Positive Solutions of Nonlinear Integral Equations Arising in Infectious Diseases

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1. Introduction

The nonlinear integral equation

\[ x(t) = \int_{t-\tau}^{t} f(s, x(s)) \, ds \tag{1} \]

can be interpreted as a model for the spread of certain infectious diseases with periodic contact rate that varies seasonally. In Eq. (1), \( x(t) \) represents the proportion of infectives in the population at time \( t \), \( f(t, x(t)) \) is the proportion of new infectives per unit time \( (f(t, 0) = 0) \), and \( \tau \) is the length of time an individual remains infectious. Obviously, \( x(t) = 0 \) is the trivial solution of Eq. (1). In [1, 2] the following three conditions are used:

\[ (H_1) \quad f(t, x) \text{ is nonnegative and continuous for } -\infty < t < +\infty \text{ and } x \geq 0. \]

\[ (H_2) \quad f(t, 0) = 0 \text{ for } -\infty < t < +\infty \text{ and there exists } \omega > 0 \text{ such that } f(t + \omega, x) = f(t, x) \text{ for all } -\infty < t < +\infty \text{ and } x \geq 0. \]

\[ (H_3) \quad \text{there exists } R > 0 \text{ such that } f(t, x) \leq R/\tau \text{ for } 0 \leq t \leq \omega \text{ and } 0 \leq x \leq R. \]
Under \((H_1)-(H_3)\) and some complicated inequalities for the function

\[ a(t) = \lim_{x \to +0} \frac{f(t, x)}{x}, \]

Leggett and Williams proved in [1, 2] that Eq. (1) has at least one nontrivial nonnegative and continuous solution with period \(\omega\).

In this paper, we shall use the fixed point index theory and the monotone technique [3] to establish three theorems about the solutions of Eq. (1). Theorem 1 asserts the existence of two nontrivial nonnegative solutions with period \(\omega\) and Theorems 2 and 3 are concerned with the existence and uniqueness of nontrivial positive solutions under conditions which are essentially different from [1, 2].

2. Main Results

Let us list two more conditions for convenience.

\((H_4)\) \(\lim_{x \to +0} (f(t, x)/x) = a_0(t)\) uniformly with respect to \(t \in [0, \omega]\) and \(\sup_{0 \leq t \leq \omega} a_0(t) < 1/\tau\).

\((H_5)\) \(\lim_{x \to +\infty} (f(t, x)/x) = a_\infty(t)\) uniformly with respect to \(t \in [0, \omega]\) and \(\sup_{0 \leq t \leq \omega} a_\infty(t) < 1/\tau\).

**Theorem 1.** Suppose that \((H_1), (H_2), (H_4), \) and \((H_5)\) are satisfied. If there exist \(a > 0\) and a nonnegative continuous function \(b(t)\) with period \(\omega\) such that

\[ f(t, x) \geq b(t), \quad \forall t \in [0, \omega], \quad x \geq a \quad (2) \]

and

\[ \int_{t-\tau}^{t} b(s) \, ds > a, \quad \forall t \in [0, \omega], \quad (3) \]

then Eq. (1) has at least two nontrivial nonnegative and continuous solutions \(x_1(t)\) and \(x_2(t)\) with period \(\omega\) and

\[ \inf_{-\infty < t < +\infty} x_1(t) > a. \quad (4) \]

**Proof.** Let \(E\) be the Banach space of all continuous and \(\omega\)-periodic functions \(x(t)\) on \(R^1\) with norm

\[ \|x\| = \sup_{-\infty < t < +\infty} |x(t)| = \max_{0 \leq t \leq \omega} |x(t)|, \]
and let \( P = \{ x \in E | x(t) \geq 0 \text{ for } t \in \mathbb{R}^1 \} \). Clearly, \( P \) is a normal cone of \( E \). It is easy to show that the nonlinear integral operator

\[
Ax(t) = \int_{t-\tau}^{t} f(s, x(s)) \, ds
\]

is completely continuous from \( P \) into \( P \). Choosing \( \mu > \tau \) such that

\[
\sup_{0 \leq r \leq \omega} a_0(r) < \frac{1}{\mu}, \quad \sup_{0 \leq t \leq \omega} a_x(t) < \frac{1}{\mu}
\]

and observing that \( a_0(t) \) and \( a_x(t) \) are \( \omega \)-periodic functions, we see from (H4) and (H3) that there exist \( 0 < r < a \) and \( l > a \) such that

\[
0 \leq f(t, x) < \frac{1}{\mu} x, \quad \forall -\infty < t < +\infty, \ 0 < x \leq r \text{ or } x \geq l.
\]

Hence

\[
0 \leq f(t, x) \leq \frac{1}{\mu} x + \beta, \quad \forall -\infty < t < +\infty, \ x > 0,
\]

where

\[
\beta = \max_{0 \leq t \leq \omega, 0 \leq x \leq l} f(t, x).
\]

Now, choose \( R > a \) such that \( (1/\mu) R + \beta < (1/\tau) R \), then from (6) and (7) it follows that

\[
0 \leq f(t, x) < \frac{R}{\mu}, \quad \forall -\infty < t < +\infty, \ 0 \leq x \leq r
\]

and

\[
0 \leq r(t, x) < \frac{R}{\tau}, \quad \forall -\infty < t < +\infty, \ 0 \leq x \leq R.
\]

Let \( U_1 = \{ x \in P \ | \ ||x|| < r \} \), \( U_2 = \{ x \in P \ | \ ||x|| < R \text{ and } \min_{0 \leq t \leq \omega} x(t) > a \} \) and \( U_3 = \{ x \in P \ | \ ||x|| < R \} \). Obviously,

\[
U_1 \subset U_2 \subset U_3, \quad U_1 \cap U_2 = \emptyset
\]

and \( \bar{U}_1 = \{ x \in P \ | \ ||x|| \leq r \} \), \( \bar{U}_2 = \{ x \in P \ | \ ||x|| \leq R \text{ and } \min_{0 \leq t \leq \omega} x(t) \geq a \} \), \( \bar{U}_3 = \{ x \in P \ | \ ||x|| \leq R \} \). For \( x \in \bar{U}_1 \), by (8) we have

\[
||Ax|| = \max_{0 \leq t \leq \omega} \frac{1}{\mu} \int_{t-\tau}^{t} r \, ds = \frac{r \tau}{\mu} < r.
\]
hence

\[ A(\overline{U}_1) \subset U_1. \quad (11) \]

In the same way, from (9) we can easily get

\[ A(\overline{U}_3) \subset U_3. \quad (12) \]

when \( x \in \overline{U}_2 \), we have \( \|x\| \leq R \) and

\[ \inf_{-\infty < t < +\infty} x(t) = \min_{0 < t < \omega} x(t) \geq a, \]

and so, by (12), \( \|Ax\| < R \). Moreover, from (2) and (3) we obtain

\[ \min_{0 < t < \omega} Ax(t) \geq \min_{0 < t < \omega} \int_{t}^{t'} b(s) \, ds > a, \]

and therefore

\[ A(\overline{U}_2) \subset U_2. \quad (13) \]

it follows from (11)-(13) that the fixed point index (see [3])

\[ i(A, U_k, P) = 1 \quad (k = 1, 2, 3). \quad (14) \]

Consequently, \( A \) has a fixed point \( x_1 \) in \( U_2 \) and it is clear that

\[ \inf_{-\infty < t < +\infty} x_1(t) = \min_{0 < t < \omega} x_1(t) > a, \]

i.e., (4) holds. On the other hand, (14) implies

\[ i(A, U_3 \setminus (\overline{U}_1 \cup \overline{U}_2), P) = i(A, U_3, P) \\
- i(A, U_1, P) - i(A, U_2, P) = -1 \neq 0, \]

and so \( A \) has a fixed point \( x_2 \) in \( U_3 \setminus (\overline{U}_1 \cup \overline{U}_2) \), and our theorem is proved. \( \blacksquare \)

Theorem 1 shows that under certain circumstances the spread of the infectious disease may have two possible states.

It is easy to give some elementary functions, which satisfy all conditions of Theorem 1. For example,

\[ f(t, x) = b(t) \sqrt{x} \ln(1 + x), \quad \forall -\infty < t + \infty, \ x \geq 0, \quad (15) \]
where $b(t)$ is a nonnegative continuous function with period $\omega$ and satisfies

$$\int_{t-\tau}^{t} b(s) \, ds > \frac{e}{2} \quad \text{for} \quad t \in [0, \omega];$$

$$f(t, x) = c(t) \left( \frac{x^2}{1 + x^2} + \sin^2 x \right), \quad \forall \, -\infty < t < +\infty, \, x \geq 0, \quad (16)$$

where $c(t)$ is a nonnegative continuous function with period $\omega$ and satisfies

$$\int_{t-\tau}^{t} c(s) \, ds > 2 \quad \text{for} \quad t \in [0, \omega].$$

For functions (15) and (16) we take $a = e^2 - 1$ and $a = 1$ in (2) and (3), respectively.

**Theorem 2.** Suppose that $(H_1)-(H_3)$ are satisfied. If there exist $0 < a < R$ and a nonnegative continuous function $b(t)$ with period $\omega$ such that

$$f(t, x) \geq b(t), \quad \forall t \in [0, \omega], \, a \leq x \leq R \quad (17)$$

and

$$\int_{t-\tau}^{t} b(s) \, ds \geq a, \quad \forall t \in [0, \omega], \quad (18)$$

then Eq. (1) has at least one positive and continuous solution $x(t)$ with period $\omega$ and

$$a \leq \inf_{-\infty < t < +\infty} x(t) \leq \sup_{-\infty < t < +\infty} x(t) \leq R. \quad (19)$$

**Proof.** Let $E$, $P$, $A$, and $U_2$ be the same as in the proof of Theorem 1. Obviously, $\bar{U}_2 = \{ x \in P \mid a \leq x(t) \leq R \text{ for } t \in [0, \omega]\}$ and $\bar{U}_2$ is a nonempty bounded closed convex set in $E$. For $x \in \bar{U}_2$, we see by $(H_3)$ and (17), (18) that

$$Ax(t) \leq \int_{t-\tau}^{t} \frac{R}{\tau} \, ds = R, \quad t \in [0, \omega],$$

and

$$\min_{0 \leq t \leq \omega} Ax(t) \geq \min_{0 \leq t \leq \omega} \int_{t-\tau}^{t} b(s) \, ds \geq a.$$
Hence $Ax \in \bar{U}_2$, and so $A$ is a completely continuous operator from $\bar{U}_2$ into $\bar{U}_2$. Consequently, by Schauder’s fixed point theorem, $A$ has a fixed point in $\bar{U}_2$.

We can also easily point out some elementary functions, which satisfy all conditions of Theorem 2. For example,

$$f(t, x) = b(t) \ln(1 + x^5) + c(t) \sqrt{x} \sin^2 \left( x + \frac{\pi}{\omega} t \right),$$

(20)

where $b(t)$ and $c(t)$ are nonnegative continuous functions with period $\omega$ and $b(t)$ satisfies

$$\int_{t-\tau}^{t'} b(s) \, ds \geq (\ln 2)^{-1}.$$

For this function we can take $a = 1$ and $R$ sufficiently large in (17) and (18).

**Theorem 3.** Let the conditions of Theorem 2 be satisfied. Suppose that $f(t, x)$ is nondecreasing in $x$ for $x \geq 0$. Then, Eq. (1) has a minimal $\omega$-periodic continuous solution $x_*(t)$ and a maximal $\omega$-periodic continuous solution $x^*(t)$ in $[a, R]$; moreover,

$$U_n(t) \to x_*(t) \quad \text{uniformly in } t \in [0, \omega] \text{ as } n \to \infty,$$

$$V_n(t) \to x^*(t) \quad \text{uniformly in } t \in [0, \omega] \text{ as } n \to \infty,$$

where

$$U_0(t) \equiv a, \quad U_n(t) = \int_{t-\tau}^{t'} f(s, U_{n-1}(s)) \, ds \quad (n = 1, 2, 3, \ldots),$$

$$V_0(t) \equiv R, \quad V_n(t) = \int_{t-\tau}^{t'} f(s, V_{n-1}(s)) \, ds \quad (n = 1, 2, 3, \ldots),$$

which satisfy

$$a \equiv U_0(t) \leq U_1(t) \leq \cdots \leq U_n(t) \leq \cdots \leq x_*(t) \leq x^*(t) \leq \cdots \leq V_n(t) \leq \cdots \leq V_1(t) \leq V_0(t) \equiv R.$$

**Proof.** We use the same notations as in the proof of Theorem 2. Let $U_0(t) \equiv a$ and $V_0(t) \equiv R$. Obviously, $A: [U_0, V_0] \to E$ is completely continuous and nondecreasing. Moreover, we have

$$AU_0(t) = \int_{t-\tau}^{t'} f(s, a) \, ds \geq \int_{t-\tau}^{t'} b(s) \, ds \geq a = U_0(t), \quad t \in [0, \omega]$$
and
\[ AV_0(t) = \int_{t-\tau}^{t} f(s, R) \, ds \leq \int_{t-\tau}^{t} \frac{R}{\tau} \, ds = R = V_0(t), \quad t \in [0, \omega]. \]

Hence, our conclusions follow from Theorem 2.1.1 in [3].

**Theorem 4.** Let the conditions of Theorem 3 be satisfied. Suppose that there exist \( 0 < \alpha < 1 \) such that
\[ f'(c_y x) \geq \gamma f(t, x), \quad \forall 0 < \gamma < 1, \quad t \in [0, \omega], \quad a \leq x \leq R. \quad (21) \]

Then, Eq. (1) has an unique \( \omega \)-periodic continuous solution \( x(t) \) in \([a, R]\) and, for any \( \omega \)-periodic continuous function \( x_0(t) \) satisfying \( a \leq x_0(t) \leq R \), we have
\[ \sup_{-\infty < t < +\infty} |x_n(t) - x(t)| = \max_{0 < t < \omega} |x_n(t) - x(t)| \rightarrow 0 \quad (22) \]
as \( n \rightarrow \infty \), where
\[ x_n(t) = \int_{t-\tau}^{t} f(s, x_{n-1}(s)) \, ds \quad (n = 1, 2, 3, \ldots). \]

**Proof:** Using the same notations as in Theorem 3 and noticing that \( A \) is nondecreasing, we find from \( \nu(t)dx(t) < V(t) \) that \( \nu(t) < x(t) < V(t) \) and, in general
\[ U_n(t) \leq x_n(t) \leq V_n(t) \quad (n = 1, 2, 3, \ldots). \]

Hence, we need only to prove \( x_n(t) \equiv x^*(t) \). Let \( \gamma_0 = \sup \{ \gamma > 0 \mid x_\gamma \geq \gamma x^* \} \). Since \( x^*(t) \geq x_\gamma(t) \geq (a/R)x^*(t) \), we have \( 0 < a/R \leq \gamma_0 \leq 1 \) and \( x_\gamma \geq \gamma_0 x^* \).

Now, we show \( \gamma_0 = 1 \). In fact, if \( 0 < \gamma_0 < 1 \), then (21) implies
\[ x_\gamma = Ax_\gamma \geq A(\gamma_0 x^*) = \int_{t-\tau}^{t} f(s, \gamma_0 x^*(s)) \, ds \geq \gamma_0^2 \int_{t-\tau}^{t} f(s, x^*(s)) \, ds = \gamma_0^2 A x^* = \gamma_0^2 x^* \]
which contradicts the definition of \( \gamma_0 \), since \( \gamma_0 x > \gamma_0 \). Hence \( \gamma_0 = 1 \) and \( x_\gamma = x^* \).

It is easy to give some elementary functions, which satisfy all conditions of Theorem 4. For example,
\[ f(t, x) = \sum_{i=1}^{n} b_i(t) x^i, \quad (23) \]
where $0 < \alpha_i < 1$ for $i = 1, 2, \ldots, n$ and $b_i(t)$ $(i = 1, 2, \ldots, n)$ are nonnegative continuous functions with period $\omega$ such that

$$\sum_{i=1}^{n} b_i(t) > 0 \quad \text{for} \quad t \in [0, \omega]$$

For function (23), condition (21) is clearly satisfied for $\alpha = \max\{\alpha_1, \ldots, \alpha_n\}$ $(0 < \alpha < 1)$. It is easy to show that other conditions of Theorem 4 are all satisfied for any numbers $a$ and $R$ which obey

$$0 < a \leq (\tau m) \frac{1}{\alpha - 1}, \quad R \geq (\tau M) \frac{1}{\alpha - 1},$$

where

$$m = \min_{0 \leq t \leq \omega} \sum_{i=1}^{n} b_i(t) > 0, \quad M = \max_{0 \leq t \leq \omega} \sum_{i=1}^{n} b_i(t).$$

Hence, by Theorem 4 we assert that for function (23), Eq. (1) has a unique $\omega$-periodic positive continuous solution $x(t)$ and (22) holds for any $\omega$-periodic positive continuous function $x_0(t)$.

Finally, it should be noticed that the conditions (17) and (18) of Theorems 2–4 are essentially different from that of [1, 2]. Indeed, (17) is concerned with $f(t, x)$ for $a \leq x < R$, where $a > 0$, and inequalities of the function

$$a(t) = \lim_{x \to +0} \frac{f(t, x)}{x}$$

deal with the values of $f(t, x)$ for sufficiently small $x > 0$.

REFERENCES