces. It is included with his kind permission.

**Theorem 7.7.** Suppose the nonstationary ideal on \( \omega_1 \) is generated by \( \kappa \) sets and that there is a (strongly) almost disjoint family of \( \kappa \) stationary sets. Then there is a (regular) space \( X \) of character \( \kappa \), having cardinality and character \( \kappa \), such that \( X \) has a universal sequence but \( X^2 \) does not.

In particular, in the model we have been considering there is such a regular space of weight and cardinality \( 2^{\omega_1} \).

**Proof.** Given any stationary costationary \( T \), either \( \kappa \) members of the almost disjoint family have stationary intersection with \( T \), or \( \kappa \) many do with \( \omega_1 - T \). Thus without loss of generality we assume the given family \( \mathcal{F} \) is composed of subsets of such a \( T \). By subtracting say the \( \alpha \)th nonstationary generator from the \( \alpha \)th and \( (\alpha + 1) \)th member of \( \mathcal{F} \), we see that we may also assume that each nonstationary set is disjoint from at least two members of \( \mathcal{F} \). The points of our space \( X \) will be the members of \( \mathcal{F} \) (labeled as \( P_{\eta} \) to avoid confusion) and the members of \( T \). Points in \( T \) are taken to be isolated, while a neighbourhood base for \( P_{\eta} \) is \( \{ (P_{\eta}) \cup (F - \alpha): \alpha < \omega_1 \} \). It is obvious that \( w(X) = |X| = \kappa \) and that \( \chi(X) = \kappa \), and it is also clear that if the members of \( \mathcal{F} \) are strongly almost disjoint, then \( X \) is Hausdorff and zero-dimensional, hence regular. Take \( \langle x_\alpha: \alpha < \omega_1 \rangle \) such that for
each $\alpha \in T$, $x_\alpha = \alpha$ and $\{\beta: x_\beta = \alpha\}$ is stationary. The latter can be arranged by partitioning $\omega_1 - T$ into $\mathfrak{r}$, disjoint stationary sets. Thus $\langle x_\alpha: \alpha < \omega_1 \rangle$ is a universal sequence for $X$. Suppose $\{\langle y_\alpha, z_\alpha \rangle: \alpha < \omega_1 \}$ were a universal sequence for $X^2$. By the Pressing Down Lemma, $N = \{y_\alpha \in T: y_\alpha > \alpha\} \cup \{z_\alpha \in T: z_\alpha > \alpha\}$ is nonstationary. Choose distinct $F, F' \in \mathcal{F}$ such that $F \cap N = F' \cap N = \emptyset$. Suppose there were a stationary $S$ such that the net $\langle \langle y_\alpha, z_\alpha \rangle: \alpha \in S \rangle$ converged to $\langle p_F, p_{F'} \rangle$. $\langle y_\alpha: \alpha \in S \rangle$ converges to $p_F$, so since $F \cap N = F' \cap N = \emptyset$, $y_\alpha < \alpha$ for all but countably many $\alpha \in S$. If $\{\alpha: y_\alpha < \alpha\}$ were stationary, then pressing down again yields a $\beta$ such that for uncountably many $\alpha$, $y_\alpha = \beta$. But then $\langle y_\alpha: \alpha \in S \rangle$ couldn't converge to $p_F$. So $y_\alpha = \alpha$ for all but nonstationarily many $\alpha \in S$. But $F$ includes a tail of the sequence $\langle y_\alpha: \alpha \in S \rangle$, so $S - F$ is nonstationary. The same argument proves $S - F'$ is nonstationary. But that contradicts $F \cap F'$ nonstationary. $\square$

Not every combinatorial principle of $L$ holds for $\omega_1$ in Baumgartner's model. In particular, $\diamondsuit^*$ does not. We establish this by an extension of Devlin's argument in [13] that the adjunction via countable conditions of $\mathfrak{r}_2$ subsets of $\omega_1$ destroys $\diamondsuit^*$.

**Definition 7.8.** $\diamondsuit^*$ is the assertion that there is a sequence $\{\mathcal{R}_\alpha: \alpha \in \omega_1\}$ such that $\mathcal{R}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{R}_\alpha| \leq \mathfrak{r}_\alpha$, and for each $X \subseteq \omega_1$, there is a closed unbounded $C \subseteq \omega_1$ such that if $\alpha \in C$ then $X \cap \alpha \in \mathcal{R}_\alpha$.

**Theorem 7.9.** Baumgartner's model for BACH is not a model for $\diamondsuit^*$ if the iteration sequence has length $> \omega_2$.

**Proof.** Arguing as in [13], assume without loss of generality that the supposed $\diamondsuit^*$ sequence lies in the ground model. We may also assume that the first generic extension adds a generic subset $G$ of $\omega_1$ via countable partial functions into 2. By Devlin's proof, in $M[G]$ the $\diamondsuit^*$ sequence fails to trap $X_G = \{\alpha \in \omega_1: (\exists p \in G)[p(\alpha) = 1]\}$. Getting the full model of BACH from $M[G]$ proceeds via countably closed forcing; it suffices therefore to show that countably closed forcing—while it may create new angels—can never restore a fallen one to grace. $\square$

I have replaced my original proof by a referee's short one.

**Lemma 7.10.** Suppose $M'$ is obtained from $M$ by countably closed forcing. If $\{\mathcal{R}_\alpha: \alpha \in \omega_1\}$ is not a $\diamondsuit^*$-sequence in $M$, then it is not one in $M'$.

**Proof.** Suppose $X \subseteq M$ is a subset of $\omega_1$ such that $\{\alpha: X \cap \alpha \notin \mathcal{R}_\alpha\}$ is stationary. Then by Lemma 7.1 it remains stationary. $\square$

It is not known whether $\Sigma$ plus $2^{\mathfrak{r}_1} > \mathfrak{r}_2$ implies $\diamondsuit^*$ fails. Baumgartner notes that $\Sigma$ alone doesn't, since $\Sigma$ holds in $L$.

The argument for obtaining $P_S$ was a prototypical example of countably closed forcing applied to $\omega_1$, wherein a key role is played by a descending sequence of $\mathfrak{r}_1$. 
conditions. BACH engenders the possibility of extending such arguments to prove results about other cardinals less than $2^{\aleph_1}$ by replacing the descending sequence of $\kappa_1$ conditions by a compatible collection of less than $2^{\aleph_1}$ conditions, forcing over a model of BACH plus $2^{\aleph_1} > \kappa_2$. We shall prove two results via this technique: the first is a technical topological one which requires two interesting set-theoretic lemmas; the second is purely set-theoretic, involving the preservation of $\Sigma^1_2$ formulas. I expect there to be additional applications.

**Theorem 7.11.** There is a model for CH plus generalized Martin's axiom for $\kappa_1$-centred countably compact partial orders plus $2^{\aleph_1} > \kappa_2$ in which every normal space of character $< 2^{\aleph_1}$ is collectionwise normal with respect to discrete collections of $\kappa_1$ Lindelöf sets, each of cardinality $< 2^{\aleph_1}$.

This was the first consistency result not using large cardinals which nontrivially enables the separation of $\kappa_1$ closed sets of cardinality greater than $\kappa_1$. (Baumgartner points out that the Lindelöf case can be reduced to that for points by collapsing the Lindelöf sets to points, provided that $\kappa < 2^{\aleph_1}$ implies $\kappa^{\kappa_0} < 2^{\aleph_1}$.) However we need not make that assumption, and in any event more complicated versions of Theorem 7.11 can be demonstrated via the same technique. Except in trivial cases, these results are not known to follow from Fleissner's stationary systems methods, even in the "$\kappa_1 - \kappa_1$" case. The proof proceeds via a modification of the countably closed forcing techniques used in [55] (or [60]), to which we shall refer. We would like to have stated the result for Baumgartner's axiom rather than for the $\kappa_1$-centred version but as we shall see, the proof of the topological result needs that whatever the version of generalized Martin's axiom we're dealing with, it's preserved by the adjunction of a single Cohen subset of $\omega_1$. A claim to that effect for BACH in [41] has been withdrawn; but Baumgartner has produced a proof for the weaker axiom which we present with his permission.

**Lemma 7.12.** Assume CH plus $(\forall \kappa < 2^{\aleph_1})(\kappa^{\kappa_0} < 2^{\aleph_1})$. Then generalized Martin's axiom for $\kappa_1$-centred countably compact partial orders is preserved by the adjunction of a Cohen subset of $\omega_1$.

Roitman [41] proved that the adjunction of a Cohen subset of $\omega$ preserves Martin's axiom for $\sigma$-linked partial orders; the difficulty in generalizing the proof lies in the well-met (or countably compact) condition. There is no reason to believe her auxiliary partial orders preserve these. However the following straightforward argument establishes Lemma 7.12.

We work in $\mathcal{B}$, a model for generalized Martin's axiom for $\kappa_1$-centred, countably compact partial orders, $(\forall \kappa < 2^{\aleph_1})(\kappa^{\kappa_0} < 2^{\aleph_1})$, and CH. We force with $\text{Fn}(\omega_1, 2, \omega_1)$, i.e., countable partial functions from $\omega_1$ into 2. This preserves the cardinality assumptions, so by Theorem 1.4 it suffices to establish $P_1$.

Assume $1 \vdash (\dot{h} : \kappa \to \mathcal{P}(\omega_1))$ and countable intersections of members of range $\dot{h}$ are uncountable. We want to find a name $\tau$ which is forced to be an uncountable
subset of $\omega_1$ almost included in each member of range $h$. Let

$$P = \{ \langle \alpha, s, X \rangle : \alpha \subseteq \omega_1, s \text{ is a partial function from } \alpha \text{ into } \text{Fn}(\omega_1, 2, \omega_1) \text{ and } X \in [\kappa]^\aleph_0 \},$$

$$\langle \alpha, s, X \rangle \leq \langle \beta, t, y \rangle \text{ if } \alpha \leq \beta, s|\beta = t, X \supseteq Y, \text{ and for each }$$

$$\gamma \in \text{dom } s - \beta, s(\gamma) \not\in \gamma \in \cap \{ h(\sigma) : \sigma \in \bar{Y} \}.$$ 

$\mathcal{P} = \langle P, \leq \rangle$ is clearly countably closed and well-met. Any finite set of conditions with the same first two coordinates has a lower bound, so by CH, $\mathcal{P}$ is $\kappa_1$-centred. For $f \in \text{Fn}(\omega_1, 2, \omega_1)$ and $\gamma \in \omega_1$, let

$$D(f, \gamma) = \{ \langle \alpha, s, X \rangle \in P : \text{there is } \sigma > \gamma, \sigma \in \text{dom } s, s(\sigma) > f \}.$$ 

For $\beta \in \kappa$, let

$$D_\beta = \{ \langle \alpha, s, X \rangle \in P : \beta \subseteq X \}.$$ 

The $D(f, \gamma)$ and the $D_\beta$ are clearly dense and there are $< 2^{\kappa_1}$ of them so there is a filter $G \subseteq P$ generic for all of them. Define an $\text{Fn}(\omega_1, 2, \omega_1)$-name

$$\tau = \{ \langle \beta, f \rangle : (\exists \langle \alpha, s, X \rangle \in G) \{ \beta \subseteq \text{dom } s \text{ and } f \subseteq s(\beta) \} \}.$$ 

By the density of the $D(f, \gamma)$, the interpretation of $\tau$ is forced to be uncountable; by the density of the $D_\beta$, it is forced to be almost included in each member of the range of $h$.

Using Lemma 7.12 we can now proceed to construct the model for Theorem 7.11. In the usual BACH iteration to get say $2^{\kappa_1} = \kappa_2$, GCH holds at initial stages; we shall want $2^{\kappa_1} > \kappa_2$ plus generalized Martin's axiom for $\kappa_1$-centred, countably compact partial orders to hold at “almost all” stages. This can be arranged by going out further. We shall also want to adjoin a Cohen subset of $\omega_1$ via countable conditions “often” at stages when the axiom holds. To do this formally, iterate with countable support $\kappa_1$-centred countably compact partial orders. Use an elementary submodel argument to get $2^{\kappa_1} > \kappa_2$ plus generalized Martin’s axiom for such orders holding at closed unboundedly many stages of the iteration, having arranged the bookkeeping so that the Cohen order is taken care of stationarily many times. At stationarily many places then, we are adjoining a Cohen subset to a model of generalized Martin’s axiom for $\kappa_1$-centred countably compact partial orders and so preserve that axiom. We are going to want to apply that axiom to the partial order that gives the remainder of the iteration. A technical difficulty arises however: in Baumgartner’s proof [3] of the consistency of BACH plus $2^{\kappa_1} > \kappa_2$, he needs and proves only that the countable support iteration of $\kappa_1$-linked countably compact partial orders is countably closed and satisfies the $\kappa_2$-chain condition, so there is no reason to believe our axiom applies to such an iteration. Now his proof that the iteration is countably closed easily generalizes to show it’s countably compact, but a new argument is needed for the chain condition. We shall show

**Lemma 7.13.** Assuming CH, the countable support iteration of $\leq 2^{\kappa_1} \kappa_1$-centred countably compact partial orders is $\kappa_1$-centred.
2^{\omega_1}$ here is of course taken in the sense of the ground model. Thus if we were interested in getting a model for Theorem 7.11 plus $2^{\omega_1} = \kappa$, we could first blow up $2^{\omega_1}$ to $\kappa$, say by adding Cohen subsets of $\omega_1$, so that the iteration would then not be too long to apply the lemma.

In order to prove Lemma 7.13 (which was also known to Baumgartner and presumably anyone else who thought about it), there are two cases to consider, depending on whether the length of the iteration is a successor or a limit ordinal. For the successor case we need only prove

**Lemma 7.14.** If $\mathcal{P}$ is $\mathcal{K}_\alpha$-centred and $\mathcal{P} \models \varnothing$ is $\mathcal{K}_\alpha$-centred, then $\mathcal{P} \ast \varnothing$ is $\mathcal{K}_\alpha$-centred.

The proof is straightforward. Without loss of generality we assume $\varnothing = \langle Q, <_\omega \rangle, Q \in V$. By the maximum principle, let $f'$ be a term such that

$$\mathcal{P} \models f : \check{Q} \to \omega_1 \land (\forall \alpha \in \omega_1)(\forall F \in [\check{Q}]^{<\omega})$$

$$((\forall x \in F)[f'(x) = \alpha \to (\exists q \in \check{Q})(\forall x \in F)[q < x]]).$$

Let $P = \bigcup_{\alpha < \omega_1} P_\alpha$, where $P_\alpha$ is centred in $\mathcal{P}$. Define $D_{\alpha\beta} = \langle p \times q, p \in \mathcal{P}, \alpha, \beta \in \omega_1 \rangle$ is dense in $\mathcal{P} \ast \varnothing$ so it suffices to show $D_{\alpha\beta}$ is centred. But if $\langle p_j, q_j \rangle \in D_{\alpha\beta}, j < k$, some $k \in \omega$, and each $p_j \models f'(q_j) = \beta$, take $r \leq \alpha$, then $r \models (\exists s \in \check{Q})(\forall j \leq k)[s < q_j]$ so $(\forall t < r)(\exists u \in r)(\exists s \in Q)(\forall j < k)[u \models s < q_j]$. Then $\langle u, s \rangle < \langle p_j, q_j \rangle$ for all $j < k$.

For the limit case we don't have to even mention forcing. All we shall deal with are the (real) partial orders $\{\mathcal{P}_\xi \}_{\xi < \alpha}$, where $\mathcal{P}_\xi$ is the partial order that gives the first $\xi$ stages of the iteration. We assume as induction hypothesis for $\alpha$ limit that the $\{\mathcal{P}_\xi \}_{\xi < \alpha}$ are $\mathcal{K}_\alpha$-centred. As noted earlier, the $\{\mathcal{P}_\xi \}_{\xi < \alpha}$ are countably compact. The proof essentially boils down to the following topological

**Sublemma 7.15.** Suppose $\{X_\alpha\}_{\alpha < 2^\omega}$ are topological spaces of density $\leq \kappa$. Then the density of the topology on $\prod(X_\alpha; \alpha < 2^\omega)$ generated countable "boxes" is $\leq \kappa^{\omega_1}$.

For a proof, see e.g. [12,3,18].

Countable compactness is necessary in Lemma 7.13 since if Lemma 7.13 held without it, one could prove the consistency of CH plus $2^{\omega_1} > \mathcal{K}_\tau$ plus generalized Martin axiom for $\mathcal{K}_\tau$-centred countably closed partial orders, but Shelah proved that conjunction is false—see Section 9. The restriction to $< 2^{\omega_1}$ factors is also necessary by standard topological arguments.

To prove Lemma 7.13, it is useful to define auxiliary partial orders that are not only $\mathcal{K}_\tau$-centred but actually have a dense set of power $\mathcal{K}_\tau$. Suppose $P = \bigcup_{\alpha < \omega_1} P_\alpha$, where each $P_\alpha$ is centred in $P$. Without loss of generality assume each $P_\alpha$ is maximal centred. Define a partial order $\leq_{p*}$ on $P^* = P \cup \{(P \times \omega_1)\}$ where $P$ is used
merely as a labeling device to distinguish this copy of \( \omega_1 \) from \( P \) and from other \( P^* \) by
\[
\leq \mu \restriction (P \times P) \leq \mu,
\]
\[
\leq \mu \restriction (((P) \times \omega_1) \times P) \subseteq \{\{\langle P, \alpha \rangle, p \subseteq P_\alpha\}\},
\]
\[
\leq \mu \restriction (P \times (\{P\} \times \omega_1)) = \emptyset,
\]
\[
\leq \mu \restriction (((P) \times \omega_1) \times (\{P\} \times \omega_1)) = \{\{\langle P, \alpha \rangle, \langle P, \alpha \rangle : \alpha < \omega_1\}\}.
\]
Observe that if a finite subset of \( P \) has a lower bound in \( P^* \), then it does in \( P \) as well.

We shall use "\( II \)" to symbolize the countable box topology on a Cartesian product, rather than the Tychonoff topology. By Sublemma 7.15, CH, and the induction hypothesis, let \( \{q_\beta\}_{\beta < \omega_1} \) be dense in \( \prod\{(P_\xi \times \omega_1) : \xi < \alpha\} \), each factor given the discrete topology. We shall use "\( \pi_\mu \)" for the \( \mu \)-th projection map. Let
\[
Q_\beta = \{p \in P_\alpha : (\forall \xi < \alpha) [\pi_\xi(q_\beta) \leq \mu p \mid \xi]\}.
\]
Then \( P_\alpha = \bigcup_{\beta < \omega_1} Q_\beta \) since \( \alpha \) is a limit and supports are countable.

Let \( p_j \in Q_\beta, j \leq k, \) some \( k \in \omega \). Let \( \Sigma = \bigcup \{\text{support } p_j : j < k\} \). Let \( i_\xi \alpha \) be the natural injection of \( P_\xi \) into \( P_\alpha \). Let \( T = \{i_\xi \alpha(p_j \mid \sigma) : j < k, \sigma < \Sigma\} \). Any finite subset \( S \) of \( T \) "lives" in some \( P_\xi \), \( \xi < \alpha \), and by hypothesis its restrictions have a lower bound \( s \) there, since \( \pi_\xi(q_\beta) \) bounds them in \( P_\xi^* \). Then \( i_\xi \alpha(s) \) bounds \( S \) in \( P_\alpha^* \). But then by the countable compactness of \( P_\alpha \), there is a lower bound \( t \) for \( T \). Then \( t \) is below each \( p_j \).

**Remark.** Lemma 7.13 is also true with "\( \kappa_1 \)-linked" replacing "\( \kappa_1 \)-centred". The same method as in the proof of Lemma 7.13 also establishes the following results which have become folklore.

**Theorem 7.16.** A finite support iteration of \( \leq 2^{\kappa_0} \sigma \)-centred (\( \sigma \)-linked) partial orders is \( \sigma \)-centred (\( \sigma \)-linked). Hence Martin's axiom for \( \sigma \)-centred (\( \sigma \)-linked) partial orders may be established by forcing with a \( \sigma \)-centred (\( \sigma \)-linked) partial order.

Finally returning to Theorem 7.11, in outline the proof proceeds by first assuming the collection is unseparated, and then arguing that all the relevant objects appear at some initial stage when a Cohen subset is adjoined. The Cohen subset then unnormalizes the collection. One then proves that it stays that way. The use of a generalized Martin's axiom is to replace the usual descending sequence determining the values of a function from \( \omega_1 \), with a filter doing the same for a function from \( \kappa \), where \( \kappa < 2^{\kappa_1} \). For the details, we assume the reader has a copy of [55] at hand. In 2.1.8 of [55], replace "\( |\mathcal{Y}^*| \leq \kappa \)" by "\( |\mathcal{Y}^*| \leq \kappa \)". In 2.1.10 and its proof, replace \( \omega_1 \) by an ordinal. In the proof of 1.8, replace \( \omega_1 \) by \( |\mathcal{Y}^*| \). Instead of constructing a descending sequence of conditions \( \{p_\alpha \alpha_\beta < \omega_1\} \), use generalized Martin's axiom plus \( 2^{\kappa_1} > |\mathcal{Y}^*| \) to get a compatible collection of conditions, one from each \( D_{\omega_1}^{\mathcal{Y}^*} \). (\( \mathcal{Y}^* \) appears at a stage when the generalized
Martin’s axiom holds, since it holds cofinally often.) By compatibility, the function $h$ can be defined and the altered 2.1.8 proved. The other nonnotational change from the proof in [55] occurs in the statement and proof of 2.1.15. The proof of 2.1.15 is modified in the same way as was 2.1.8, replacing a descending sequence by a compatible collection. It is crucial here that the adjunction of a Cohen subset in this context preserves the version of the generalized Martin’s axiom in question. To finish the proof, one argues as in 2.0.12(a), (f).

**Remark.** The fact that the tail of the iteration does not normalize an unnormalized collection is actually a special case of the corollary to a more general result:

**Theorem 7.17.** Suppose $\mathcal{M}[G]$ is obtained via $\kappa$-centred countably compact forcing over a model $\mathcal{M}$ of generalized Martin’s axiom for $\kappa$-centred countably compact partial orders plus $2^\kappa > \lambda$. Then any $\Sigma_1^\lambda(\lambda)$ sentence holding in $\mathcal{M}[G]$ holds in $\mathcal{M}$.

(We allow subsets of $\lambda$ which lie in $\mathcal{M}$ as parameters.)

**Corollary 7.18.** Suppose $\mathcal{M}$, $\mathcal{M}[G]$ are as in the theorem. Then any $\Sigma_1^\lambda(\lambda)$ sentence holding in $\mathcal{M}$ holds in $\mathcal{M}[G]$. For countably closed forcing over an arbitrary $\mathcal{M}$, the result for $\omega_1$ is due to Silver. The corollary is immediate, since if $\Sigma_1^1$ goes down, $\Pi_1^1$ goes up; so then does $\Sigma_1^1$. We leave to the reader of [55] the coding necessary to verify the remark.

**Proof of Theorem 7.17.** Suppose $\Phi$ is $\Sigma_1^\lambda(\lambda)$. We may assume $\Phi$ is of the form $(\exists S)(\Phi(S))$, where $\Phi$ is first-order over $\lambda$ and $S$ ranges over subsets of $\lambda$. Suppose there is a $p \in G$ such that $p \not\vDash \Phi$ (we really mean “$p$ forces $\Psi$ relativized to $\langle \lambda, \mathcal{P}(\lambda), R, \pi, \in, (\alpha)_{\alpha < \lambda} \rangle$”, where $R \subseteq \lambda$ and $\pi \subseteq (\lambda \times \lambda) \times \lambda$ is a pairing function, but we shall be sloppy). By the maximum principle there is a name $\dot{S}$ such that $p \vDash \Phi(\dot{S})$. For reasons of convenience, without loss of generality we assume $\Phi$ has only “$\sim$”, “&” and “$\exists$” as logical operators, no terms other than variables and constants, and only “$\in$” as a predicate. For each subformula $\theta$ of $\Phi$ and each finite sequence $u$ of elements of $\lambda$ we define a dense set $D_{\theta,u}$ as follows:

1. If $\theta[u]$ is “$\dot{\alpha} \in \dot{\beta}$”, $\alpha, \beta \in \lambda$, $D_{\theta,u} = \{q: q \vDash \dot{\alpha} \in \dot{\beta} \}$.
2. If $\theta[u]$ is “$\dot{\alpha} \in \dot{R}$”, $\alpha \in \lambda$, $D_{\theta,u} = \{q: q \vDash \dot{\alpha} \in \dot{R} \}$.
3. If $\theta[u]$ is “$\dot{\alpha} \notin \dot{S}$”, $\alpha \in \lambda$, $D_{\theta,u} = \{q: q \vDash \dot{\alpha} \notin \dot{S} \}$.
4. If $\theta$ is “$\sigma_1 \& \sigma_2$”, $D_{\theta,u} = \{q: q \vDash \sigma_1[\dot{u}]$ and $q \vDash \sigma_2[\dot{u}] \}$.
5. If $\theta$ is “$\sim \sigma$”, $D_{\theta,u} = \{q: q \vDash \sigma[\dot{u}]$ or $q \vDash \sim \sigma[\dot{u}] \}$.
6. If $\theta$ is “$\exists \beta(\sigma(\beta))$”, $D_{\theta,u} = \{q: (\text{for some } \beta \in \lambda, q \vDash \sigma(\beta)[\dot{u}]$ or $q \vDash \sim (\exists \beta)[\sigma(\beta)[\dot{u}]) \}$.

Since $\lambda < 2^{\kappa_1}$, by generalized Martin’s axiom there is a filter $H$ below $p$ which meets each $D_{\theta,u}$. Pick $p_{\theta,u} \in H \cap D_{\theta,u}$. Let $T = \{\alpha: p_{\theta,\alpha} \vDash \dot{\alpha} \in \dot{S}\}$. Claim $\Phi(T)$ holds in $\mathcal{M}$. It suffices to prove by induction that for every subformula $\theta$ of $\Phi$ and every finite sequence $u$ of elements of $\lambda$, $\mathcal{M} \vDash \theta(T)[\dot{u}]$ if and only if there is a $q \in H$ such that $q \vDash \theta(\dot{S})[\dot{u}]$.\[\]
(1) and (2) are by absoluteness. Clause (3) is taken care of by $p_{(a \in \mathcal{A})}$. Clause (4) works since $H$ is a filter. For clause (5), $\mathfrak{A} \models \sigma[u]$ if and only if there is no $q \in H$ such that $q \models \sigma[\check{u}]$ if and only if some $q \in H$ forces $\sim \sigma[u]$ (since $H \cap D_{\sigma}[u] \neq \emptyset$). For clause (6), $\mathfrak{A} \models (\exists \beta)[\sigma(\beta)[\check{u}]]$ if and only if for some $q \in H$, $q \models \sigma(\beta)[\check{u}]$ if and only if some $q \in H$, $q \neq \sigma(\beta)[\check{u}]$. The final backwards implication is because under the hypothesis, $p_{(\exists \beta)[\sigma(\beta)[u]]}$ is the required $q$.

I do not know whether the generic filter given by generalized Martin’s axiom can always without loss of generality be taken to be countably closed; if $(\forall \mu < 2^{\omega_1})[\mu^{\aleph_0} < 2^{\omega_1}]$ is assumed however, this can be done. We sketch the proof. First argue that without loss of generality we may assume the given partial order is separative, i.e., if $p \leq q$, there is an $r < p$ such that $r$ is incompatible with $q$. (See [25, p. 152].) Then do the usual Lüwenheim–Skolem argument (using the cardinality assumption to claim that without loss of generality we also may assume the partial order has cardinality $< 2^{\omega_1}$). We then meet not only the desired dense sets in $\mathcal{P}$, but also for each countable descending sequence $R = (r_n)_{n<\omega}$ in $\mathcal{P}$, the dense set $D_R = \{q: (\forall n)(q \not\leq r_n) \text{ or } (\exists n) [q \text{ is incompatible with } r_n]\}$. This assures the generic set is countably compact.

**Remark.** Arbitrary countably closed forcing over BACH plus $2^{\omega_1} > \mathfrak{K}_2$ does not preserve $\Sigma_1^1(\omega_2)$, in fact not even $\Pi_1^1(\omega_2)$. Recall that in Section 5 we constructed from these hypotheses a normal space of character $\mathfrak{K}_1$ in which the $F_\alpha$ are closed but which is not $\mathfrak{K}_2$-collectionwise Hausdorff. Force with countable disjoint collections of closures of open sets, each containing one element of the unseparated closed discrete subspace. This notion of forcing is obviously countably closed. Since the $F_\alpha$ are closed, the sets needed to assure a generic set yields a separation are dense. Because the space has character $\leq \mathfrak{K}_2$ and cardinality $\leq \mathfrak{K}_2$, the statement that it is not $\mathfrak{K}_2$-collectionwise Hausdorff can be coded as $\Pi_1^1(\omega_2)$. A similar argument to that for Theorem 7.17 and Corollary 7.18 shows that $\Sigma_1^1(\lambda)$ statements are preserved by countable chain condition forcing over a model of Martin’s axiom plus $2^{\aleph_0} > \lambda$, or indeed that $\Sigma_1^1(\omega_1)$ statements are preserved by proper forcing over a model of the Proper Forcing Axiom.

I should like to thank Jim Baumgartner for filling a gap in my earlier proof of Theorem 7.17 and for noting that the result holds as well for the language $L_{\omega_1,\omega_1}$, if we assume $\lambda^{\aleph_0} < 2^{\omega_1}$. There are several points to observe when extending the previous proof. Note that by countable closure $L_{\omega_1,\omega_1}$ is the same in the extension as in the ground model. By $\lambda^{\aleph_0} < 2^{\omega_1}$, there are not too many formulas. At first sight it would appear that one needs the generic filter to be countably closed to take care of infinite conjunction, but this can be avoided by forcing with the complete Boolean algebra $\mathcal{B}$ associated with the forcing partial order $\mathcal{P}$. As far as forcing is concerned, nothing changes; however we may now in $\mathcal{B}$ close the generic filter $H$ under infs for countable subsets. The result—by countable compactness—is a countably closed filter.
8. Kurepa trees

The reader will have noticed two reasons why we had to use the model or strong BACH to obtain the desired results in Section 7. One is that to show e.g. that a set is stationary, one has to meet \( 2^{\aleph_1} \) dense sets. The other is that in the model the description of the sets involved the use of names in the forcing language. One interesting example of countably closed forcing using \(< 2^{\aleph_1}\) dense sets, all describable without reference to names, is—if \( 2^{\aleph_1} > \aleph_2 \) is assumed—that for obtaining a Kurepa tree, viz. a tree of height \( \omega_1 \) with countable levels and at least \( \aleph_2 \) cofinal branches. There are some difficulties encountered in trying to construct such a tree from BACH plus \( 2^{\aleph_1} > \aleph_2 \). Several quite similar partial orders can be used to force a Kurepa tree. The standard proofs that these partial orders have the \( \aleph_2 \)-chain condition proceed via \( \Delta \)-system arguments and it is not obvious that these partial orders are in fact \( \aleph_1 \)-linked. However my student M. Dahroug was able, following a suggestion of Kunen, to prove that they are, assuming CH. The other difficulty is that the usual partial order (see e.g. [54] or [9]) is not well-met. In an earlier version of this paper I claimed it was countably compact, but Kunen produced a counterexample. However, as several people pointed out, various minor modifications of this partial order are well-met. We will use one here that Lee Stanley suggested.

**Theorem 8.1.** BACH plus \( 2^{\aleph_1} > \aleph_2 \) implies there is a Kurepa tree.

**Proof.** Let \( \mathcal{P} = \langle P, \leq \rangle \) where \( P \) consists of all pairs \( \langle T, f \rangle \) such that

1. \( T = \langle T, \leq \rangle \) is a normal \( \alpha \)-tree for some \( \alpha < \omega_1 \),
2. \( f \) is a function from a countable subset of \( \omega_2 \) with range included in the set of branches of \( T \).

Let \( \langle T', f' \rangle \leq \langle T, f \rangle \) if

1. \( T' \) end extends \( T \),
2. \( \text{dom } f' \supseteq \text{dom } f \),
3. for every \( \rho \in \text{dom } f \), \( f'(\rho) \supseteq f(\rho) \).

This formulation differs from that in [25] in that \( f \) is not required to be a bijection and its range need not consist of \( \alpha \)-branches. However, by meeting \( \aleph_2 \) dense sets we may assure that the generic function has range of cardinality \( \aleph_2 \); while by CH, the \( \aleph_2 \) branches it determines must include \( \aleph_2 \) \( \omega_1 \)-branches since the generic tree has only \( \aleph_1 \) countable branches. The partial order is countably closed as usual; to see that it is well met, note that if \( \langle T, f \rangle \) and \( \langle T', f' \rangle \) are compatible, then \( T \) end extends \( T' \) or vice versa, say e.g. the former. Then \( \langle T, f \cup f' \rangle \) is the desired inf. \( \square \)

To prove \( \mathcal{P} \) is \( \aleph_1 \)-linked, it suffices by CH to show that for fixed \( T \), the collection of all conditions with first coordinate \( T \) is the union of \( \aleph_1 \) compatible subcollections. If \( \langle T, f \rangle \) and \( \langle T, f' \rangle \) are conditions such that \( f \) and \( f' \) agree on
their common domain, then they are compatible. Without loss of generality assume
the elements of the trees are countable ordinals. For a fixed $\mathcal{F}$ then, a branch is
determined by a countable ordinal, namely the set of its $\mathcal{F}$-predecessors. Thus we
may consider $f$ and $f'$ as countable partial functions from $\omega_2$ into $\omega_1$. Then $f$ and
$f'$ agree on their common domain if and only if they are compatible in the usual
extension order. It therefore suffices to show that this extension order is $\aleph_1$-linked.
Finally, it suffices to show that the density of the countable box topology on the
product of $\aleph_2$ copies of the discrete space of power $\aleph_1$ is $\aleph_1$. But this follows from
CH by Sublemma 7.15. We of course have actually shown the partial order is
$\aleph_1$-centred. Since only $\aleph_2$ dense sets are involved, we have the surprising

**Theorem 8.2.** $P_1$ plus CH implies there is a Kurepa tree.

This situation calls for a direct proof, since both $P_1$ and the existence of a
Kurepa tree are combinatorial statements about $\omega_1$. Stepráns found one (given in
[67]) which is somewhat less indirect. He called attention to the (yet another)
partial order used by Juhász to get a Kurepa tree from BACH in his SETOP
lectures [29]. Stepráns observed that Juhász’ use of BACH in the latter’s proof that
his partial order yielded a Kurepa tree was only to get a function from $\omega_1$ to $\omega_1$
dominating except on a countable set each of $\aleph_2$ functions from $\omega_1$ to $\omega_1$. But that
is an easy consequence of $P_1$ plus $2^{\aleph_1} > \aleph_2$.

In Section 2 we referred to the combinatorial principle $W(\kappa)$ which (for our
purposes) says there is a particularly nice Kurepa tree with $\kappa$ branches. As
Dahroug observed, all that is required to derive $W(\kappa)$ from BACH plus $2^{\aleph_1} > \kappa$
is to modify the standard proof (see e.g. [9]) in the same way as one does for the
Kurepa tree partial order. We leave the details to the reader but we do define

$W(\kappa)$ is the proposition that there exists a Kurepa tree $\mathcal{F}$ with $\kappa$ branches and a
function $W$ with domain $\omega_1$ such that for each $\alpha < \omega_1$, $W(\alpha)$ is a countable
family of subsets of $\mathcal{F} \setminus \alpha$, and for any countable collection $\mathcal{C}$ of branches of
$\mathcal{F}$, there is a $\gamma < \omega_1$ such that for any $\beta$ with $\gamma < \beta < \omega_1$, the set of nodes of
members of $\mathcal{C}$ on level $\beta$ is an element of $W(\beta)$.

**Theorem 8.3.** BACH plus $2^{\aleph_1} > \kappa$ implies $W(\kappa)$.

A question not decided by BACH plus $2^{\aleph_1} > \aleph_2$ is whether there is a Kurepa
tree with $2^{\aleph_1}$ branches. One can start with a model of CH in which there is such a
tree and $2^{\aleph_1} = \aleph_3 = 2^{\aleph_2}$ and extend to a model of BACH plus $2^{\aleph_1} = \aleph_3$. The tree
and its branches will be preserved. On the other hand, if we start with $2^{\aleph_1} = \aleph_2$ and
obtain BACH via an iteration sequence of length $\omega_3$, $2^{\aleph_1} = \aleph_2$ will hold at each
initial stage. Every Kurepa tree will appear at some initial stage and will have at
most $\aleph_2$ branches there. By [51], countably closed forcing adds no new branches to
$\omega_1$-trees, so the Kurepa tree will still have $\aleph_2 < 2^{\aleph_1}$ branches in the final model.
9. Generalizing Martin’s axiom

For a while, the research program for generalizing Martin’s axiom (say to $\mathfrak{K}_\delta$, as with BACH) seemed relatively clear. Laver, Baumgartner, and Shelah had each succeeded in getting a weak version of generalized Martin’s axiom by strengthening both the countably closed and $\mathfrak{K}_\delta$-chain condition requirements. It was widely thought one could do better, but that large cardinals would be needed to get an axiom sufficiently strong to imply the $\mathfrak{K}_\delta$-Souslin hypothesis, and possibly even to get the $\mathfrak{K}_\delta$-chain condition, countably closed version of the axiom. Note that $\mathfrak{K}_\delta$-Souslin trees need not be countably closed, and so are not obviously destroyed by that version. Indeed by assuming the existence of a measurable cardinal, Laver was able to prove the consistency of CH plus $2^{\aleph_1} > \aleph_2$ plus the $\mathfrak{K}_\delta$-Souslin hypothesis. Shelah improved this to assume only a weakly compact [37]. Laver was then able to get BACH holding as well, again assuming a weakly compact. The necessity of a large cardinal assumption was then demonstrated by Shelah and Stanley [50] who proved that if CH and the $\mathfrak{K}_\delta$-Souslin hypothesis hold, then $\mathfrak{K}_\delta$ is inaccessible in $L$. Shelah and Stanley also proved that BACH plus $2^{\aleph_1} > \aleph_2$ plus the (weak) combinatorial principle $\square_\omega_1$ imply there is a $\mathfrak{K}_\delta$-Souslin tree. $\square_\omega_1$ can be obtained via a countably closed $\mathfrak{K}_\delta$-chain condition partial order, so it follows that a generalized Martin’s axiom strong enough to prove $\square_\omega_1$ cannot also be strong enough to yield the $\mathfrak{K}_\delta$-Souslin hypothesis, and vice versa. Shelah and Stanley (and independently, Kunen [34]) cooked up generalized Martin’s axioms sufficiently strong to yield both BACH and $\square_\omega_1$, but the statements are so technical as not to be worth mentioning here. See [50] for both. It would be interesting to have a reasonable axiom implying both BACH and the $\mathfrak{K}_\delta$-Souslin hypothesis.

The most surprising result contained in [50] is that the well-met condition cannot in fact be removed from (say) Baumgartner’s axiom: under CH there is a countably closed $\mathfrak{K}_\delta$-linked (even $\mathfrak{K}_\delta$-centred) partial order for which one cannot meet $\mathfrak{K}_\delta$ dense sets.

It seems then that there may be no all-purpose generalized Martin’s axiom, but rather a collection for various applications. However, a question that remains is: keeping well-met, how far can $\mathfrak{K}_\delta$-linked be weakened? Shelah [49] weakens it just enough to preserve the proof that the iteration has the $\mathfrak{K}_\delta$-chain condition but it is by no means evident that it can’t be weakened further, even to the $\mathfrak{K}_\delta$-chain condition.

References


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