Strong continuity of generalized Feynman–Kac semigroups: Necessary and sufficient conditions

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Abstract
Let (E, D(E)) be a strongly local, quasi-regular symmetric Dirichlet form on L^2(E; m) and ((X_t)_{t≥0}, (P_x)_{x∈E}) the diffusion process associated with (E, D(E)). For u ∈ D(E)_c, u has a quasi-continuous version ˜u and ˜u(X_t) has Fukushima’s decomposition: ˜u(X_t) − ˜u(X_0) = M^u_t + N^u_t, where M^u_t is the martingale part and N^u_t is the zero energy part. In this paper, we study the strong continuity of the generalized Feynman–Kac semigroup defined by

P^u_t f(x) = E_x[e^{N^u_t} f(X_t)], t ≥ 0.

Two necessary and sufficient conditions for (P^u_t)_{t≥0} to be strongly continuous are obtained by considering the quadratic form (Q^u, D(E)_b), where Q^u(f, f) := E(f, f) + E(u, f^2) for f ∈ D(E)_b, and the energy measure μ_⟨u⟩ of u, respectively. An example is also given to show that (P^u_t)_{t≥0} is strongly continuous when μ_⟨u⟩ is not a measure of the Kato class but of the Hardy class with the constant δμ_⟨u⟩(E) ≤ 1/2 (cf. Definition 4.5).

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1. Introduction

Let $E$ be a metrizable Lusin space, i.e. a space topologically isomorphic to a Borel subset of a complete separable metric space, $\mathcal{B}(E)$ the Borel $\sigma$-field of $E$ and $m$ a $\sigma$-finite measure on $(E, \mathcal{B}(E))$. Suppose that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E; m)$. Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is associated with a right-continuous Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, (X_t)_{t \geq 0}, (P_t)_{t \geq 0})$ in the sense that $T_t f = P_t f$ $m$-a.e. for any $f \in B_b(E) \cap L^2(E; m)$ and $t \geq 0$, where $B_b(E)$ is the set of all bounded $\mathcal{B}(E)$-measurable functions on $E$, $(T_t)_{t \geq 0}$ the $L^2$-semigroup corresponding to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and $(P_t)_{t \geq 0}$ the transition semigroup of $X$. We refer the readers to [8] for the theory of regular Dirichlet forms.

Hence most results originally stated and proved for regular Dirichlet forms also hold for quasi-regular Dirichlet forms. It is proved in [4] that a Dirichlet form is quasi-regular if and only if it is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space. Hence most results originally stated and proved for regular Dirichlet forms also hold for quasi-regular Dirichlet forms. We refer the readers to [8] for the theory of regular Dirichlet forms.

The notations and terminologies of this paper follow [8,11]. We use $(\cdot, \cdot)_m$ to denote the inner product of $L^2(E; m)$. Let $U \subset E$ be an open set. Denote

$$\mathcal{D}(\mathcal{E})_{U^c} = \{ u \in \mathcal{D}(\mathcal{E}) | u = 0 \text{ m-a.e. on } U \}.$$ 

An increasing sequence $\{F_k\}_{k \geq 1}$ of closed subsets of $E$ is called an $\mathcal{E}$-nest if $\bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_{F_k}$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to (w.r.t.) the $\mathcal{E}_1^{1/2}$-norm, where $\mathcal{E}_1(u, u) := \mathcal{E}(u, u) + (u, u)_m$. Equivalently,

$$\{F_k\}_{k \geq 1} \text{ is an } \mathcal{E}\text{-nest } \text{ if and only if } \quad P_m \left( \lim_{k \to \infty} \tau(F_k) < \zeta \right) = 0, \quad (1.1)$$

where $\zeta$ is the lifetime of $X$ and $\tau(B) := \inf\{t > 0 | X_t \notin B\}$ is the first exit time of $B$ for any $B \in \mathcal{B}(E)$. Let $F$ be a subset of $E$. Then, $F$ is called $\mathcal{E}$-exceptional if there exists an $\mathcal{E}$-nest $\{F_k\}_{k \geq 1}$ such that $F \subset \bigcap_{k \geq 1} F_k^c$. A property about $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ or $X$ is said to hold quasi-everywhere (q.e. for abbreviation) if it holds for any point in $E$ except for an $\mathcal{E}$-exceptional set. Denote by $(\mathcal{E}, \mathcal{D}(\mathcal{E})_e)$ the extended Dirichlet space of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Hereafter $\mathcal{D}(\mathcal{E})_e$ is the family of $\mathcal{B}(E)$-measurable functions $u$ on $E$ that is finite $m$-a.e. and there is an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E})$ such that $\lim_{n \to \infty} u_n = u$ $m$-a.e. on $E$. It is known that every element $u \in \mathcal{D}(\mathcal{E})_e$ admits an $\mathcal{E}$-quasi-continuous $m$-version $\tilde{u}$, where $\mathcal{E}$-quasi-continuous means that there exists an $\mathcal{E}$-nest $\{F_k\}_{k \geq 1}$ such that $\tilde{u}|_{F_k}$ is continuous on $F_k$ for each $k$.

Following [8], we say that a subset $A \subset E$ is quasi-open (respectively, quasi-closed) if there exists an $\mathcal{E}$-nest $\{F_k\}_{k \geq 1}$ such that $F_k \cap A$ is relatively open (respectively, relatively closed) in $F_k$ for each $k$. Let $u$ be an $m$-a.e. defined function on $E$. Then there exists a smallest (up to an $\mathcal{E}$-exceptional set) quasi-closed set $F$, which is called the quasi-support of $u$ and is denoted by $\text{supp}_q[u]$, such that $\int_{E \setminus F} |u(x)| m(dx) = 0$. We use the same notation for a function $f$ ($m$-a.e. defined) on $E$ and for the $m$-equivalence class of functions represented by $f$, if there is no risk of confusion. From now on till the end of this paper, we assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strongly local, quasi-regular symmetric Dirichlet form on $L^2(E; m)$, where “strongly local” means that $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{D}(\mathcal{E})$ and $u$ is constant $\mathcal{E}$-q.e. on a quasi-open set containing $\text{supp}_q[v]$.

For $u \in \mathcal{D}(\mathcal{E})_e$, we denote by $\mu_{\langle u \rangle}$ the energy measure of $u$. Then
\[ \mathcal{E}(u, u) = \frac{1}{2} \int_{E} \mu_{(u)}(dx). \]

It is well known that \( \tilde{u}(X_t) \) has Fukushima’s decomposition

\[ \tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u, \quad (1.2) \]

where \( M_t^u \) is a square-integrable martingale additive functional (AF) and \( N_t^u \) is a continuous AF of zero energy. This paper is concerned with the following generalized Feynman–Kac semigroup

\[ P_t^u f(x) := E_x \left[ e^{N_t^u} f(X_t) \right], \quad f \geq 0 \text{ and } t \geq 0, \]

where \( E_x \) is the expectation w.r.t. \( P_x \) for \( x \in E \).

Note that in general \( (N_t^u)_{t \geq 0} \) is not of finite variation (cf. [5,8]). Hence the classical results on the strong continuity of \( (P_t^u)_{t \geq 0} \) (see [2]) do not apply directly. Under the assumption that \( X \) is the standard \( d \)-dimensional Brownian motion, \( u \) is a bounded continuous function on \( \mathbb{R}^d \) and \( |\nabla u|^2 \) belongs to the Kato class, Glover et al. proved in [9] that \( (P_t^u)_{t \geq 0} \) is a strongly continuous semigroup on \( L^2(\mathbb{R}^d; dx). \) Moreover, they gave the explicit representation for the closed quadratic form corresponding to \( (P_t^u)_{t \geq 0} \). In [14], Zhang generalized the results of [9] to symmetric Lévy processes on \( \mathbb{R}^d \) and removed the assumption that \( u \) is bounded continuous. Furthermore, in the remarkable paper [5], Chen and Zhang established the corresponding results for general symmetric right Markov processes via the Girsanov transformation. They proved that if \( \mu_{(u)} \) is a measure of the Kato class, then \( (P_t^u)_{t \geq 0} \) is a strongly continuous semigroup on \( L^2(E; m) \). Also, they characterized the closed quadratic form corresponding to \( (P_t^u)_{t \geq 0} \) (cf. [5, Theorem 1.2]). Recently, Fitzsimmons and Kuwae studied the strong continuity of \( (P_t^u)_{t \geq 0} \) under the assumption that \( X \) is a symmetric diffusion process and \( \mu_{(u)} \) is a measure of the Hardy class with the constant \( \delta_{\mu_{(u)}}(\mathcal{E}) < \frac{1}{2} \) (see [7, Example 5.3]).

All the results mentioned above give only sufficient conditions for \( (P_t^u)_{t \geq 0} \) to be strongly continuous, where \( \mu_{(u)} \) is assumed to be either of the Kato class or of the Hardy class with the constant \( \delta_{\mu_{(u)}}(\mathcal{E}) < \frac{1}{2} \). To the best of our knowledge, no necessary and sufficient condition has been presented for the strong continuity of \( (P_t^u)_{t \geq 0} \). In this paper, we give two necessary and sufficient conditions for \( (P_t^u)_{t \geq 0} \) to be strongly continuous by virtue of the quadratic form \( (Q^u, \mathcal{D}(\mathcal{E})_b) \), where \( Q^u(f, f) := \mathcal{E}(f, f) + \mathcal{E}(u, f^2) \) for \( f \in \mathcal{D}(\mathcal{E})_b \), and \( \mu_{(u)} \) (cf. Theorems 4.3 and 4.6). We also discuss the relation between \( (Q^u, \mathcal{D}(\mathcal{E})_b) \) and the closed quadratic form corresponding to \( (P_t^u)_{t \geq 0} \) when \( (P_t^u)_{t \geq 0} \) is strongly continuous. Moreover, we give some sufficient conditions for \( (P_t^u)_{t \geq 0} \) to be strongly continuous. In particular, we present an example showing that \( (P_t^u)_{t \geq 0} \) is strongly continuous when \( \mu_{(u)} \) is not a measure of the Kato class but of the Hardy class with the constant \( \delta_{\mu_{(u)}}(\mathcal{E}) \leq \frac{1}{2} \). Finally, we present an example in which \( (P_t^u)_{t \geq 0} \) is not strongly continuous since \( (Q^u, \mathcal{D}(\mathcal{E})_b) \) is not lower semibounded. The main idea of this work is to transfer the case of infinite variation into the case of finite variation via the Girsanov transformation for symmetric diffusion process, perturbation of Dirichlet form and \( h \)-transformation for quadratic form.

The rest part of this paper is organized as follows. In Section 2, we consider the Girsanov transformation for \( X \) and the Dirichlet form associated with the transformed process. In Section 3, we discuss perturbation of the Dirichlet form by a signed smooth measure and the \( h \)-transformation for a quadratic form. In Section 4, we present the main results on the strong continuity of the generalized Feynman–Kac semigroup. In Section 5, we give some examples.
2. Girsanov transformation for symmetric diffusion process and associated Dirichlet form

Let $M^u$ be the square-integrable martingale AF given in (1.2) and $(M^u_t)$ the quadratic variation of $M^u$. We define
\[ Z^u_t := e^{-M^u_t - \frac{1}{2}(M^u)_t}. \]
Then $(Z^u_t, \mathcal{F}_t)_{t \geq 0}$ is a positive local martingale and hence a positive supermartingale. By [13, Section 62] (cf. also [3,5]), we know that
\[ d\hat{P}_x \bigg|_{\mathcal{F}_t} := Z^u_t, \quad x \in E, \]
uniquely determine a family of probability measures $(\hat{P}_x)_{x \in E}$ on $(\Omega, \mathcal{F}_\infty)$, where $\mathcal{F}_\infty := \bigcup_{t \geq 0} \mathcal{F}_t$. Moreover, $X$ is a Markov process on $E$ under $(\hat{P}_x)_{x \in E}$. Denote by $\hat{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, (\hat{X}_t)_{t \geq 0}, (\hat{P}_x)_{x \in E})$ the transformed process. Here $\hat{X}_t = X_t$ but we use $\hat{X}_t$ for emphasis when working with $\hat{P}_x$. Let $\hat{P}_t$ be the transition semigroup of $X_t$ under $\hat{P}_x$, that is,
\[ \hat{P}_t f(x) := \hat{E}_x [ f(\hat{X}_t)] = E_x [Z^u_t f(X_t)], \]
where $\hat{E}_x$ is the expectation w.r.t. $\hat{P}_x$ for $x \in E$. Similar to [5, Lemma 3.2 and Theorem 3.3], we get

**Theorem 2.1.**

(i) $\hat{P}_t$ is symmetric on $L^2(E; e^{-2u} m)$.

(ii) Let $(A_t)_{t \geq 0}$ be a positive continuous AF (PCAF) of $X$ with Revuz measure $\mu$, then the Revuz measure for $(A_t)_{t \geq 0}$ as a PCAF of $\hat{X}$ is $e^{-2u} \mu$.

Let $(\hat{\mathcal{E}}, \mathcal{D}(\hat{\mathcal{E}}))$ be the symmetric Dirichlet form on $L^2(E; e^{-2u} m)$ associated with $\hat{X}$. Since $\hat{u}$ is an $\mathcal{E}$-quasi-continuous function, there exists an $\mathcal{E}$-nest (for $X$) $\{F_k\}_{k \geq 1}$ such that $\hat{u}|_{F_k}$ is continuous on $F_k$ for each $k$. In the following, we fix such an $\mathcal{E}$-nest $\{F_k\}_{k \geq 1}$. By (1.1), $\{F_k\}_{k \geq 1}$ is also an $\hat{\mathcal{E}}$-nest (for $\hat{X}$). Define
\[ \mathcal{D}(\mathcal{E})_{F_k,b} = \mathcal{D}(\mathcal{E})_{F_k} \cap L^\infty(E; m) \quad \text{and} \quad \hat{\mathcal{D}}(\hat{\mathcal{E}})_{F_k,b} = \hat{\mathcal{D}}(\hat{\mathcal{E}})_{F_k} \cap L^\infty(E; m), \]
where $L^\infty(E; m)$ is the set of all essentially bounded functions on $E$ with the supremum norm $\| \cdot \|_\infty$.

**Theorem 2.2.**

(i) If $f \in \mathcal{D}(\mathcal{E})_{F_k,b}$, then $f \in \mathcal{D}(\hat{\mathcal{E}})_{F_k,b}$ and
\[ \hat{\mathcal{E}}(f, f) = \frac{1}{2} \int_E e^{-2u(x)} \mu(f)(dx). \]

(ii) $\bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_{F_k,b} \subseteq \bigcup_{k \geq 1} \mathcal{D}(\hat{\mathcal{E}})_{F_k,b}$. 
Proof. (i) The basic idea of the proof comes from [5, Theorem 3.4], however, necessary modifications should be made since $u$ can be an unbounded function. For $f \in D(\mathcal{E})_{F_k,b}$, we have the following Lyons–Zheng’s forward–backward martingale decomposition:

$$\tilde{f}(X_t) - \tilde{f}(X_0) = \frac{1}{2} (M^f_t - M^f_t \circ r_t), \quad P_m \text{-a.s.,}$$  \hfill (2.1)

where $M^f_t$ is the martingale part in Fukushima’s decomposition for $\tilde{f}(X_t)$ and $r_t$ is a time reversal operator. By Girsanov’s theorem (cf. [10]),

$$K_t := M^f_t - \int_0^t \frac{1}{Z_s^t} d[M^f, Z^u]_s = M^f_t - [M^f, - M^u]_t$$

is a martingale AF under $\hat{P}_x$ and

$$(K)_t(\hat{P}_x) = [M^f]_t(P_x).$$ \hfill (2.2)

Since the Revuz measure for $[M^f]_t$ as a PCAF of $X$ is $\mu_{(f)}$, the Revuz measure for $(K)_t$ as a PCAF of $\hat{X}$ is $e^{-2u} \mu_{(f)}$ by Theorem 2.1(ii) and (2.2). Noting that

$$[M^f, - M^u]_t = [M^f, - M^u]_t \circ r_t,$$

we get by (2.1) that

$$\tilde{f}(X_t) - \tilde{f}(X_0) = \frac{1}{2} (K_t - K_t \circ r_t), \quad P_m \text{-a.s.}$$ \hfill (2.3)

Denote $\nu = e^{-2u} \mu$ and

$$\hat{P}_\nu(\cdot) = \int_E \hat{P}_x(\cdot) \nu(dx).$$

By (2.3), we get

$$\lim_{t \to 0} \frac{1}{t} \hat{E}_\nu \left[ (f(\hat{X}_t) - f(\hat{X}_0))^2 \right] \leq \lim_{t \to 0} \frac{1}{2t} \left\{ \hat{E}_\nu \left[ K_t^2 \right] + \hat{E}_\nu \left[ (K_t \circ r_t)^2 \right] \right\}$$

$$= \lim_{t \to 0} \frac{1}{t} \hat{E}_\nu \left[ (K)_t \right]$$

$$= \int_E e^{-2u(x)} \mu_{(f)}(dx).$$

Note that $f = 0$ m-a.e. on $F_k^c$ and $e^{-u}$ is a bounded function on $F_k$. Denote $C_k := \|e^{-u}\|_{F_k} \|_\infty$. Then

$$\int_E e^{-2u(x)} \mu_{(f)}(dx) \leq C_k^2 \int_E \mu_{(f)}(dx)$$
and hence
\[ \lim_{t \to 0} \frac{1}{t} \hat{E}_v \left[ (f(\hat{X}_t) - f(\hat{X}_0))^2 \right] < \infty. \]

Therefore
\[ \lim_{t \to 0} \frac{1}{t} \int \left( f(x) - \hat{P}_t f(x) \right) f(x) \nu(dx) = \lim_{t \to 0} \frac{1}{t} \left\{ \frac{1}{2} \hat{E}_v \left[ (f(\hat{X}_t) - f(\hat{X}_0))^2 \right] \right\} < \infty, \]

which implies that \( f \in \mathcal{D}(\hat{E}) \) and hence \( f \in \mathcal{D}(\hat{E})_{F_k,b} \).

Since \( f \in \mathcal{D}(\hat{E}) \cap \mathcal{D}(E) \), \( f \) admits two Fukushima’s decompositions
\[
\tilde{f}(\hat{X}_t) - \tilde{f}(\hat{X}_0) = \hat{M}_t^f + \hat{N}_t^f \quad \text{under } \hat{P}_x,
\]
\[
\tilde{f}(X_t) - \tilde{f}(X_0) = M_t^f + N_t^f \quad \text{under } P_x.
\]

By Girsanov’s theorem again, \( K_t = M_t^f - \langle M^f, -M^\mu \rangle_t \) is a martingale AF under \( \hat{P}_x \). Then
\[ \hat{M}_t^f = K_t, \quad \forall t \geq 0. \quad (2.4) \]

By (2.4),
\[ \hat{E}(f, f) = \lim_{t \to 0} \frac{1}{2t} \hat{E}_v \left[ (f(\hat{X}_t) - f(\hat{X}_0))^2 \right] = \lim_{t \to 0} \frac{1}{2t} \hat{E}_v \left[ (\hat{M}_t^f)^2 \right] = \lim_{t \to 0} \frac{1}{2t} \hat{E}_v \left[ \langle K \rangle_t \right] \]
\[ = \frac{1}{2} \int \frac{e^{-\alpha(u(x))}}{\mu(f)} (dx). \]

(ii) Direct consequence of (i). \( \square \)

**Corollary 2.3.** If \( f \in \mathcal{D}(E)_{F_k,b} \), then \( fe^u, fe^{-u} \in \mathcal{D}(E)_{F_k,b} \) and hence \( fe^u, fe^{-u} \in \mathcal{D}(\hat{E})_{F_k,b} \).

**Proof.** Denote \( c = \|u|_{F_k}\|_\infty \). Then \( u_c := ((-c) \vee u) \wedge c \in \mathcal{D}(E) \) and \( (e^{u_c} - 1) \in \mathcal{D}(E) \). Let \( \{g_n\}_{n \geq 1} \) be an approximating sequence of \( (e^{u_c} - 1) \) in \( (E, \mathcal{D}(E)) \). Without loss of generality, we assume that \( \|g_n\|_\infty \leq C, \forall n \geq 1 \), for some constant \( C > 0 \). Then,
\[
\mathcal{E}^{1/2}(fg_n, fg_n) \leq \|f\|_\infty \mathcal{E}^{1/2}(g_n, g_n) + \|g_n\|_\infty \mathcal{E}^{1/2}(f, f)
\leq \|f\|_\infty \mathcal{E}^{1/2}(g_n, g_n) + C \mathcal{E}^{1/2}(f, f)
\rightarrow \|f\|_\infty \mathcal{E}^{1/2}(e^{u_c} - 1, e^{u_c} - 1) + C \mathcal{E}^{1/2}(f, f) \quad \text{as } n \to \infty
\]
and hence \( f(e^{u_c} - 1) \in \mathcal{D}(E) \) by [11, Lemma I.2.12]. Therefore \( fe^u = fe^{u_c} = f(e^{u_c} - 1) + f \in \mathcal{D}(E)_{F_k,b} \). Similarly, we can show that \( fe^{-u} \in \mathcal{D}(E)_{F_k,b} \). The remainder of the proof follows by Theorem 2.2. \( \square \)

Set
\[
\hat{M}_t^u := M_t^u + [M^u]_t(P_x). \quad (2.5)
\]
Then $\tilde{M}_t^u$ is a local martingale AF of $\hat{X}$ by Girsanov’s theorem. Let $\tilde{Z}_t^u$ be the solution of the stochastic differential equation

$$\tilde{Z}_t^u = 1 + \int_0^t \tilde{Z}_s^u \, d\tilde{M}_s^u.$$ 

Then, $\tilde{Z}_t^u$ is a positive local martingale on $[0, \infty)$ and hence a positive supermartingale.

$$d\tilde{P}_x \frac{d\tilde{P}_x}{d\tilde{P}_x} \bigg|_{\mathcal{F}_t} := \tilde{Z}_t^u, \quad x \in E,$$

uniquely determine a family of probability measures $(\tilde{P}_x)_x \in E$ on $(\Omega, \mathcal{F}_\infty)$. Moreover, $X$ is a Markov process on $E$ under $(\tilde{P}_x)_x \in E$. Denote by $\tilde{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, (\tilde{X}_t)_{t \geq 0}, (\tilde{P}_x)_x \in E)$ the transformed process. Here $\tilde{X}_t = X_t$ but we use $\tilde{X}_t$ for emphasis when working with $\tilde{P}_x$. Let $\tilde{P}_t$ be the transition semigroup of $X_t$ under $\tilde{P}_x$, that is,

$$\tilde{P}_t f(x) := \tilde{E}_x \left[ f(\tilde{X}_t) \right] = E_x \left[ \tilde{Z}_t^u f(X_t) \right],$$

where $\tilde{E}_x$ is the expectation w.r.t. $\tilde{P}_x$ for $x \in E$.

By the Doleans–Dade formula (cf. [10, Theorem 9.39]), (2.5) and noting that $\langle \tilde{M}_t^u \rangle_t = \langle M_t^u \rangle_t$, we get

$$\tilde{Z}_t = e^{\tilde{M}_t^u - \frac{1}{2} \langle \tilde{M}_t^u \rangle_t} = e^{M_t^u + \frac{1}{2} \langle M_t^u \rangle_t} = \frac{1}{\tilde{Z}_t^u}.$$ 

Thus

$$\tilde{P}_t f(x) = \tilde{E}_x \left[ \tilde{Z}_t f(\tilde{X}_t) \right] = E_x \left[ Z_t^u \tilde{Z}_t f(X_t) \right] = E_x \left[ f(X_t) \right].$$

Therefore, the process $\tilde{X}$ is the same as the process $X$. Similar to Theorem 2.2 and Corollary 2.3, we obtain the following theorem.

**Theorem 2.4.**

$$\bigcup_{k \geq 1} \mathcal{D}(\hat{\mathcal{E}})_{F_k,b} \subseteq \bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_{F_k,b}.$$ 

By Theorems 2.2 and 2.4, we get

**Theorem 2.5.**

$$\bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_{F_k,b} = \bigcup_{k \geq 1} \mathcal{D}(\hat{\mathcal{E}})_{F_k,b}.$$ 

3. Perturbation of Dirichlet form $(\hat{\mathcal{E}}, \mathcal{D}(\hat{\mathcal{E}}))$ and $h$-transformation for quadratic form

Since $\mu_u := \frac{1}{2} \mu_{(u)}$ is the Revuze measure for $\frac{1}{2} \langle M_u \rangle_t$ as a PCAF of $X$, $e^{-2u} \mu_u$ is the Revuze measure for $\frac{1}{2} \langle M_u \rangle_t$ as a PCAF of $\hat{X}$ by Theorem 2.1(ii). We consider now the perturbation of Dirichlet form $(\hat{\mathcal{E}}, \mathcal{D}(\hat{\mathcal{E}}))$ by the signed measure $-e^{-2u} \mu_u$:
\[ \hat{E}_{\mu u}(f, f) := \hat{E}(f, f) - \langle f, f \rangle_{e^{-2u \mu u}}, \quad \forall f \in \mathcal{D}(\hat{E}_{\mu u}), \]
\[ \mathcal{D}(\hat{E}_{\mu u}) := \mathcal{D}(\hat{E}) \cap L^2(E; e^{-2u \mu u}), \]

and its corresponding semigroup
\[ \hat{T}_{\mu u}^t f(x) := \hat{E}_x[\frac{1}{2}(M^u_t) f(\hat{X}_t)] = E_x[\exp(-M^u_t) f(X_t)], \quad t \geq 0, \]

provided the right-hand side makes sense.

Let \( \tilde{m} \) be a \( \sigma \)-finite measure on \( E \). Recall that a quadratic form \((A, \mathcal{D}(A))\) on \( L^2(E; \tilde{m})\) is said to be lower semibounded if there exists a constant \( \alpha > 0 \) such that
\[ A(f, f) \geq -\alpha \langle f, f \rangle_{\tilde{m}}, \quad \forall f \in \mathcal{D}(A). \]

By [2, Theorem 4.1], we get

**Theorem 3.1.** \((\hat{E}_{\mu u}, \mathcal{D}(\hat{E}_{\mu u}))\) is lower semibounded if and only if \((\hat{T}_{\mu u}^t)_{t \geq 0}\) is a strongly continuous semigroup on \( L^2(E; e^{-2u \mu u}) \). Furthermore, if \((\hat{T}_{\mu u}^t)_{t \geq 0}\) is a strongly continuous semigroup on \( L^2(E; e^{-2u \mu u}) \), then the closed quadratic form corresponding to \((\hat{T}_{\mu u}^t)_{t \geq 0}\) is the largest closed quadratic form that is smaller than \((\hat{E}_{\mu u}, \mathcal{D}(\hat{E}_{\mu u}))\).

**Proposition 3.2.** \( \bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_{F_k, b} \subseteq \mathcal{D}(\hat{E}_{\mu u}). \)

**Proof.** Let \( f \in \mathcal{D}(\mathcal{E})_{F_k, b} \) for some \( k \). Then \( f \in \mathcal{D}(\hat{E}) \) by Theorem 2.2. Since
\[ \int_E f^2(x) e^{-2u(x) \mu u} dx \leq \frac{1}{2}(\| f \|_\infty e^{-u} \| f \|_{F_k, \infty})^2 \mu(u)(F_k) < \infty, \]
\[ f \in \mathcal{D}(\hat{E}_{\mu u}). \]

Throughout this paper, we set \( h = e^u \). Let \( \tilde{m} \) be a \( \sigma \)-finite measure on \( E \) and \((S, \mathcal{D}(S))\) a quadratic form on \( L^2(E; \tilde{m}) \). Then, the quadratic form \((S^h, \mathcal{D}(S^h))\) on \( L^2(E; h^2 \tilde{m}) \) defined by
\[ S^h(f, f) := S(fh, fh), \quad f \in \mathcal{D}(S^h), \]
\[ \mathcal{D}(S^h) := \{ f \in L^2(E; h^2 \tilde{m}) \mid fh \in \mathcal{D}(S) \} \]
is called the \( h \)-transformation for \((S, \mathcal{D}(S))\) (cf. [12]).

**Remark 3.3.** It is not hard to prove the following assertions:

(i) \((S^h, \mathcal{D}(S^h))\) is positive definite if and only if \((S, \mathcal{D}(S))\) is positive definite.
(ii) \((S^h, \mathcal{D}(S^h))\) is lower semibounded if and only if \((S, \mathcal{D}(S))\) is lower semibounded.
(iii) \((S^h, \mathcal{D}(S^h))\) is closed if and only if \((S, \mathcal{D}(S))\) is closed.
(iv) Suppose that \((S, \mathcal{D}(S))\) is a closed quadratic form on \(L^2(E; \tilde{m})\) and \((L_t^h)_{t \geq 0}\) is the semigroup corresponding to \((S, \mathcal{D}(S))\). Let \((L_t^h)_{t \geq 0}\) be the semigroup on \(L^2(E; h^2 \tilde{m})\) corresponding to \((S^h, \mathcal{D}(S^h))\). Then

\[
L_t^h f = h^{-1} L_t(f h) \quad h^2 \tilde{m}\text{-a.e.,} \quad \forall f \in L^2(E; h^2 \tilde{m}) \text{ and } t \geq 0.
\]

**Proposition 3.4.** \(\bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_{F_k,b} \subseteq \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h})\).

**Proof.** Let \(f \in \mathcal{D}(\mathcal{E})_{F_k,b}\) for some \(k\). Then, we have \(f e^u \in \mathcal{D}(\mathcal{E})_{F_k,b}\) by Corollary 2.3. Hence \(f e^u \in \mathcal{D}(\hat{\mathcal{E}}_{\mu_u})\) by Proposition 3.2. Since \(f \in \mathcal{D}(\mathcal{E})_{F_k,b}\), \(f \in L^2(E; h^2 e^{-2u} m)\) and thus \(f \in \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h})\). \(\square\)

**Theorem 3.5.**

(i) \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\) is lower semibounded if and only if \((\hat{\mathcal{E}}_{\mu_u}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u}))\) is lower semibounded.

(ii) \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\) if and only if \((\hat{T}_t^{\mu_u})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; e^{-2u} m)\).

(iii) Suppose that \((\hat{T}_t^{\mu_u})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; e^{-2u} m)\). Then the closed quadratic form corresponding to \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) is the \(h\)-transformation for the closed quadratic form corresponding to \((\hat{T}_t^{\mu_u})_{t \geq 0}\).

**Proof.** (i) It is obvious by Remark 3.3(ii).

(ii) Let \(f \in L^2(E; e^{-2u} m)\). Then \(\hat{f} := f h^{-1} \in L^2(E; m)\). Recalling that \(h = e^u\), we get

\[
\| \hat{T}_t^{\mu_u,h} \hat{f} - \hat{f} \|_{L^2(E; m)} = \| h^{-1} \hat{T}_t^{\mu_u}(f h) - fh^{-1} \|_{L^2(E; m)} = \| \hat{T}_t^{\mu_u} f - f \|_{L^2(E; e^{-2u} m)}.
\]

The assertion follows immediately.

(iii) Suppose that \((\hat{T}_t^{\mu_u})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; e^{-2u} m)\). Denote by \((B, \mathcal{D}(B))\) and \((A, \mathcal{D}(A))\) the closed quadratic forms corresponding to \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) and \((\hat{T}_t^{\mu_u})_{t \geq 0}\), respectively. For \(f \in \mathcal{D}(A^h)\), we have that \(fh \in \mathcal{D}(A)\) and

\[
\lim_{t \downarrow 0} \frac{1}{t} (f - T_t^{\mu_u,h} f, f)_{h^2 m} = \lim_{t \downarrow 0} \frac{1}{t} (f - h^{-1} T_t^{\mu_u}(f h), f)_{h^2 m} = \lim_{t \downarrow 0} \frac{1}{t} (f h - T_t^{\mu_u}(f h), f h)_{h^2 m} = A(f h, f h) = A^h(f, f).
\]

Thus \(f \in \mathcal{D}(B)\) and \(B(f, f) = A^h(f, f)\). For \(f \in \mathcal{D}(B)\), we have that

\[
\lim_{t \downarrow 0} \frac{1}{t} (f h - T_t^{\mu_u}(f h), f h) = \lim_{t \downarrow 0} \frac{1}{t} (f - h^{-1} T_t^{\mu_u}(f h), f)_{h^2 m} = \lim_{t \downarrow 0} \frac{1}{t} (f - T_t^{\mu_u,h}(f), f)_{h^2 m} = B(f, f).
\]

Thus \(fh \in \mathcal{D}(A)\) and \(A(f h, f h) = B(f, f)\), that is, \(f \in \mathcal{D}(A^h)\) and \(A^h(f, f) = B(f, f)\). The proof is completed. \(\square\)
Theorem 3.6. \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\) is lower semibounded if and only if \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\). Furthermore, if \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\), then the closed quadratic form corresponding to \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) is the largest closed quadratic form that is smaller than \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\).

Proof. By Theorems 3.1 and 3.5, we know that \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\) is lower semibounded if and only if \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\). We now suppose that \((\hat{T}_t^{\mu_u,h})_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\). Denote by \((A, \mathcal{D}(A))\) the closed quadratic form corresponding to \((\hat{T}_t^{\mu_u})_{t \geq 0}\). By Theorem 3.5, to complete the proof, it is sufficient to show that \((A^h, \mathcal{D}(A^h))\) is the largest closed quadratic form that is smaller than \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\). Let \(f \in \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h})\). Then \(f h \in \mathcal{D}(\hat{\mathcal{E}}_{\mu_u})\). Since the closed quadratic form \((A, \mathcal{D}(A))\) corresponding to \((\hat{T}_t^{\mu_u})_{t \geq 0}\) is the largest closed quadratic form that is smaller than \((\hat{\mathcal{E}}_{\mu_u}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u}))\), we get \(f h \in \mathcal{D}(A)\) and \(A(f h, f h) \leq \hat{\mathcal{E}}_{\alpha}^\mu(u, f h, f h)\), where \(\alpha\) is a lower semibound of \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\). Thus

\[
\mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}) \subseteq \mathcal{D}(A^h),
\]

\[
A^h(f, f) \leq \hat{\mathcal{E}}_{\alpha}^\mu(u, f, f), \quad \forall f \in \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}).
\]

Therefore, \((A^h, \mathcal{D}(A^h))\) is smaller than \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\).

Let \((J, \mathcal{D}(J))\) be another closed quadratic form that is smaller than \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\). Then \((J^h^{-1}, \mathcal{D}(J^h^{-1}))\) is a closed quadratic form on \(L^2(E; e^{-2\mu}m)\) by Remark 3.3(iii). Let \(g \in \mathcal{D}(\hat{\mathcal{E}}_{\mu_u})\). Then \(gh^{-1} \in \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}) \subseteq \mathcal{D}(J)\) and hence \(g \in \mathcal{D}(J^h^{-1})\). Since \(g\) is arbitrary, \(\mathcal{D}(\hat{\mathcal{E}}_{\mu_u}) \subseteq \mathcal{D}(J^h^{-1})\). Moreover,

\[
J^h^{-1}(g, g) = J(gh^{-1}, gh^{-1}) \leq \hat{\mathcal{E}}_{\alpha}^\mu(u, g, g).
\]

Then, \((J^h^{-1}, \mathcal{D}(J^h^{-1}))\) is a closed quadratic form that is smaller than \((\hat{\mathcal{E}}_{\mu_u}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u}))\) and hence is smaller than \((A, \mathcal{D}(A))\). Thus \((J, \mathcal{D}(J))\) is smaller than \((A^h, \mathcal{D}(A^h))\) and therefore \((A^h, \mathcal{D}(A^h))\) is the largest closed quadratic form that is smaller than \((\hat{\mathcal{E}}_{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h}))\). The proof is completed. \(\square\)

4. Strong continuity of the generalized Feynman–Kac semigroup

We define the quadratic form \((Q^u, \mathcal{D}(\mathcal{E})_b)\) on \(L^2(E; m)\) as follows:

\[
Q^u(f, f) := \mathcal{E}(f, f) + \mathcal{E}(u, f^2), \quad \forall f \in \mathcal{D}(\mathcal{E})_b,
\]

\[
\mathcal{D}(\mathcal{E})_b := \mathcal{D}(\mathcal{E}) \cap L^\infty(E; m).
\]

Proposition 4.1.

(i) \(\hat{T}_t^{\mu_u,h} f = P_t^u f, \quad \forall f \in L^2(E; m)\).

(ii) \(\bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_F_k, b \subseteq \mathcal{D}(\hat{\mathcal{E}}_{\mu_u,h})\) and \(\hat{\mathcal{E}}_{\mu_u,h}(f, f) = Q^u(f, f), \forall f \in \bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_F_k, b\).
Proof. (i) Let \( f \in L^2(E; m) \). Note that \( h = e^u \), we get

\[
P_t^u f(x) = E_x\left[e^{N_t^u f(X_t)}\right]
\]

\[
= E_x\left[e^{\tilde{u}(X_t) - \tilde{u}(X_0) - M_t^u f(X_t)}\right]
\]

\[
= e^{-u(x)} E_x\left[e^{-M_t^u (f e^u)(X_t)}\right]
\]

\[
= e^{-u(x)} E_x\left[Z_t^{\frac{1}{2}}(M_u)_t (f e^u)(X_t)\right]
\]

\[
h^{-1} \tilde{E}_x\left[e^{\frac{1}{2} \langle M_u \rangle_t(f h)(X_t)}\right]
\]

\[
h^{-1} \hat{T}^{\mu_u,h}_t f(x).
\]

(ii) Let \( f \in \mathcal{D}(\mathcal{E})_{F_k,b} \) for some \( k \). Then \( fh \in \mathcal{D}(\hat{\mathcal{E}}) \) by Corollary 2.3. It is easy to see that \( fh \in L^2(E; e^{-2u} \mu_u) \). Hence \( f \in \mathcal{D}(\hat{\mathcal{E}}^{\mu_u,h}) \) and

\[
\hat{\mathcal{E}}^{\mu_u,h}(f, f) = \hat{\mathcal{E}}(f h, f h) - (fh, fh) e^{-2u} \mu_u
\]

\[
= \frac{1}{2} \int e^{-2u(x)} \mu_{(fh)}(dx) - \frac{1}{2} \int (fh)^2(x) e^{-2u(x)} \mu_{(u)}(dx)
\]

\[
= \frac{1}{2} \int \mu_{(f)}(dx) + \frac{1}{2} \int \mu_{(u, f^2)}(dx)
\]

\[
= \mathcal{E}(f, f) + \mathcal{E}(u, f^2)
\]

\[
= Q^u(f, f).
\]

The proof is completed. \( \square \)

Proposition 4.2. \((Q^u, \mathcal{D}(\mathcal{E})_b)\) is lower semibounded if and only if \((\hat{\mathcal{E}}^{\mu_u,h}, \mathcal{D}(\hat{\mathcal{E}}^{\mu_u,h}))\) is lower semibounded.

Proof. Suppose that \((Q^u, \mathcal{D}(\mathcal{E})_b)\) is lower semibounded. Let \( f \in \mathcal{D}(\hat{\mathcal{E}}^{\mu_u,h}) \). Then \( fh \in \mathcal{D}(\hat{\mathcal{E}}) \cap L^2(E; e^{-2u} \mu_u) \). Without loss of generality, we assume that \( f \geq 0 \). Then there exist \( \{g_n\}_{n \geq 1} \subset \bigcup_{k \geq 1} \mathcal{D}(\hat{\mathcal{E}})_{F_k,b} \) such that \( g_n \) converges to \( fh \) w.r.t. the \( \hat{\mathcal{E}}^{1/2}_1 \)-norm as \( n \to \infty \). Set \( f_n = (0 \lor g_n) \land fh \) for \( n \geq 1 \). Then \( f_n \in \bigcup_{k \geq 1} \mathcal{D}(\hat{\mathcal{E}})_{F_k,b} \) and \( f_n \) converges to \( fh \) w.r.t. the \( \hat{\mathcal{E}}^{1/2}_1 \)-norm as \( n \to \infty \). Set \( f_n = \tilde{f}_n h^{-1} \) for \( n \geq 1 \). Then \( f_n h \) converges to \( fh \) w.r.t. the \( \hat{\mathcal{E}}^{1/2}_1 \)-norm as \( n \to \infty \). Without loss of generality, we assume that \( f_n \to f \) a.s. as \( n \to \infty \) (taking a subsequence if necessary). By Theorem 2.4, Corollary 2.3 and Proposition 4.1(ii), we get \( f_n \in \bigcup_{k \geq 1} \mathcal{D}(\mathcal{E})_{F_k,b} \) and \( f_n \in L^2(E; \mu_u) \) for \( n \geq 1 \). Hence, by the dominated convergence theorem,

\[
\lim_{n \to 0} \int_E f_n^2(x) \mu_u(dx) = \int_E f^2(x) \mu_u(dx).
\]

By Proposition 4.1(ii) and (4.1), we get
\[ \hat{E}^{\mu_u,h}(f, f) = \hat{E}(f h, f h) - \langle f h, f h \rangle e^{-2u \mu_u} \]
\[ = \lim_{n \to 0} \left[ \hat{E}(f_n h, f_n h) - \langle f_n h, f_n h \rangle e^{-2u \mu_u} \right] \]
\[ = \lim_{n \to 0} \hat{E}^{\mu_u,h}(f_n, f_n) \]
\[ = \lim_{n \to 0} Q^u(f_n, f_n) \]
\[ \geq -\alpha \lim_{n \to 0} (f_n, f_n)_m \]
\[ = -\alpha(f, f)_m. \]

where \( \alpha \) is a lower semibound of \( (Q^u, \mathcal{D}(E)_b) \). This completes the proof of the “necessary” part.

Now suppose that \( (\hat{E}^{\mu_u,h}, \mathcal{D}(\hat{E}^{\mu_u,h})) \) is lower semibounded. Let \( f \in \mathcal{D}(E)_b \). Then there exist \( \{f_n\}_{n \geq 1} \subset \bigcup_{k \geq 1} \mathcal{D}(E)_{F_k,b} \) such that \( f_n \) converges to \( f \) w.r.t. the \( \mathcal{E}_1^{1/2} \)-norm as \( n \to \infty \). Without loss of generality, we assume that \( \{f_n\}_{n \geq 1} \) is uniformly bounded. Then \( f_n^2 \) converges to \( f^2 \) in \( L^2(E; m) \) as \( n \to \infty \). Since \( \mathcal{E}_1^{1/2}(f_n^2, f_n^2) \leq 2\|f_n\|_\infty \mathcal{E}_1^{1/2}(f_n, f_n), \sup_{n \geq 1} \mathcal{E}_1^{1/2}(f_n^2, f_n^2) < \infty \). By [11, Lemma I.2.12], one finds that \( \mathcal{E}(u, f_n^2) \to \mathcal{E}(u, f^2) \) as \( n \to \infty \). Therefore,

\[ Q^u(f, f) = \mathcal{E}(f, f) + \mathcal{E}(u, f^2) \]
\[ = \lim_{n \to \infty} \left[ \mathcal{E}(f_n, f_n) + \mathcal{E}(u, f_n^2) \right] \]
\[ = \lim_{n \to \infty} Q^u(f_n, f_n) \]
\[ = \lim_{n \to \infty} \hat{E}^{\mu_u,h}(f_n, f_n) \]
\[ \geq \lim_{n \to \infty} -\alpha(f_n, f_n)_m \]
\[ = -\alpha(f, f)_m. \]

where \( \alpha \) is a lower semibound of \( (\hat{E}^{\mu_u,h}, \mathcal{D}(\hat{E}^{\mu_u,h})) \). This completes the proof of the “sufficient” part. \( \square \)

**Theorem 4.3.** \( (P_t^u)_{t \geq 0} \) is a strongly continuous semigroup on \( L^2(E; m) \) if and only if \( (Q^u, \mathcal{D}(E)_b)_{t \geq 0} \) is lower semibounded. Furthermore, if \( (P_t^u)_{t \geq 0} \) is a strongly continuous semigroup on \( L^2(E; m) \), then the closed quadratic form corresponding to \( (P_t^u)_{t \geq 0} \) is the largest closed quadratic form that is smaller than \( (Q^u, \mathcal{D}(E)_b) \).

**Proof.** The first part of the theorem follows by Theorem 3.6, Propositions 4.1 and 4.2. In the following, we prove the second part of the theorem. Suppose that \( (P_t^u)_{t \geq 0} \) is a strongly continuous semigroup on \( L^2(E; m) \) and \( (B, \mathcal{D}(B)) \) is the closed quadratic form corresponding to \( (P_t^u)_{t \geq 0} \). Let \( f \in \mathcal{D}(E)_b \). Then, there exist \( \{f_n\}_{n \geq 1} \subset \bigcup_{k \geq 1} \mathcal{D}(E)_{F_k,b} \) such that \( f_n \to f \) w.r.t. the \( \mathcal{E}_1^{1/2} \)-norm and \( \mathcal{E}(u, f_n^2) \to \mathcal{E}(u, f^2) \) as \( n \to \infty \) (cf. the proof of Proposition 4.2). By Theorem 3.6 and Proposition 4.1(i), we know that \( (B, \mathcal{D}(B)) \) is smaller than \( (\hat{E}^{\mu_u,h}, \mathcal{D}(\hat{E}^{\mu_u,h})) \). By Proposition 4.1(ii), we get
where $\alpha$ is a lower semibound of $(Q^\mu, D(E)_b)$. Note that $f_n \to f$ in $L^2(E; m)$ as $n \to \infty$ and $(B, D(B))$ is a closed quadratic form, we obtain by [11, Lemma I.2.12] that $f \in D(B)$ and

$$B(f, f) \leq \liminf_{n \to \infty} B(f_n, f_n) \leq \liminf_{n \to \infty} \hat{E}_{\alpha}^{\mu_u, h}(f_n, f_n) = \liminf_{n \to \infty} Q^\mu_{\alpha}(f_n, f_n) = \hat{E}_{\mu_u, h}(f, f).$$

Hence $(B, D(B))$ is smaller than $(Q^\mu, D(E)_b)$.

Suppose that $(J, D(J))$ is another closed quadratic form that is smaller than $(Q^\mu, D(E)_b)$. Let $f \in D(\hat{E}_{\mu_u, h})$. Then, there exist $\{f_n\}_{n \geq 1} \subset \bigcup_{k \geq 1} D(E)_{F_k, b}$ such that $f_n \to f$ in $L^2(E; m)$ and $\hat{E}_{\mu_u, h}(f_n, f_n) \to \hat{E}_{\mu_u, h}(f, f)$ as $n \to \infty$ (cf. the proof of Proposition 4.2). Since $(J, D(J))$ is smaller than $(Q^\mu, D(E)_b)$, we obtain by Proposition 4.1(ii) that

$$\sup_{n \geq 1} J(f_n, f_n) \leq \sup_{n \geq 1} Q^\mu_{\alpha}(f_n, f_n) = \hat{E}_{\mu_u, h}(f, f).$$

By [11, Lemma I.2.12] again, one finds that $f \in D(J)$ and

$$J(f, f) \leq \liminf_{n \to \infty} J(f_n, f_n) \leq \liminf_{n \to \infty} Q^\mu_{\alpha}(f_n, f_n) = \liminf_{n \to \infty} \hat{E}_{\alpha}^{\mu_u, h}(f_n, f_n) = \hat{E}_{\mu_u, h}(f, f).$$

Then $(J, D(J))$ is a closed quadratic form that is smaller than $(\hat{E}_{\mu_u, h}, D(\hat{E}_{\mu_u, h}))$ and hence is smaller than $(B, D(B))$, since $(B, D(B))$ is the largest closed quadratic form that is smaller than $(\hat{E}_{\mu_u, h}, D(\hat{E}_{\mu_u, h}))$. The proof is completed. \hfill \Box

**Corollary 4.4.** $(P_1^\mu)_{t \geq 0}$ is a strongly continuous semigroup on $L^2(E; m)$ if and only if there exist constants $0 \leq a \leq 1$ and $A \geq 0$ such that

$$-\mathcal{E}(u, f^2) \leq a \mathcal{E}(f, f) + A \int_E f^2(x)m(dx), \quad \forall f \in D(E)_b.$$ 

Furthermore, if there exist constants $0 \leq a < 1$ and $A \geq 0$ such that

$$|\mathcal{E}(u, f^2)| \leq a \mathcal{E}(f, f) + A \int_E f^2(x)m(dx), \quad \forall f \in D(E)_b,$$

then $(Q^\mu, D(E)_b)$ is closable and $(P_1^\mu)_{t \geq 0}$ is the strongly continuous semigroup on $L^2(E; m)$ corresponding to the closure of $(Q^\mu, D(E)_b)$. 

Proof. The first part of the proof follows by Theorem 4.3. We now prove the second part. Suppose that (4.2) holds for some constants \(0 \leq a < 1\) and \(A \geq 0\). Let \(f \in \mathcal{D}(E)_b\). Then, we get by (4.2) that

\[
(1-a)\mathcal{E}(f,f) \leq \mathcal{Q}^u(f,f) + A \int_E f^2(x)m(dx) \leq (1+a)\mathcal{E}(f,f) + 2A \int_E f^2(x)m(dx).
\]

Thus \((\mathcal{Q}^u, \mathcal{D}(E)_b)\) is closable and the proof follows by Theorem 4.3.

Definition 4.5. (Cf. [7]) Let \(\mu\) be a smooth measure on \(E\) and \((B, \mathcal{D}(B))\) a quadratic form on \(L^2(E; m)\). Then \(\mu\) is said to be of the Hardy class for \((B, \mathcal{D}(B))\) and denoted by \(\mu \in \mathcal{SH}(B)\) if there are constants \(\delta_{\mu(B)} \geq 0\) and \(A_{\mu(B)} \geq 0\) such that

\[
\int_E f^2(x)\mu(dx) \leq \delta_{\mu(B)} B(f,f) + A_{\mu(B)} \int_E f^2(x)m(dx), \quad \forall f \in \mathcal{D}(B).
\]

We define the non-negative quadratic form \((\mathcal{E}^u, \mathcal{D}(E)_b)\) on \(L^2(E; m)\) by

\[
\mathcal{E}^u(f,f) := \hat{\mathcal{E}}(f^u_h, f^u_h) = \mathcal{E}(f,f) + \mathcal{E}(u, f^2) + \frac{1}{2} \langle f, f \rangle_{\mu(\mu)} \mathcal{E}(u), \quad \forall f \in \mathcal{D}(E)_b,
\]

where \(\langle f, f \rangle_{\mu(\mu)} := \int f^2(x)\mu(\mu)(dx)\).

Theorem 4.6. \((P^u_t)_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\) if and only if \(\hat{\mathcal{E}}(\mu(\mu)) \in \mathcal{SH}(\hat{\mathcal{E}}(\mu(\mu)))\) with \(\delta_{\mu(\mu)(\mathcal{E}^u)} \leq 2\). Moreover, if \(\delta_{\mu(\mu)}(\mathcal{E}^u) < 2\), then \((\mathcal{Q}^u, \mathcal{D}(E)_b)\) is closable and \((P^u_t)_{t \geq 0}\) is the strongly continuous semigroup on \(L^2(E; m)\) corresponding to the closure of \((\mathcal{Q}^u, \mathcal{D}(E)_b)\).

Proof. By Theorems 3.5(i), 3.6 and Proposition 4.1, we know that \((P^u_t)_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\) if and only if \(\hat{\mathcal{E}}(\mu(\mu)) \in \mathcal{SH}(\mathcal{E}^u)\) with \(\delta_{\mu(\mu)}(\mathcal{E}^u) \leq 2\). Moreover, if \(\delta_{\mu(\mu)}(\mathcal{E}^u) < 2\), then \((\mathcal{Q}^u, \mathcal{D}(E)_b)\) is closable and \((P^u_t)_{t \geq 0}\) is the strongly continuous semigroup on \(L^2(E; m)\) corresponding to the closure of \((\mathcal{Q}^u, \mathcal{D}(E)_b)\)
\[
\int_E f^2(x)\mu_{(u)}(dx) \leq 2\delta \mathcal{E}^u(f, f) + 2A \int_E f^2(x)m(dx), \quad \forall f \in \mathcal{D}(\mathcal{E}_b),
\]
which shows that (4.3) holds if and only if \(\mu_{(u)} \in S_H(\mathcal{E}^u)\) with \(\delta_{\mu_{(u)}}(\mathcal{E}^u) \leq 2\).

Suppose that \(\delta_{\mu_{(u)}}(\mathcal{E}^u) < 2\), that is, \(\delta < 1\) in (4.3). By (4.3), we have that for \(\hat{f} \in \mathcal{D}(\hat{\mathcal{E}})\),

\[
(1 - \delta)\hat{\mathcal{E}}(\hat{f}, \hat{f}) \leq \hat{\mathcal{E}}(\hat{f}, \hat{f}) - \langle \hat{f}, \hat{f} \rangle e^{-2\mu_{(u)}} + A \int_E \hat{f}^2(x)e^{-2\mu_{(u)}}m(dx) \leq \hat{\mathcal{E}}_A(\hat{f}, \hat{f}),
\]
that is,

\[
(1 - \delta)\hat{\mathcal{E}}(\hat{f}, \hat{f}) \leq \hat{\mathcal{E}}_{\mu_{(u)}}(\hat{f}, \hat{f}) + A \int_E \hat{f}^2(x)e^{-2\mu_{(u)}}m(dx) \leq \hat{\mathcal{E}}_A(\hat{f}, \hat{f}).
\]

Thus \((\hat{\mathcal{E}}_{\mu_{(u)}}, \mathcal{D}(\hat{\mathcal{E}}_{\mu_{(u)}}))\) is lower semibounded and closable. The remainder of the proof follows by Theorem 3.5(i), Proposition 4.2 and Theorem 4.3. \(\square\)

**Corollary 4.7.** If \(\mu_{(u)} \in S_H(\mathcal{E})\) with \(\delta_{\mu_{(u)}}(\mathcal{E}) \leq \frac{1}{2}\), then \((P_t^u)_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\). Moreover, if \(\delta_{\mu_{(u)}}(\mathcal{E}) < \frac{1}{2}\), then \((Q_t^u, \mathcal{D}(Q_t^u))\) is closable and \((P_t^u)_{t \geq 0}\) is the strongly continuous semigroup on \(L^2(E; m)\) corresponding to the closure of \((Q_t^u, \mathcal{D}(Q_t^u))\).

**Proof.** Suppose that \(\mu_{(u)} \in S_H(\mathcal{E})\). That is,

\[
\int_E f^2(x)\mu_{(u)}(dx) \leq \delta_{\mu_{(u)}}(\mathcal{E})\mathcal{E}(f, f) + A_{\mu_{(u)}}(\mathcal{E}) \int_E f^2(x)m(dx), \quad \forall f \in \mathcal{D}(\mathcal{E}). \tag{4.5}
\]

Since

\[
|\mathcal{E}(u, f^2)| = \left| \frac{1}{2} \int_E \mu_{(u, f^2)}(dx) \right| = \left| \int_E f(x)\mu_{(u, f)}(dx) \right| \\
\leq \left( \int_E f^2(x)\mu_{(u)}(dx) \right)^{1/2} \left( \int_E \mu_{(f)}(dx) \right)^{1/2} \\
\leq \int_E f^2(x)\mu_{(u)}(dx) + \frac{1}{4} \int_E \mu_{(f)}(dx) \\
= \int_E f^2(x)\mu_{(u)}(dx) + \frac{1}{2} \mathcal{E}(f, f),
\]
we obtain by (4.5) and (4.6) that
\[
2\delta_{\mu(u)}(\mathcal{E})\mathcal{E}u(f, f) + A_{\mu(u)}(\mathcal{E}) \int_E f^2(x)\mu(dx) \\
= 2\delta_{\mu(u)}(\mathcal{E})[\mathcal{E}(f, f) + \mathcal{E}(u, f^2) + \frac{1}{2} \int_E f^2(x)\mu(dx)] + A_{\mu(u)}(\mathcal{E}) \int_E f^2(x)m(dx) \\
\geq 2\delta_{\mu(u)}(\mathcal{E})[\frac{1}{2}\mathcal{E}(f, f) - \frac{1}{2} \int_E f^2(x)\mu(dx)] + A_{\mu(u)}(\mathcal{E}) \int_E f^2(x)m(dx) \\
\geq \delta_{\mu(u)}(\mathcal{E})(f, f) - \delta_{\mu(u)}(\mathcal{E}) \int_E f^2(x)\mu(dx) + A_{\mu(u)}(\mathcal{E}) \int_E f^2(x)m(dx) \\
\geq [1 - \delta_{\mu(u)}(\mathcal{E})] \int_E f^2(x)\mu(dx).
\]

Hence \(\mu(u) \in S_H(\mathcal{E}^u)\) with \(\delta_{\mu(u)}(\mathcal{E}^u) = 2\delta_{\mu(u)}(\mathcal{E})/(1 - \delta_{\mu(u)}(\mathcal{E}))\). The remainder of the proof follows by Theorem 4.6. \(\square\)

**Remark 4.8.** Corollary 4.7 generalizes a result of Fitzsimmons and Kuwae. In [7, Example 5.3], they proved that if \(\delta_{\mu(u)}(\mathcal{E}) < \frac{1}{2}\), then \((\mathcal{Q}^u, \mathcal{D}(\mathcal{E})_b)\) is closable and \((P_t^u)_{t \geq 0}\) is the strongly continuous semigroup on \(L^2(E; m)\) corresponding to the closure of \((\mathcal{Q}^u, \mathcal{D}(\mathcal{E})_b)\). The closability of \((\mathcal{Q}^u, \mathcal{D}(\mathcal{E})_b)\) when \(\delta_{\mu(u)}(\mathcal{E}) = \frac{1}{2}\) deserves further analysis.

**Corollary 4.9.** If \(\mu(u)\) is a measure of the Hardy class for \((\mathcal{Q}^u, \mathcal{D}(\mathcal{E})_b)\), then \((P_t^u)_{t \geq 0}\) is a strongly continuous semigroup on \(L^2(E; m)\). Moreover, \((\mathcal{Q}^u, \mathcal{D}(\mathcal{E})_b)\) is closable and \((P_t^u)_{t \geq 0}\) is the strongly continuous semigroup on \(L^2(E; m)\) corresponding to the closure of \((\mathcal{Q}^u, \mathcal{D}(\mathcal{E})_b)\).

**Proof.** Suppose that \(\mu(u) \in S_H(\mathcal{Q}^u)\). Then there exist constants \(\delta = \delta_{\mu(u)}(\mathcal{Q}^u) \geq 0\) and \(A = A_{\mu(u)}(\mathcal{Q}^u) \geq 0\) such that

\[
\int_E f^2(x)\mu(dx) \leq \delta \mathcal{Q}^u(f, f) + A \int_E f^2(x)m(dx), \quad \forall f \in \mathcal{D}(\mathcal{E})_b,
\]

Taking \(\tilde{\delta} = \frac{\delta}{2+\delta} < 1\) and noting that \(\mathcal{E}^u(f, f) = \mathcal{Q}^u(f, f) + \frac{1}{2} \int_E f^2(x)\mu(dx)\), we get

\[
\int_E f^2(x)\mu(dx) \leq 2\tilde{\delta}\mathcal{E}^u(f, f) + \frac{2A}{2+\delta} \int_E f^2(x)m(dx), \quad \forall f \in \mathcal{D}(\mathcal{E})_b.
\]

Therefore \(\delta_{\mu}(\mathcal{E}^u) = 2\tilde{\delta} < 2\) and the proof follows by Theorem 4.6. \(\square\)

**Definition 4.10.** (See [2]) A smooth measure \(\mu\) on \(E\) is said to be of the Kato class if its associated PCAF \((A_t)_{t \geq 0}\) satisfies

\[
\lim_{t \downarrow 0} ||E[A_t]||_q = 0,
\]
where

\[ \| f \|_q := \inf_{N \subset E, \text{Cap}(N) = 0} \sup_{x \in E \setminus N} |f(x)|, \quad f \in \mathcal{B}(E). \]

**Corollary 4.11.** If \( \mu_{(u)} \) is of the Kato class, then \((Q^u, \mathcal{D}(\mathcal{E}))_b)\) is lower semibounded and closable. Moreover, \((P^u_t)_{t \geq 0}\) is the strongly continuous semigroup on \(L^2(E; m)\) corresponding to the closure of \((Q^u, \mathcal{D}(\mathcal{E}))_b)\).

**Proof.** Suppose that \( \mu_{(u)} \) is of the Kato class. Then, for any \( 0 < \delta < \frac{1}{2} \), there exists \( A > 0 \) such that (cf. [2, Proposition 3.1])

\[ \int_E f^2(x) \mu_{(u)}(dx) \leq \delta \mathcal{E}(f, f) + A \int_E f^2(x) m(dx), \quad \forall f \in \mathcal{D}(\mathcal{E}). \]

Therefore the proof follows by Corollary 4.7. \( \square \)

### 5. Examples

**Example 5.1.** Let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be the classical Dirichlet form on \(L^2(R^d; dx)\) defined by

\[ \mathcal{E}(f, g) = \frac{1}{2} \int_{R^d} \nabla f(x) \cdot \nabla g(x) \, dx, \]

\[ \mathcal{D}(\mathcal{E}) = H^1(R^d) := \{ f \in L^2(R^d; dx) \mid |\nabla f| \in L^2(R^d; dx) \}, \]

where \( dx \) is the Lebesgue measure on \( R^d \). It is well known that the Markov process \( X \) associated with \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) is the standard Brownian motion on \( R^d \). Suppose that \( d \geq 3 \) and \( 0 < \varepsilon < 1 \). Take \( \varphi \in C^\infty(0, \infty) \) such that the support of \( \varphi \) is bounded, \( 0 < \varphi \leq \varepsilon \), \( \varphi \equiv \varepsilon \) on \([0, 1/2]\) and \( \varphi(s) \leq \varepsilon/s^d \) for \( s \in [1, \infty] \). Similar to [7, Example 5.5] (cf. also [1, 6]), one finds that \( \bar{u}(x) := \varphi^2(|x|)/|x|^2 \) does not belong to the Kato class and for \( f \in \mathcal{D}(\mathcal{E}) \),

\[ \int_{R^d} f^2(x) \frac{\varphi^2(|x|)}{|x|^2} \, dx \leq \varepsilon^2 \int_{R^d} f^2(x) \frac{1}{|x|^2} \, dx \leq \frac{4\varepsilon^2}{(d-2)^2} \int_{R^d} |\nabla f(x)|^2 \, dx \]

\[ = \frac{8\varepsilon^2}{(d-2)^2} \mathcal{E}(f, f). \quad (5.1) \]

(i) Define \( u(x) := \int_{|x|^2}^{\infty} (\varphi(s^{1/2})/s) \, ds, \quad x > 0 \). We claim that \( u \in \mathcal{D}(\mathcal{E}) \). In fact, if \( 0 < |x|^2 < 1 \), then

\[ u(x) = \int_{0}^{\sqrt{x}} \frac{\varphi(s^{1/2})}{s} \, ds + \int_{\sqrt{x}}^{\infty} \frac{\varphi(s^{1/2})}{s} \, ds \leq \varepsilon \left( \int_{|x|^2}^{1} \frac{1}{s} \, ds + \int_{1}^{\infty} \frac{1}{s^{(d+2)/2}} \, ds \right) \]

\[ = \varepsilon \left( - \ln |x|^2 + \frac{2}{d} \right). \quad (5.2) \]
If \( |x|^2 \geq 1 \), then

\[
 u(x) = \int_{|x|^2}^\infty \frac{\varphi(s^{1/2})}{s} \, ds \leq \int_{|x|^2}^\infty \frac{\epsilon}{s^{(d+2)/2}} \, ds = \frac{2\epsilon}{d} |x|^{-d}. \tag{5.3}
\]

Since

\[
 \int_{|x|^2 < 1} (-\ln |x|^2)^2 \, dx = C_1 \int_0^1 (\ln r)^2 r^{d-1} \, dr < \infty
\]

and

\[
 \int_{|x|^2 \geq 1} \frac{|x|}{-d} \, dx = C_2 \int_1^\infty r^{-(d+1)} \, dr < \infty,
\]

where \( C_1, C_2 > 0 \) are constants, we obtain by (5.2) and (5.3) that

\[
 \int_{\mathbb{R}^d} u^2(x) \, dx = \int_{|x|^2 < 1} u^2(x) \, dx + \int_{|x|^2 \geq 1} u^2(x) \, dx < \infty.
\]

Hence \( u \in L^2(\mathbb{R}^d; dx) \).

Note that \( \partial u(x)/\partial x_i = -2\varphi(|x|)x_i/|x|^2 \), \( 1 \leq i \leq d \). We get

\[
 \int_{\mathbb{R}^d} \left( \frac{\partial u}{\partial x_i} \right)^2 \, dx \leq 4\epsilon^2 \left( \int_{|x|^2 < 1} \frac{x_i^2}{|x|^4} \, dx + \int_{|x|^2 \geq 1} \frac{x_i^2}{|x|^{2(d+2)}} \, dx \right) \leq C_3 \left( \int_0^{d-3} r^d \, dr + \int_1^{\infty} \frac{1}{r^{d+3}} \, dr \right) < \infty,
\]

where \( C_3 > 0 \) is a constant. Then \( \partial u/\partial x_i \in L^2(\mathbb{R}^d; dx) \), \( 1 \leq i \leq d \), and hence \( u \in \mathcal{D}(\mathcal{E}) \).

Note that the energy measure of \( u \) is

\[
 \mu_{(u)}(dx) = |\nabla u(x)|^2 \, dx = \frac{4\varphi^2(|x|)}{|x|^2} \, dx = 4\bar{u}(x) \, dx. \tag{5.4}
\]

Then \( \mu_{(u)} \) is not a measure of the Kato class. By (5.1) and (5.4), for \( f \in \mathcal{D}(\mathcal{E}) \), we get

\[
 \int_{\mathbb{R}^d} f^2(x) \mu_{(u)}(dx) = 4 \int_{\mathbb{R}^d} f^2(x) \frac{\varphi^2(|x|)}{|x|^2} \, dx \leq \frac{32\epsilon^2}{(d-2)^2} \mathcal{E}(f, f).
\]
Hence \( \mu(u) \in S_H(\mathcal{E}) \) with \( \delta_{\mu(u)}(\mathcal{E}) = \frac{32\epsilon^2}{(d-2)^2} \). By Corollary 4.7 and Theorem 4.3, if \( \frac{32\epsilon^2}{(d-2)^2} \leq \frac{1}{2} \), then \( (P_t^u)_{t \geq 0} \) is the strongly continuous semigroup on \( L^2(\mathbb{R}^d; dx) \) corresponding to the largest closed quadratic form that is smaller than \( (Q^u, \mathcal{D}(\mathcal{E})_b) \); if \( \frac{32\epsilon^2}{(d-2)^2} < \frac{1}{2} \), then \( (Q^u, \mathcal{D}(\mathcal{E})_b) \) is closable and \( (P_t^u)_{t \geq 0} \) is the strongly continuous semigroup on \( L^2(\mathbb{R}^d; dx) \) corresponding to the closure of \( (Q^u, \mathcal{D}(\mathcal{E})_b) \).

(ii) Note that \( \bar{u} \in L^2(\mathbb{R}^d; dx) \). Fix \( \alpha_0 > 0 \) and define

\[
u(x) := -G_{\alpha_0} \bar{u}(x) = -\int_0^\infty e^{-\alpha_0 t} E_x[\bar{u}(X_t)] dt,
\]

where \( (G_\alpha)_{\alpha > 0} \) are the resolvents on \( L^2(\mathbb{R}^d; dx) \) corresponding to \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \). Then \( u \in \mathcal{D}(\mathcal{E}) \) and we obtain by (5.1) that

\[
-\mathcal{E}(u, f^2) = \mathcal{E}_{\alpha_0}(G_{\alpha_0} \bar{u}, f^2) - \alpha_0 \int_{\mathbb{R}^d} (G_{\alpha_0} \bar{u})(x) f^2(x) dx \\
\leq \int_{\mathbb{R}^d} \bar{u}(x) f^2(x) dx \leq \frac{8\epsilon^2}{(d-2)^2} \mathcal{E}(f, f).
\]

By Corollary 4.4, if \( \frac{8\epsilon^2}{(d-2)^2} \leq 1 \), then \( (P_t^u)_{t \geq 0} \) is the strongly continuous semigroup on \( L^2(\mathbb{R}^d; dx) \) corresponding to the largest closed quadratic form that is smaller than \( (Q^u, \mathcal{D}(\mathcal{E})_b) \). Note that \( (N_t^u)_{t \geq 0} \) is of bounded variation in this case. Hence the strong continuity of \( (P_t^u)_{t \geq 0} \) can also be obtained by considering the Revuz measure for \( (N_t^u)_{t \geq 0} \) (cf. [2]).

**Example 5.2.** Denote \( E := \bigcup_{m=2}^\infty (m, m + \frac{2}{m^4}) \). We consider the following Dirichlet form \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \) on \( L^2(E; dx) \) defined by

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_E f'(x)g'(x) dx,
\]

\[
\mathcal{D}(\mathcal{E}) = H^1(E) := \{ f \in L^2(E; dx) \mid f' \in L^2(E; dx) \},
\]

where \( dx \) is the Lebesgue measure on \( \mathbb{R} \). Define \( u(x) = \frac{1}{m^3} - m(x - m) \) for \( x \in (m, m + \frac{1}{m^4}) \), \( \forall m \geq 2 \), and \( u(x) = 0 \), elsewhere. Then one can check that \( u \in \mathcal{D}(\mathcal{E}) \). For each \( n \geq 2 \), define \( f_n(x) = 1 + (x - n) \) for \( x \in (n, n + \frac{1}{n^4}) \), \( f_n(x) = 1 + \frac{2}{n^4} + (n - x) \) for \( x \in (n + \frac{1}{n^4}, n + \frac{2}{n^4}) \) and \( f_n(x) = 0 \), elsewhere. Then \( \{f_n\}_{n \geq 2} \subset \mathcal{D}(\mathcal{E})_b \). Note that, for \( n \geq 2 \),

\[
\mathcal{E}(f_n, f_n) + \mathcal{E}(u, f_n^2) \leq \frac{1}{n^4} - \frac{1}{n^3} \quad \text{and} \quad (f_n, f_n)_dx = \int_E f_n^2(x) dx \leq \frac{8}{n^3}.
\]
For $\alpha > 0$, we define $n_\alpha = \lfloor 8\alpha \rfloor + 2$, where $\lfloor \cdot \rfloor$ denotes the integer part of a real number. Then

$$E(f_{n_\alpha}, f_{n_\alpha}) + E(u, f_{n_\alpha}^2) \leq \frac{1}{n_\alpha^4} - \frac{1}{n_\alpha^3} < -\frac{8\alpha}{n_\alpha^4} \leq -\alpha(f_{n_\alpha}, f_{n_\alpha})dx.$$ 

Since $\alpha > 0$ is arbitrary, $(\mathcal{Q}^u, \mathcal{D}(\mathcal{E}))_b$ is not lower semibounded and hence $(P_t^u)_{t \geq 0}$ is not strongly continuous on $L^2(E; dx)$ by Theorem 4.3.

**Concluding remark.** In this paper, we give two necessary and sufficient conditions for the generalized Feynman–Kac semigroup of a strongly local, quasi-regular symmetric Dirichlet form to be strongly continuous (cf. Theorems 4.3 and 4.6). Our method can be modified to extend the corresponding results to the general quasi-regular symmetric Dirichlet form setting. We plan to study this problem in a forthcoming paper.

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