NORTH-HOLLAND

The Arithmetic – Geometric – Harmonic-Mean and Related Matrix Inequalities

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ABSTRACT

Recently, Sagae and Tanabe defined a geometric mean of positive definite matrices and proved the harmonic–geometric–arithmetic-mean inequality. Here, we give a reversal of these results. © 1997 Elsevier Science Inc.

1. INTRODUCTION

Let $w_1, \dots, w_r$ be positive numbers such that $w_1 + \cdots + w_r = 1$, and let $C_1, \dots, C_r$ be $n \times n$ positive definite Hermitian matrices. Consider weighted power means of the matrices C_i, defined by

$$M_s = \left(w_1 C_1^s + \cdots + w_r C_r^s\right)^{1/s}, \qquad s \neq 0 \tag{1.1}$$

$$M_0 = G = C_r^{1/2}\Big(C_r^{-1/2} C_{r-1}^{1/2} \cdots \Big(C_3^{-1/2} C_2^{1/2}\big(C_2^{-1/2} C_1 C_2^{-1/2}\big)^{u_1} \times C_2^{1/2} C_3^{-1/2}\Big)^{u_2} \cdots C_{r-1}^{1/2} C_r^{-1/2}\Big)^{u_{r-1}} C_r^{1/2},$$

where $u_i = 1 - w_{i+1}/\sum_{k=1}^{i+1} w_k$ for $i = 1, \dots, r-1$. We shall also use the notation $A = M_1$ and $H = M_{-1}$.

The following results were proved in [1]:

$$\begin{aligned} G \exp(G^{-1}A - I) &= \exp(AG^{-1} - I)\, G \\ &\geqslant \tfrac{1}{2}(AG^{-1}A + G) \\ &\geqslant A \geqslant G \geqslant H \\ &\geqslant 2(H^{-1}GH^{-1} + G^{-1})^{-1} \\ &\geqslant G \exp(I - H^{-1}G) = \exp(I - GH^{-1})\, G \end{aligned} \tag{1.2}$$

and

$$
\begin{aligned}
H \exp(H^{-1}A - I) &= \exp(AH^{-1} - I)\, H \\
&\geqslant \frac{1}{2}(AH^{-1}A + H) \\
&\geqslant A \geqslant G \geqslant H \\
&\geqslant 2(H^{-1}AH^{-1} + A^{-1})^{-1} \\
&\geqslant A \exp(I - H^{-1}A) \\
&= \exp(I - AH^{-1})\, A,
\end{aligned}
\tag{1.3}
$$

where all the inequalities are strict unless $C_1 = \cdots = C_r$, in which case all the equalities hold. Note that $A \geqslant G$ means that $A - G$ is positive semidefinite.

In fact, the main results from [1] are the inequalities

$$H \leqslant G \leqslant A. \tag{1.4}$$

In this paper, we shall prove reverse inequalities to (1.4). For our reverse results, we relax the requirement that all the weights w_i must be positive.

2. THE REVERSE ARITHMETIC-GEOMETRIC-HARMONIC-MEAN MATRIX INEQUALITIES

In this section, we shall use the notation A_r, G_r, and H_r for A, G, and H respectively.

We begin with the following:

LEMMA 2.1. *If $\alpha \in (0, 1)$, then*

$$G_2 \equiv C_2^{1/2}\left(C_2^{-1/2}C_1C_2^{-1/2}\right)^{\alpha}C_2^{1/2} \leqslant \alpha C_1 + (1 - \alpha)C_2 \equiv A_2, \tag{2.1}$$

but if either $\alpha < 0$ or $\alpha > 1$, the reverse inequality, i.e.

$$G_2 \geqslant A_2, \tag{2.2}$$

is valid.

Proof. The following generalization of Bernoulli's inequality is well known (see e.g. [2, p. 34], or [3, p. 65]): For $-1 < x \neq 0$

$$(1+x)^{\alpha} > 1 + \alpha x \qquad \text{if} \quad \alpha > 1 \text{ or } \alpha < 0$$

and

$$(1+x)^{\alpha} < 1 + \alpha x \qquad \text{if} \quad 0 < \alpha < 1. \tag{2.3}$$

For $1 + x = t$, this is equivalent to

$$t^{\alpha} > \alpha t + 1 - \alpha \qquad \text{if} \quad \alpha > 1 \text{ or } \alpha < 0 \tag{2.4}$$

and

$$t^{\alpha} < \alpha t + 1 - \alpha \qquad \text{if} \quad 0 < \alpha < 1. \tag{2.5}$$

For $t = 1$, we have equality.

If the positive definite matrix C has the representation $C = \Gamma D_{\lambda} \Gamma^{*}$ when Γ is unitary and $D_{\lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_1, \dots, \lambda_n$ are characteristic roots of C, then it follows from (2.4) and (2.5) that

$$D_{\lambda}^{\alpha} > \alpha D_{\lambda} + (1 - \alpha) I \qquad \text{if} \quad \alpha > 1 \text{ or } \alpha < 0$$

and

$$D_{\lambda}^{\alpha} < \alpha D_{\lambda} + (1 - \alpha) I \qquad \text{if} \quad 0 < \alpha < 1.$$

Pre- and postmultiplication by Γ and Γ^{*} respectively, yields

$$C^{\alpha} > \alpha C + (1 - \alpha) I \qquad \text{if} \quad \alpha > 1 \text{ or } \alpha < 0$$

and

$$C^{\alpha} < \alpha C + (1 - \alpha) I \qquad \text{if} \quad 0 < \alpha < 1$$

with equalities iff $C = I$.

If we now set $C = C_2^{-1/2} C_1 C_2^{-1/2}$, we obtain

$$\left(C_2^{-1/2} C_1 C_2^{-1/2}\right)^{\alpha} > \alpha C_2^{-1/2} C_1 C_2^{-1/2} + (1 - \alpha) I$$

if $\alpha > 1$ or $\alpha < 0$, and

$$\left(C_2^{-1/2}C_1C_2^{-1/2}\right)^{\alpha} < \alpha C_2^{-1/2}C_1C_2^{-1/2} + (1-\alpha)I$$

for $0 < \alpha < 1$. Pre- and postmultiplication by $C_2^{1/2}$ now yields (2.2) and (2.1) respectively. Equality holds iff $C_2^{-1/2}C_1C_2^{-1/2} = I$ or equivalently $C_1 = C_2$. ∎

Sagae and Tanabe [1] used (2.1) and mathematical induction to prove the inequality $G_r \leqslant A_r$ for positive weights. Similarly, we shall use (2.2).

THEOREM 2.1. *Let w_i, $i = 1,\dots,r$, be real numbers such that*

$$w_1 > 0,\ w_i < 0, \quad i = 2,\dots,r; \qquad w_1 + \cdots + w_r = 1.$$

Then

$$A_r \leqslant G_r. \tag{2.6}$$

If we also have $w_1C_1^{-1} + \cdots + w_rC_r^{-1} > 0$, then

$$G_r \leqslant H_r. \tag{2.7}$$

Equality holds in (2.6) *and* (2.7) *iff $C_1 = \cdots = C_r$.*

Proof. For $r = 2$, (2.6) is proved in Lemma 2.1, i.e., it is the inequality (2.2). Suppose $r > 1$ and (2.6) holds for $r-1$.

Let A_{r-1} and G_{r-1} be weighted arithmetic and geometric means of matrices $C_1,\dots,C_{r-1}$ with weights $\tilde{w}_i = w_i/\sum_{i=1}^{r-1} w_i$ for $i = 1,\dots,r-1$. Note that $\tilde{w}_1 > 0$, $\tilde{w}_i < 0$, $i = 2,\dots,r-1$; $\tilde{w}_1 + \cdots + \tilde{w}_{r-1} = 1$; $\tilde{u}_i = 1 - \tilde{w}_{i+1}/\sum_{j=1}^{i+1}\tilde{w}_j = 1 - w_{i+1}/\sum_{j=1}^{i+1} w_j = u_i$ for $i = 1,\dots,r-2$; and $u_{r-1} = 1 - w_r$ (> 1). So we have

$$\begin{aligned} A_r &= \sum_{i=1}^{r-1} w_iC_i + w_rC_r = (1-w_r)A_{r-1} + w_rC_r \\ &\leqslant (1-w_r)G_{r-1} + w_rC_r \qquad \text{(by the inductive hypothesis)} \\ &\leqslant C_r^{1/2}\left(C_r^{-1/2}G_{r-1}C_r^{-1/2}\right)^{1-w_r}C_r^{1/2} = G_r \qquad [\text{by (2.2)}]. \end{aligned}$$

The equality $A_r = G_r$ holds only when all the equalities are valid simultaneously. Equality in the first inequality holds if $A_{r-1} = G_{r-1}$, i.e. $C_1 = C_2 = \cdots = C_{r-1}$, by induction for $r-1$; and equality in the second inequality holds if $G_{r-1} = C_r$, by the conditions for equality for $r = 2$. Therefore, the equality $A_r = G_r$ holds iff $A_r = G_{r-1} = C_r$, i.e. $C_1 = C_2 = \cdots = C_r$.

Now by the substitutions $C_i^{-1} \to C_i$, $i = 1,\ldots,r$, we get (2.7) from (2.6). ∎

REMARK. It is interesting that (2.6), i.e., the reverse arithmetic–geometric-mean inequality, is stronger than the arithmetic–geometric-mean inequality itself, in the sense that we can obtain the second inequality in (1.4) from (2.6).

Let us consider (2.6) with nonnormalized weights, i.e., if

$$w_1 > 0, \qquad w_i < 0, \quad i = 2,\ldots,r, \quad \text{and} \quad W_r = \sum_{i=1}^{r} w_i > 0, \quad (2.8)$$

then

$$\frac{1}{W_r}\sum_{i=1}^{r} w_i C_i \leqslant C_r^{1/2}\Big(C_r^{-1/2}C_{r-1}^{1/2} \cdots \Big(C_3^{-1/2}C_2^{1/2}\big(C_2^{-1/2}C_1C_2^{-1/2}\big)^{u_1} \times C_2^{1/2}C_3^{-1/2}\Big)^{u_2} \cdots C_{r-1}^{1/2}C_r^{-1/2}\Big)^{u_{r-1}} C_r^{1/2}, \quad (2.9)$$

where u_i is defined as for (1.1). Now, let p_i, $i = 1,\ldots,r$, be positive weights with $P_r = \sum_{k=1}^{r} p_k$, and let D_i, $i = 1,\ldots,r$, be positive definite Hermitian matrices.

We are going to prove the following inequality:

$$\frac{1}{P_r}\sum_{i=1}^{r} p_i D_i \geqslant D_r^{1/2}\Big(D_{r-1}^{1/2} \cdots \Big(D_3^{-1/2}D_2^{1/2}\big(D_2^{-1/2}D_1 \times D_2^{-1/2}\big)^{\bar{u}_1} D_2^{1/2}\Big)^{\bar{u}_2} \cdots D_{r-1}^{1/2}D_r^{-1/2}\Big)^{\bar{u}_{r-1}} D_r^{1/2},$$

where

$$\bar{u}_i = 1 - \frac{p_{i+1}}{\sum_{k=1}^{i+1} p_k} \qquad (i = 1,\ldots,r-1).$$

To this end we make the substitutions in (2.8) and (2.9)

$$w_1 = P_r, \qquad C_1 = \frac{1}{P_r} \sum_{j=1}^{r} p_j D_j,$$

$$w_i = -p_{r-i+2}, \quad C_i = D_{r-i+2}, \qquad i = 2, \ldots, r.$$

Now the conditions (2.8) are satisfied with $W_r = p_1$, and

$$\frac{1}{W_r} \sum_{i=1}^{r} w_i C_i = D_1.$$

Then (2.9) implies

$$D_2^{1/2}\left(D_2^{-1/2} D_3^{1/2} \cdots \left(D_{r-1}^{-1/2} D_r^{1/2}\left(D_r^{-1/2}\left(\frac{1}{P_r} \sum_{j=1}^{r} p_j D_j\right) D_r^{-1/2}\right)^{u_1}\right.\right.$$

$$\left.\left. \times D_r^{1/2} D_{r-1}^{-1/2}\right)^{u_2} \cdots D_3^{1/2} D_2^{-1/2}\right)^{u_{r-1}} D_2^{1/2} \geqslant D_1,$$

from which we have

$$\left(D_2^{-1/2} D_3^{1/2} \cdots \left(D_{r-1}^{-1/2} D_r^{1/2}\left(D_r^{-1/2}\left(\frac{1}{P_r} \sum_{j=1}^{r} p_j D_j\right) D_r^{-1/2}\right)^{u_1}\right.\right.$$

$$\left.\left. \times D_r^{1/2} D_{r-1}^{-1/2}\right)^{u_2} \cdots D_3^{1/2} D_2^{-1/2}\right)^{u_{r-1}} \geqslant D_2^{-1/2} D_1 D_2^{-1/2}.$$

Since $1/u_{r-1} = \bar{u}_1 < 1$, by the Loewner theorem the last inequality implies

$$D_2^{-1/2} D_3^{1/2} \cdots \left(D_{r-1}^{-1/2} D_r^{1/2} \left(D_r^{-1/2} \left(\frac{1}{P_r} \sum_{j=1}^{r} p_j D_j \right) D_r^{-1/2} \right)^{u_1} \times D_r^{1/2} D_{r-1}^{-1/2} \right)^{u_2} \cdots D_3^{1/2} D_2^{-1/2} \geqslant \left(D_2^{-1/2} D_1 D_2^{-1/2} \right)^{\bar{u}_1},$$

$$\left(D_3^{-1/2} D_4^{1/2} \cdots \left(D_{r-1}^{-1/2} D_r^{1/2} \left(D_r^{-1/2} \left(\frac{1}{P_r} \sum_{j=1}^{r} p_j D_j \right) D_r^{-1/2} \right)^{u_1} \times D_r^{1/2} D_{r-1}^{-1/2} \right)^{u_2} \cdots D_4^{1/2} D_3^{-1/2} \right)^{u_{r-2}}$$

$$\geqslant D_3^{-1/2} D_2^{1/2} \left(D_2^{-1/2} D_1 D_2^{-1/2} \right)^{\bar{u}_1} D_2^{1/2} D_3^{-1/2}.$$

If we continue this procedure, we get

$$\frac{1}{P_r} \sum_{j=1}^{r} p_j D_j \geqslant D_r^{1/2} \left(D_r^{-1/2} D_{r-1}^{1/2} \cdots \left(D_3^{-1/2} D_2^{1/2} \left(D_2^{-1/2} D_1 D_2^{-1/2} \right)^{\bar{u}_1} \times D_2^{1/2} D_3^{-1/2} \right)^{\bar{u}_2} \cdots D_{r-1}^{1/2} D_r^{-1/2} \right)^{\bar{u}_{r-1}} D_r^{1/2},$$

the arithmetic–geometric-mean inequality.

REFERENCES

1 M. Sagae and K. Tanabe, Upper and lower bounds for the arithmetic-geometric-harmonic means of positive definite matrices, *Linear and Multilinear Algebra* 37:279–282 (1994).

2 D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York, 1970.

3 D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Boston, 1993.

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