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RESEARCH ARTICLE

Subtraction Menger algebras

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Abstract We give an abstract characterization of algebras of partial functions from A^n to A endowed with the operations of the Menger superposition and the set-theoretic difference of functions as subsets of A^{n+1}.

Keywords Menger algebra · Algebra of multiplace functions · Subtraction algebra

1. Let A^n be the n-th Cartesian power of a set A. Any partial mapping from A^n into A is called a *partial n-place function*. The set of all such mappings is denoted by $\mathcal{F}(A^n, A)$. On $\mathcal{F}(A^n, A)$ we define the *Menger superposition* (composition) of n-place functions $\mathrm{O}\colon\ (f, g_1, \ldots, g_n) \mapsto f[g_1 \ldots g_n]$ as follows:

$$(\bar{a}, c) \in f[g_1 \ldots g_n] \longleftrightarrow (\exists \bar{b})\big((\bar{a}, b_1) \in g_1 \wedge \cdots \wedge (\bar{a}, b_n) \in g_n \wedge (\bar{b}, c) \in f\big) \quad (1)$$

for all $\bar{a} \in A^n$, $\bar{b} = (b_1, \ldots, b_n) \in A^n$, $c \in A$.

Each subalgebra (Φ, O), where $\Phi \subset \mathcal{F}(A^n, A)$, of the algebra $(\mathcal{F}(A^n, A), \mathrm{O})$ is a Menger algebra of rank n in the sense of [2–4, 8]. Menger algebras of partial n-place functions are partially ordered by the set-theoretic inclusion, i.e., such algebras can be considered as algebras of the form $(\Phi, \mathrm{O}, \subset)$. The first abstract characterization of such algebras was given in [9]. Later, in [10, 11] there have been found abstract characterizations of Menger algebras of n-place functions closed with respect to the

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set-theoretic intersection and union of functions, i.e., Menger algebras of the form $(\Phi, \mathrm{O}, \cap)$, $(\Phi, \mathrm{O}, \cup)$ and $(\Phi, \mathrm{O}, \cap, \cup)$.

As is well known, the set-theoretic inclusion $\subset$ and the operations $\cap$, $\cup$ can be expressed via the set-theoretic difference (subtraction) in the following way:

$$A \subset B \longleftrightarrow A \setminus B = \varnothing, \qquad A \cap B = A \setminus (A \setminus B),$$
$$A \cup B = C \setminus \big((C \setminus A) \cap (C \setminus B)\big),$$

where A, B, C are arbitrary sets such that $A \subset C$ and $B \subset C$.

Thus it makes sense to examine sets of functions closed with respect to the subtraction of functions. Such sets of functions are called *difference semigroups*, while their abstract analogs are called *subtraction semigroups*. Some properties of subtraction semigroups can bee found in [1]. The investigation of difference semigroups was initiated by Schein [7].

Below we present a generalization of Schein's results to the case of Menger algebras of n-place functions, i.e., to the case of algebras $(\Phi, \mathrm{O}, \setminus, \varnothing)$, where $\Phi \subset \mathcal{F}(A^n, A)$, $\varnothing \in \Phi$. Such algebras will be called *difference Menger algebras*.

2. A *Menger algebra of rank n* is a non-empty set G with one $(n+1)$-ary operation $o(x, y_1, \ldots, y_n) = x[y_1 \ldots y_n]$ satisfying the identity:

$$x[y_1 \ldots y_n][z_1 \ldots z_n] = x\big[y_1[z_1 \ldots z_n] \ldots y_n[z_1 \ldots z_n]\big]. \tag{2}$$

A Menger algebra of rank 1 is a semigroup. A Menger algebra (G, o) of rank n is called *unitary* if it contains *selectors*, i.e., elements $e_1, \ldots, e_n \in G$ such that $x[e_1 \ldots e_n] = x$ and $e_i[x_1 \ldots x_n] = x_i$ for all $x, x_1, \ldots, x_n \in G$, $i = 1, \ldots, n$. One can prove (see [2, 3]) that every Menger algebra (G, o) of rank n can be isomorphically embedded into a unitary Menger algebra (G^*, o^*) of the same rank with selectors $e_1, \ldots, e_n \notin G$ such that $G \cup \{e_1, \ldots, e_n\}$ is a generating set of (G^*, o^*).

Let (G, o) be a Menger algebra of rank n. Consider the alphabet $G \cup \{[\,,\,], x\}$, where the symbols $[\,,\,], x$ do not belong to G, and construct the set $T_n(G)$ of *polynomials* over this alphabet by the following rules:

(a) $x \in T_n(G)$;
(b) if $i \in \{1, \ldots, n\}$, $a, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in G$, $t \in T_n(G)$, then $a[b_1 \ldots b_{i-1} t\, b_{i+1}, \ldots b_n] \in T_n(G)$;
(c) $T_n(G)$ contains those and only those polynomials which are constructed by (a) and (b).

A binary relation $\rho \subset G \times G$, where (G, o) is a Menger algebra of rank n, is

- *stable* if for all $x, y, x_i, y_i \in G$, $i = 1, \ldots, n$

$$(x, y), (x_1, y_1), \ldots, (x_n, y_n) \in \rho \longrightarrow \big(x[x_1 \ldots x_n], y[y_1 \ldots y_n]\big) \in \rho;$$

- *l-regular*, if for any $x, y, z_i \in G$, $i = 1, \ldots, n$

$$(x, y) \in \rho \longrightarrow \big(x[z_1 \ldots z_n], y[z_1 \ldots z_n]\big) \in \rho;$$

- *v-regular*, if for all $x_i, y_i, z \in G$, $i = 1, \ldots, n$

$$(x_1, y_1), \ldots, (x_n, y_n) \in \rho \longrightarrow \big(z[x_1 \ldots x_n], z[y_1 \ldots y_n]\big) \in \rho;$$

- *i-regular* $(1 \le i \le n)$, if for all $u, x, y \in G$, $\bar{w} \in G^n$

$$(x, y) \in \rho \longrightarrow \big(u[\bar{w}|_i x], u[\bar{w}|_i y]\big) \in \rho;$$

- *weakly steady* if for all $x, y, z \in G$, $t_1, t_2 \in T_n(G)$

$$(x, y), \big(z, t_1(x)\big), \big(z, t_2(y)\big) \in \rho \longrightarrow \big(z, t_2(x)\big) \in \rho,$$

where $\bar{w} = (w_1, \ldots, w_n)$ and $u[\bar{w}|_i x] = u[w_1 \ldots w_{i-1} x w_{i+1} \ldots w_n]$. It is clear that a quasiorder[1] on a Menger algebra is v-regular if and only if it is i-regular for every $i = 1, \ldots, n$. A quasiorder is stable if and only if it is both v-regular and l-regular.

A subset H of a Menger algebra (G, o) is called

- *stable* if

$$g, g_1, \ldots, g_n \in H \longrightarrow g[g_1 \ldots g_n] \in H;$$

- an *l-ideal*, if for all $x, h_1, \ldots, h_n \in G$

$$(h_1, \ldots, h_n) \in G^n \setminus (G \setminus H)^n \longrightarrow x[h_1 \ldots h_n] \in H;$$

- an *i-ideal* $(1 \le i \le n)$, if for all $h, u \in G$, $\bar{w} \in G^n$

$$h \in H \longrightarrow u[\bar{w}|_i h] \in H.$$

Clearly, H is an l-ideal if and only if it is an i-ideal for every $i = 1, \ldots, n$.

Definition 1 An algebra $(G, -, 0)$ of type $(2, 0)$ is called a *subtraction algebra* if it satisfies the following identities:

$$x - (y - x) = x, \tag{3}$$

$$x - (x - y) = y - (y - x), \tag{4}$$

$$(x - y) - z = (x - z) - y, \tag{5}$$

$$0 - 0 = 0. \tag{6}$$

Proposition 1 (Abbott [1]) *Every subtraction algebra satisfies the identity*

$$0 = x - x. \tag{7}$$

Proof Below we give a short proof of this identity:

[1] Recall that a *quasiorder* is a reflexive and transitive binary relation.

$$0 \overset{(3)}{=} 0-\big((0-(x-x))-0\big) \overset{(5)}{=} 0-\big((0-0)-(x-x)\big) \overset{(6)}{=} 0-\big(0-(x-x)\big)$$
$$\overset{(4)}{=} (x-x)-\big((x-x)-0\big) \overset{(5)}{=} (x-x)-\big((x-0)-x\big)$$
$$\overset{(5)}{=} \big(x-((x-0)-x)\big)-x \overset{(3)}{=} x-x,$$

as required. □

From (7), by using (3), we obtain the following two identities:

$$x-0=x, \qquad 0-x=0. \tag{8}$$

Similarly, from (4), (5), (7) and (8) we can deduce the identities

$$\big((x-y)-(x-z)\big)-(z-y)=0, \tag{9}$$
$$\big(x-(x-y)\big)-y=0. \tag{10}$$

Thus, subtraction algebras are implicative BCK-algebras (cf. [5, 6]).

Definition 2 An algebra $(G, o, -, 0)$ of type $(n+1, 2, 0)$ is called a *subtraction Menger algebra* of rank n, if (G, o) is a Menger algebra of rank n, $(G, -, 0)$ is a subtraction algebra and the conditions

$$(x-y)[z_1 \dots z_n]=x[z_1 \dots z_n]-y[z_1 \dots z_n], \tag{11}$$
$$u\big[\bar{w}|_i \big(x-(x-y)\big)\big]=u[\bar{w}|_i\, x]-u[\bar{w}|_i\,(x-y)], \tag{12}$$
$$x-y=0 \wedge z-t_1(x)=0 \wedge z-t_2(y)=0 \longrightarrow z-t_2(x)=0 \tag{13}$$

hold for all $x, y, z, u, z_1, \dots, z_n \in G$, $\bar{w} \in G^n$, $i=1, \dots, n$ and $t_1, t_2 \in T_n(G)$.

By putting $n=1$ in the above definition we obtain the notion of a *weak subtraction semigroup*[2] studied by Schein (cf. [7]). Such semigroups are isomorphic to some subtraction semigroups of the form $(\Phi, \circ, \setminus)$.

3. Now we can present the first result of our paper.

Theorem 1 *Each difference Menger algebra of n-place functions is a subtraction Menger algebra of rank n.*

Proof Let $(\Phi, \mathrm{O}, \setminus, \varnothing)$ be a difference Menger algebra of n-place functions defined on A. Since, as it is proved in [2], the superposition O satisfies (2), the algebra (Φ, O) is a Menger algebra of rank n. From the results proved in [1] it follows that the operation $\setminus$ satisfies (3), (4) and (5). Hence $(\Phi, \setminus, \varnothing)$ is a subtraction algebra. Thus,

[2] A weak subtraction semigroup $(S, \cdot, -)$ is a semigroup $(S, \cdot)$ satisfying the identities (3), (4), (5), $x(y-z)=xy-xz$ and $(x-(x-y))z=xz-(x-y)z$.

$(\Phi, \mathrm{O}, \setminus, \varnothing)$ will be a subtraction Menger algebra if (11), (12) and (13) will be satisfied.

To verify (11) observe that for each $(\bar{a}, c) \in (f \setminus g)[h_1 \ldots h_n]$, where $f, g, h_1, \ldots, h_n \in \Phi$, $\bar{a} \in A^n$, $c \in A$ there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{b}, c) \in f \setminus g$ and $(\bar{a}, b_i) \in h_i$ for each $i = 1, \ldots, n$. Consequently, $(\bar{b}, c) \in f$ and $(\bar{b}, c) \notin g$. Thus, $(\bar{a}, c) \in f[h_1 \ldots h_n]$. If $(\bar{a}, c) \in g[h_1 \ldots h_n]$, then there exists $\bar{d} = (d_1, \ldots, d_n) \in A^n$ such that $(\bar{d}, c) \in g$ and $(\bar{a}, d_i) \in h_i$ for every $i = 1, \ldots, n$. Since $h_1, \ldots, h_n$ are functions, we obtain $b_i = d_i$ for all $i = 1, \ldots, n$. Thus $\bar{b} = \bar{d}$. Therefore $(\bar{b}, c) \in g$, which is impossible. Hence $(\bar{a}, c) \notin g[h_1 \ldots h_n]$. This means that $(\bar{a}, c) \in f[h_1 \ldots h_n] \setminus g[h_1 \ldots h_n]$. So, the following implication

$$(\bar{a}, c) \in (f \setminus g)[h_1 \ldots h_n] \longrightarrow (\bar{a}, c) \in f[h_1 \ldots h_n] \setminus g[h_1 \ldots h_n]$$

is valid for any $\bar{a} \in A^n$, $c \in A$, i.e., $(f \setminus g)[h_1 \ldots h_n] \subset f[h_1 \ldots h_n] \setminus g[h_1 \ldots h_n]$.

Conversely, let $(\bar{a}, c) \in f[h_1 \ldots h_n] \setminus g[h_1 \ldots h_n]$. Then $(\bar{a}, c) \in f[h_1 \ldots h_n]$ and $(\bar{a}, c) \notin g[h_1 \ldots h_n]$. Thus, there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{b}, c) \in f$, $(\bar{b}, c) \notin g$ and $(\bar{a}, b_i) \in h_i$ for each $i = 1, \ldots, n$. Hence, $(\bar{b}, c) \in f \setminus g$ and $(\bar{a}, c) \in (f \setminus g)[h_1 \ldots h_n]$. So,

$$(\bar{a}, c) \in f[h_1 \ldots h_n] \setminus g[h_1 \ldots h_n] \longrightarrow (\bar{a}, c) \in (f \setminus g)[h_1 \ldots h_n]$$

for any $\bar{a} \in A^n$, $c \in A$, i.e., $f[h_1 \ldots h_n] \setminus g[h_1 \ldots h_n] \subset (f \setminus g)[h_1 \ldots h_n]$. Thus,

$$(f \setminus g)[h_1 \ldots h_n] = f[h_1 \ldots h_n] \setminus g[h_1 \ldots h_n],$$

which proves (11).

Now, let $(\bar{a}, c) \in u[\bar{\omega}|_i (f \setminus (f \setminus g))] = u[\bar{\omega}|_i (f \cap g)]$, where $f, g, u \in \Phi$, $\bar{\omega} \in \Phi^n$, $\bar{a} \in A^n$, $c \in A$. Then there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{a}, b_i) \in f \cap g$, $(\bar{a}, b_j) \in \omega_j$, $j \in \{1, \ldots, n\} \setminus \{i\}$ and $(\bar{b}, c) \in u$. Since $(\bar{a}, b_i) \in f \cap g$ implies $(\bar{a}, b_i) \notin f \setminus g$, we have $(\bar{a}, c) \in u[\bar{\omega}|_i f]$ and $(\bar{a}, c) \notin u[\bar{\omega}|_i (f \setminus g)]$. Therefore $(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i (f \setminus g)]$. Thus, we have shown that for any $\bar{a} \in A^n$, $c \in A$ holds the implication

$$(\bar{a}, c) \in u\big[\bar{\omega}|_i \big(f \setminus (f \setminus g)\big)\big] \longrightarrow (\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u\big[\bar{\omega}|_i (f \setminus g)\big],$$

which is equivalent to the inclusion $u[\bar{\omega}|_i (f \setminus (f \setminus g))] \subset u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i (f \setminus g)]$.

Conversely, let $(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i (f \setminus g)]$. Then $(\bar{a}, c) \in u[\bar{\omega}|_i f]$ and $(\bar{a}, c) \notin u[\bar{\omega}|_i (f \setminus g)]$. The first of these two conditions means that there exists $\bar{b} = (b_1, \ldots, b_n) \in A^n$ such that $(\bar{a}, b_i) \in f$, $(\bar{a}, b_j) \in \omega_j$ for each $j \in \{1, \ldots, n\} \setminus \{i\}$ and $(\bar{b}, c) \in u$. It is easy to see that the second condition $(\bar{a}, c) \notin u[\bar{\omega}|_i (f \setminus g)]$ is equivalent to the implication

$$(\forall \bar{d}) \left((\bar{a}, d_i) \in f \wedge \bigwedge_{j=1, j \neq i}^{n} (\bar{a}, d_j) \in \omega_j \wedge (\bar{d}, c) \in u \longrightarrow (\bar{a}, d_i) \in g \right), \tag{14}$$

where $\bar{d} = (d_1, \ldots, d_n) \in A^n$. From this implication for $\bar{d} = \bar{b}$, we obtain

$$(\bar{a}, b_i) \in f \wedge \bigwedge_{j=1, j \neq i}^{n} (\bar{a}, b_j) \in \omega_j \wedge (\bar{b}, c) \in u \longrightarrow (\bar{a}, b_i) \in g,$$

which gives $(\bar{a}, b_i) \in g$. Therefore $(\bar{a}, b_i) \in f \cap g = f \setminus (f \setminus g)$. This means that $(\bar{a}, c) \in u[\bar{\omega}|_i (f \setminus (f \setminus g))]$. So, the implication

$$(\bar{a}, c) \in u[\bar{\omega}|_i f] \setminus u\big[\bar{\omega}|_i (f \setminus g)\big] \longrightarrow (\bar{a}, c) \in u\big[\bar{\omega}|_i \big(f \setminus (f \setminus g)\big)\big]$$

is valid for all $\bar{a} \in A^n$, $c \in A$. Hence $u[\bar{\omega}|_i f] \setminus u[\bar{\omega}|_i (f \setminus g)] \subset u[\bar{\omega}|_i (f \setminus (f \setminus g))]$. Thus

$$u\big[\bar{\omega}|_i \big(f \setminus (f \setminus g)\big)\big] = u[\bar{\omega}|_i f] \setminus u\big[\bar{\omega}|_i (f \setminus g)\big].$$

This proves (12).

To prove (13) suppose that for some $f, g, h \in \Phi$ and $t_1, t_2 \in T_n(\Phi)$ we have $f \setminus g = \varnothing$, $h \setminus t_1(f) = \varnothing$ and $h \setminus t_2(g) = \varnothing$. Then $f \subset g$, $h \subset t_1(f)$ and $h \subset t_2(g)$. Hence $f = g \circ \Delta_{\mathrm{pr}_1 f}$ and $\mathrm{pr}_1 h \subset \mathrm{pr}_1 f$, where $\mathrm{pr}_1 f$ denotes the domain of f and $\Delta_{\mathrm{pr}_1 f}$ is the identity binary relation on $\mathrm{pr}_1 f$.

From the inclusion $h \subset t_2(g)$ we obtain

$$h = h \circ \Delta_{\mathrm{pr}_1 f} \subset t_2(g) \circ \Delta_{\mathrm{pr}_1 f} = t_2(g \circ \Delta_{\mathrm{pr}_1 f}) = t_2(f),$$

which means that (13) is also satisfied. This completes the proof that $(\Phi, \mathrm{O}, \setminus, \varnothing)$ is a subtraction Menger algebra of rank n. □

To prove the converse statement, we should first consider a number of properties of subtraction Menger algebras of rank n, introduce some definitions and prove a few auxiliary propositions.

4. Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n.

Proposition 2 *In every subtraction Menger algebra of rank n we have*

$$0[x_1 \ldots x_n] = 0, \qquad x[x_1 \ldots x_{i-1} 0\, x_{i+1} \ldots x_n] = 0$$

for all $x, x_1, \ldots, x_n \in G$, $i = 1, \ldots, n$.

Proof Indeed, using (7) and (11) we obtain

$$0[x_1 \ldots x_n] = (0 - 0)[x_1 \ldots x_n] = 0[x_1 \ldots x_n] - 0[x_1 \ldots x_n] = 0.$$

Similarly, applying (12) and (7) we get

$$u[\bar{w}|_i\, 0] = u\big[\bar{w}|_i \big(0 - (0 - 0)\big)\big] = u[\bar{w}|_i\, 0] - u\big[\bar{w}|_i (0 - 0)\big] = u[\bar{w}|_i\, 0] - u[\bar{w}|_i\, 0] = 0,$$

which was to show. □

Let ω be a binary relation defined on $(G, o, -, 0)$ in the following way:

$$\omega = \big\{(x, y) \in G \times G \mid x - y = 0\big\}.$$

Using (7), (8) and (9) it is easy to see that this is an order, i.e., a reflexive, transitive and antisymmetric relation. In connection with this fact we will sometimes write $x \leq y$ instead of $(x, y) \in \omega$. Using this notation it is not difficult to verify that

$$0 \leq x, \quad x - y \leq x, \tag{15}$$

$$x \leq y \longleftrightarrow x - (x - y) = x, \tag{16}$$

$$x \leq y \longrightarrow x - z \leq y - z, \tag{17}$$

$$x \leq y \longrightarrow z - y \leq z - x, \tag{18}$$

$$x \leq y \wedge u \leq v \longrightarrow x - v \leq y - u \tag{19}$$

holds for all $x, y, z, u, v \in G$.

Moreover, in a subtraction algebra the following two identities

$$(x - y) - y = x - y, \tag{20}$$

$$(x - y) - z = (x - z) - (y - z) \tag{21}$$

are valid (cf. [1, 5, 6]).

Proposition 3 *The relation* ω *on the algebra* $(G, o, -, 0)$ *is stable and weakly steady.*

Proof Let $x \leq y$ for some $x, y \in G$. Then $x - y = 0$ and

$$(x - y)[z_1 \dots z_n] = 0[z_1 \dots z_n] = (0 - 0)[z_1 \dots z_n] = 0[z_1 \dots z_n] - 0[z_1 \dots z_n] = 0$$

for all $z_1, \dots, z_n \in G$. This, by (11), implies

$$x[z_1 \dots z_n] - y[z_1 \dots z_n] = 0,$$

i.e., $x[z_1 \dots z_n] \leq y[z_1 \dots z_n]$. Thus, ω is l-regular.

Moreover, from $x \leq y$, using (8), we obtain $x - (x - y) = x$, which together with (4), gives $y - (y - x) = x$. Consequently, for any $u \in G$, $\bar{w} \in G^n$ we have $u[\bar{w}|_i (y - (y - x))] = u[\bar{w}|_i\, x]$. This and (11) give $u[\bar{w}|_i\, y] - u[\bar{w}|_i (y - x)] = u[\bar{w}|_i\, x]$. Hence, according to (15), we obtain $u[\bar{w}|_i\, x] \leq u[\bar{w}|_i\, y]$. Thus, ω is i-regular for every $i = 1, \dots, n$. Since ω is a quasiorder, this means that ω is v-regular. But ω also is l-regular, hence it is stable.

It is clear that ω is weakly steady if and only if it satisfies (13).[3] □

Proposition 4 *The axiom* (12) *is equivalent to each of the following conditions*:

$$x \leq y \longrightarrow u\big[\bar{w}|_i\, (y - x)\big] = u[\bar{w}|_i\, y] - u[\bar{w}|_i\, x], \tag{22}$$

$$x \leq y \longrightarrow t(y - x) = t(y) - t(x), \tag{23}$$

[3] In the case of semigroups the fact that ω is weakly steady can be deduced directly from the axioms of a weak subtraction semigroup (cf. [7]).

$$t\big(x-(x-y)\big)=t(x)-t(x-y) \tag{24}$$

for all $x,y,u\in G$, $\bar{w}\in G^n$, $i=1,\dots,n$, $t\in T_n(G)$.

Proof (12) → (22). Suppose that the condition (12) is satisfied and $x\le y$ for some $x,y\in G$. Then, according to (16), we have $x-(x-y)=x$. Hence, by (4), we obtain $y-(y-x)=x$. Thus, $y-x=y-(y-(y-x))$, which, in view of (12), gives $u[\bar{w}|_i\,(y-x)]=u[\bar{w}|_i\,(y-(y-(y-x)))]=u[\bar{w}|_i\,y]-u[\bar{w}|_i(y-(y-x))]=u[\bar{w}|_i\,y]-u[\bar{w}|_i x]$. This means that (12) implies (22).

(22) → (23). From (22) it follows that for $x\le y$ and all polynomials $t\in T_n(G)$ of the form $t(x)=u[\bar{w}|_i x]$ the condition (23) is satisfied. To prove that (23) is satisfied by an arbitrary polynomial from $T_n(G)$ suppose that it is satisfied by some $t'\in T_n(G)$. Since the relation ω is stable on the algebra $(G,o,-,0)$, from $x\le y$ it follows $t'(x)\le t'(y)$, which in view of (22), implies

$$u\big[\bar{w}|_i\,\big(t'(y)-t'(x)\big)\big]=u\big[\bar{w}|_i\,t'(y)\big]-u\big[\bar{w}|_i\,t'(x)\big].$$

But according to the assumption on t' for $x\le y$ we have $t'(y)-t'(x)=t'(y-x)$, so the above equation can be written as

$$u\big[\bar{w}|_i\,t'(y-x)\big]=u\big[\bar{w}|_i\,t'(y)\big]-u\big[\bar{w}|_i\,t'(x)\big].$$

Thus, (23) is satisfied by polynomials of the form $t(x)=u[\bar{w}|_i t'(x)]$.

From the construction of $T_n(G)$ it follows that (23) is satisfied by all polynomials $t\in T_n(G)$. Therefore (22) implies (23).

(23) → (24). Since, by (15), $x-y\le x$ holds for all $x,y\in G$, from (23) it follows $t(x-(x-y))=t(x)-t(x-y)$ for any polynomial $t\in T_n(G)$. Thus, (23) implies (24).

(24) → (12). By putting $t(x)=u[\bar{w}|_i\,x]$ we obtain (12). □

On a subtraction Menger algebra $(G,o,-,0)$ of rank n we can define a binary operation $\curlywedge$ by putting:

$$x\curlywedge y\overset{def}{=}x-(x-y). \tag{25}$$

By using this operation the conditions (11), (16), (24) can be written in a more useful form:

$$u\big[\bar{w}|_i(x\curlywedge y)\big]=u[\bar{w}|_i\,x]-u\big[\bar{w}|_i(x-y)\big], \tag{26}$$

$$x\le y\longleftrightarrow x\curlywedge y=x, \tag{27}$$

$$t(x\curlywedge y)=t(x)-t(x-y), \tag{28}$$

where $x,y,u\in G$, $\bar{w}\in G^n$, $i=1,\dots,n$, $t\in T_n(G)$. Moreover, from (11) and (25), we can deduce the identity:

$$(x\curlywedge y)[z_1\dots z_n]=x[z_1\dots z_n]\curlywedge y[z_1\dots z_n]. \tag{29}$$

The algebra $(G, \curlywedge)$ is a lower semilattice. Directly from the conditions (3)–(10) we obtain (cf. [1]) the following properties:

$$x \leq y \wedge x \leq z \longrightarrow x \leq y \curlywedge z, \tag{30}$$

$$x \leq y \longrightarrow x \curlywedge z \leq y \curlywedge z, \tag{31}$$

$$x \curlywedge y = 0 \longrightarrow x - y = x, \tag{32}$$

$$(x - y) \curlywedge y = 0, \tag{33}$$

$$x \curlywedge (y - z) = (x \curlywedge y) - (x \curlywedge z), \tag{34}$$

$$x - y = x - (x \curlywedge y), \tag{35}$$

$$(x \curlywedge y) - (y - z) = x \curlywedge y \curlywedge z, \tag{36}$$

$$(x \curlywedge y) - z = (x - z) \curlywedge (y - z), \tag{37}$$

$$(x \curlywedge y) - z = (x - z) \curlywedge y \tag{38}$$

for all $x, y, z \in G$.

Proposition 5 *In a subtraction Menger algebra* $(G, o, -, 0)$ *of rank n the following conditions*

$$t(x - y) = t(x) - t(x \curlywedge y), \tag{39}$$

$$t(x) - t(y) \leq t(x - y) \tag{40}$$

are valid for each $t \in T_n(G)$ *and* $x, y \in G$.

Proof From (35) we obtain $t(x - y) = t(x - (x \curlywedge y))$ for every $t \in T_n(G)$. (25) and (15) imply $x \curlywedge y \leq x$, which together with (23) gives $t(x - (x \curlywedge y)) = t(x) - t(x \curlywedge y)$. Hence, $t(x - y) = t(x) - t(x \curlywedge y)$. This proves (39).

Since $x \curlywedge y \leq y$, the stability of ω implies $t(x \curlywedge y) \leq t(y)$ for every $t \in T_n(G)$. From this, by applying (15) and (18), we obtain $t(x) - t(y) \leq t(x) - t(x \curlywedge y) = t(x - y)$, which proves (40). □

By $[0, a]$ we denote the *initial segment* of the algebra $(G, -, 0)$, i.e., the set of all $x \in G$ such that $0 \leq x \leq a$. According to [7], on any $[0, a]$ we can define a binary operation $\curlyvee$ by putting:

$$x \curlyvee y \overset{def}{=} a - \big((a - x) \curlywedge (a - y)\big) \tag{41}$$

for all $x, y \in [0, a]$. It is not difficult to see that this operation is idempotent and commutative, and 0 is its neutral element, i.e., $x \curlyvee x = x$, $x \curlyvee y = y \curlyvee x$, $x \curlyvee 0 = x$ for all $x, y \in [0, a]$.

Proposition 6 *For any* $x, y \in [0, b] \subset [0, a]$, *where* $a, b \in G$, *we have*

$$b - \big((b - x) \curlywedge (b - y)\big) = a - \big((a - x) \curlywedge (a - y)\big). \tag{42}$$

Proof Note first that $b = b \curlywedge a$ because $b \le a$. Moreover, from $x \le b$ and $y \le b$, according to (18), we obtain $a - b \le a - x$ and $a - b \le a - y$. This together with (30) gives $a - b \le (a - x) \curlywedge (a - y)$. Thus, $(a - b) - ((a - x) \curlywedge (a - y)) = 0$.

By (15) we have $b - ((a - x) \curlywedge (a - y)) \le b$, which implies

$$b \curlywedge \big(b - \big((a - x) \curlywedge (a - y)\big)\big) = b - \big((a - x) \curlywedge (a - y)\big). \tag{43}$$

Obviously $b = b \curlywedge b = b \curlywedge a$, $x = b \curlywedge x$, $y = b \curlywedge y$. Therefore:[4]

$$\begin{aligned}
& b - \big((b - x) \curlywedge (b - y)\big) \\
& \quad = b \curlywedge b - \big((b \curlywedge a - b \curlywedge x) \curlywedge (b \curlywedge a - b \curlywedge y)\big) \\
& \quad \overset{(34)}{=} b \curlywedge b - \big(b \curlywedge (a - x) \curlywedge b \curlywedge (a - y)\big) = b \curlywedge b - b \curlywedge \big((a - x) \curlywedge (a - y)\big) \\
& \quad \overset{(34)}{=} b \curlywedge \big(b - \big((a - x) \curlywedge (a - y)\big)\big) \overset{(42)}{=} b - \big((a - x) \curlywedge (a - y)\big) \\
& \quad = a \curlywedge b - \big((a - x) \curlywedge (a - y)\big) \overset{(25)}{=} \big(a - (a - b)\big) - \big((a - x) \curlywedge (a - y)\big) \\
& \quad \overset{(21)}{=} \big(a - \big((a - x) \curlywedge (a - y)\big)\big) - \big((a - b) - \big((a - x) \curlywedge (a - y)\big)\big) \\
& \quad = \big(a - \big((a - x) \curlywedge (a - y)\big)\big) - 0 \overset{(8)}{=} a - \big((a - x) \curlywedge (a - y)\big),
\end{aligned}$$

which completes the proof. □

Corollary 1 *The condition* (42) *is valid for all* $x, y \in [0, a] \cap [0, b]$.

Proof Since $[0, a] \cap [0, b] = [0, a \curlywedge b] \subset [0, a] \cup [0, b]$, by Proposition 6, for all $x, y \in [0, a] \cap [0, b]$ we have:

$$\begin{aligned}
a - \big((a - x) \curlywedge (a - y)\big) &= a \curlywedge b - \big((a \curlywedge b - x) \curlywedge (a \curlywedge b - y)\big), \\
b - \big((b - x) \curlywedge (b - y)\big) &= a \curlywedge b - \big((a \curlywedge b - x) \curlywedge (a \curlywedge b - y)\big).
\end{aligned}$$

This implies (42). □

From the above corollary it follows that the value of $x \curlyvee y$, if it exists, does not depend on the choice of the interval $[0, a]$ containing the elements x and y. In [1] it is proved that for $x, y, z \in [0, a]$ we have:

$$x \curlywedge (x \curlyvee y) = x, \tag{44}$$

$$x \curlyvee (x \curlywedge y) = x, \tag{45}$$

$$(x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z), \tag{46}$$

$$x \curlywedge (y \curlyvee z) = (x \curlywedge y) \curlyvee (x \curlywedge z), \tag{47}$$

$$x \curlyvee (y \curlywedge z) = (x \curlyvee y) \curlywedge (x \curlyvee z), \tag{48}$$

[4] To reduce the number of brackets we will write $x \curlywedge y - z$ instead of $(x \curlywedge y) - z$.

$$(x \curlyvee y) - z = (x - z) \curlyvee (y - z), \tag{49}$$

$$x \leq z \wedge y \leq z \longrightarrow x \curlyvee y \leq z, \tag{50}$$

$$y \leq x \longrightarrow x = (x - y) \curlyvee y, \tag{51}$$

$$x = (x \curlyvee y) - (y - x), \tag{52}$$

$$x = (x \curlywedge y) \curlyvee (x - y). \tag{53}$$

From (44) it follows $x \leq x \curlyvee y$.

Proposition 7 *If for some $x, y \in G$ there exists $x \curlyvee y$, then for all $u \in G$, $\bar{z}, \bar{w} \in G^n$, $i = 1, \ldots, n$ there are also elements $x[\bar{z}] \curlyvee y[\bar{z}]$ and $u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]$, and the following identities are satisfied*:

$$(x \curlyvee y)[\bar{z}] = x[\bar{z}] \curlyvee y[\bar{z}], \tag{54}$$

$$u\big[\bar{w}|_i (x \curlyvee y)\big] = u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]. \tag{55}$$

Proof Suppose that the element $x \curlyvee y$ exists. Then $x \leq a$ and $y \leq a$ for some $a \in G$, which, by the l-regularity of the relation ω, implies $x[\bar{z}] \leq a[\bar{z}]$ and $y[\bar{z}] \leq a[\bar{z}]$ for any $\bar{z} \in G^n$. This means that $x[\bar{z}] \curlyvee y[\bar{z}]$ exists and

$$\begin{aligned}(x \curlyvee y)[\bar{z}] &\overset{(41)}{=} \big(a - \big((a - x) \curlywedge (a - y)\big)\big)[\bar{z}] \overset{(11)}{=} a[\bar{z}] - \big((a - x) \curlywedge (a - y)\big)[\bar{z}] \\ &\overset{(29)}{=} a[\bar{z}] - \big((a - x)[\bar{z}] \curlywedge (a - y)[\bar{z}]\big) \\ &\overset{(11)}{=} a[\bar{z}] - \big(\big(a[\bar{z}] - x[\bar{z}]\big) \curlywedge \big(a[\bar{z}] - y[\bar{z}]\big)\big) \overset{(41)}{=} x[\bar{z}] \curlyvee y[\bar{z}].\end{aligned}$$

This proves (54).

Further, from $x \leq a$, $y \leq a$ and the i-regularity of ω we obtain $u[\bar{w}|_i\, x] \leq u[\bar{w}|_i\, a]$ and $u[\bar{w}|_i\, y] \leq u[\bar{w}|_i\, a]$. Hence, the element $u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]$ exists. Since $x \leq x \curlyvee y$ and $y \leq x \curlyvee y$, we also have $u[\bar{w}|_i\, x] \leq u[\bar{w}|_i (x \curlyvee y)]$ and $u[\bar{w}|_i\, y] \leq u[\bar{w}|_i (x \curlyvee y)]$, which, according to (50), gives

$$u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y] \leq u\big[\bar{w}|_i (x \curlyvee y)\big]. \tag{56}$$

On the other side, the existence of $u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]$ implies

$$u[\bar{w}|_i\, x] \leq u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y] \quad \text{and} \quad u[\bar{w}|_i\, y] \leq u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y].$$

Moreover,

$$u\big[\bar{w}|_i (x \curlyvee y)\big] - u\big[\bar{w}|_i (y - x)\big] \overset{(40)}{\leq} u\big[\bar{w}|_i \big((x \curlyvee y) - (y - x)\big)\big] \overset{(52)}{=} u[\bar{w}|_i\, x].$$

Consequently,

$$u\big[\bar{w}|_i (x \curlyvee y)\big] - u\big[\bar{w}|_i (y - x)\big] \leq u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]. \tag{57}$$

But $y - x \le y$, so, $u[\bar{w}|_i (y - x)] \le u[\bar{w}|_i\, y]$ and

$$u\big[\bar{w}|_i (y - x)\big] \le u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y].$$

This and (57) guarantee the existence of the element

$$\big(u\big[\bar{w}|_i (x \curlyvee y)\big] - u\big[\bar{w}|_i (y - x)\big]\big) \curlyvee u\big[\bar{w}|_i (y - x)\big]$$

such that

$$\big(u\big[\bar{w}|_i (x \curlyvee y)\big] - u\big[\bar{w}|_i (y - x)\big]\big) \curlyvee u\big[\bar{w}|_i (y - x)\big] \le u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y]. \tag{58}$$

Since $u[\bar{w}|_i (y - x)] \le u[\bar{w}|_i\, y] \le u[\bar{w}|_i (x \curlyvee y)]$, the last inequality and (51) imply

$$\big(u\big[\bar{w}|_i (x \curlyvee y)\big] - u\big[\bar{w}|_i (y - x)\big]\big) \curlyvee u\big[\bar{w}|_i (y - x)\big] = u\big[\bar{w}|_i (x \curlyvee y)\big],$$

which together with (58) gives

$$u\big[\bar{w}|_i (x \curlyvee y)\big] \le u[\bar{w}|_i\, x] \curlyvee u[\bar{w}|_i\, y].$$

Comparing this inequality with (56) we obtain (55). □

Corollary 2 *If for some $x, y \in G$ an element $x \curlyvee y$ exists, then for any polynomial $t \in T_n(G)$ an element $t(x) \curlyvee t(y)$ also exists and $t(x \curlyvee y) = t(x) \curlyvee t(y)$.*

Proposition 8 *For all $x, y \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ we have:*

$$t_1(x \curlywedge y) \curlywedge t_2(x - y) = 0.$$

Proof Let $t_1(x \curlywedge y) \curlywedge t_2(x - y) = h$. Obviously $h \le t_1(x \curlywedge y)$ and $h \le t_2(x - y)$. Since $t_2(x - y) \le t_2(x)$, we have $h \le t_2(x)$. Thus, $x \curlywedge y \le x$, $h \le t_1(x \curlywedge y)$ and $h \le t_2(x)$. This, in view of Proposition 3 and (13), gives $h \le t_2(x \curlywedge y)$. Consequently,

$$h \le t_2(x - y) \curlywedge t_2(x \curlywedge y). \tag{59}$$

Further,

$$\begin{aligned} t_2(x - y) - t_2(x \curlywedge y) &\overset{(39)}{=} \big(t_2(x) - t_2(x \curlywedge y)\big) - t_2(x \curlywedge y) \\ &\overset{(20)}{=} t_2(x) - t_2(x \curlywedge y) \overset{(39)}{=} t_2(x - y). \end{aligned}$$

Therefore,

$$\begin{aligned} t_2(x - y) \curlywedge t_2(x \curlywedge y) &\overset{(25)}{=} t_2(x - y) - \big(t_2(x - y) - t_2(x \curlywedge y)\big) \\ &= t_2(x - y) - t_2(x - y) = 0, \end{aligned}$$

which together with (59) implies $h \le 0$. Hence $h = 0$. This completes the proof. □

Proposition 9 *For all $x, y, z, g \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ the following conditions are valid*:

$$t_1(x \curlywedge y) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y), \tag{60}$$

$$t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) \le t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z), \tag{61}$$

$$g \le t_1(x \curlywedge y) \wedge g \le t_2(y \curlywedge z) \longrightarrow g \le t_2(x \curlywedge y \curlywedge z). \tag{62}$$

Proof To prove (60) observe first that for $z = t_1(x \curlywedge y) \curlywedge t_2(y)$ we have $z \le t_1(x \curlywedge y)$ and $z \le t_2(y)$. Since the relation ω is weakly steady and $x \curlywedge y \le y$, from the above we conclude $z \le t_2(x \curlywedge y)$, i.e., $t_1(x \curlywedge y) \curlywedge t_2(y) \le t_2(x \curlywedge y)$. This, by (31), implies $t_1(x \curlywedge y) \curlywedge t_2(y) \le t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y)$.

On the other side, the stability of ω and $x \curlywedge y \le y$ imply $t_2(x \curlywedge y) \le t_2(y)$ for every $t_2 \in T_n(G)$. Hence, $t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y) \le t_1(x \curlywedge y) \curlywedge t_2(y)$ by (31). This completes the proof of (60).

Further: $t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) = t_1((x \curlywedge z) \curlywedge y) \curlywedge t_2(y) \overset{(60)}{=} t_1((x \curlywedge z) \curlywedge y) \curlywedge t_2((x \curlywedge z) \curlywedge y) \le t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z)$ proves (61).

Finally, let $g \le t_1(x \curlywedge y)$ and $g \le t_2(y \curlywedge z)$. Then

$$\begin{aligned}
g &\le t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z)\\
&\overset{(28)}{=} t_1(x \curlywedge y) \curlywedge \big(t_2(y) - t_2(y - z)\big)\\
&\overset{(34)}{=} \big(t_1(x \curlywedge y) \curlywedge t_2(y)\big) - \big(t_1(x \curlywedge y) \curlywedge t_2(y - z)\big)\\
&\overset{(60)}{=} \big(t_1(x \curlywedge y) \curlywedge t_2(x \curlywedge y)\big) - \big(t_1(x \curlywedge y) \curlywedge t_2(y - z)\big)\\
&\overset{(34)}{=} t_1(x \curlywedge y) \curlywedge \big(t_2(x \curlywedge y) - t_2(y - z)\big) \le t_2(x \curlywedge y) - t_2(y - z)\\
&\overset{(40)}{\le} t_2\big((x \curlywedge y) - (y - z)\big) \overset{(36)}{=} t_2(x \curlywedge y \curlywedge z).
\end{aligned}$$

This proves (62) and completes the proof of our proposition. □

Corollary 3 *For all $x, y, z \in G$ and all polynomials $t_1, t_2 \in T_n(G)$ we have*:

$$t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) = t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z). \tag{63}$$

Proof We have $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \le t_1(x \curlywedge y)$ and $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \le t_2(y \curlywedge z)$, so by (62) we obtain $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \le t_1(x \curlywedge y \curlywedge z)$. Considering now that $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \le t_2(y \curlywedge z) \le t_2(y)$, by (30), we get $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \le t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y)$. Taking now into account the condition (61) we obtain (63). □

5. Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n.

Definition 3 By a *determining pair* of a subtraction Menger algebra $(G, o, -, 0)$ of rank n we mean an ordered pair (ε^*, W), where ε is a v-regular equivalence relation defined on (G, o), $\varepsilon^* = \varepsilon \cup \{(e_1, e_1), \ldots, (e_n, e_n)\}$, $e_1, \ldots, e_n$ are the selectors of the unitary extension (G^*, o^*) of (G, o) and W is the empty set or an l-ideal of (G, o) which is an ε-class.

Definition 4 A non-empty subset F of a subtraction Menger algebra $(G, o, -, 0)$ of rank n is called a *filter* if:

(1) $0 \notin F$;
(2) $x \in F \wedge x \leq y \longrightarrow y \in F$;
(3) $x \in F \wedge y \in F \longrightarrow x \curlywedge y \in F$

for all $x, y \in G$.

If $a, b \in G$ and $a \nleq b$, then $[a) = \{x \in G \,|\, a \leq x\}$ is a filter with $a \in [a)$ and $b \notin [a)$. By Zorn's Lemma the collection of filters which contain an element a, but do not contain an element b, has a maximal element which is denoted by $F_{a,b}$. Using this filter we define the following three sets:

$$W_{a,b} = \big\{x \in G \,|\, \big(\forall t \in T_n(G)\big)\, t(x) \notin F_{a,b}\big\},$$
$$\varepsilon_{a,b} = \big\{(x, y) \in G \times G \,|\, x \curlywedge y \notin W_{a,b} \vee x, y \in W_{a,b}\big\},$$
$$\varepsilon^*_{a,b} = \varepsilon_{a,b} \cup \big\{(e_1, e_1), \ldots, (e_n, e_n)\big\}.$$

Proposition 10 *For any $a, b \in G$, the pair $(\varepsilon^*_{a,b}, W_{a,b})$ is the determining pair of the algebra $(G, o, -, 0)$.*

Proof First we show that $\varepsilon_{a,b}$ is an equivalence relation on G. It is clear that this relation is reflexive and symmetric. To prove its transitivity let $(x, y), (y, z) \in \varepsilon_{a,b}$. We have four possibilities:

(a) $x \curlywedge y \notin W_{a,b} \wedge y \curlywedge z \notin W_{a,b}$,
(b) $x \curlywedge y \notin W_{a,b} \wedge y, z \in W_{a,b}$,
(c) $x, y \in W_{a,b} \wedge y \curlywedge z \notin W_{a,b}$,
(d) $x, y \in W_{a,b} \wedge y, z \in W_{a,b}$.

In the case (a) we have $t_1(x \curlywedge y), t_2(y \curlywedge z) \in F_{a,b}$ for some $t_1, t_2 \in T_n(G)$. Since $F_{a,b}$ is a filter, then, obviously, $t_1(x \curlywedge y) \curlywedge t_2(y \curlywedge z) \in F_{a,b}$. This, according to (63), implies $t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) \in F_{a,b}$. But $t_1(x \curlywedge y \curlywedge z) \curlywedge t_2(y) \leq t_1(x \curlywedge z)$, hence also $t_1(x \curlywedge z) \in F_{a,b}$, i.e., $x \curlywedge z \notin W_{a,b}$. Thus, $(x, z) \in \varepsilon_{a,b}$.

In the case (b) from $x \curlywedge y \notin W_{a,b}$ it follows $t(x \curlywedge y) \in F_{a,b}$ for some polynomial $t \in T_n(G)$. But $x \curlywedge y \leq y$, and consequently $t(x \curlywedge y) \leq t(y)$. Thus $t(y) \in F_{a,b}$, i.e., $y \notin W_{a,b}$, which is a contradiction. Hence the case (b) is impossible. Analogously we can show that also the case (c) is impossible. The case (d) is obvious, because in this case $x, z \in W_{a,b}$ which means that $(x, z) \in \varepsilon_{a,b}$. This completes the proof that $\varepsilon_{a,b}$ is transitive.

Moreover, if $x \in W_{a,b}$, then $t(x) \notin F_{a,b}$ for every $t \in T_n(G)$. In particular, for all $t(x) = t'(u[\bar{w}|_i\, x]) \in T_n(G)$ we have $t'(u[\bar{w}|_i\, x]) \notin F_{a,b}$. Thus, $u[\bar{w}|_i\, x] \in W_{a,b}$ for every $i = 1, \ldots, n$. Hence, $W_{a,b}$ is an i-ideal of (G, o), and consequently, an l-ideal. It is clear that $W_{a,b}$ is an $\varepsilon_{a,b}$-class.

Next, we prove that the relation $\varepsilon_{a,b}$ is v-regular. Let $x \equiv y(\varepsilon_{a,b})$. Then $x \curlywedge y \notin W_{a,b}$ or $x, y \in W_{a,b}$. In the case $x, y \in W_{a,b}$ we obtain $u[\bar{w}|_i\, x], u[\bar{w}|_i\, y] \in W_{a,b}$ because $W_{a,b}$ is an l-ideal of (G, o). Thus, $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. In the case $x \curlywedge y \notin W_{a,b}$ elements $u[\bar{w}|_i\, x]$, $u[\bar{w}|_i\, y]$ belong or not belong to $W_{a,b}$ simultaneously. Indeed, if $u[\bar{w}|_i\, x], u[\bar{w}|_i\, y] \in W_{a,b}$, then obviously $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. Now, if $u[\bar{w}|_i\, x] \notin W_{a,b}$, then $t(u[\bar{w}|_i\, x]) \in F_{a,b}$ for some $t \in T_n(G)$. Since $x \curlywedge y \notin W_{a,b}$, then also $t_1(x \curlywedge y) \in F_{a,b}$ for some $t_1 \in T_n(G)$. Thus $t_1(x \curlywedge y) \curlywedge t(u[\bar{w}|_i\, x]) \in F_{a,b}$, which, by (60), implies $t_1(x \curlywedge y) \curlywedge t(u[\bar{w}|_i(x \curlywedge y)]) \in F_{a,b}$. But $t_1(x \curlywedge y) \curlywedge t(u[\bar{w}|_i(x \curlywedge y)]) \le t(u[\bar{w}|_i\, y])$, hence $t(u[\bar{w}|_i\, y]) \in F_{a,b}$, i.e., $u[\bar{w}|_i\, y] \notin W_{a,b}$. So, we have shown that $x \curlywedge y \notin W_{a,b}$ and $u[\bar{w}|_i\, x] \notin W_{a,b}$ imply $u[\bar{w}|_i\, y] \notin W_{a,b}$. Similarly we can show that $x \curlywedge y \notin W_{a,b}$ and $u[\bar{w}|_i\, y] \notin W_{a,b}$ imply $u[\bar{w}|_i\, x] \notin W_{a,b}$. Therefore, we have proved that in the case $x \curlywedge y \notin W_{a,b}$ elements $u[\bar{w}|_i\, x], u[\bar{w}|_i\, y]$ belong or not belong to $W_{a,b}$ simultaneously.

So, if for $x \curlywedge y \notin W_{a,b}$ we have $u[\bar{w}|_i\, x], u[\bar{w}|_i\, y] \in W_{a,b}$, then clearly $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. Therefore assume that $u[\bar{w}|_i\, x] \notin W_{a,b}$ (hence $u[\bar{w}|_i\, y] \notin W_{a,b}$). Thus, $x \curlywedge y \notin W_{a,b}$, $u[\bar{w}|_i\, x] \notin W_{a,b}$, i.e., $t(x \curlywedge y) \in F_{a,b}$, $t_1(u[\bar{w}|_i\, x]) \in F_{a,b}$ for some $t, t_1 \in T_n(G)$. Hence, $t(y \curlywedge x \curlywedge y) \curlywedge t_1(u[\bar{w}|_i\, x]) \in F_{a,b}$. From this, according to (63), we obtain $t(y \curlywedge x) \curlywedge t_1(u[\bar{w}|_i(x \curlywedge y)]) \in F_{a,b}$. This implies $t_1(u[\bar{w}|_i(x \curlywedge y)]) \in F_{a,b}$. Since $u[\bar{w}|_i(x \curlywedge y)] \le u[\bar{w}|_i\, x]$ and $u[\bar{w}|_i(x \curlywedge y)] \le u[\bar{w}|_i\, y]$, we have $u[\bar{w}|_i(x \curlywedge y)] \le u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]$, which, by the stability of ω gives $t_1(u[\bar{w}|_i(x \curlywedge y)]) \le t_1(u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y])$. Consequently, $t_1(u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y]) \in F_{a,b}$, so $u[\bar{w}|_i\, x] \curlywedge u[\bar{w}|_i\, y] \notin W_{a,b}$, i.e., $u[\bar{w}|_i\, x] \equiv u[\bar{w}|_i\, y](\varepsilon_{a,b})$. In this way we have proved that the relation $\varepsilon_{a,b}$ is i-regular for every $i = 1, \ldots, n$. Thus it is v-regular. □

Proposition 11 *All equivalence classes of* $\varepsilon_{a,b}$*, except of* $W_{a,b}$*, are filters.*

Proof Indeed, let $H \ne W_{a,b}$ be an arbitrary class of $\varepsilon_{a,b}$. If $x \in H$ and $x \le y$, then $x \curlywedge y = x \notin W_{a,b}$, consequently, $(x, y) \in \varepsilon_{a,b}$. Hence, $y \in H$. Further, let $x, y \in H$, then $(x, y) \in \varepsilon_{a,b}$. Thus $x \curlywedge y \notin W_{a,b}$, i.e., $t(x \curlywedge y) \in F_{a,b}$ for some $t \in T_n(G)$. But $x \curlywedge y = x \curlywedge (x \curlywedge y)$, hence, $t(x \curlywedge (x \curlywedge y)) \in F_{a,b}$ and $x \curlywedge (x \curlywedge y) \notin W_{a,b}$. So $x \equiv x \curlywedge y(\varepsilon_{a,b})$. This implies $x \curlywedge y \in H$. Thus, we have shown that H is a filter. □

Proposition 12 *If* $x \curlyvee y$ *exists for some* $x, y \in W_{a,b}$*, then* $x \curlyvee y \in W_{a,b}$*.*

Proof Let $x \curlyvee y$ exists for some $x, y \in W_{a,b}$. If $x \curlyvee y \notin W_{a,b}$, then $t(x \curlyvee y) \in F_{a,b}$ for some $t \in T_n(G)$, and, according to Corollary 2, $t(x \curlyvee y) = t(x) \curlyvee t(y)$. If $t(x) \notin F_{a,b}$, then $F_{a,b}$ is a proper subset of the set

$$U = \{u \in G \mid (\exists z \in F_{a,b})\, z \curlywedge t(x) \le u\}$$

because $t(x) \in U$.

We show that U is a filter. $0 \notin U$ because, by (15), we have $0 \le z \curlywedge t(x)$ for any $z \in F_{a,b}$. Let $s \in U$ and $s \le r$. Then $z \curlywedge t(x) \le s$ for some $z \in F_{a,b}$. Consequently, $z \curlywedge t(x) \le r$, so $r \in U$. Now let $s \in U$ and $r \in U$, i.e., $z_1 \curlywedge t(x) \le s$ and $z_2 \curlywedge t(x) \le r$ for some $z_1, z_2 \in F_{a,b}$. Since $F_{a,b}$ is a filter, we have $z_1 \curlywedge z_2 \in F_{a,b}$. Hence, $(z_1 \curlywedge z_2) \curlywedge t(x) \le s \curlywedge r$, which implies $s \curlywedge r \in U$. Thus U is a filter. But by assumption $F_{a,b} \subset U$ is a maximal filter, which does not contain b, so $b \in U$. Consequently,

$z_1 \curlywedge t(x) \le b$ for some $z_1 \in F_{a,b}$. Similarly, if $t(y) \notin F_{a,b}$, then $z_2 \curlywedge t(y) \le b$ for some $z_2 \in F_{a,b}$. This implies $z \curlywedge t(x) \le b$ and $z \curlywedge t(y) \le b$ for $z = z_1 \curlywedge z_2$. Hence $(z \curlywedge t(x)) \curlyvee (z \curlywedge t(y))$ exists and

$$\big(z \curlywedge t(x)\big) \curlyvee \big(z \curlywedge t(y)\big) = z \curlywedge \big(t(x) \curlyvee t(y)\big) = z \curlywedge t(x \curlyvee y) \in F_{a,b}$$

by (47). But by (50) we have $(z \curlywedge t(x)) \curlyvee (z \curlywedge t(y)) \le b$, so $z \curlywedge t(x \curlyvee y) \le b$. Since $z \curlywedge t(x \curlyvee y) \in F_{a,b}$, then, obviously, $b \in F_{a,b}$, which is impossible. So, $t(x) \in F_{a,b}$ or $t(y) \in F_{a,b}$, hence $x \notin W_{a,b}$ or $y \notin W_{a,b}$, contrary to the assumption that $x, y \in W_{a,b}$. Thus, the assumption that $x \curlyvee y \notin W_{a,b}$ is incorrect. Therefore $x \curlyvee y \in W_{a,b}$. □

6. Each homomorphism of a Menger algebra (G, o) of rank n into a Menger algebra $(\mathcal{F}(A^n, A), \mathrm{O})$ is called a *representation by n-place functions*. Thus, $P : G \to \mathcal{F}(A^n, A)$ is a representation, if

$$P\big(x[y_1 \dots y_n]\big) = P(x)\big[P(y_1) \dots P(y_n)\big]$$

for all $x, y_1, \dots, y_n \in G$. A representation which is an isomorphism is called *faithful* (cf. [2–4, 8]). A representation P of (G, o) is a representation of $(G, o, -, 0)$ if

$$P(x - y) = P(x) \setminus P(y) \quad \text{and} \quad P(0) = \varnothing$$

for all $x, y \in G$.

Let $(P_i)_{i \in I}$ be the family of representations of a subtraction Menger algebra $(G, o, -, 0)$ of rank n by n-place functions defined on pairwise disjoint sets $(A_i)_{i \in I}$. By the *sum* of the family $(P_i)_{i \in I}$ we mean the map $P\colon g \mapsto P(g)$, denoted by $\sum_{i \in I} P_i$, where $P(g)$ is an n-place function on $A = \bigcup_{i \in I} A_i$ defined by $P(g) = \bigcup_{i \in I} P_i(g)$. It is clear (cf. [2, 3]) that P is a representation of $(G, o, -, 0)$.

Similarly as in [2, 3] with each determining pair (ε^*, W) we can associate the so-called *simplest representation* $P_{(\varepsilon^*, W)}$ of (G, o) which assigns to each element $g \in G$ the n-place function $P_{(\varepsilon^*, W)}(g)$ defined on $\mathcal{H} = \mathcal{H}_0 \cup \{\{e_1\}, \dots, \{e_n\}\}$, where $\mathcal{H}_0$ is the set of all ε-classes of G different from W such that

$$(H_1, \dots, H_n, H) \in P_{(\varepsilon, W)}(g) \longleftrightarrow g[H_1 \dots H_n] \subset H,$$

for $(H_1, \dots, H_n) \in \mathcal{H}_0^n \cup \{(\{e_1\}, \dots, \{e_n\})\}$ and $H \in \mathcal{H}$.

Theorem 2 *Each subtraction Menger algebra of rank n is isomorphic to some difference Menger algebra of n-place functions.*

Proof Let $(G, o, -, 0)$ be a subtraction Menger algebra of rank n. Then the sum

$$P = \sum_{a,b \in G,\, a \nleq b} P_{(\varepsilon^*_{a,b}, W_{a,b})}$$

of the family $(P_{(\varepsilon^*_{a,b}, W_{a,b})})_{a,b \in G,\, a \nleq b}$ of simplest representations of (G, o) is a representation of (G, o).

Now we show that P is a representation of $(G, o, -, 0)$. Let $\mathcal{H}_0$ be the set of all $\varepsilon_{a,b}$-classes of G different from $W_{a,b}$. Consider $H_1, \ldots, H_n, H \in \mathcal{H}$, where $\mathcal{H} = \mathcal{H}_0 \cup \{\{e_1\}, \ldots, \{e_n\}\}$, such that $(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2)$ for some $g_1, g_2 \in G$. Then, obviously, $(g_1 - g_2)[H_1 \ldots H_n] \subset H \neq W_{a,b}$. Thus $(g_1 - g_2)[\bar{x}] \in H$ for each $\bar{x} \in H_1 \times \cdots \times H_n$, which, by (11), gives $g_1[\bar{x}] - g_2[\bar{x}] \in H$. But $g_1[\bar{x}] - g_2[\bar{x}] \leq g_1[\bar{x}]$ and H is a filter (Proposition 11), hence $g_1[\bar{x}] \in H$. Thus $(g_1[\bar{x}] - g_2[\bar{x}]) \curlywedge g_2[\bar{x}] = 0$, by (33). Consequently, $(g_1[\bar{x}] - g_2[\bar{x}]) \curlywedge g_2[\bar{x}] \in W_{a,b}$, because the other $\varepsilon_{a,b}$-classes as filters do not contain 0. This means that $g_1[\bar{x}] - g_2[\bar{x}] \not\equiv g_2[\bar{x}](\varepsilon_{a,b})$. Hence, $g_2[\bar{x}] \notin H$. Therefore $g_1[H_1 \ldots H_n] \subset H$ and $g_2[h_1 \ldots H_n] \cap H = \varnothing$, which implies

$$(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2).$$

In this way, we have proved the inclusion

$$P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2) \subset P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2). \tag{64}$$

To show the reverse inclusion let

$$(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2).$$

Then $(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1)$ and $(H_1, \ldots, H_n, H) \notin P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2)$, i.e., $g_1[H_1 \ldots H_n] \subset H$ and $g_2[H_1 \ldots H_n] \cap H = \varnothing$. Thus $g_1[\bar{x}] \in H$ and $g_2[\bar{x}] \notin H$ for all $\bar{x} \in H_1 \times \cdots \times H_n$. Since from $g_1[\bar{x}] \curlywedge g_2[\bar{x}] \notin W_{a,b}$, it follows $g_1[\bar{x}] \equiv g_2[\bar{x}](\varepsilon_{a,b})$ and $g_2[\bar{x}] \in H$, which is a contradiction, we conclude that $g_1[\bar{x}] \curlywedge g_2[\bar{x}] \in W_{a,b}$.

If $g_1[\bar{x}] - g_2[\bar{x}] \in W_{a,b}$, then, by (53) and Proposition 12, we obtain $g_1[\bar{x}] = (g_1[\bar{x}] \curlywedge g_2[\bar{x}]) \curlyvee (g_1[\bar{x}] - g_2[\bar{x}]) \in W_{a,b}$. Consequently, $g_1[\bar{x}] \in W_{a,b}$, which is impossible because $g_1[\bar{x}] \in H$. Thus, $(g_1[\bar{x}] - g_2[\bar{x}]) \curlywedge g_1[\bar{x}] = g_1[\bar{x}] - g_2[\bar{x}] \notin W_{a,b}$. Hence, $g_1[\bar{x}] - g_2[\bar{x}] \equiv g_1[\bar{x}](\varepsilon_{a,b})$. This implies $(g_1 - g_2)[\bar{x}] = g_1[\bar{x}] - g_2[\bar{x}] \in H$. Therefore, $(g_1 - g_2)[H_1 \ldots H_n] \subset H$, i.e., $(H_1, \ldots, H_n, H) \in P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2)$. So, we have proved

$$P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2) \subset P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2).$$

This together with (64) proves

$$P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1 - g_2) = P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_1) \setminus P_{(\varepsilon^*_{a,b}, W_{a,b})}(g_2),$$

which means that $P(g_1 - g_2) = P(g_1) \setminus P(g_2)$ for $g_1, g_2 \in G$. Further, $P(0) = P(0 - 0) = P(0) \setminus P(0) = \varnothing$. So, P is a representation of $(G, o, -, 0)$ by n-place functions.

We show that this representation is faithful. Let $P(g_1) = P(g_2)$ for some $g_1, g_2 \in G$. If $g_1 \neq g_2$, then both inequalities $g_1 \leq g_2$ and $g_2 \leq g_1$ at the same time are impossible. Suppose that $g_1 \nleq g_2$. Then $g_1 \in F_{g_1,g_2}$ and, consequently,

$$\big(\{e_1\}, \ldots, \{e_n\}, F_{g_1,g_2}\big) \in P_{(\varepsilon^*_{g_1,g_2}, W_{g_1,g_2})}(g_2).$$

Since $P_{(\varepsilon^*_{g_1,g_2},W_{g_1,g_2})}(g_1) = P_{(\varepsilon^*_{g_1,g_2},W_{g_1,g_2})}(g_2)$, then, obviously,

$$\big(\{e_1\},\ldots,\{e_n\}, F_{g_1,g_2}\big) \in P_{(\varepsilon^*_{g_1,g_2},W_{g_1,g_2})}(g_2).$$

Thus $\{g_2\} = g_2[\{e_1\}\ldots\{e_n\}] \subset F_{g_1,g_2}$, hence $g_2 \in F_{g_1,g_2}$. This is a contradiction because F_{g_1,g_2} is a filter containing g_1 but not containing g_2. The case $g_2 \nleq g_1$ is analogous. So, the supposition $g_1 \neq g_2$ is not true. Hence $g_1 = g_2$ and P is a faithful representation. The theorem is proved. □

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