Computers and Mathematics with Applications 58 (2009) 1662–1671

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A class of nth-order BVPs with nonlocal conditions

John R. Graef[a,*], Toufik Moussaoui[b]

[a] Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
[b] Department of Mathematics, E.N.S., PO. Box 92, 16050 Kouba, Algiers, Algeria

ARTICLE INFO

Article history:
Received 25 February 2009
Received in revised form 8 July 2009
Accepted 8 July 2009

Keywords:
BVPs
Existence
Nonlocal conditions
Fixed point theorems

ABSTRACT

The aim of this paper is to present some existence results for a nonlinear nth-order boundary value problem with nonlocal conditions. Various fixed point theorems are used in the proofs. Examples are included to illustrate the results.

1. Introduction

In [1], Eloe and Ahmad studied the problem

$$u^{(n)} + f(t, u) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = 0, \quad u'(0) = 0, \ldots, u^{(n-2)}(0) = 0, \quad \alpha u(\eta) = u(1), \tag{1.2}$$

where $0 < \eta < 1$, $0 < \alpha\eta^{n-1} < 1$, and $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and they proved the existence of solutions for the problem (1.1)–(1.2) when f is either sublinear or superlinear. In this paper, we are interested in the existence of both sign changing solutions and positive solutions for this problem under conditions different from those imposed in [1]. Our approach is based on the application of three different fixed point theorems. In the last section of the paper we extend the results obtained for this problem to the boundary value problem consisting of Eq. (1.1) and the multipoint boundary condition

$$u(0) = 0, \quad u'(0) = 0, \ldots, u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i).$$

First, we introduce some notation and preliminary facts that are used throughout the paper. Let $E = C^0([0, 1], \mathbb{R})$ be the Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ endowed with the norm

$$\|u\| = \sup\{|u(t)| : 0 \le t \le 1\},$$

and let $L^1([0, 1], \mathbb{R})$ be the Banach space of integrable functions on $[0, 1]$ with the norm

$$|u|_1 = \int_0^1 |u(s)|\, ds.$$

We need the following lemmas in the sequel. The first three of these are due to Eloe and Ahmad.

* Corresponding author. Tel.: +1 423 425 4545; fax: +1 423 425 4586.
E-mail addresses: John-Graef@utc.edu (J.R. Graef), moussaoui@ens-kouba.dz (T. Moussaoui).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2009.07.009

Lemma 1.1 ([1, Lemma 2.2]). *Let $0 < \alpha\eta^{n-1} < 1$. If $u \in E$ satisfies the differential inequality $u^{(n)}(t) \le 0$ for $0 < t < 1$ and the boundary conditions (1.2), then $u \ge 0$ on $[0, 1]$.*

Lemma 1.2 ([1, Lemma 2.3]). *Let $0 < \alpha\eta^{n-1} < 1$. If u satisfies $u^{(n)}(t) \le 0$ for $0 < t < 1$ and the boundary conditions (1.2), then*

$$\inf_{t\in[\eta,1]} u(t) \ge \gamma \|u\|,$$

where $\gamma = \min\left\{\alpha\eta^{n-1}, \alpha(1-\eta)(1-\alpha\eta)^{-1}, \eta^{n-1}\right\}$.

Lemma 1.3 ([1]). *The Green's function $G(t, s)$ associated with the problem (1.1)–(1.2) is defined by*

$$G(t,s) = \begin{cases} \dfrac{a(s)t^{n-1}}{(n-1)!}, & \text{if } 0 \le t \le s \le 1, \\ \dfrac{a(s)t^{n-1} + (t-s)^{n-1}}{(n-1)!}, & \text{if } 0 \le s \le t \le 1, \end{cases} \tag{1.3}$$

where

$$a(s) = \begin{cases} -\dfrac{(1-s)^{n-1}}{1-\alpha\eta^{n-1}}, & \eta \le s, \\ -\dfrac{(1-s)^{n-1} - \alpha(\eta-s)^{n-1}}{1-\alpha\eta^{n-1}}, & s \le \eta. \end{cases}$$

It is easy to prove the following result.

Lemma 1.4. *The Green's function $G(t, s)$ associated with the problem (1.1)–(1.2) is negative on $(0, 1) \times (0, 1)$ and satisfies*

$$|G(t,s)| \le \frac{1}{(n-1)!}\left[1 + \frac{1+\alpha\eta^{n-1}}{1-\alpha\eta^{n-1}}\right] \equiv \sigma_n \quad \text{for all } t, s \in [0, 1].$$

Remark 1.1. In [1], there were a couple of misprints that have led to some confusion. For example, the α in the second part of the expression above for $a(s)$ was missing. Also, although it was not stated explicitly, it is implied in Lemma 1.2 that $\alpha\eta < 1$ in order to ensure that γ is positive.

2. Existence of sign changing solutions

In this section, we are concerned with the existence of sign changing solutions for the problem (1.1)–(1.2) under different kinds of growth conditions on the nonlinear function f. Such problems were considered by, for example, Infante [2] and Infante and Webb [3] for the case $n = 2$. Our main result in this section is the following.

Theorem 2.1. *Assume that one of the following hypotheses is satisfied.*

(a) *There exist $q \in L^1([0, 1], \mathbb{R}^+)$, a continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}^+$, and a constant $r_0 > 0$ such that*

$$|f(t,u)| \le q(t)F(u) \quad \text{for all } (t,u) \in [0,1] \times \mathbb{R}$$

and

$$\max_{|y|\le r_0} F(y) \le \frac{r_0}{\sigma_n |q|_1}. \tag{2.1}$$

(b) *There exist $q \in L^1([0, 1], \mathbb{R}^+)$ and a continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that*

$$|f(t,u)| \le q(t)F(u) \quad \text{for all } (t,u) \in [0,1] \times \mathbb{R}$$

and

$$\sigma_n F_0 |q|_1 < 1,$$

where

$$\lim_{s\to 0} \frac{F(s)}{s} = F_0.$$

(c) *There exists a continuous function $F : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}^+$ that is nondecreasing in its second argument and a constant $r_1 > 0$ such that*

$$|f(t,u)| \le F(t,u) \quad \text{for all } (t,u) \in [0,1] \times \mathbb{R}$$

and

$$\int_0^1 F(s, r_1)\,\mathrm{d}s \le \frac{r_1}{\sigma_n}.$$

Then the problem (1.1)–(1.2) has at least one solution.

The proof of this theorem makes use of the following result known as Schauder's fixed point theorem.

Theorem A ([4]). *Let E be a Banach space and let $C \subset E$ be a bounded, closed, and convex subset of E. Let $\mathcal{A} : C \longrightarrow C$ be a completely continuous operator. Then $\mathcal{A}$ has a fixed point in C.*

Proof of Theorem 2.1. Set $C = \{u \in E : \|u\| \le r_0\}$ and define the operator $\mathcal{A} : C \longrightarrow E$ by

$$\mathcal{A}u(t) = -\int_0^1 G(t,s) f(s, u(s))\,\mathrm{d}s.$$

We can easily verify that fixed points of the operator $\mathcal{A}$ are solutions of the problem (1.1)–(1.2). A standard application of the Arzela–Ascoli theorem guarantees that $\mathcal{A}$ is completely continuous. All we need to show is that $\mathcal{A}(C) \subset C$.

(a) For $t \in [0, 1]$, we have

$$\begin{aligned} |\mathcal{A}u(t)| &\le \int_0^1 |G(t,s)| q(s) F(u(s))\,\mathrm{d}s \\ &\le \sigma_n \int_0^1 q(s) F(u(s))\,\mathrm{d}s \\ &\le \sigma_n |q|_1 \max_{|y| \le r_0} F(y) \\ &\le r_0 \end{aligned}$$

for all $u \in C$. Thus, $\|\mathcal{A}u\| \le r_0$, and so $\mathcal{A}(C) \subset C$.

(b) Choose $\epsilon > 0$ so that $\sigma_n(F_0 + \epsilon)|q|_1 \le 1$. There exists $r_0 > 0$ such that

$$|v| < r_0 \quad \text{implies} \quad \left| \frac{F(v)}{v} - F_0 \right| < \epsilon.$$

For $t \in [0, 1]$,

$$\begin{aligned} |\mathcal{A}u(t)| &\le \int_0^1 |G(t,s)| q(s) F(u(s))\,\mathrm{d}s \\ &\le \sigma_n (F_0 + \epsilon) |q|_1 |u(s)| \\ &\le \sigma_n (F_0 + \epsilon) |q|_1 r_0 \\ &\le r_0 \end{aligned}$$

for all $u \in C$. Hence, $\|\mathcal{A}u\| \le r_0$ and so $\mathcal{A}(C) \subset C$.

(c) For $t \in [0, 1]$, we have

$$\begin{aligned} |\mathcal{A}u(t)| &\le \int_0^1 |G(t,s)| |f(s, u(s))|\,\mathrm{d}s \\ &\le \sigma_n \int_0^1 F(s, |u(s)|)\,\mathrm{d}s \\ &\le \sigma_n \int_0^1 F(s, \|u\|)\,\mathrm{d}s \\ &\le \sigma_n \int_0^1 F(s, r_1)\,\mathrm{d}s \\ &\le r_1 \end{aligned}$$

for all $u \in C$. Therefore, $\|\mathcal{A}u\| \le r_1$ and so $\mathcal{A}(C) \subset C$. This completes the proof of the theorem. □

We will illustrate the above theorem with a couple of examples.

Example 2.1. Consider the nonlinear third-order boundary value problem

$$u''' + t^4(u+1)^5/10 = 0, \quad t \in (0,1), \tag{2.2}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(1/2) = u(1). \tag{2.3}$$

Note that

$$|G(t,s)| \le \sigma_3 = 4/3 \quad \text{for all } t, s \in [0,1].$$

By taking $q(t) = t^4/10$ and $F(u) = |u+1|^5$, we then have $|q|_1 = 1/50$, so condition (2.1) holds with $r_0 = 1$. From part (a) of Theorem 2.1, the problem (2.2)–(2.3) has at least one nontrivial solution.

Example 2.2. Consider the problem

$$u''' + t^3\sqrt{|u+1|} = 0, \quad t \in (0,1), \tag{2.4}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(1/2) = u(1). \tag{2.5}$$

By taking $q(t) = t^3$ and $F(u) = \sqrt{|u+1|}$, then from part (c) of Theorem 2.1 with $r_1 = 1$, the problem (2.4)–(2.5) admits at least one nontrivial solution.

3. The sublinear case

In this section, we are concerned with the existence of solutions for the problem (1.1)–(1.2) under a sublinear condition on the function f. Our main result of this type is the following.

Theorem 3.1. *Assume that there are functions $q_1, q_2 \in L^1([0,1], \mathbb{R}^+)$ such that*

$$|f(t,u)| \le q_1(t)|u| + q_2(t) \quad \textit{for all } t \in [0,1] \quad \textit{and} \quad u \in \mathbb{R}.$$

If $\sigma_n |q_1|_1 < 1$, then the problem (1.1)–(1.2) has at least one solution.

The proof of this theorem will make use of the following Leray–Schauder nonlinear alternative.

Theorem B ([5,4]). *Let E be a Banach space and let Ω be a bounded open subset of E with $0 \in \Omega$, and let $\mathcal{A}: \overline{\Omega} \longrightarrow E$ be a completely continuous operator. Then either there exist $u \in \partial\Omega$ and $\lambda > 1$ such that $\mathcal{A}u = \lambda u$, or there exists a fixed point $u^* \in \overline{\Omega}$ of the mapping $\mathcal{A}$.*

Proof of Theorem 3.1. Let Ω is a bounded open set to be chosen later, let $u \in \partial\Omega$, and let $\lambda > 1$ be such that $\mathcal{A}u = \lambda u$. For $t \in [0,1]$, we have

$$\begin{aligned} |u(t)| &\le \frac{1}{\lambda}\int_0^1 |G(t,s)||f(s,u(s))|\,ds \\ &\le \sigma_n \int_0^1 (q_1(s)|u(s)| + q_2(s))\,ds \\ &\le \sigma_n (|q_1|_1 \|u\| + |q_2|_1). \end{aligned}$$

Hence, $\|u\| \le \sigma_n(|q_1|_1\|u\| + |q_2|_1)$, or $\|u\| \le \left(\frac{\sigma_n |q_2|_1}{1-\sigma_n|q_1|_1}\right) \equiv M$. With $\Omega = \{u \in E : \|u\| < M+1\}$, the second alternative in Theorem B is not satisfied, so the existence of a solution to the problem (1.1)–(1.2) follows. □

Example 3.1. Consider the nonlinear third-order problem

$$u''' + t^3\sqrt{|u|} + t + 1 = 0, \quad t \in (0,1), \tag{3.1}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(1/2) = u(1). \tag{3.2}$$

Taking $f(t,u) = t^3\sqrt{|u|} + t + 1$, $q_1(t) = t^3$, and $q_2(t) = t+1$, Theorem 3.1 implies that the problem (3.1)–(3.2) has at least one nontrivial solution.

As a final remark in this section, we should point out that since the trivial solution satisfies the boundary conditions, it is possible that the solution whose existence is guaranteed in Theorems 2.1 and 3.1 may in fact be the zero solution. The added condition that $f(t,0) \not\equiv 0$ ensures that this is not the case (see Examples 2.1, 2.2 and 3.1).

4. Existence of positive solutions

Under suitable assumptions on the nonlinear function f, we shall prove the existence of a positive solution to the problem (1.1)–(1.2). The proofs rely on the Guo–Krasnosel'skii fixed point theorem in cones given in Theorem C. This approach has been used by many authors in the last ten years. In a very nice paper that recently appeared, Webb [6] used the fixed point index to prove the existence of positive solutions. He made use of a careful analysis of the Green's function.

Theorem C ([7,8]). *Let E be a Banach space and $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are two bounded open sets in E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $\mathcal{A} : K \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that either:*

(i) $\|\mathcal{A}u\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{A}u\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|\mathcal{A}u\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|\mathcal{A}u\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then $\mathcal{A}$ has at least one fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

We will prove three existence theorems; the first one is the following.

Theorem 4.1. *Suppose that $f : [0, 1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and there exist positive constants M_1, M_2, C_1, and C_2 such that*

(i) $f(t, u) \le M_1$ *for* $t \in [0, 1]$ *and* $0 \le u \le C_1$, *and*
(ii) $f(t, u) \ge M_2$ *for* $t \in [\eta, 1]$ *and* $\gamma C_2 \le u \le C_2$,

where $C_1 = \sigma_n M_1$ and $C_2 = M_2 \int_\eta^1 |G(\eta, s)| \mathrm{d}s$. Then the problem (1.1)–(1.2) has at least one positive solution.

Proof. Without loss of generality, we assume that $C_1 < C_2$ and set

$$\Omega_1 = \{u \in E \ : \ \|u\| < C_1\} \quad \text{and} \quad \Omega_2 = \{u \in E \ : \ \|u\| < C_2\}.$$

We take as our cone the set

$$K = \left\{u \in E \ : \ u \ge 0 \ \text{and} \ \min_{t\in[\eta,1]} u(t) \ge \gamma\|u\|\right\}.$$

A standard application of the Arzela–Ascoli theorem shows that the operator $\mathcal{A}$ is completely continuous. It also follows by Lemma 1.2 that $\mathcal{A}$ maps K into K.

Let $u \in K \cap \partial\Omega_1$, i.e., $u \in K$ and $\|u\| = C_1$. We then have

$$\begin{aligned} 0 \le \mathcal{A}u(t) &\le \int_0^1 |G(t,s)| f(s, u(s))\,\mathrm{d}s \\ &\le \sigma_n M_1 \\ &= C_1 \\ &= \|u\|. \end{aligned}$$

Taking the supremum, we obtain

$$\|\mathcal{A}u\| \le \|u\| \quad \text{on } K \cap \partial\Omega_1.$$

Let $u \in K \cap \partial\Omega_2$, i.e., $u \in K$ and $\|u\| = C_2$. Using the fact that $\min_{t\in[\eta,1]} u(t) \ge \gamma\|u\|$, we have

$$\begin{aligned} \mathcal{A}u(\eta) &= \int_0^1 |G(\eta,s)| f(s, u(s))\,\mathrm{d}s \\ &\ge \int_\eta^1 |G(\eta,s)| f(s, u(s))\,\mathrm{d}s \\ &\ge M_2 \int_\eta^1 |G(\eta,s)|\,\mathrm{d}s \\ &= C_2 \\ &= \|u\|. \end{aligned}$$

Thus,

$$\|\mathcal{A}u\| \ge \|u\| \quad \text{on } K \cap \partial\Omega_2.$$

By Krasnosel'skii's fixed point theorem, Theorem C, the problem (1.1)–(1.2) has a positive solution $u(t)$ such that $C_1 \le \|u\| \le C_2$. □

To illustrate this result, we have the following example.

Example 4.1. Consider the nonlinear third-order problem

$$u''' + f(t,u) = 0, \quad t \in (0,1), \tag{4.1}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(1/2) = u(1), \tag{4.2}$$

where

$$f(t,u) = \begin{cases} \frac{3}{4}\sqrt{u}t^3, & \text{if } t \in [0,1], u \in \left[0, \frac{4}{3}\right] \\ t^3\left[\left(27,648 - \frac{3\sqrt{3}}{4}\right)u - 36,864 + \frac{3\sqrt{3}}{2}\right], & \text{if } t \in [0,1], u \in \left[\frac{4}{3}, 2\right], \\ 1152t^3u^4, & \text{if } t \in \left[\frac{1}{2}, 1\right], u \in [2, +\infty). \end{cases}$$

It is easy to see that $\int_{\frac{1}{2}}^{1} |G(\frac{1}{2}, s)|ds = \frac{1}{144}$ and $\gamma = \frac{1}{4}$. Theorem 4.1 is satisfied with $M_1 = 1$, $C_1 = 4/3$, $M_2 = 1152$, and $C_2 = 8$, so the problem (4.1)–(4.2) has at least one positive solution $u(t)$ with $4/3 \le \|u\| \le 8$.

Our second existence theorem in this section employs a type of growth condition on the nonlinear function f very different to those used in our previous theorems.

Theorem 4.2. *Suppose that $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function and there are continuous functions $F_1, F_2, G_1, G_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, where F_1 and G_1 are nonincreasing and $\frac{F_2}{F_1}$ and $\frac{G_2}{G_1}$ are nondecreasing, such that*

$$G_1(u) + G_2(u) \le f(t,u) \le F_1(u) + F_2(u) \quad \text{for all } (t,u) \in [0,1] \times \mathbb{R}^+.$$

If there exist constants $r_0, R_0 > 0$, $r_0 \neq R_0$, such that

$$\sigma_n F_1(0)\left(1 + \frac{F_2(r_0)}{F_1(r_0)}\right) \le r_0 \quad \text{and} \quad G_1(R_0)\left(1 + \frac{G_2(\gamma R_0)}{G_1(\gamma R_0)}\right)\int_\eta^1 |G(\eta,s)|ds \ge R_0, \tag{4.3}$$

then the problem (1.1)–(1.2) has at least one positive solution.

Proof. With no loss in generality, we may assume that $r_0 < R_0$ and set

$$\Omega_1 = \{u \in E : \|u\| < r_0\} \quad \text{and} \quad \Omega_2 = \{u \in E : \|u\| < R_0\}.$$

If $u \in K \cap \partial\Omega_1$, then

$$\begin{aligned} 0 \le \mathcal{A}u(t) &\le \sigma_n \int_0^1 (F_1(u(s)) + F_2(u(s)))\,ds \\ &\le \sigma_n F_1(0)\left(1 + \frac{F_2(r_0)}{F_1(r_0)}\right) \\ &\le r_0 \\ &= \|u\|. \end{aligned}$$

We then have

$$\|\mathcal{A}u\| \le \|u\| \quad \text{on } K \cap \partial\Omega_1.$$

If $u \in K \cap \partial\Omega_2$, then

$$\begin{aligned} \mathcal{A}u(\eta) &= \int_0^1 |G(\eta,s)|f(s,u(s))\,ds \\ &\ge \int_\eta^1 |G(\eta,s)|f(s,u(s))\,ds \\ &\ge \int_\eta^1 |G(\eta,s)|\,(G_1(u(s)) + G_2(u(s)))\,ds \\ &= \int_\eta^1 |G(\eta,s)|G_1(u(s))\left(1 + \frac{G_2(u(s))}{G_1(u(s))}\right)\,ds \end{aligned}$$

$$\begin{aligned}
&\geq \int_\eta^1 |G(\eta, s)| G_1(\|u\|) \left(1 + \frac{G_2(\gamma\|u\|)}{G_1(\gamma\|u\|)}\right) \mathrm{d}s \\
&= G_1(R_0)\left(1 + \frac{G_2(\gamma R_0)}{G_1(\gamma R_0)}\right) \int_\eta^1 |G(\eta, s)|\, \mathrm{d}s \\
&\geq R_0 \\
&= \|u\|,
\end{aligned}$$

so

$$\|\mathcal{A}u\| \geq \|u\| \quad \text{on } K \cap \partial\Omega_2.$$

By Krasnosel'skii's fixed point theorem, the problem (1.1)–(1.2) has at least one positive solution $u(t)$ with $r_0 \leq \|u\| \leq R_0$. □

As an example of Theorem 4.2 we have the following.

Example 4.2. Consider the third-order problem

$$u''' + \frac{t^2+1}{t^2+2}\left[e^{-u/40} + e^{u/10}\right], \quad t \in (0, 1), \tag{4.4}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(1/2) = u(1). \tag{4.5}$$

The first inequality in (4.3) is satisfied with $r_0 = 4$. The second inequality in (4.3) reduces to $1 + e^{R_0/32} \geq 288 R_0 e^{R_0/40}$, which is clearly satisfied for sufficiently large R_0. Theorem 3.1 then shows that the problem (4.4)–(4.5) has at least one nontrivial solution $u(t)$ with $4 \leq \|u\| \leq R_0$.

Our final existence theorem is the following.

Theorem 4.3. *Suppose that $f : [0, 1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and there are $q_1, q_2 \in L^1([0, 1], \mathbb{R}^+)$ and a continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that*

$$\begin{aligned}
&q_1(t)F(u) \leq f(t, u) \leq q_2(t)F(u) \quad \text{for all } (t, u) \in [0, 1] \times \mathbb{R}^+, \\
&\sigma_n F_0 |q_2|_1 < 1, \quad \text{and} \quad \gamma F_\infty q_1^* > 1,
\end{aligned}$$

where

$$F_0 = \lim_{s \to 0^+} \frac{F(s)}{s} \in (0, +\infty), \qquad F_\infty = \lim_{s \to +\infty} \frac{F(s)}{s} \in (0, +\infty),$$

and

$$q_1^* = \int_\eta^1 |G(\eta, s)| q_1(s)\, \mathrm{d}s > 0.$$

Then the problem (1.1)–(1.2) has at least one positive solution.

Proof. Choose $\epsilon > 0$ such that $\sigma_n(F_0 + \epsilon)|q_2|_1 \leq 1$. There exists $H_1 > 0$ such that

$$F(x) \leq (F_0 + \epsilon)x \quad \text{for } 0 < x < H_1.$$

Let

$$\Omega_1 = \{u \in E : \|u\| < H_1\}$$

and take $u \in K \cap \partial\Omega_1$. Then,

$$\begin{aligned}
0 \leq \mathcal{A}u(t) &\leq \sigma_n \int_0^1 q_2(s) F(u(s))\, \mathrm{d}s \\
&\leq \sigma_n \int_0^1 q_2(s) u(s)(F_0 + \epsilon)\, \mathrm{d}s \\
&\leq \sigma_n (F_0 + \epsilon)\|u\| |q_2|_1 \\
&\leq \|u\|.
\end{aligned}$$

Hence,

$$\|\mathcal{A}u\| \leq \|u\| \quad \text{on } K \cap \partial\Omega_1.$$

Now choose $\epsilon > 0$ so that $(F_\infty - \epsilon)q_1^* \geq 1$. There exists $\overline{H_2} > 0$ such that

$$F(x) \geq (F_\infty - \epsilon)x \quad \text{for } x \geq \overline{H_2}.$$

Let $H_2 = \max\left\{2H_1, \overline{H_2}/\gamma\right\}$ and set

$$\Omega_2 = \{u \in E \ : \ \|u\| < H_2\}.$$

For $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}\mathcal{A}u(\eta) &= \int_0^1 |G(\eta, s)|f(s, u(s))\,\mathrm{d}s \\ &\geq \int_\eta^1 |G(\eta, s)|f(s, u(s))\,\mathrm{d}s \\ &\geq \int_\eta^1 |G(\eta, s)|q_1(s)F(u(s))\,\mathrm{d}s \\ &\geq \int_\eta^1 |G(\eta, s)|(F_\infty - \epsilon)|u(s)|q_1(s)\,\mathrm{d}s \\ &\geq \int_\eta^1 |G(\eta, s)|(F_\infty - \epsilon)\gamma\|u\|q_1(s)\,\mathrm{d}s \\ &\geq \|u\|.\end{aligned}$$

Thus,

$$\|\mathcal{A}u\| \geq \|u\| \quad \text{on } K \cap \partial\Omega_2.$$

Krasnosel'skii's theorem again implies that the problem (1.1)–(1.2) has at least one positive solution $u(t)$ with $H_1 \leq \|u\| \leq H_2$. □

As an example of this theorem, we give the following.

Example 4.3. Consider the nonlinear third-order problem

$$u''' + \frac{1}{8}(t^3 + 1)\,(u + \sin u + 2940\,u \arctan u) = 0, \quad t \in (0, 1), \tag{4.6}$$

$$u(0) = 0, \quad u'(0) = 0, \quad u(1/2) = u(1). \tag{4.7}$$

Taking $q_1 \equiv 1$, $q_2(t) = t^3 + 1$, and $F(u) = \frac{1}{8}\,(u + \sin u + 2940\,u \arctan u)$, the hypotheses of Theorem 4.3 are satisfied, so the problem (4.6)–(4.7) has at least one positive solution.

We can also prove the following corollary.

Corollary 4.4. *Suppose that $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and satisfies*

$$f_0 = \lim_{s\to 0^+} \frac{f(s)}{s} \in (0, +\infty) \quad \textit{and} \quad f_\infty = \lim_{s\to+\infty} \frac{f(s)}{s} \in (0, +\infty),$$

with

$$\sigma_n f_0 |q|_1 < 1 \quad \textit{and} \quad \gamma f_\infty q_* > 1,$$

where

$$q_* = \int_\eta^1 |G(\eta, s)|q(s)\,\mathrm{d}s > 0.$$

Then the problem

$$u^{(n)} + q(t)f(u) = 0, \quad t \in (0, 1), \tag{4.8}$$

$$u(0) = 0, \quad u'(0) = 0, \ldots, u^{(n-2)} = 0, \quad \alpha u(\eta) = u(1), \tag{4.9}$$

has at least one positive solution.

Remark 4.1. In the same way as is done in [1], we can prove the following:

If either
(a) $f_0 = \lim_{s\to 0^+} \frac{f(s)}{s} = 0$ and $f_\infty = \lim_{s\to+\infty} \frac{f(s)}{s} = +\infty$, or
(b) $f_0 = \lim_{s\to 0^+} \frac{f(s)}{s} = +\infty$ and $f_\infty = \lim_{s\to+\infty} \frac{f(s)}{s} = 0$,
then the problem (4.8)–(4.9) has at least one positive solution.

5. Extensions

The results above can be extended to the m-point boundary value problem

$$u^{(n)} + f(t,u) = 0, \quad t \in (0,1), \tag{5.1}$$

$$u(0) = 0, \quad u'(0) = 0, \dots, u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{5.2}$$

where now $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\alpha_i > 0$, and $D = \sum_{i=1}^{m-2} \alpha_i \eta_i^{n-1} < 1$. A direct calculation (or see, for example, Pang et al. [9]) shows that the Green's function associated with this problem is

$$H(t,s) = \begin{cases} -\dfrac{t^{n-1}(1-s)^{n-1}}{(n-1)!(1-D)} + \displaystyle\sum_{s\le\eta_i} \alpha_i \dfrac{(\eta_i - s)^{n-1} t^{n-1}}{(n-1)!(1-D)}, & \text{if } 0 \le t \le s \le 1, \\ -\dfrac{t^{n-1}(1-s)^{n-1}}{(n-1)!(1-D)} + \dfrac{(t-s)^{n-1}}{(n-1)!} + \displaystyle\sum_{s\le\eta_i} \alpha_i \dfrac{(\eta_i - s)^{n-1} t^{n-1}}{(n-1)!(1-D)}, & \text{if } 0 \le s \le t \le 1. \end{cases} \tag{5.3}$$

We then have that u is a solution of the problem (5.1)–(5.2) if and only if

$$u(t) = -\int_0^1 H(t,s) f(s,u(s)) \mathrm{d}s,$$

and moreover the following estimate is satisfied:

$$|H(t,s)| \le \frac{1}{(n-1)!}\left[1 + \frac{1+D}{1-D}\right] \equiv \rho_n.$$

Theorems 2.1 and 3.1 hold with σ_n replaced by ρ_n; this is also the only change that needs to be made in the proofs. In order to extend Theorems 4.1–4.3 to the problem (5.1)–(5.2), we will need the following lemma.

Lemma 5.1. *Let $0 < \alpha_{m-2}\eta_{m-2} < 1$. If u satisfies $u^{(n)}(t) \le 0$ for $0 < t < 1$ and the boundary conditions (5.2), then*

$$\inf_{t\in[\eta_{m-2},1]} u(t) \ge \Gamma \|u\|,$$

where

$$\Gamma = \min\left\{ \frac{\alpha_{m-2}(1-\eta_{m-2})}{(1-\alpha_{m-2}\eta_{m-2})}, \ \alpha_{m-2}\eta_{m-2}^{n-1}, \ \eta_{m-2}^{n-1} \right\}.$$

Proof. The proof of this lemma is similar in some respects to the proof of Lemma 1.2 above, i.e., Lemma 2.3 in [1], so we only sketch the details. Let $\hat{t}$ denote the zero of $u'(t)$ in $(0,1)$. Then $u'(\hat{t}) = 0$ and $\|u\| = u(\hat{t})$. We consider two cases.

Case 1. Assume $\hat{t} \le \eta_{m-2}$. The line joining the points $(1, u(1))$ and $(\eta_{m-2}, u(\eta_{m-2}))$ is given by

$$v(t) = u(1) + \frac{u(1) - u(\eta_{m-2})}{1 - \eta_{m-2}}(t-1).$$

Observe that

$$u(\hat{t}) \le v(\hat{t}) = u(1) + \frac{u(1) - u(\eta_{m-2})}{1-\eta_{m-2}}(\hat{t} - 1).$$

Now $u(1) = \min_{t\in[\eta_{m-2},1]} u(t)$ and $\alpha_{m-2} u(\eta_{m-2}) < u(1)$, so

$$u(\hat{t}) \le u(1) - \frac{u(1) - u(\eta_{m-2})}{1-\eta_{m-2}} \le u(1)\left[\frac{1-\alpha_{m-2}\eta_{m-2}}{\alpha_{m-2}(1-\eta_{m-2})}\right].$$

Thus,

$$u(1) \ge \left[\frac{\alpha_{m-2}(1-\eta_{m-2})}{1-\alpha_{m-2}\eta_{m-2}}\right] u(\hat{t}),$$

so

$$\min_{t\in[\eta_{m-2},1]} u(t) \geq \left[\frac{\alpha_{m-2}(1-\eta_{m-2})}{1-\alpha_{m-2}\eta_{m-2}}\right]\|u\|. \tag{5.4}$$

Case 2. Assume that $\eta_{m-2} < \hat{t}$. There are two possible subcases.

Case 2(a): $u(\eta_{m-2}) \geq u(1)$. In this case $\min_{t\in[\eta_{m-2},1]} u(t) = u(1)$. An argument like the one in [1] yields

$$u(t) > u(\hat{t})t^{n-1} \quad \text{for } 0 < t < \hat{t},$$

so

$$u(\eta_{m-2}) > \eta_{m-2}^{n-1}\|u\|.$$

Thus,

$$u(t) \geq u(1) \geq \alpha_{m-2}u(\eta_{m-2}) \geq \alpha_{m-2}\eta_{m-2}^{n-1}\|u\|,$$

i.e.,

$$u(t) \geq \alpha_{m-2}\eta_{m-2}^{n-1}\|u\| \quad \text{for } \eta_{m-2} \leq t \leq 1. \tag{5.5}$$

Case 2(b): $u(\eta_{m-2}) < u(1)$. Now we have $\min_{t\in[\eta_{m-2},1]} u(t) = u(\eta_{m-2})$. Like in Case 2(a), we have

$$u(t) \geq u(\eta_{m-2}) \geq \eta_{m-2}^{n-1}\|u\|. \tag{5.6}$$

The conclusion of the lemma then follows from (5.4)–(5.6).

The results in Section 4 now hold for the problem (5.1)–(5.2) with γ, η, and σ_n replaced by Γ, η_{m-2}, and ρ_n, respectively. □

Acknowledgment

The authors would like to thank the referee for making several suggestions for improvements to the paper.

References

[1] P.W. Eloe, B. Ahmad, Positive solutions of a nonlinear *n*th order boundary value problem with nonlocal conditions, Appl. Math. Lett. 18 (2005) 521–527.
[2] G. Infante, Eigenvalues of some non-local boundary value problems, Proc. Edinb. Math. Soc. 46 (2003) 75–86.
[3] G. Infante, J. Webb, Three point boundary value problems with solutions that change sign, J. Integral Equations Appl. 15 (2003) 37–57.
[4] E. Zeidler, Nonlinear Functional Analysis and its Applications. Vol. I: Fixed Point Theorems, Springer-Verlag, New York, 1986.
[5] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, Heidelberg, 1985.
[6] J.R.L. Webb, Nonlocal conjugate type boundary value problems of higher order, Nonlinear Anal. 71 (2009) 1933–1940.
[7] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.
[8] M.A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations, Cambridge University Press, New York, 1964.
[9] C. Pang, W. Dong, Z. Wei, Green's function and positive solutions of *n*th order *m*-point boundary value problem, Appl. Math. Comput. 182 (2006) 1231–1239.