Solutions of Superlinear Sturm–Liouville Problems in Banach Spaces

Bendong Lou

Department of Mathematics, Shandong University, Jinan 250100, People’s Republic of China

Submitted by V. Lakshmikantham

Received June 20, 1995

1. INTRODUCTION

Let $E$ be a Banach space and $I = [0, 1]$. Consider the Sturm–Liouville problem

$$L \phi = f(x, \phi), \quad x \in I,$$

$$\alpha_0 \phi(0) - \beta_0 \phi'(0) = \theta, \quad \alpha_1 \phi(1) + \beta_1 \phi'(1) = \theta,$$

where $L \phi = -(p(x)\phi'(x))' + q(x)\phi(x)$, $p(x) \in C^1(I)$, $p(x) > 0 (x \in I)$, $q(x) \in C(I)$, $q(x) \geq 0 (x \in I)$, $f \in C[I \times E, E]$, $\alpha_i, \beta_i (i = 0, 1)$ are non-negative constants, and $\theta$ is the zero element of $E$. In this paper, we shall use the topological degree theory and the cone theory to investigate the existence of nontrivial and positive solutions of (1), (2). Finally, an example for infinite system of equations is given.

2. SEVERAL LEMMAS

Consider the linear Sturm–Liouville problem,

$$Lh = \lambda a(x)h, \quad x \in I,$$

$$\alpha_0 h(0) - \beta_0 h'(0) = 0, \quad \alpha_1 h(1) + \beta_1 h'(1) = 0,$$

where $Lh = -(p(x)h'(x))' + q(x)h(x)$, $p, q, \alpha_i, \beta_i (i = 0, 1)$ are as above, $a(x) \in C(I)$ with $a(x) > 0 (x \in I)$ and $h \in C^2(I)$. In this paper, we
always assume that 0 is not an eigenvalue of (3), (4); then it is known that
the BVP (3), (4) is equivalent to the integral equation
\begin{equation}
  h(x) = \lambda \int_I k(x, y) a(y) h(y) \, dy = \lambda Bh(x),
\end{equation}
where the Green function \( k(x, y) \) is defined by
\begin{equation}
  k(x, y) = \begin{cases} 
    (1/w)u(x)v(y), & 0 \leq x \leq y \leq 1, \\
    (1/w)u(y)v(x), & 0 \leq y \leq x \leq 1,
  \end{cases}
\end{equation}
where \( w \neq 0 \) is a constant and \( u, v \) satisfy
\begin{align}
  Lu &= 0, \quad u(0) = \beta_0, \quad u'(0) = \alpha_0, \\
  Lv &= 0, \quad v(1) = \beta_1, \quad v'(1) = -\alpha_1.
\end{align}

**Lemma 1.** Let \( \alpha_0 \geq 0, \beta_0 \geq 0, \alpha_1 \geq 0, \beta_1 \geq 0, \alpha_0 \alpha_1 + \alpha_0 \beta_1 + \beta_0 \alpha_1 > 0 \). Then there exist \( h \in C(I) \) with \( h \geq 0, h \neq 0 \) and a constant \( t > 0 \) such that
\begin{align}
  r(B)h(x) &= \int_I k(y, x) a(x) h(y) \, dy, \\
  h(y) &\geq tk(x, y) a(y) \quad \forall x, y \in I,
\end{align}
where \( r(B) \) denotes the spectral radius of \( B \).

**Proof.** By the Sturm–Liouville theory of ODE we know that \( k(x, y) \geq 0 \) for \( x, y \in I \) and there exists an eigenvector of (3), (4) \( h_i(x) \) with respect to the first eigenvalue \( r(B)^{-1} \) such that \( h_i(x) > 0 \) for \( x \in (0, 1) \), and, if \( h_i(0) = 0 \), then \( h_i'(0) > 0 \); if \( h_i(1) = 0 \), then \( h_i'(1) < 0 \). It is easy to verify by (6), (7), and (8) that there exists \( t > 0 \) such that
\begin{equation}
  h_i(x) \geq tk(x, y), \quad x, y \in I.
\end{equation}

Set \( h(x) = a(x)h_i(x) \). Observing \( k(x, y) = k(y, x) \) \( \forall x, y \in I \), we get
\begin{equation}
  r(B)h(x) = \int_I k(y, x) a(x) h(y) \, dy.
\end{equation}
Also, by (11) we get
\begin{equation}
  h(x) \geq tk(y, x) a(x), \quad x, y \in I.
\end{equation}
This completes the proof.
We also need the following lemma.

**Lemma 2.** Let $P$ be a cone of $E$ and $B: P \to P$ a $k$-set-contraction ($k < 1$). Assume that $B$ is order-preserving and positively homogeneous of degree 1 (see [1]) and that there exist $u \in P \setminus \{\varnothing\}$, $\lambda > 1$, such that $Bu \geq \lambda u$. Then $r(B) > 1$.

**Proof.** The proof follows from Corollary 2.1 in [1].

### 3. Main Theorems

In what follows, the norms in spaces $E$ and $C[I, E]$ are denoted by $\| \cdot \|$ and $\| \cdot \|_C$, respectively; i.e., $\| \phi \|_C = \max_{x \in I} \| \phi(x) \|$ for any $\phi \in C[I, E]$. The closed balls in spaces $E$ and $C[I, E]$ are denoted by $T_i = \{ u \in E \mid \| u \| \leq l \}$ ($l > 0$) and $U_i = \{ \phi \in C[I, E] \mid \| \phi \|_C \leq l \}$ ($l > 0$), respectively. Let $P$ be a normal cone of $E$ and $N$ the normal constant of $P$; i.e., $\theta \leq u \leq v$ implies $\| u \| \leq N \| v \|$. Clearly, $K = \{ \phi(x) \in C[I, E] \mid \phi(x) \in P \text{ for } x \in I \}$ is a cone of $C[I, E]$. Set $M = \max_{x, y \in I} |k(x, y)|$, where $k$ is defined by (6), then $M > 0$.

Let us list some conditions for convenience:

- **(H.1)** $f \in C[I \times E, E]$. For any $l > 0$, $f$ is uniformly continuous on $I \times T_i$ and there exists a constant $L_i$ with $0 \leq L_i < 1/2M$ such that
  \[ \alpha(f(x, D)) \leq L_i \alpha(D) \quad \forall x \in I, D \subset T_i, \]
  where $\alpha$ denotes the Kuratowski measure of noncompactness.

- **(H.1)′** $f \in C[I \times P, P]$. For any $l > 0$, $f$ is uniformly continuous on $I \times (P \cap T_i)$ and there exists a constant $L_i$ with $0 \leq L_i < 1/2M$ such that
  \[ \alpha(f(x, D)) \leq L_i \alpha(D) \quad \forall x \in I, D \subset P \cap T_i. \]

- **(H.2)** There exist $a(x) \in C(I)$ with $a(x) > 0$ ($x \in I$) and $b(x) \in C[I, E]$ such that
  \[ f(x, u) \geq a(x)u - b(x) \quad \forall x \in I, u \in E. \]

- **(H.2)′** There exist $a(x) \in C(I)$ with $a(x) > 0$ ($x \in I$) and $b(x) \in C[I, E]$ such that
  \[ f(x, u) \geq a(x)u - b(x) \quad \forall x \in I, u \in P. \]

- **(H.3)** There exist a constant $r > 0$ and $c(x) \in C(I)$ with $c(x) \geq 0$, $\max_{x \in I} c(x) < 1/M$ such that
  \[ \| f(x, u) \| \leq c(x) \| u \| \quad \forall x \in I, \| u \| \leq r. \]
There exist a constant \(r > 0\) and \(c(x) \in C(I)\) with \(c(x) \geq 0\), \(\max_{x \in I} c(x) < 1/M\) such that
\[
\|f(x, u)\| \leq c(x)\|u\| \quad \forall x \in I, u \in P, \|u\| \leq r.
\]

There exists \(d \in E\) such that
\[
f(x, u) \geq -d \quad \forall x \in I, u \in E.
\]

Let \(L_0, L_\infty\) be two linear operators defined by
\[
L_0 h(x) = \int_I k(x, y)c(y)h(y) \, dy, \quad h \in C(I),
\]
\[
L_\infty h(x) = \int_I k(x, y)a(y)h(y) \, dy, \quad h \in C(I),
\]
where \(k\) is defined by (6). Clearly \(L_0\) and \(L_\infty\) map \(C(I)\) into \(C(I)\). Denote by \(r(L_0)\) and \(r(L_\infty)\) the spectral radii of \(L_0\) and \(L_\infty\), respectively.

**Theorem 1.** Let \(\alpha_0 \geq 0\), \(\beta_0 \geq 0\), \(\alpha_1 \geq 0\), \(\beta_1 \geq 0\), \(\alpha_0 \alpha_1 + \alpha_0 \beta_1 + \alpha_1 \beta_0 > 0\). Suppose that conditions (H_1), (H_2), (H_3), and (H_4) are satisfied, \(r(L_0) \leq 1 < r(L_\infty)\). Then the Sturm–Liouville problem (1), (2) has at least one nontrivial solution.

**Proof.** It is known that the Sturm–Liouville problem (1), (2) is equivalent to the integral equation
\[
\phi(x) = \int_I k(x, y)f(y, \phi(y)) \, dy = A\phi(x),
\]
where \(k\) is defined by (6). By Lemma 1, there exist \(h \in C(I)\) with \(\hat{g} \geq 0\), \(h \neq 0\), and a constant \(t > 0\) such that
\[
r(L_\infty)h(x) = \int_I k(y, x)a(x)h(y) \, dy,
\]
\[
h(y) \geq tk(x, y)a(y) \quad \forall x, y \in I.
\]
Set
\[
K(h, \delta) = \left\{ \phi \in K \left| \int_I h(x) \phi(x) \, dx \geq \delta \phi(y) \forall y \in I \right. \right\},
\]
where \(0 < \delta \leq \min\{j, h(x) \, dx, r(L_\infty)\} \) is a constant. It is clear that \(K(h, \delta) \setminus \{0\} \neq \emptyset\) and \(K(h, \delta) \subset K\) is a convex closed set. Define an
operator \( B_\phi \) by
\[
B_\phi \phi(x) = \int_I k(x, y) a(y) \phi(y) \, dy, \quad \phi \in C[I, E].
\]
Then \( B_\phi \) maps \( K \) into \( K(h, \delta) \). In fact, for any \( \phi \in K \), by (17) and (18) we get
\[
\int_I h(x) B_\phi \phi(x) \, dx = \int_I \phi(y) \, dy \left( \int_I k(x, y) a(y) h(x) \, dx \right)
= r(L_\phi) \int_I \phi(y) \, dy
\geq r(L_\phi) t \int_I k(x, y) a(y) \phi(y) \, dy
\geq \delta B_\phi \phi(x) \quad \forall x \in I,
\]
which implies \( B_\phi \phi(x) \in K(h, \delta) \).

Choose \( R > r \) such that
\[
R > \frac{N}{\epsilon r(L_\phi) \delta} \left\| \epsilon r(L_\phi) \int_I h(x) B_\phi \frac{d}{dx} h(x) - \epsilon r(L_\phi) \delta B_\phi \frac{d}{dx} h(x) \right\| \infty
+ \int_I \int_I k(x, y) h(x) b(y) \, dx \, dy \right\| \infty, \quad (20)
\]
where \( \epsilon = 1 - r(L_\phi)^{-1} > 0 \), \( N \) is the normal constant of \( P \).

We now prove that for any given \( \phi_0 \in K(h, \delta) \setminus \{0\} \),
\[
\forall \phi \in \partial U_R, \lambda \geq 0. \quad (21)
\]
In fact, if there exist \( \phi_1 \in \partial U_R, \lambda_1 \geq 0 \), such that
\[
\phi_1 - A \phi_1 = \lambda_1 \phi_0, \quad (22)
\]
then by (H2) and (17),
\[
\theta \geq -\lambda_1 \int_I h(x) \phi_0(x) \, dx = \int_I h(x) \left[ A \phi_1(x) - \phi_1(x) \right] \, dx
\geq \int_I h(x) B_\phi \phi_1(x) \, dx - \int_I k(x, y) h(x) b(y) \, dx \, dy - \int_I h(x) \phi_1(x) \, dx
= \int_I h(x) \left( r(L_\phi)^{-1} + \epsilon \right) B_\phi \phi_1(x) \, dx - \int_I h(x) \phi_1(x) \, dx
- \int_I \int_I k(x, y) h(x) b(y) \, dx \, dy
\]
\[
= \varepsilon r(L) \int h(x) \phi_2(x) \, dx - \int \int k(x, y) h(x) b(y) \, dx \, dy
\]
\[
= \varepsilon r(L) \int h(x) \left[ \phi_2(x) + B_\varepsilon \frac{d}{a(x)} \right] \, dx - \int \int k(x, y) h(x) b(y) \, dx \, dy
\]
\[
- \varepsilon r(L) \int h(x) B_\varepsilon \frac{d}{a(x)} \, dx. \tag{23}
\]
By (H.4), we get \( f(x, \phi_1(x)) + d \in K \). Since \( B_\varepsilon \) maps \( K \) into \( K(h, \delta) \), we get
\[
B_\varepsilon \left( \frac{f(x, \phi_1(x)) + d}{a(x)} \right) \in K(h, \delta).
\]
Consequently, by (22),
\[
\phi_1(x) + B_\varepsilon \frac{d}{a(x)} = A \phi_1(x) + B_\varepsilon \frac{d}{a(x)} + \lambda_1 \phi_0(x)
\]
\[
= B_\varepsilon \left( \frac{f(x, \phi_1(x)) + d}{a(x)} \right) + \lambda_1 \phi_0(x) \in K(h, \delta);
\]
i.e.,
\[
\int h(x) \left[ \phi_1(x) + B_\varepsilon \frac{d}{a(x)} \right] \, dx \geq \delta \left[ \phi_1(y) + B_\varepsilon \frac{d}{a(y)} \right] \quad \forall y \in I. \tag{24}
\]
It follows from (23) and (24) that
\[
\theta \geq \varepsilon r(L) \delta \left[ \phi_1(y) + B_\varepsilon \frac{d}{a(y)} \right] - \int \int k(x, y) h(x) b(y) \, dx \, dy
\]
\[
- \varepsilon r(L) \int h(x) B_\varepsilon \frac{d}{a(x)} \, dx. \tag{25}
\]
Set
\[
\xi(y) = \varepsilon r(L) \int h(x) B_\varepsilon \frac{d}{a(x)} \, dx - \varepsilon r(L) \delta B_\varepsilon \frac{d}{a(y)}
\]
\[
+ \int \int k(x, y) h(x) b(y) \, dx \, dy.
\]
Then (25) implies
\[
\xi(y) \geq \varepsilon r(L) \delta \phi_2(y) \geq \theta \quad \forall y \in I.
\]
and so
\[ \| \xi(y) \|_C \geq \| \xi(y) \|_C \geq \frac{er(L_\alpha) \delta}{N} \| \phi_1(y) \| \quad \forall y \in I. \tag{26} \]

Since (26) is true for any \( y \in I \), we have
\[ \| \xi \|_C \geq \frac{er(L_\alpha) \delta}{N} \| \phi_1 \|_C = \frac{er(L_\alpha) \delta R}{N}; \]
i.e., \( R \leq N \frac{er(L_\alpha) \delta}{\delta R} \| \xi \|_C \), in contradiction with (20). Thus, (21) is true.

In the same way as proving Lemma 2 in [2], we can prove by (H3) that, for any \( I > 0 \), operator \( A \) is a strict-set-contraction on \( U \), By (21) and the homotopy invariance of the topological degree we get
\[ \deg(I - A, U_R, \theta) = 0. \tag{27} \]

Without loss of generality, assume that
\[ A \phi \neq \phi \quad \forall \phi \in \partial U, \tag{28} \]

We now prove that
\[ A \phi \neq \lambda \phi \quad \forall \phi \in \partial U, \lambda > 1. \tag{29} \]

In fact, if there exist \( \phi_2 \in \partial U, \) and \( \lambda_2 > 1 \) such that \( A \phi_2 = \lambda_2 \phi_2 \), then by (H3) we get
\[ \lambda_2 \| \phi_2(x) \| = \| A \phi_2(x) \| \leq \int_I k(x, y) \| f(y, \phi_2(y)) \| \, dy \]
\[ \leq \int_I k(x, y) c(y) \| \phi_2(y) \| \, dy = L_0(\| \phi_2(x) \|) \quad \forall x \in I. \tag{30} \]

It follows from \( \max_{x \in I} c(x) < 1/M \) that \( L_0 \) is a strict-set-contraction. Consequently, (30) and Lemma 2 imply \( r(L_0) > 1 \), in contradiction with \( r(L_0) \leq 1 \). Thus, (29) is true. Observing (28) and (29) and using the homotopy invariance of the topological degree, we get
\[ \deg(I - A, U, \theta) = 1. \tag{31} \]

Since \( \overline{U} \subseteq U_R \), it follows from the additivity of the topological degree and (27), (31) that
\[ \deg(I - A, U_R \setminus \overline{U}, \theta) = -1, \]
which implies \( A \) has at least one fixed point in \( U_R \setminus \bigcup_i U_i \); i.e., the Sturm–Liouville problem (1), (2) has at least one nontrivial solution. This completes the proof.

**Theorem 2.** Let \( \alpha_0 \geq 0, \beta_0 \geq 0, \alpha_1 \geq 0, \beta_1 \geq 0, \alpha_0 \alpha_1 + \alpha_0 \beta_1 + \alpha_1 \beta_0 > 0. \) Suppose that conditions \((H_1)', (H_2)', \) and \((H_3)'\) are satisfied, \( r(L_n) > 1 \geq r(L_0). \) Then the Sturm–Liouville problem (1), (2) has at least one nontrivial solution.

**Proof.** As in the proof of Theorem 1, (1), (2) is equivalent to (16); there exist \( h \in C(I) \) with \( h \geq 0, h \neq 0, \) and a constant \( \varepsilon > 0 \) such that (17), (18) hold. Let \( K(h, \delta) \) be defined by (19). Set \( \varepsilon = r(L_n) - 1 > 0; \) choose a constant \( R > r \) such that

\[
R > \frac{N}{\varepsilon} \left\| \int_I \int_I k(x, y) h(x) b(y) \, dx \, dy \right\|.
\]

Then, in the same way as establishing (21), we can show that, for any given \( \phi_0 \in K(h, \delta) \setminus \{0\}, \)

\[
\phi - A \phi \neq \lambda \phi_0 \quad \forall \phi \in K, \left\| \phi \right\|_C = R, \lambda \geq 0.
\]

It follows from Lemma 2 in [2] and \((H_1)'\) that, \( A: K \cap B_i \rightarrow K \) is a strict-set-contraction. Consequently, by (32) and the homotopy invariance of the fixed-point index, we get

\[
i(A, K \cap U_R, K) = 0,
\]

where \( i(A, K \cap U_R, K) \) denotes the fixed-point index of \( A \) over \( K \cap U_R \) with respect to \( K \) (see [3]).

On the other hand, as in the proof of Theorem 1, we can show that

\[
A \phi \neq \lambda \phi \quad \forall \phi \in K, \left\| \phi \right\|_C = r, \lambda \geq 1,
\]

which implies, by the homotopy invariance of the fixed-point index,

\[
i(A, K \cap U_R, K) = 1.
\]

Finally, Theorem 2 follows from (33) and (34).

**Example.** Consider the BVP of an infinite system for second-order ordinary differential equations

\[
\begin{align*}
-u''_n(x) &= u'_n(x) - \frac{u_1^2(x)}{1 + u_1^2(x) + u_1(x) + u_2(x)}, \quad x \in I, \\
-u''_n(x) &= u'_n(x) - \frac{u_2^2(x)}{1 + u_2^2(x) + u_2(x) + u_3(x)}, \quad x \in I, (n = 2, 3, 4, \ldots) \\
u'_n(0) &= 0, \quad 2u_n(1) + u'_n(1) = 0 \quad (n = 1, 2, 3, \ldots).
\end{align*}
\]
Conclusion. BVP (35) has at least one solution \( \{ u_0(x) \} \) satisfying \( u_0(x) \geq u_0(x) \geq 0 \) \((n = 2, 3, 4, \ldots)\) and \( u_0(x) \neq 0 \).

Proof. Let \( E = \{ u = (u_1, u_2, \ldots, u_n, \ldots) | \sum_{n=1}^{\infty} |u_n| / 2^n < +\infty \}, \|u\| = \sum_{n=1}^{\infty} |u_n| / 2^n \) for \( u \in E \). It is easy to verify that \( E \) is a Banach space with norm \( \| \cdot \| \). Set \( P = \{ u \in E | u_1 \geq u_n \geq 0, \ n = 2, 3, 4, \ldots \} \). Clearly \( P \) is a normal cone in \( E \) and the system (35) may be regarded as a BVP in \( E \),

\[
-u'' = f(x, u), \quad x \in I, \\
u'(0) = \theta, \quad 2u(1) + u'(1) = \theta, 
\]

where \( u = (u_1, u_2, \ldots, u_n, \ldots), \ u' = (u'_1, u'_2, \ldots, u'_n, \ldots), \) and \( f = (f_1, f_2, \ldots, f_n, \ldots) \) with

\[
f_1(x, u) = u_1^2 - \frac{u_1^2}{1 + u_1^2 + u_1 + u_2},
\]

\[
f_n(x, u) = u_n^2 - \frac{u_n^2}{1 + u_n^2 + u_{n+1}} \quad (n = 2, 3, 4, \ldots).
\]

It is known that the BVP (36) is equivalent to the integral equation

\[
u(x) = \int_I k(x, y)f(y, u(y)) \, dy = Au(x),
\]

where \( k \) is the corresponding Green function, i.e.,

\[
k(x, y) = \begin{cases} 
3 - 2y, & 0 \leq y \leq 1, \\
2, & 0 \leq x \leq 1, \\
3 - 2x, & 0 \leq x \leq 1.
\end{cases}
\]

It is easy to verify that \( f \) is uniformly continuous on \( I \times (P \cap T) \) for any \( l > 0 \), where \( T_l = \{ u \in E | \|u\| \leq l \} \).

We now prove that, for any \( D \subseteq P \cap T \), \( x \in I, f(x, D) \) is relatively compact in \( P \). In fact, let \( \{ u^{(m)} \} \subseteq D \subseteq P \cap T \), then \( 0 \leq u_1^{(m)} \leq u_1^{(m)} \leq 2l \) \((m, n = 1, 2, 3, \ldots)\). By the diagonal method, we can choose a subsequence \( \{ u^{(m')} \} \subseteq \{ u^{(m)} \} \) such that

\[
f_n(x, u^{(m)}) \to w_n \quad \text{as} \ m \to +\infty; \ n = 1, 2, 3, \ldots
\]

where \( x \) is fixed. Since \( 0 \leq u_n^{(m)} \leq 2l \) \((m, n = 1, 2, 3, \ldots)\), by (37) we get that \( f_n(x, u^{(m)}) \), \( w_n \in [0, 4l^2] \) \((m, n = 1, 2, 3, \ldots)\). Hence \( w = (w_1, w_2, \ldots, w_n, \ldots) \in E \) and, for any given \( \epsilon > 0 \), we can choose a positive
integer $n_0$ sufficiently large such that
\[
\sum_{n=n_0+1}^{\infty} \frac{4l^2}{2^n} < \frac{\epsilon}{2}.
\]  
(41)

By virtue of (40), there exists a positive integer $m_0$ such that
\[
\sum_{n=1}^{n_0} \left| f_n(x, u^{(m_0)}) - w_n \right| \frac{1}{2^n} < \frac{\epsilon}{2}, \quad m > m_0.
\]  
(42)

It follows from (41) and (42) that
\[
\| f(x, u^{(m_0)}) - w \| \leq \sum_{n=1}^{n_0} \left| f_n(x, u^{(m_0)}) - w_n \right| \frac{1}{2^n} + \sum_{n=n_0+1}^{\infty} \frac{4l^2}{2^n} < \epsilon,
\]
\[
m > m_0.
\]

This means $f(x, u^{(m_0)}) \to w$ in $E$ as $m \to \infty$ and, therefore, $f(x, D)$ is relatively compact. For any $x \in I$, $u \in P$, $f_n(x, u) \geq f_n(x, u) \geq 0$ ($n = 2, 3, 4, \ldots$) and, consequently, $(H_2)'$ is satisfied for $f$.

Let $L_0, L_\infty$ be defined by (14) and (15), respectively, where $k$ is defined by (39), $c(x) = \frac{1}{x}$, $a(x) = 3(x \in I)$. Then
\[
\| L_0 \| = \max_{x \in I} \left( \frac{1}{4} \int_I k(x, y) \, dy \right) = \frac{1}{4} < 1;
\]
hence $r(L_0) < 1$. For any given $e \in R^3$,
\[
L_\infty e = 3e \int_I k(x, y) \, dy = 3e \left( 1 - \frac{x^2}{2} \right) \geq \frac{3}{2} e,
\]
which implies by Lemma 1 that $r(L_\infty) > 1$.

Moreover, choose $h(x) = (7, \frac{13}{2}, \frac{13}{4}, \ldots, \frac{13}{4}, \ldots) \in C(I, E)$, $r = \frac{1}{2}$; then it is easy to verify that $(H_2)'$ and $(H_3)'$ are satisfied for $f$.

Finally, choose $\alpha_0 = 0$, $\beta_0 = 1$, $\alpha_3 = 2$, $\beta_3 = 1$. Then Theorem 2 shows that the BVP (36) has at least one positive solution; i.e., BVP (35) has at least one solution $(u_n(x))$ satisfying $u_1(x) \geq u_n(x) \geq 0$ ($n = 2, 3, 4, \ldots$) and $u_1(x) \not= 0$. 

REFERENCES