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Function Spaces with a Projective Limit Structure

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Vector spaces of functions and equivalence classes of functions for which a natural projective limit structure exists are studied in a systematic manner. The theory is illustrated by a series of examples arising from specific applications to the stability of feedback systems and the theory of hereditary differential systems.

1. INTRODUCTION

Function spaces with an inductive limit structure have known a great deal of popularity in the theory of partial differential equations and hence benefited from a sustained attention. Function spaces with a projective limit structure also have some interesting applications. They can be used in the Theory of Differential Equations where one is interested in working in a space (preferably a Fréchet space) where global solutions can be considered. Within this framework one can also study the stability of the solutions and their behaviour at infinity. The *extended spaces* introduced by Sandberg [1, 2] and Zames [3, 4] in the study of the stability of feedback controlled systems, and, the *local spaces* used in the theory of hereditary differential systems as developed by Delfour and Mitter [5, 6] are examples of such spaces. Here they are Fréchet spaces with a natural projective limit structure.

In Sections 2 and 3 we establish some fundamental results which do not appear to be available in the literature and in Section 4 we consider some examples useful in applications.

2. DEFINITION OF THE SPACES $\mathscr{V}(X, E)$ AND $V(X, E)$

Let X be an arbitrary set, E a real vector space and $\mathbf{N}$ the set of all positive nonzero integers. We shall construct a space $\mathscr{V}(X, E)$ of maps $X \to E$ and a space $V(X, E)$ of equivalence classes of elements of $\mathscr{V}(X, E)$. We are given a family of subsets of X with the following properties:

DEFINITION 2.1. A family $\mathscr{S}(X)$ of nonempty subsets of X is said to be *saturated* if

(i) $S,\ T \in \mathscr{S}(X) \Rightarrow S \cup T \in \mathscr{S}(X)$

(ii) $\cup \{S \mid S \in \mathscr{S}(X)\} = X$.

A saturated subfamily $\mathscr{R}(X)$ of $\mathscr{S}(X)$ is *equivalent to* $\mathscr{S}(X)$ if $\forall S \in \mathscr{S}(X)$ $\exists R \in \mathscr{R}(X)$ such that $S \subset R$.

A family $\mathscr{S}(X)$ is *denumerably saturated* if it is saturated and equivalent to a saturated subfamily with a denumerable number of elements. ▮

Remark 1. Ordered by inclusion, $\mathscr{S}(X)$ is a directed set.

Remark 2. If there exists a denumerable subfamily $(D_n)_{n \in \mathbf{N}}$ of $\mathscr{S}(X)$ such that

(i) $D_n \subset D_{n+1}$, $n \in \mathbf{N}$

(ii) $\forall S \in \mathscr{S}(X)\, \exists n \in \mathbf{N}$ for which $S \subset D_n$,

then $\mathscr{S}(X)$ is denumerably saturated.

EXAMPLES. The family $\mathscr{P}(X)$ of all nonempty subsets of a set X is saturated. When X is a topological space the family $\mathscr{F}(X)$ of all nonempty closed subsets of X is saturated. When X is Hausdorff the family $\mathscr{K}(X)$ of all nonempty compact subsets of X is saturated; if in addition X is σ-compact (or locally compact denumerable at infinity) $\mathscr{K}(X)$ is a denumerably saturated family [7, Chap. 1, Section 9, no. 9, Prop. 15 and Cor. 1, pp. 106–107]. When $X \subset R$ the families

$$\mathscr{E}(-\infty, +\infty) = \{]-\infty, t] \mid t \in R\}$$

and

$$\mathscr{E}(t_0, +\infty) = \{[t_0, t] \mid t > t_0\} \qquad (t_0 \in R)$$

are denumerably saturated (using Remark 2). The family $\mathscr{K}([t_0, +\infty[)$ and $\mathscr{E}(t_0, +\infty)$ are equivalent.

We associate with the saturated family $\mathscr{S}(X)$ a family $\{\mathscr{V}(S, E)\}$ of real vector spaces of maps $S \to E$ indexed by $S \in \mathscr{S}(X)$; it is assumed that the following hypotheses hold:

HYPOTHESES 2.2.

(i) *for all* $S \in \mathscr{S}(X)$ $\mathscr{V}(S, E)$ *is a complete real locally convex topological vector space the topology of which is defined by the single seminorm* q_S,

(ii) *for all pairs* $R \subset S$ *in* $\mathscr{S}(X)$ *the restriction map*

$$\pi_{RS}: \mathscr{V}(S, E) \to \mathscr{V}(R, E)$$

is well defined and

$$q_R(\pi_{RS}(f)) \leqslant q_S(f) \qquad \forall f \in \mathscr{V}(S, E). \quad \blacksquare \tag{2.1}$$

DEFINITION 2.3. $\mathscr{V}(X, E)$ denotes the space of all maps $f: X \to E$ for which the restriction $\pi_S(f)$ of f to S belongs to $\mathscr{V}(S, E)$ for all $S \in \mathscr{S}(X)$. $\mathscr{V}(X, E)$ is endowed with the initial topology for the family $(\pi_S)_{S\in\mathscr{S}(X)}$. ▮

PROPOSITION 2.4. *$\mathscr{V}(X, E)$ is a real topological vector space with the following properties:*

(i) *the restriction map* $\pi_S: \mathscr{V}(X, E) \to \mathscr{V}(S, E)$ *is linear and continuous,*

(ii) *given a family* $(f_S)_{S\in\mathscr{S}(X)}$, $f_S \in \mathscr{V}(S, E)$, *such that for all* $R \subset S$ *in* $\mathscr{S}(X)$ *we have* $\pi_{RS}(f_S) = f_R$ *there exists a unique* $f \in \mathscr{V}(X, E)$ *such that* $\pi_S(f) = f_S$ *for all* $S \in \mathscr{S}(X)$,

(iii) $\forall f \neq 0$ *in* $\mathscr{V}(X, E) \exists S \in \mathscr{S}(X)$ *such that* $\pi_S(f) \neq 0$,

(iv) *for all pairs* $R \subset S$ *in* $\mathscr{S}(X)$ $\pi_R = \pi_{RS} \circ \pi_S$. ▮

The space $\mathscr{V}(X, E)$ we constructed has a very interesting structure. Let $\mathscr{V} = \text{proj}(\mathscr{V}(S, E), \pi_{RS})$ be the *projective limit* of the spaces $(\mathscr{V}(S, E))_{S\in\mathscr{S}(X)}$ for the maps (π_{RS}) [8, Chap. 3, Section 7, no. 1, p. 76]. Let v_S be the canonical map $\mathscr{V} \to \mathscr{V}(S, E)$.

PROPOSITION 2.5. *$\mathscr{V}(X, E)$ is isomorphic to $\mathscr{V}$.*

Proof. By Proposition 2.4 there is a unique map $u: \mathscr{V}(X, E) \to \mathscr{V}$ such that $v_S = \pi_S \circ u$ for $S \in \mathscr{S}(X)$, when u is defined by

$$u(y) = (\pi_S(y))_{S\in\mathscr{S}(X)} \in \prod_{S\in\mathscr{S}(X)} \mathscr{V}(S, E)$$

[8, Chap. 3, No. 2, Prop. 1, p. 77]. The map u is linear since the π_S's are linear. It is injective since for any $f \neq g$ in $\mathscr{V}(X, E)$ there exists $S \in \mathscr{S}(X)$ for which $\pi_S(f) \neq \pi_S(g)$ [8, Chap. 3, No. 2, Prop. 1, p. 77]. It is surjective by Prop. 2.4(ii). To show this let $x \in \mathscr{V}$ and consider $(v_S(x))_{S\in\mathscr{S}(X)}$. By definition of the projective limit $\mathscr{V}$ for all pairs $R \subset S$ in $\mathscr{S}(X)$, $\pi_{RS}(v_S(x)) = v_R(x)$. Proposition 2.4(ii) asserts the existence of an element y in $\mathscr{V}(X, E)$ such that $\pi_S(y) = v_S(x)$ and

$$u(y) = (\pi_S(y))_{S\in\mathscr{S}(X)} = (v_S(x))_{S\in\mathscr{S}(X)} = x.$$

This proves that u is an algebraic isomorphism. By definition the projective limit topology in $\mathscr{V}$ is the initial topology for the maps v_S. Hence

$$u\colon \mathscr{V}(X, E) \to \mathscr{V} \text{ is continuous } \Leftrightarrow v_S \circ u \text{ is continuous } \forall S \in \mathscr{S}(X)$$
$$u^{-1} \text{ is continuous } \Leftrightarrow \pi_S \circ u^{-1} \text{ is continuous } \forall S \in \mathscr{S}(X).$$

But by definition $\pi_S = v_S \circ u$, $v_S = \pi_S \circ u^{-1}$ and the maps π_S and v_S are continuous for all $S \in \mathscr{S}(X)$. This proves the proposition. ∎

Consider the linear subspace $\mathscr{N}(X, E)$,

$$\mathscr{N}(X, E) = \{f \in \mathscr{V}(X, E) \mid q_S(f) = 0, \forall S \in \mathscr{S}(X)\},$$

of $\mathscr{N}(X, E)$ and the quotient space $V(X, E) = \mathscr{V}(X, E)/\mathscr{N}(X, E)$. We now add some hypotheses on $\mathscr{V}(X, E)$ in order to obtain a projective limit structure for $V(X, E)$.

HYPOTHESES 2.6. *In addition to Hypotheses 2.2 we assume that*

(i) *$\mathscr{S}(X)$ is a denumerably saturated family,*

(ii) *given a family $(f_S)_{S \in \mathscr{S}(X)}$, $f_S \in \mathscr{V}(S, E)$, such that for all $R \subset S$ in $\mathscr{S}(X)$ we have $q_R(\pi_{RS}(f_S) - f_R) = 0$, then there exists an $f \in \mathscr{V}(X, E)$ such that $q_S(f_S - \pi_S(f)) = 0$ for all $S \in \mathscr{S}(X)$,*

(iii) *for all $S \in \mathscr{S}(X)$, $\pi_S(\mathscr{N}(X, E)) = \mathscr{N}(S, E)$ where $\mathscr{N}(S, E)$ is the linear subspace of all $f \in \mathscr{V}(S, E)$ such that $q_S(f) = 0$.* ∎

We also define the quotient space $V(S, E) = \mathscr{V}(S, E)/\mathscr{N}(S, E)$ and the canonical surjections $j\colon \mathscr{V}(X, E) \to V(X, E)$ and $j_S\colon \mathscr{V}(S, E) \to V(S, E)$.

DEFINITION 2.7. For $S \in \mathscr{S}(X)$ the map $\bar{\pi}_S\colon V(X, E) \to V(S, E)$ is the unique linear map making the following diagram commutative

$$(\ker j \subset \ker(j_S \circ \pi_S))$$

$$\begin{array}{ccc} \mathscr{V}(X, E) & \xrightarrow{\;j\;} & V(X, E) \\ \Big\downarrow{\scriptstyle \pi_S} & & \Big\downarrow{\scriptstyle \bar{\pi}_S} \\ \mathscr{V}(S, E) & \xrightarrow{\;j_S\;} & V(S, E). \end{array} \tag{2.2}$$

Also for all pairs $R \subset S$ in $\mathscr{S}(X)$ the map $\bar{\pi}_{RS}\colon V(S, E) \to V(R, E)$ is the unique linear map for which the following diagram commutes

$$(\ker j_S \subset \ker(j_R \circ \pi_{RS}))$$

$$\begin{array}{ccc} \mathscr{V}(S, E) & \xrightarrow{\;j_S\;} & V(S, E) \\ \Big\downarrow{\scriptstyle \pi_{RS}} & & \Big\downarrow{\scriptstyle \bar{\pi}_{SR}} \\ \mathscr{V}(R, E) & \xrightarrow{\;j_R\;} & V(R, E). \end{array} \tag{2.3}$$

∎

For all $S \in \mathscr{S}(X)$ $V(S, E)$ is a Banach space since its topology is Hausdorff and defined by a unique seminorm (hence a norm). The quotient norm is written $\bar{q}_S$. We now summarize the properties of $V(X, E)$ and $V(S, E)$ which can be directly obtained from the properties of $\mathscr{V}(X, E)$ and $\mathscr{V}(S, E)$.

PROPOSITION 2.8.

(i) *For all* $S \in \mathscr{S}(X)$ $V(S, E)$ *is a Banach space endowed with the quotient norm* $\bar{q}_S$ *as constructed from* q_S.

(ii) *For all pairs* $R \subset S$ *in* $\mathscr{S}(X)$ $\bar{\pi}_R = \bar{\pi}_{RS} \circ \bar{\pi}_S$.

(iii) *Given* $f \neq 0$ *in* $V(X, E)$ *there exists* $S \in \mathscr{S}(X)$ *such that* $\bar{\pi}_S(f) \neq 0$.

(iv) *Given a family* $(\bar{f}_S)_{S \in \mathscr{S}(X)}$, $\bar{f}_S \in V(S, E)$, *for which* $\bar{\pi}_{RS}(\bar{f}_S) = \bar{f}_R$ *for all pair* $R \subset S$ *in* $\mathscr{S}(X)$, *then there exists an* $\bar{f}$ *in* $V(X, E)$ *such that* $\bar{f}_S = \bar{\pi}_S(\bar{f})$ *for all* $S \in \mathscr{S}(X)$.

(v) *For all pairs* $R \subset S$ *in* $\mathscr{S}(X)$

$$\bar{q}_R(\bar{\pi}_{RS}(f)) \leqslant \bar{q}_S(f) \qquad \forall f \in V(S, E).$$

Proof. (i) is clair. (ii) Given $f \in V(X, E)$ there exists $g \in \mathscr{V}(X, E)$ such that $j(g) = f$. We know that $\pi_R = \pi_{RS} \circ \pi_S$ and hence

$$(j_R \circ \pi_R)(g) = (j_R \circ \pi_{RS} \circ \pi_S)(g).$$

But

$$(j_R \circ \pi_R)(g) = (\bar{\pi}_R \circ j)(g) = \bar{\pi}_R(f)$$

and

$$\begin{aligned}(j_R \circ \pi_{RS} \circ \pi_S)(g) &= (\bar{\pi}_{RS} \circ j_S \circ \pi_S)(g) = (\bar{\pi}_{RS} \circ \bar{\pi}_S \circ j)(g) \\ &= (\bar{\pi}_{RS} \circ \bar{\pi}_S)(j(g)) = (\bar{\pi}_{RS} \circ \bar{\pi}_S)(f).\end{aligned}$$

Finally

$$\bar{\pi}_R(f) = (\bar{\pi}_{RS} \circ \bar{\pi}_S)(f).$$

(iii) Assume there exists no $S \in \mathscr{S}(X)$ such that $\bar{\pi}_S(f) \neq 0$. For some $g \in \mathscr{V}(X, E)$ $j(g) = f$ and $\bar{\pi}_S(j(g)) = 0$ for all S. But $j_S \circ \pi_S = \bar{\pi}_S \circ j$ and hence $\pi_S(g) \in \mathscr{N}(S, E)$ for all S. This means that $g \in \mathscr{N}(X, E)$ and $f = j(g) = 0$, which contradicts the fact that $f \neq 0$.

(iv) There exists a family $(f_S)_{S \in \mathscr{S}(X)}$, $f_S \in \mathscr{V}(S, E)$ such that $j_S(f_S) = \bar{f}_S$. For all $R \subset S$ in $\mathscr{S}(X)$ $\bar{\pi}_{RS}(\bar{f}_S) = \bar{f}_R$. But

$$(j_R \circ \pi_{RS})(f_S) = (\bar{\pi}_{RS} \circ j_S)(f_S) = j_R(f_R)$$

and hence $q_R(\pi_{RS}(f_S) - f_R) = 0$. By hypotheses 2.6(ii) there exists an

$f \in \mathscr{V}(X, E)$ such that $q_S(f_S - \pi_S(f)) = 0$ for all S. This implies that $f_S - \pi_S(f) \in \mathscr{N}(S, E)$ and

$$\bar{f}_S = j_S(f_S) = (j_S \circ \pi_S)(f) = (\bar{\pi}_S \circ j)(f) = \bar{\pi}_S(j(f))$$

for all $S \in \mathscr{S}(X)$. We can now pick $\bar{f} = j(f)$.

(v) By definition of the quotient norm. ∎

PROPOSITION 2.9. (i) *$V(X, E)$ is isomorphic to the projective limit*

$$V = \underleftarrow{\text{proj.}}(V(S, E), \bar{\pi}_{RS}).$$

(ii) *The initial topology $\bar{\mathscr{T}}_i$ is coarser than the quotient topology $\bar{\mathscr{T}}$ in $V(X, E)$.*

Proof. (i) Directly from Proposition 2.8 (ii) to (iv) by techniques similar to the ones used in the proof of Proposition 2.5.

(ii) Since the diagram (2.2) is commutative $\bar{\pi}_S \circ j = j_S \circ \pi_S$ and $\bar{\pi}_S \circ j$ is continuous for all $S \in \mathscr{S}(X)$. So j is continuous for the topology $\bar{\mathscr{T}}_i$. As for $\bar{\mathscr{T}}$, it is the finest topology for which the map j is continuous. Hence $\bar{\mathscr{T}}_i$ is coarser than $\bar{\mathscr{T}}$. ∎

Remark 3. Notice that Hypotheses 2.6(i), and (iii) have not yet been used. They will be used in Section 3 to show that $V(X, E)$ is a Fréchet space. In particular we shall see that the equivalence of the topologies $\bar{\mathscr{T}}_i$ and $\bar{\mathscr{T}}$ follows from Hypothesis 2.6(iii).

3. PROPERTIES OF THE SPACES $\mathscr{V}(X, E)$ AND $V(X, E)$

In this section we study the properties of the space $\mathscr{V}(X, E)$ of Definition 2.3 under Hypotheses 2.2 and the space $V(X, E)$ constructed from $\mathscr{V}(X, E)$ under Hypotheses 2.6. We have already established that there exists an isomorphism between $\mathscr{V}(X, E)$ (resp. $V(X, E)$) and the projective limit $\mathscr{V}$ (rep. V).

THEOREM 3.1.

(i) *$V(X, E)$ is a Fréchet space and $\mathscr{V}(X, E)$ is a complete locally convex topological space with a denumerable fundamental system of neighborhoods at each point.*

(ii) *The initial topology $\mathscr{T}_i$ (resp. $\bar{\mathscr{T}}_i$) in $\mathscr{V}(X, E)$ (resp. $V(X, E)$) is equivalent to the topology $\mathscr{T}_p$ (resp. $\bar{\mathscr{T}}_p$) defined by the saturated family $(p_S)_{S \in \mathscr{S}(X)}$ (resp. $(\bar{p}_S)_{S \in \mathscr{S}(X)}$ of seminorms, where $p_S = q_S \circ \pi_S$ (resp. ($\bar{p}_S = \bar{q}_S \circ \bar{\pi}_S$).*

(iii) *The quotient topology* $\bar{\mathscr{T}}$ *and the initial topology* $\bar{\mathscr{T}}_i$ *are equivalent in* $V(X, E)$.

(iv) *Assume* $\mathscr{R}(X)$ *is a saturated subfamily of* $\mathscr{S}(X)$ *equivalent to* $\mathscr{S}(X)$ *the initial topologies in* $\mathscr{V}(X, E)$ *(resp.* $V(X, E)$*) with respect to* $\mathscr{R}(X)$ *and* $\mathscr{S}(X)$ *are equivalent.* ∎

The above theorem summarizes our results; it will be proved via several propositions.

Remark. The isomorphism between $\mathscr{V}$ (resp. V) and $\mathscr{V}(X, E)$ (resp. $V(X, E)$) makes it possible to use the well developed theory of projective limit spaces in order to obtain the properties of $\mathscr{V}(X, E)$ (resp. $V(X, E)$). In the remainder we shall not distinguish between $\mathscr{V}$ and $\mathscr{V}(X, E)$ (resp. V and $V(X, E)$).

PROPOSITION 3.2. *The initial topology* $\mathscr{T}_i$ *(resp.* $\bar{\mathscr{T}}_i$*) on* $\mathscr{V}(X, E)$ *(resp.* $V(X, E)$*) for the family* $(\bar{\pi}_S)_{S\in\mathscr{S}(X)}$ *(resp.* $(\pi_S)_{S\in\mathscr{S}(X)}$*) is equivalent to the topology* $\mathscr{T}_p$ *(resp.* $\bar{\mathscr{T}}_p$*) defined by the saturated family* $(p_S)_{S\in\mathscr{S}(X)}$ *(resp.* $(\bar{p}_S)_{S\in\mathscr{S}(X)}$*) of seminorms, where* $p_S = q_S \circ \pi_S$ *(resp.* $\bar{p}_S = \bar{q}_S \circ \bar{\pi}_S$*).*

Proof. Again we only prove that $\mathscr{T}_i$ and $\mathscr{T}_p$ are equivalent. The proof of the equivalence of $\bar{\mathscr{T}}_i$ and $\bar{\mathscr{T}}_p$ proceeds along identical lines. For all $S \in \mathscr{S}(X)$ the composite map $p_S = q_S \circ \pi_S$ is a continuous seminorm for the topology $\mathscr{T}_i$; hence $\mathscr{T}_p$ is coarser than $\mathscr{T}_i$. To show the converse pick a neighborhood U of 0 in $(\mathscr{V}(X, E), \mathscr{T}_i)$. By definition of the initial topology there exists a finite subfamily $\{S_i\}_{i=1}^n$ of $\mathscr{S}(X)$ and for each S_i a neighborhood V_i of 0 in $\mathscr{V}(S_i, E)$ such that

$$\bigcap_{i=1}^{n} \pi_{S_i}^{-1}(V_i) \subset U.$$

For each V_i there exists $r_i > 0$ such that

$$R_i = \{x \in \mathscr{V}(S_i, E) \mid q_{S_i}(x) < r_i\} \subset V_i;$$

thus

$$\pi_{S_i}^{-1}(R_i) \subset \pi_{S_i}^{-1}(V_i).$$

But

$$\pi_{S_i}^{-1}(R_i) = \{y \in \mathscr{V}(X, E) \mid p_{S_i}(y) < r_i\}.$$

Hence there exists a neighborhood $\tilde{U}$,

$$\tilde{U} = \bigcap_{i=1}^{n} \{y \in \mathscr{V}(X, E) \mid p_{S_i}(y) < r_i\},$$

of the origin in $(\mathscr{V}(X, E), \mathscr{T}_p)$ such that

$$\tilde{U} \subset \bigcap_{i=1}^{n} \pi_{S_i}^{-1}(V_i) \subset U.$$

This establishes that $\mathscr{T}_p$ is finer than $\mathscr{T}_i$.

By the definition of a saturated family $\mathscr{S}(X)$ the corresponding family of seminorms $(p_S)_{S \in \mathscr{S}(X)}$ is saturated by Hypothesis 2.4(iii) (Eq. (2.1)). ∎

PROPOSITION 3.3. *The quotient topology $\bar{\mathscr{T}}$ and the initial topology $\bar{\mathscr{T}}_i$ are equivalent in $V(X, E)$.*

Proof. We use Hypotheses 2.6(iii) to prove the proposition. Since the topology $\mathscr{T}_i$ on $\mathscr{V}(X, E)$ can be defined in terms of a saturated family $(p_S)_{S \in \mathscr{S}(X)}$ of seminorms, the family $(\dot{p}_S)_{S \in \mathscr{S}(X)}$ ($\dot{p}_S$, the quotient seminorm on $V(X, E)$ constructed from p_S) defines the quotient topology $\bar{\mathscr{T}}$ on the space $V(X, E)$ [9, p. 106]. By definition

$$\dot{p}_S(\bar{f}) = \inf_{g \in \bar{f}} p_S(g) = \inf_{n \in \mathscr{N}(X,E)} p_S(f + n)$$

where $j(f) = \bar{f}$; but

$$p_S(f + n) = q_S(\pi_S(f) + \pi_S(n))$$

and since

$$\pi_S(\mathscr{N}(X, E)) = \mathscr{N}(S, E)$$

$$\inf_{n \in \mathscr{N}(X,E)} p_S(f + n) = \inf_{n_S \in \mathscr{N}(S,E)} q_S(\pi_S(f) + n_S)$$

$$= \bar{q}_S(j_S(\pi_S(f))$$

(the diagram (1.3) is commutative)

$$= \bar{q}_S(\bar{\pi}_S(j(f)))$$

$$= \bar{p}_S(j(f)) = \bar{p}_S(\bar{f}).$$

This shows that for all $S \in \mathscr{S}(X)$ $\bar{p}_S = \dot{p}_S$ and the topologies $\bar{\mathscr{T}}$ and $\bar{\mathscr{T}}_i$ are equivalent. ∎

PROPOSITION 3.4. *Given an equivalent saturated subfamily $\mathscr{R}(X)$ of $\mathscr{S}(X)$ the initial topologies $\mathscr{T}_i(\mathscr{R})$ and $\mathscr{T}_i(\mathscr{S})$ in $\mathscr{V}(X, E)$ with respect to $\mathscr{R}(X)$ and $\mathscr{S}(X)$ are equivalent. This is also true of $V(X, E)$.*

Proof. The proposition will only be proved for $\mathscr{V}(X, E)$. By definition the initial topology $\mathscr{T}_i(\mathscr{R})$ is weaker than $\mathscr{T}_i(\mathscr{S})$. By Proposition 3.2 both topologies can be defined in terms of their respective family of seminorms $(p_R)_{R\in\mathscr{R}(X)}$ and $(p_S)_{S\in\mathscr{S}(X)}$. But since $\mathscr{R}(X)$ is equivalent to $\mathscr{S}(X)$ for all $S \in \mathscr{S}(X)$ there exists $R \in \mathscr{R}(X)$ such that $S \subset R$. Hence for all $f \in \mathscr{V}(X, E)$ and $S \in \mathscr{S}(X)$ there exists $R \in \mathscr{R}(X)$ such that

$$p_S(f) = q_S(\pi_S(f)) = q_S(\pi_{SR}(\pi_R(f))) \leqslant q_R(\pi_R(f)) = p_R(f)$$

by Hypothesis 2.2(i). Thus $\mathscr{T}_i(\mathscr{R})$ is finer than $\mathscr{T}_i(\mathscr{S})$ since the two families of seminorms are saturated [9, p. 96 and Prop. 2, p. 97]. This proves the proposition. ∎

PROPOSITION 3.5.

(i) *$\mathscr{V}(X, E)$ and $V(X, E)$ are locally convex topological vector spaces with a denumerable fundamental system of neighborhoods at each point.*

(ii) *In addition $V(X, E)$ is Hausdorff and complete (hence Fréchet) and $\mathscr{V}(X, E)$ is complete.*

Proof.

(i) By definition $\mathscr{V}(X, E)$ is a locally convex topological vector space. Since $\mathscr{S}(X)$ is a denumerable saturated family there exists a denumerable subfamily $\mathscr{D}(X)$ which is equivalent to $\mathscr{S}(X)$ and generates the initial topology $\mathscr{T}_i(\mathscr{D})$ equivalent to $\mathscr{T}_i(\mathscr{S})$ (Proposition 3.4). But the topology $\mathscr{T}_i(\mathscr{D})$ can be defined in terms of a denumerable family of seminorms since $\mathscr{D}(X)$ is denumerable (Proposition 3.2). Thus each point in $\mathscr{V}(X, E)$ has a denumerable fundamental system of neighborhoods. The same arguments can be repeated for $V(X, E)$.

(ii) $V(S, E)$ is Hausdorff and complete ($S \in \mathscr{S}(X)$) and for each $f \neq 0$ in $V(X, E)$ there exists $S \in \mathscr{S}(X)$ such that $\bar{\pi}_S(f) \neq 0$ by parts (i) and (iii) of Proposition 2.8. Hence $V(X, E)$ is Hausdorff and complete [9, p. 152 and Proposition 3, p. 153]. Combining the above results with (i) $V(X, E)$ is a Fréchet space. Finally $\mathscr{V}(X, E)$ is complete by the properties of the canonical surjection j since $V(X, E)$ is complete and the initial and quotient topologies $\bar{\mathscr{T}}$ and $\bar{\mathscr{T}}_i$ are equivalent in $V(X, E)$ (Proposition 3.3). ∎

4. EXAMPLES

In this section we assume X is σ-compact (or locally compact denumerable at infinity [7, Chap. 1, Section 9, no. 9]. Let $\mathscr{M}$ be a σ-algebra of subsets of X containing all the Borel sets and μ a positive regular measure on X. Assume also that $\mathscr{S}(X)$ is a denumerably saturated family of nonempty closed subsets of X. We also assume that E is a Banach space.

4.1. *The Spaces $C_{\text{loc}}(X, \mathscr{S}; E)$, $\mathscr{L}^p_{\text{loc}}(\mu, \mathscr{S}; E)$ and $L^p_{\text{loc}}(\mu, \mathscr{S}; E)$.*

Let $C(S, E)$, $S \in \mathscr{S}(X)$, be the Banach space of all continuous maps $f: X \to E$ for which

$$q_S(f) = \sup\{|f(x)|: x \in S\} < \infty.$$

$C_{\text{loc}}(X, \mathscr{S}; E)$ will denote the real vector space of all continuous maps $f: X \to E$ such that $\pi_S(f) \in C(S, E)$ for all $S \in \mathscr{S}(X)$. Here

$$V(X, E) = \mathscr{V}(X, E) = C_{\text{loc}}(X, \mathscr{S}; E)$$

is a Fréchet space.

Let $(X, \mathscr{M}, \mu)$ be a measured space [10, p. 229], where X, $\mathscr{M}$ and μ are as defined at the beginning of this section. Since $S \in \mathscr{S}(X)$ is closed it is measurable and we denote by $\mathscr{L}^p(\mu \mid S, E)$ the real vector space of all μ-measurable [10, p. 232] maps $X \to E$ which are p-integrable in S, $1 < p < \infty$, or essentially bounded in S, $p = \infty$. $\mathscr{L}^p(\mu \mid S, E)$ is endowed with the seminorm

$$q_S(f) = \begin{cases} \left[\int_S |f|^p \, d\mu\right]^{1/p} & \text{if} \quad 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_S |f| & \text{if} \quad p = \infty. \end{cases}$$

$\mathscr{L}^p_{\text{loc}}(\mu, \mathscr{S}; E)$ will denote the real vector space of all μ-measurable maps $f: X \to E$ such that $\pi_S(f) \in \mathscr{L}^p(\mu \mid S, E))$ for all $S \in \mathscr{S}(X)$. Corresponding to $\mathscr{L}^p_{\text{loc}}(\mu, \mathscr{S}; E)$ (resp. $\mathscr{L}^p(\mu \mid S, E)$) we define the natural quotient space $L^p_{\text{loc}}(\mu, \mathscr{S}; E)$ (resp. $\mathscr{L}^p(\mu \mid S, E)$. Here the conclusions of Propositions 2.4, 2.8, and Theorem 3.1 apply with

$$\mathscr{V}(X, E) = \mathscr{L}^p_{\text{loc}}(\mu, \mathscr{S}; E) \qquad \text{and} \qquad V(X, E) = L^p_{\text{loc}}(\mu, \mathscr{S}; E).$$

4.2. *Spaces with the Topology of Compact Convergence.*

Here we further specialize the results of Section 4.1 to the situation where $\mathscr{S}(X)$ is $\mathscr{K}(X)$ the family of all nonempty subsets of a σ-compact space X. Since $\mathscr{K}(X)$ is a denumerably saturated family (see examples in Section 2), the results of Section 4.1 apply. It is customary to denote the spaces $C_{\text{loc}}(X, \mathscr{K}; E)$, $\mathscr{L}^p_{\text{loc}}(\mu, \mathscr{K}; E)$ and $L^p_{\text{loc}}(\mu, \mathscr{K}; E)$ by $C(X, E)$, $\mathscr{L}^p_{\text{loc}}(\mu, E)$ and $L^p_{\text{loc}}(\mu, E)$ [13, p. 222]. The following propositions are interesting in the sense that they are proved via the projective limit structure of the spaces in presence.

PROPOSITION 4.1. *With X, $\mathscr{M}$, μ and E as defined earlier, $C(X, E)$ is dense in $L^p_{\text{loc}}(\mu, E)$ for $1 \leqslant p < \infty$.*

Proof. For each $K \in \mathscr{K}(X)$ we know that $C(K, E)$ is dense in $L^p(\mu \mid K, E)$. Denote by $j_K: C(K, E) \to L^p(\mu \mid K, E)$ the continuous injection. This generates the unique continuous injection j making the following diagram commutative for all $K \in \mathscr{K}(X)$:

$$\begin{array}{ccc} C(X, E) & \xrightarrow{\ j\ } & L^p_{\mathrm{loc}}(X, E) \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \pi_K} \\ C(K, E) & \xrightarrow{\ j_K\ } & L^p(\mu \mid K, E). \end{array}$$

The spaces $j_K(C(K, E))$ form a projective system with respect to the family of maps $(\bar{\pi}_{KL})$:

$$j(C(X, E)) \approx \varprojlim\, (j_K(C(K, E), \bar{\pi}_{KL}).$$

Moreover

$$\overline{j(C(X, E))} \approx \overline{\varprojlim(j_K(C(K, E), \bar{\pi}_{KL})} = \varprojlim(\overline{j_K(C(K, E)}, \bar{\pi}_{KL})$$

[7, Chap. 1, Section 4, p. 52] and

$$\varprojlim(\overline{j_K(C(K, E))}, \bar{\pi}_{KL}) = \varprojlim(L^p(K, E), \bar{\pi}_{KL}) \approx L^p_{\mathrm{loc}}(X, E).$$

Hence $C(X, E)$ is dense in $L^p_{\mathrm{loc}}(X, E)$. ∎

PROPOSITION 4.2. *The injection $L^p_{\mathrm{loc}}(\mu, E) \to L^{p'}_{\mathrm{loc}}(\mu, E)$ is continuous for all p' and p such that $1 \leqslant p' \leqslant p \leqslant \infty$.*

Proof. For all $K \in \mathscr{K}(X)$ the injection map $j_K: L^p(\mu \mid K, E) \to L^{p'}(\mu \mid K, E)$ is continuous. For all $K \subset L$ in $\mathscr{K}(X)$ the following diagram is commutative

$$\begin{array}{ccc} L^p(\mu \mid L, E) & \xrightarrow{\ j_L\ } & L^{p'}(\mu \mid L, E) \\ {\scriptstyle \pi_{KL}}\downarrow & & \downarrow{\scriptstyle \pi'_{KL}} \\ L^p(\mu \mid K, E) & \xrightarrow{\ j_K\ } & L^{p'}(\mu \mid K, E). \end{array}$$

This establishes the existence and uniqueness of the continuous injection $j: L^p_{\mathrm{loc}}(\mu, E) \to L^{p'}_{\mathrm{loc}}(\mu, E)$ making the following diagram commutative:

$$\begin{array}{ccc} L^p_{\mathrm{loc}}(\mu, E) & \xrightarrow{\ j\ } & L^{p'}_{\mathrm{loc}}(\mu, E) \\ \downarrow{\scriptstyle \pi_K} & & \downarrow{\scriptstyle \pi_K'} \\ L^p(\mu \mid K, E) & \xrightarrow{\ j_K\ } & L^{p'}(\mu \mid K, E). \end{array}$$ ∎

When $X \subset R^n$ it is often more convenient to work with a saturated family equivalent to $\mathscr{K}(X)$. For instance let $X = [t_0, t_1[$, $t_0 \in R$ and $t_1 \in]t_0, +\infty]$, then the denumerably saturated subfamily $\mathscr{E}(t_0, t_1)$ (see Examples in Section 2) of $\mathscr{K}(t_0, t_1) = \mathscr{K}([t_0, t_1[)$ is equivalent to the latter. It is usually more convenient to test a property for all $t \in]t_0, t_1[$ rather than for all compact subsets of $[t_0, t_1[$ (Thm. 3.1(iv)).

4.3. *The extended Spaces of Sandberg and Zames*

Let $X = R$ (resp. $X = [t_0, \infty[$ for some $t_0 \in R$). Consider the denumerably saturated family $\mathscr{E}(-\infty, \infty)$ (resp. $\mathscr{E}(t_0, \infty)$) as defined in the examples of Section 2. Let m be the complete Lebesgue measure in R. The extended spaces of Sandberg and Zames are

$$L_e^p(-\infty, \infty; E) = L^p_{\mathscr{E}(-\infty,\infty)}(m, E)$$

and

$$L_e^p(t_0, \infty; E) = L^p_{\mathscr{E}(t_0,\infty)}(m, E).$$

However in their original definition the extended spaces were not topologized. This was attempted later in a effort to generalize the notion of an extended space; for this matter the reader is referred to J. C. Willems [11] who makes use of such spaces to study the stability of feedback systems [12]. Notice that the family $\mathscr{E}(t_0, +\infty)$ is an equivalent subfamily of

$$\mathscr{K}(t_0, +\infty) = \mathscr{K}([t_0, +\infty[)$$

and that

$$L_e^p(t_0, \infty; E) = L^p_{\mathscr{K}(t_0,\infty)}(m, E).$$

However this is not true of $L_e^p(-\infty, \infty; E)$ and $L^p_{\mathscr{K}(-\infty,\infty)}(m, E)$ since $\mathscr{E}(-\infty, \infty)$ and $\mathscr{K}(-\infty, \infty) = \mathscr{K}(]-\infty, \infty[)$ are not equivalent saturated families (they are not even equal).

As can be seen from the above examples the projective limit structure is very natural in engineering problems. As a result this detailed study opens the way to the solution of such problems by functional analysis techniques.

4.4 *The Spaces* $AC^p_{\text{loc}}(t_0, t_1; E)$.

Let $t_0 \in R$, $t_1 \in]t_0, \infty]$, $X = [t_0, t_1[$ and $1 \leqslant p \leqslant \infty$. For all $t \in]t_0, t_1[$ let $AC^p(t_0, t; E)$ be the vector space of all maps $f: [t_0, t] \to E$ which are differentiable almost everywhere in $[t_0, t]$ with derivative df/dt in $\mathscr{L}^p(t_0, t; E)$ and such that

$$f(s) = f(t_0) + \int_{t_0}^{s} \frac{df}{dr}(r)\, dr, \qquad s \in [t_0, t].$$

In simpler terms $AC^p(t_0, t; E)$ is the vector space of all absolutely continuous maps defined on $[t_0, t]$ with values in E which have a derivative in $\mathscr{L}^p(t_0, t; E)$. This space naturally arises in the study of differential equations since it is precisely the space in which solutions are sought ([5], [6]). It is a Banach space when it is endowed with the norms:

$$q_t(f) = \begin{cases} \left[|f(t_0)|^p + \int_{t_0}^{t} \left|\frac{df(s)}{ds}\right|^p ds\right]^{1/p}, & 1 \leqslant p < \infty, \\ \max\left\{|f(t_0)|, \operatorname*{ess\,sup}_{[t_0,t]} \left|\frac{df(s)}{ds}\right|\right\}, & p = \infty. \end{cases}$$

We now define the vector space $AC^p_{\text{loc}}(t_0, t_1; E)$ of all absolutely continuous maps $[t_0, t_1[\to E$ with a derivative in $\mathscr{L}^p_{\text{loc}}(t_0, t_1; E)$. It is easy to verify that this space and the saturated family $\mathscr{K}(t_0, t_1)$ satisfy Definition 2.1 and Hypotheses 2.2 and 2.6; here $\mathscr{V}(X, E) = V(X, E)$ and $AC^p_{\text{loc}}(t_0, t_1; E)$ is a Fréchet space.

Let $\text{St}(t_0, t; E)$ be the vector space of all step maps $[t_0, t] \to E$ (with respect to the Lebesque measure), where $t \in]t_0, t_1[$. We define the vector space $\text{St}_{\text{loc}}(t_0, t_1; E)$ of all maps $f: [t_0, t_1[\to E$ such that $\pi_t(f) \in \text{St}(t_0, t; E)$ for all $t \in]t_0, t_1[$. By techniques similar to the ones of Proposition 4.1 it is easy to show that $\text{St}_{\text{loc}}(t_0, t_1; E)$ is dense in $L^p_{\text{loc}}(t_0, t_1; E)$ for $1 \leqslant p < \infty$. It is also readily seen that $AC^p(t_0, t; E)$ (resp. $AC^p_{\text{loc}}(t_0, t_1; E)$) is isometrically isomorphic to $E \times L^p(t_0, t; E)$ (resp. $E \times L^p_{\text{loc}}(t_0, t_1; E)$). An immediate consequence of all this is that the vector space of all maps $f: [t_0, t_1[\to E$ of the form

$$f(t) = x_0 + \int_{t_0}^{t} s(r)\, dr, \qquad t \in [t_0, t_1[,$$

where $x_0 \in E$ and $s \in \text{St}_{\text{loc}}(t_0, t_1; E)$, is dense in $AC^p_{\text{loc}}(t_0, t_1; E)$ for $1 \leqslant p < \infty$. This means that we can approximate elements in AC^p_{loc} by piecewise linear maps.

The space AC^p_{loc} and L^p_{loc} were used in the theory of hereditary differential systems ([5], [6]). As a specific example we quote the following theorem from [5].

THEOREM 4.3. *Let $N \geqslant 1$ be an integer, let $a > 0$ and*

$$-a = \theta_N < \cdots < \theta_1 < \theta_0 = 0$$

be reals, let E be a Banach space, let $1 \leqslant p < \infty$ and let

$$-\infty < t_0 < t_1 \leqslant +\infty.$$

Assume that the map $f: [t_0, t_1[\times (E^N \times M^p(-b, 0; E))^1 \to E$ *satisfy the following properties:*

(CAR–1) *The map* $t \mapsto f(t, z)$ *is m-measurable for all* $z \in E^N \times M^p(-b, 0; E)$;

(LIP) *There exists a positive function n in* $L^q_{loc}(t_0, t_1; R)$, $p^{-1} + q^{-1} = 1$,

such that for all z_1 *and* z_2 *in* $E^N \times M^p(-b, 0; E)$

$$|f(t, z_1) - f(t, z_2)| \leqslant n(t) \qquad \| z_1 - z_2 \|_{E^N \times M^p}$$

a.e. in $[t_0, t_1[$;

(BC) *The map* $t \mapsto f(t, 0)$ *is an element of* $L^1_{loc}(t_0, t_1; E)$.

Then there exists a unique global solution x_h *in* $AC^1_{loc}(t_0, t_1; E)$ *to the Cauchy problem*

$$\frac{d\tilde{x}}{dt}(t) = f(t, \tilde{x}(t + \theta_N), \ldots, \tilde{x}(t + \theta_1), \tilde{x}_t) \qquad p.p.\ [t_0, t_1[$$

$$\tilde{x}(t_0 + \theta) = h(\theta), \qquad \theta \in I(-b, 0), \quad h \in M^p(-b, 0; E),$$

where

$$\tilde{x}(s) = \begin{cases} h(s - t_0), & s \in I(t_0 - b, t_0) \\ x_h(s), & s \in [t_0, t_1[\end{cases}$$

and $\tilde{x}_t \in M^p(-b, 0; E)$ *is defined by* $\tilde{x}_t(\theta) = \tilde{x}(t + \theta)$. *Moreover the map*

$$h \mapsto x_h: M^p(-b, 0; E) \to AC^1_{loc}(t_0, t_1; E)$$

is continuous and for all t in $]t_0, t_1[$

$$\| \pi_t(x_h - x_k) \|_{AC^1} \leqslant d(p, t - t_0) \| h - k \|_{M^p}$$

for some constant $d(p, t - t_0) > 0$ ($\pi_t x$ *is the restriction to* $[t_0, t]$ *of a map* $x: [t_0, t_1[\to E$).

Now that we have solution in the quite large space $AC^1_{loc}(t_0, t_1; E)$ we can study the various properties of families of solutions.

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[1] $M^p(-b, 0; E)$ is isometrically isomorphic to $E \times L^p(-b, 0; E)$. See (5) for details.

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