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Asymptotic behaviors of solutions for time dependent damped wave equations ☆

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ABSTRACT

In this paper, we consider the Cauchy problem for the wave equation with time dependent damping $b(t)u_t$ and absorbed semilinear term $|u|^{\rho-1}u$. Here, $b(t) = b_0(1+t)^{-\beta}$ with $-1 < \beta < 1$ and $b_0 > 0$. Using the weighted energy method, we obtain the L^1 and L^2 decay rates of the solution, which coincide to those for self-similar solutions to the corresponding parabolic equation when $1 < \rho < \rho_F(N) := 1 + \frac{2}{N}$.

1. Introduction

We consider the Cauchy problem for the time dependent damped wave equation

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t + |u|^{\rho-1}u = 0, & \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & \forall x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Here u is real-valued, $b(t) = b_0(1+t)^{-\beta}$, $b_0 > 0$, $-1 < \beta < 1$, $\rho > 1$ and $N \geqslant 1$.

When $b(t)$ is a positive constant, i.e. $\beta = 0$, the problem (1.1) is reduced to

$$\begin{cases} u_{tt} - \Delta u + u_t + |u|^{\rho-1}u = 0, & \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & \forall x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

The solution u to (1.2) can be expected to behave as the solution to the problem for the corresponding heat equation

$$\begin{cases} -\Delta \phi + \phi_t + |\phi|^{\rho-1}\phi = 0, & \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \phi(0,x) = \phi_0(x), & \forall x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

In fact, Kawashima, Nakao and Ono [11] showed that there exists a unique time-global solution

$$u \in X := C\big([0,\infty); H^1\big) \cap C^1\big([0,\infty); L^2\big)$$

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to (1.2) for any data $(u_0, u_1) \in X_0 := H^1 \times L^2$ where

$$1 < \rho < \frac{N+2}{[N-2]_+} = \begin{cases} \infty, & N = 1, 2, \\ \frac{N+2}{N-2}, & N \geqslant 3, \end{cases}$$

and that, when $1 + \frac{4}{N} < \rho < \frac{N+2}{[N-2]_+}$, the global solution u decays as

$$\|u(t)\|_{L^2} = O\big(t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{2})}\big)$$

for $(u_0, u_1) \in X_0 \cap (L^r \times L^r)$, $1 \leqslant r \leqslant 2$. After this result, in the supercritical case

$$\rho > \rho_F(N) := 1 + \frac{2}{N},$$

it is shown in [4,8,10,16] that the asymptotic profile of the solution u is $\theta_0 G(t, x)$, that is,

$$u(t, x) \sim \theta_0 G(t, x) \quad \text{as } t \to \infty,$$

where

$$G(t, x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$$

and

$$\theta_0 = \int_{\mathbb{R}^N} (u_0 + u_1)\, dx - \int_0^\infty \int_{\mathbb{R}^N} |u|^{\rho-1} u \, dx\, dt.$$

In the critical case $\rho = \rho_F(N)$, see Hayashi, Kaikina and Naumkin[6,7]. In the subcritical case $\rho < \rho_F(N)$, applying the weighted L^2-energy method, Nishihara and Zhao [17] showed that the solution u of (1.2) uniquely exists, which satisfies for $t \geqslant 0$ and $1 < \rho \leqslant \frac{N}{[N-2]_+}$,

$$\|u(t, \cdot)\|_{L^2} \leqslant C I_0 (1 + t)^{-\frac{1}{\rho-1} + \frac{N}{4}}, \tag{1.4}$$

with the assumption that $I_0^2 := \int_{\mathbb{R}^N} e^{\delta|x|^2} (u_1^2 + |\nabla u_0|^2 + u_0^{\rho+1})\, dx < \infty$ for some $\delta > 0$. The decay rate (1.4) is same as that of the self-similar solution

$$w_0(t, x) = (t + 1)^{-\frac{1}{\rho-1}} f\left(\frac{x}{\sqrt{t+1}}\right)$$

to (1.3) when $1 < \rho < \rho_F(N)$. So (1.4) works effectively in the subcritical case. In fact, when ρ is near to $\rho_F(N)$, the self-similar solution $w_0(t, x)$ was proved to be an asymptotic profile in Hayashi, Kaikina and Naumkin [5].

The aim of this paper is to estimate the decay rate of solutions to (1.1) in general case of $b(t) = b_0(1 + t)^{-\beta}$ with $-1 < \beta < 1$, which is effective in the subcritical case. In the supercritical case the solution will behave as that of the corresponding linear equation. For the linear problem, using the Fourier transform method, J. Wirth [23,24] got several sharp $L^p - L^q$ estimates of the solution u for $-1 < \beta < 1$.

Theorem 1.1. *Suppose* $1 < \rho < \frac{N+2}{[N-2]_+}$, $-1 < \beta < 1$ *and* $(u_0, u_1) \in H^1 \times L^2$ *with compact support* $\operatorname{supp}\{(u_0, u_1)\} \subset B_L := \{x;\ |x| \leqslant L\}$. *If* $u \in C([0, \infty); H^1) \cap C^1([0, \infty); L^2)$ *is a weak solution to* (1.1)*, then the following decay estimates hold*:

$$\|u(t)\|_{L^1} \leqslant C(t + 1)^{-(\frac{1}{\rho-1} - \frac{N}{2})(1+\beta)}, \qquad \|u(t)\|_{L^2} \leqslant C(t + 1)^{-(\frac{1}{\rho-1} - \frac{N}{4})(1+\beta)},$$

where the constant C depends on $\|u_0\|_{H^1}$, $\|u_1\|_{L^2}$ *and the size L of the support.*

Notice that for

$$1 < \rho < \frac{N+2}{[N-2]_+}$$

there exists a unique weak solution to (1.1) for some $T > 0$ in the space

$$X_T := C\big([0, T); H^1\big) \cap C^1\big([0, T); L^2\big)$$

whose support is in B_{t+L} (see [19]). Hence, if we obtain the decay estimates for $u \in X_T$, then we have both the global existence of solution and decay rates when $1 < \rho < \frac{N+2}{[N-2]_+}$.

In the case of a semilinear source term, instead of the absorbing one in (1.1), the first author showed the unique global existence of solution with small data in [14] in the supercritical case $\rho > \rho_F(N)$, and some blow-up results in case of

$1+\frac{2\beta}{N}\leqslant\rho\leqslant 1+\frac{1+\beta}{N}$ with $0\leqslant\beta<1$. When $\beta=0$, see [12,15,20,27,28] for details. For the space dependent damped wave equation see [9,13,21].

Theorem 1.1 is proved by the weighted L^2-energy estimates in the next section. In Section 3, we give some results on both the self-similar solutions to the related time dependent semi-linear parabolic equation and decay properties of the solution to the time dependent linear parabolic equation, as well as the discussion about the critical exponent. In the final section we summarize our results and future considerations.

2. Weighted L^2-energy estimates

For the solution $u\in X_T$ to (1.1) with compact support it is sufficient to obtain the decay estimates with the constant C independent of T for the proof of Theorem 1.1.

First we take $\beta\in[0,1)$. Multiplying (1.1) by $e^{2\psi}u_t$, since

$$e^{2\psi}u_t\cdot b(t)u_t=e^{2\psi}b(t)|u_t|^2,$$

we get that

$$\begin{aligned}0&=\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}\big(|u_t|^2+|\nabla u|^2\big)+\frac{e^{2\psi}}{\rho+1}|u|^{\rho+1}\right]-\nabla\cdot\big(e^{2\psi}u_t\nabla u\big)\\&\quad+e^{2\psi}\left[\left\{\left(b(t)-\frac{|\nabla\psi|^2}{-\psi_t}\right)-\psi_t\right\}|u_t|^2+\frac{-2\psi_t}{\rho+1}|u|^{\rho+1}\right]+\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u-u_t\nabla\psi|^2.\end{aligned}\tag{2.1}$$

And multiplying (1.1) by $e^{2\psi}u$, since

$$\begin{aligned}e^{2\psi}u\cdot b(t)u_t&=b_0e^{2\psi}(1+t)^{-\beta}\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)\\&=\frac{\partial}{\partial t}\left(\frac{b_0}{2}e^{2\psi}(1+t)^{-\beta}u^2\right)+\left(-b_0\psi_t(1+t)^{-\beta}+\frac{\beta b_0}{2}(1+t)^{-\beta-1}\right)e^{2\psi}u^2,\end{aligned}$$

we obtain

$$\begin{aligned}0&=\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t+\frac{b(t)}{2}u^2\right)\right]-\nabla\cdot\big(e^{2\psi}u\nabla u\big)+e^{2\psi}\left(|\nabla u|^2+\left(-\psi_t+\frac{\beta}{2(1+t)}\right)b(t)u^2+|u|^{\rho+1}\right)\\&\quad+e^{2\psi}\big(-2\psi_tuu_t-|u_t|^2+2u\nabla\psi\cdot\nabla u\big).\end{aligned}\tag{2.2}$$

Here, we choose

$$\psi(t,x)=a\frac{|x|^2}{(t+t_0)^{1+\beta}}\tag{2.3}$$

for suitable small parameter $a>0$ and large $t_0\geqslant 1$. Thus,

$$\psi_t=-a(1+\beta)\frac{|x|^2}{(t+t_0)^{2+\beta}},\qquad\nabla\psi=a\frac{2x}{(t+t_0)^{1+\beta}}.$$

So it is easy to see that

$$\frac{|\nabla\psi|^2}{-\psi_t}=\frac{4a}{1+\beta}\frac{1}{(t+t_0)^{\beta}}\leqslant\frac{4a}{(1+\beta)b_0}b(t).\tag{2.4}$$

Multiplying (2.1) by $(t_0+t)^{\beta}$ to cover the bad term $-e^{2\psi}|u_t|^2$ in (2.2), we can get that

$$\begin{aligned}0&\geqslant\frac{\partial}{\partial t}\left[\frac{e^{2\psi}(t_0+t)^{\beta}}{2}\big(|u_t|^2+|\nabla u|^2\big)+\frac{e^{2\psi}(t_0+t)^{\beta}}{\rho+1}|u|^{\rho+1}\right]-\nabla\cdot\big(e^{2\psi}(t_0+t)^{\beta}u_t\nabla u\big)\\&\quad+e^{2\psi}\left[\left\{\left(b_0-\frac{4a}{(1+\beta)b_0}-\frac{\beta}{(t_0+t)^{1-\beta}}\right)-(t_0+t)^{\beta}\psi_t\right\}u_t^2\right]+e^{2\psi}\frac{-2\psi_t(t_0+t)^{\beta}}{\rho+1}|u|^{\rho+1}\\&\quad-\frac{\beta e^{2\psi}}{(t_0+t)^{1-\beta}}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{\rho+1}|u|^{\rho+1}\right)+\frac{e^{2\psi}(t_0+t)^{\beta}}{-\psi_t}|\psi_t\nabla u-u_t\nabla\psi|^2.\end{aligned}\tag{2.5}$$

Since

$$\begin{aligned}\frac{1}{-\psi_t}|\psi_t\nabla u-u_t\nabla\psi|^2&=\frac{1}{-\psi_t}\big((\psi_t)^2|\nabla u|^2-2\psi_tu_t\nabla u\cdot\nabla\psi+|\nabla\psi|^2|u_t|^2\big)\\&\geqslant\frac{-\psi_t}{2}|\nabla u|^2+\frac{|\nabla\psi|^2}{\psi_t}|u_t|^2,\end{aligned}$$

the sum of (2.5) and $\nu\cdot(2.2)$ with $\nu>0$ yields

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{(t_0+t)^\beta}{2}|u_t|^2+\nu uu_t+\frac{\nu b_0}{2(1+t)^\beta}u^2\right)+e^{2\psi}\left(\frac{(t_0+t)^\beta}{2}|\nabla u|^2+\frac{(t_0+t)^\beta}{\rho+1}|u|^{\rho+1}\right)\right]\\
&\quad-\nabla\cdot\left(e^{2\psi}(t_0+t)^\beta u_t\nabla u+\nu e^{2\psi}u\nabla u\right)\\
&\quad+e^{2\psi}\left[\left\{\left(b_0-\frac{4a}{(1+\beta)b_0}-\frac{\beta}{(t_0+t)^{1-\beta}}-\nu\right)-\frac{1}{2}(t_0+t)^\beta\psi_t\right\}|u_t|^2\right.\\
&\quad+\left(\nu-\frac{\beta}{2(t_0+t)^{1-\beta}}-\frac{1}{2}\psi_t(t_0+t)^\beta\right)|\nabla u|^2+\nu\left(-\psi_t+\frac{\beta}{2(1+t)}\right)b(t)u^2\\
&\quad\left.+\left(\nu-\frac{\beta}{(\rho+1)(t_0+t)^{1-\beta}}+\frac{-2\psi_t(t_0+t)^\beta}{\rho+1}\right)|u|^{\rho+1}\right]\\
&\quad+e^{2\psi}(-2\nu\psi_t uu_t+2\nu u\nabla\psi\cdot\nabla u)\\
&\leqslant 0. \qquad (2.6)
\end{aligned}$$

Then, we choose $\nu=\frac{b_0}{8}$, $0<a\ll 1$ and $t_0\gg 1$ such that $b_0-\frac{2a}{(1+\beta)b_0}-\frac{\beta}{t_0^{1-\beta}}-\nu\geqslant\frac{b_0}{2}$, $\nu-\frac{\beta}{2t_0^{1-\beta}}\geqslant\frac{b_0}{16}$, and $\nu-\frac{\beta}{(\rho+1)t_0^{1-\beta}}\geqslant\frac{b_0}{16}$. Since

$$\begin{aligned}
|-2\nu\psi_t uu_t|&=\left|2\nu\left(\frac{-\psi_t}{b(t)}\right)^{\frac12}u_t\cdot\left(-\psi_t b(t)\right)^{\frac12}u\right|\\
&\leqslant-\frac{\nu}{2}\psi_t b(t)u^2-4\nu\frac{\psi_t}{b(t)}|u_t|^2\\
&=-\frac{\nu}{2}\psi_t b(t)u^2-\frac{4\nu}{b_0}\psi_t(1+t)^\beta|u_t|^2,
\end{aligned}$$

and

$$|2\nu u\nabla\psi\cdot\nabla u|\leqslant\frac{\nu}{4}|\nabla u|^2+4\nu|\nabla\psi|^2u^2\leqslant\frac{\nu}{4}|\nabla u|^2-4\nu\cdot\frac{4a}{(1+\beta)b_0}b(t)\psi_t u^2,$$

we integrate (2.6) over $\mathbb{R}^N$ to get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}e^{2\psi}\left(\frac{(t_0+t)^\beta}{2}|u_t|^2+\frac{b_0}{8}uu_t+\frac{b_0}{16}b(t)u^2+\frac{(t_0+t)^\beta}{2}|\nabla u|^2+\frac{(t_0+t)^\beta}{\rho+1}|u|^{\rho+1}\right)dx\\
&\quad+c_0\int_{\mathbb{R}^N}e^{2\psi}\left\{\left(1-\psi_t(t_0+t)^\beta\right)|u_t|^2+\left(1-\psi_t(t_0+t)^\beta\right)|\nabla u|^2-\psi_t b(t)u^2\right.\\
&\quad\left.+\left(1-\psi_t(t_0+t)^\beta\right)|u|^{\rho+1}+|u|^{\rho+1}\right\}dx\\
&=:\frac{d}{dt}\widetilde{E}_\psi(t;u)+H_\psi(t;u)\leqslant 0. \qquad (2.7)
\end{aligned}$$

Define

$$\bar{E}_\psi(t;u)=\int_{\mathbb{R}^N}e^{2\psi}\left\{(t_0+t)^\beta\left(|u_t|^2+|\nabla u|^2+|u|^{\rho+1}\right)+b(t)u^2\right\}dx. \qquad (2.8)$$

Then we have

$$c_1\bar{E}_\psi(t;u)\leqslant\widetilde{E}_\psi(t;u)\leqslant C_1\bar{E}_\psi(t;u). \qquad (2.9)$$

Since

$$H_\psi(t;u)=c_0\int_{\mathbb{R}^N}e^{2\psi}\left\{\left(1-\psi_t(t_0+t)^\beta\right)\left(|u_t|^2+|\nabla u|^2+|u|^{\rho+1}\right)+|u|^{\rho+1}-\psi_t b(t)u^2\right\}dx,$$

we multiply (2.7) by $(t_0+t)^k$ to obtain

$$\frac{d}{dt}\left\{(t+t_0)^k\widetilde{E}_\psi(t;u)\right\}+(t+t_0)^k\underbrace{\left(H_\psi(t;u)-\frac{k}{t+t_0}\widetilde{E}_\psi(t;u)\right)}_{(*)}\leqslant 0. \qquad (2.10)$$

We can estimate $(*)$ as

$$\begin{aligned}
(*) &\geqslant H_\psi(t;u) - \frac{kC_1}{t+t_0}\overline{E}_\psi(t;u) \\
&= \int_{\mathbb{R}^N} e^{2\psi}\left\{c_0\big(1-\psi_t(t_0+t)^\beta\big) - \frac{kC_1(t_0+t)^\beta}{t_0+t}\right\}\big(|u_t|^2+|\nabla u|^2+|u|^{\rho+1}\big)\,dx \\
&\quad + \int_{\mathbb{R}^N} e^{2\psi}\left\{c_0\big(|u|^{\rho+1}-\psi_t b(t)u^2\big) - \frac{kC_1}{t+t_0}b(t)u^2\right\}dx \\
&=: I_1 + I_2. \qquad (2.11)
\end{aligned}$$

Choose t_0 large enough such that $\frac{1}{2}c_0 > \frac{kC_1}{t_0^{1-\beta}}$. Thus,

$$I_1 \geqslant \frac{c_0}{2}\int_{\mathbb{R}^N} e^{2\psi}\big(1-\psi_t(t_0+t)^\beta\big)\big(|u_t|^2+|\nabla u|^2+|u|^{\rho+1}\big)\,dx. \qquad (2.12)$$

To estimate I_2, denoting

$$\Omega_\kappa := \left\{x \;\middle|\; \frac{|x|^2}{t^{1+\beta}} \geqslant \kappa\right\}, \quad \text{and} \quad \Omega_\kappa^c = \mathbb{R}^N \setminus \Omega_\kappa,$$

we divide I_2 into to two parts:

$$I_2 = \int_{\Omega_\kappa} + \int_{\Omega_\kappa^c} =: I_{21} + I_{22}. \qquad (2.13)$$

Here we choose $\kappa \gg 1$, then

$$I_{22} \geqslant \int_{\Omega_\kappa^c} e^{2\psi}\left(c_0\frac{a(1+\beta)\kappa}{t+t_0} - \frac{kC_1}{t+t_0}\right)b(t)u^2\,dx + c_0\int_{\Omega_\kappa^c} e^{2\psi}|u|^{\rho+1}\,dx \geqslant c_0\int_{\Omega_\kappa^c} e^{2\psi}|u|^{\rho+1}\,dx \geqslant 0 \qquad (2.14)$$

and using Young's inequality with $\frac{1}{\frac{\rho+1}{2}} + \frac{1}{\frac{\rho+1}{\rho-1}} = 1$, we get

$$\begin{aligned}
I_{21} &\geqslant \int_{\Omega_\kappa} e^{2\psi}\left(c_0|u|^{\rho+1} - \frac{kC_1b_0}{(t+1)^{1+\beta}}u^2\right)dx \\
&\geqslant \int_{\Omega_\kappa} e^{2\psi}\left(c_0|u|^{\rho+1} - \frac{c_0}{2}|u|^{\rho+1} - C(t+1)^{-(1+\beta)\frac{\rho+1}{\rho-1}}\right)dx \\
&\geqslant \frac{c_0}{2}\int_{\Omega_\kappa} e^{2\psi}|u|^{\rho+1}\,dx - C(t+1)^{-(1+\beta)\frac{\rho+1}{\rho-1}}\int_{\Omega_\kappa} dx \\
&\geqslant \frac{c_0}{2}\int_{\Omega_\kappa} e^{2\psi}|u|^{\rho+1}\,dx - C(t+1)^{-(1+\beta)\frac{\rho+1}{\rho-1}+\frac{(1+\beta)N}{2}}. \qquad (2.15)
\end{aligned}$$

Combining (2.12)–(2.15), we have

$$\begin{aligned}
&(t+t_0)^k\left(H_\psi(t;u) - \frac{k}{t+t_0}\widetilde{E}_\psi(t;u)\right) \\
&\quad \geqslant c_2(t+t_0)^k\int_{\mathbb{R}^N} e^{2\psi}\big(1-\psi_t(1+t)^\beta\big)\big(|u_t|^2+|\nabla u|^2+|u|^{\rho+1}\big)\,dx - C_2(t+t_0)^k(t+1)^{-(1+\beta)\frac{\rho+1}{\rho-1}+\frac{(1+\beta)N}{2}}. \qquad (2.16)
\end{aligned}$$

It follows from (2.10) and (2.16) that

$$\begin{aligned}
&\frac{d}{dt}\big\{(t_0+t)^k\widetilde{E}_\psi(t;u)\big\} + c_2(t+t_0)^k\int_{\mathbb{R}^N} e^{2\psi}\big(1-\psi_t(1+t)^\beta\big)\big(|u_t|^2+|\nabla u|^2+|u|^{\rho+1}\big)\,dx \\
&\quad \leqslant C_2(t+t_0)^k(t+1)^{-(1+\beta)\frac{\rho+1}{\rho-1}+\frac{(1+\beta)N}{2}}. \qquad (2.17)
\end{aligned}$$

Noting that $\frac{C^{-1}}{t+t_0} \leqslant \frac{1}{t+1} \leqslant \frac{C}{t+t_0}$, for some $0<\epsilon<1$ we choose

$$k-(1+\beta)\frac{\rho+1}{\rho-1}+\frac{(1+\beta)N}{2}=-1+\epsilon,$$

i.e.,

$$k=(1+\beta)\left(\frac{\rho+1}{\rho-1}-\frac{N}{2}\right)-1+\epsilon. \tag{2.18}$$

Thus, integrating (2.17) over $[0,t]$, we obtain

$$(t+1)^k\overline{E}_\psi(t;u)+\int_0^t(\tau+1)^k\int_{\mathbb{R}^N}e^{2\psi}\left(|u_\tau|^2+|\nabla u|^2+|u|^{\rho+1}\right)dx\,d\tau \leqslant C+C_2(t+1)^\epsilon.$$

It follows that

$$\overline{E}_\psi(t;u)\leqslant C(t+1)^{-(1+\beta)(\frac{\rho+1}{\rho-1}-\frac{N}{2})+1}.$$

In particular, we have

$$\int_{\mathbb{R}^N}e^{2\psi}b(t)u^2\,dx\leqslant C(t+1)^{-(1+\beta)(\frac{\rho+1}{\rho-1}-\frac{N}{2})+1},$$

which implies that

$$\int_{\mathbb{R}^N}e^{2\psi}u^2\,dx\leqslant C(t+1)^{-(1+\beta)(\frac{\rho+1}{\rho-1}-\frac{N}{2})+1+\beta}=C(t+1)^{-(1+\beta)(\frac{2}{\rho-1}-\frac{N}{2})}.$$

Thus, we obtain

$$\left\|u(t)\right\|_{L^2}\leqslant C(t+1)^{-(\frac{1}{\rho-1}-\frac{N}{4})(1+\beta)}. \tag{2.19}$$

Moreover, since

$$\left\|u(t)\right\|_{L^1}=\int_{\mathbb{R}^N}e^\psi|u|\cdot e^{-\psi}\,dx\leqslant\left(\int_{\mathbb{R}^N}e^{2\psi}|u|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^N}e^{-2\psi}\,dx\right)^{\frac12}\leqslant C(t+t_0)^{-(\frac{1}{\rho-1}-\frac{N}{4})(1+\beta)+\frac{N(1+\beta)}{4}},$$

we have

$$\left\|u(t)\right\|_{L^1}\leqslant C(t+1)^{-(\frac{1}{\rho-1}-\frac{N}{2})(1+\beta)}. \tag{2.20}$$

So, in case of $\beta\in[0,1)$, we proved Theorem 1.1 by (2.19) and (2.20). In case of $\beta\in(-1,0)$, we only need to modify the proof mentioned above. Instead of (2.5)–(2.2), we multiply (1.1) by $\frac{1}{(b(t))^2}e^{2\psi}u_t$ and $\frac{1}{b(t)}e^{2\psi}u$, respectively, to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2(b(t))^2}\left(|u_t|^2+|\nabla u|^2\right)+\frac{e^{2\psi}}{(\rho+1)(b(t))^2}|u|^{\rho+1}\right]-\nabla\cdot\frac{e^{2\psi}u_t\nabla u}{b(t)}\\
&\quad+e^{2\psi}\left\{\left(1+\frac{|\nabla\psi|^2}{\psi_t b(t)}-\frac{\psi_t}{b(t)}\right)\frac{u_t^2}{b(t)}-\frac{2\psi_t}{(\rho+1)(b(t))^2}|u|^{\rho-1}\right.\\
&\quad\left.+\frac{-2\beta}{(t+1)(b(t))^2}\left(\frac{e^{2\psi}}{2}\left(|u_t|^2+|\nabla u|^2\right)+\frac{e^{2\psi}}{\rho+1}|u|^{\rho+1}\right)\right\}\\
&\quad-\frac{e^{2\psi}}{\psi_t(b(t))^2}|\psi_t\nabla u-u_t\nabla\psi|^2=0,
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{uu_t}{b(t)}+\frac{u^2}{2}\right)\right]-\nabla\cdot\frac{e^{2\psi}u\nabla u}{b(t)}+e^{2\psi}\left\{\frac{|\nabla u|^2}{b(t)}-\psi_t u^2+\frac{|u|^{\rho+1}}{b(t)}\right\}\\
&\quad+e^{2\psi}\left\{\left(-2\psi_t+\frac{-\beta}{t+1}\right)\frac{uu_t}{b(t)}-\frac{|u_t|^2}{b(t)}+\frac{2u\nabla\psi\cdot\nabla u}{b(t)}\right\}=0.
\end{aligned} \tag{2.22}$$

Now we take

$$\psi(t,x)=a\frac{|x|^2}{(t+1)^{1+\beta}}$$

and note that

$$\frac{|\nabla\psi|^2}{-\psi_t}=\frac{4a}{1+\beta}(t+1)^{-\beta}=\frac{4a}{b_0(1+\beta)}b(t),$$

instead of (2.4). The remaining proof is exactly same as the proof mentioned above.

3. Discussions on the critical exponent

In this section we want to discuss both the critical exponent β for the effectivity of the damping term $+b(t)u_t$ and the critical exponent ρ when the damping term is effective.

As shown in Wirth [22–24], $\beta=1$ is critical whether the damping is effective or not. We observe this fact from the point of the decay rates of solutions to the corresponding linear parabolic equation

$$\phi_t-\frac{1}{b(t)}\Delta\phi=0,\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^N \tag{3.1}$$

with the initial data

$$\phi(0,x)=\phi_0(x),\quad x\in\mathbb{R}^N. \tag{3.2}$$

The solution $\phi(t,x)$ of (3.1)–(3.2) is represented by

$$\phi(t,x)=:\left(e^{(\int_0^t\frac{d\tau}{b(\tau)})\Delta}\phi_0\right)(x)=\int\limits_{\mathbb{R}^N}G_B(t,x-y)\phi_0(y)\,dy, \tag{3.3}$$

where

$$G_B(t,x)=\left(4\pi B(t)\right)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4B(t)}},\qquad B(t):=\int\limits_0^t\frac{d\tau}{b(\tau)}. \tag{3.4}$$

Note that $G_B(t,x)$ is the Gauss kernel when $b(t)\equiv 1$. By the Hausdorff and Young inequality

$$\|f*g\|_{L^p}\leqslant\|f\|_{L^r}\|g\|_{L^q},\quad 1\leqslant p,q,r\leqslant\infty\text{ with }\frac{1}{p}=\frac{1}{r}+\frac{1}{q}-1,$$

we have for $\gamma\in\mathbb{N}_0^N$, $\mathbb{N}_0=\{0,1,2,\dots\}$, $1\leqslant q\leqslant p\leqslant\infty$ and $t>0$,

$$\left\|\partial_x^\gamma\phi(t,\cdot)\right\|_{L^p}\leqslant C\|\phi_0\|_{L^q}\left(B(t)\right)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{|\gamma|}{2}}, \tag{3.5}$$

and

$$\begin{aligned}&\left\|\partial_t\partial_x^\gamma\phi(t,\cdot)\right\|_{L^p}\leqslant C\|\phi_0\|_{L^q}\left(B(t)\right)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{|\gamma|}{2}}\cdot\frac{|B'(t)|}{B(t)},\\&\left\|\partial_t^2\partial_x^\gamma\phi(t,\cdot)\right\|_{L^p}\leqslant C\|\phi_0\|_{L^q}\left(B(t)\right)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{|\gamma|}{2}}\cdot\left[\left(\frac{|B'(t)|}{B(t)}\right)^2+\frac{|B''(t)|}{B(t)}\right].\end{aligned} \tag{3.6}$$

When, in particular, $b(t)=b_0(1+t)^{-\beta}$,

$$B(t)=\begin{cases}\frac{1}{b_0(1+\beta)}[(1+t)^{1+\beta}-1], & \beta\neq-1,\\ \frac{1}{b_0}\log_e(1+t), & \beta=-1,\end{cases}$$

and, as $t\to\infty$,

$$B(t)=\begin{cases}O(t^{1+\beta}), & \beta>-1,\\ O(\log_e t), & \beta=-1,\\ O(1), & \beta<-1.\end{cases} \tag{3.7}$$

Hence, for $t \geqslant t_0 > 0$,

$$\left\|\partial_x^\gamma \phi(t,\cdot)\right\|_{L^p} \leqslant \begin{cases} C\|\phi_0\|_{L^q} t^{-\frac{(1+\beta)N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{(1+\beta)|\gamma|}{2}}, & \beta > -1, \\ C\|\phi_0\|_{L^q} [\log_e t]^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{|\gamma|}{2}}, & \beta = -1, \\ C\|\phi_0\|_{L^q}, & \beta < -1, \end{cases} \tag{3.8}$$

$$\left\|\partial_t \partial_x^\gamma \phi(t,\cdot)\right\|_{L^p} \leqslant \begin{cases} C\|\phi_0\|_{L^q} t^{-\frac{(1+\beta)N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{(1+\beta)|\gamma|}{2}-1}, & \beta > -1, \\ C\|\phi_0\|_{L^q} [\log_e t]^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{|\gamma|}{2}-1} t^{-1}, & \beta = -1, \\ C\|\phi_0\|_{L^q} t^{\beta}, & \beta < -1 \end{cases} \tag{3.9}$$

and

$$\left\|\partial_t^2 \partial_x^\gamma \phi(t,\cdot)\right\|_{L^p} \leqslant \begin{cases} C\|\phi_0\|_{L^q} t^{-\frac{(1+\beta)N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{(1+\beta)|\gamma|}{2}-2}, & \beta > -1, \\ C\|\phi_0\|_{L^q} [\log_e t]^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{|\gamma|}{2}-1} t^{-2}, & \beta = -1, \\ C\|\phi_0\|_{L^q} t^{\beta-1}, & \beta < -1. \end{cases} \tag{3.10}$$

Now, let $u(t,x)$ be a solution to the linear damped wave equation

$$u_{tt} - \Delta u + b(t)u_t = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \tag{3.11}$$

with the initial data

$$(u,u_t)(0,x) = (u_0,u_1)(x), \quad x \in \mathbb{R}^N. \tag{3.12}$$

Then, if the solution u is assumed to behave as the solution ϕ to (3.1)–(3.2) as $t \to \infty$, then the L^1-norms of both Δu and $b(t)u_t$ decay with the same rate $t^{-1-\beta}$ when $\beta > -1$, while the L^1-norm of u_{tt} decays with the rate t^{-2}. Since u_{tt} should decay faster than Δu and $b(t)u_t$ for the diffusion phenomena, β should be less than 1. In fact, when $-1 < \beta < 1$, Wirth has shown in [24] that the solution u and its derivatives ∇u and u_t behave samely as ϕ, $\nabla\phi$ and ϕ_t, respectively, for p with $\frac{1}{p}+\frac{1}{q}=1$, $1 \leqslant q \leqslant 2$ and for the data of (3.12) with suitable regularity. The diffusion phenomena is also shown when $-1/3 < \beta < 1$. On the other hand, when $\beta > 1$, Eq. (3.11) is governed by the wave part, not the parabolic part. See also [18,25,26]. When $\beta \leqslant -1$, we do not know how the solution u behaves.

Next, when $-1 < \beta < 1$ or the damping is effective, we consider the self-similar solution to the semilinear problem (1.1). The related time dependent semi-linear parabolic equation including the sourced semilinear term is

$$-\Delta\phi + b(t)\phi_t \pm |\phi|^{\rho-1}\phi = 0, \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N. \tag{3.13$_\pm$}$$

To avoid the effect of the constant b_0, we change the time scale with $\tau = ct$, and the equation for

$$\bar\phi(\tau,x) = \phi(t,x)$$

is written as

$$-\Delta\bar\phi + c^{1+\beta} b_0 (c+\tau)^{-\beta} \bar\phi_\tau \pm |\bar\phi|^{\rho-1}\bar\phi = 0, \quad (\tau,x) \in \mathbb{R}^+ \times \mathbb{R}^N. \tag{3.14$_\pm$}$$

To seek self-similar solutions

$$\bar\phi(\tau,x) = (c+\tau)^{-\frac{1+\beta}{\rho-1}} F\left(\frac{x}{(c+\tau)^{\frac{1+\beta}{2}}}\right),$$

it is easy to see that $F(y)$ satisfies

$$\Delta F + \frac{c^{1+\beta} b_0 (1+\beta)}{2} y \cdot \nabla F + \frac{c^{1+\beta} b_0 (1+\beta)}{\rho-1} F = \pm |F|^{\rho-1} F, \quad \forall y \in \mathbb{R}^N. \tag{3.15$_\pm$}$$

For the existence of self-similar solutions we have the following proposition.

Proposition 3.1. *Choose the constant $c > 0$ as $c^{1+\beta} b_0 (1+\beta) = 1$.*

(1) *If $1 < \rho < \rho_F(N) = 1 + \frac{2}{N}$, then there is a unique smooth radial symmetric solution $f(|y|) = F(y)$ to Eq. (3.15)$_+$ with*

$$f(r) > 0, \quad \text{on } [0,\infty); \qquad f'(0) = 0; \quad \text{and} \quad \lim_{r\to\infty} r^{\frac{2}{\rho-1}} f(r) = 0.$$

(2) *There are infinitely many radially decreasing solutions to Eq. (3.15)$_-$ when $\rho < \frac{N+2}{[N-2]_+}$, and the solutions are positive if and only if $\rho > \rho_F(N)$.*

Proof. The result (1) is proved by H. Brezis, L.A. Peletier and D. Terman in [1]. For (2), the results can be obtained from A. Haraux and F.B. Weissler [3] as well as M. Escobedo and O. Kavian [2]. □

By the Proposition 3.1 our problem $(3.13)_+$ has the self-similar solution

$$w_0(t,x) = \left[c(1+t)\right]^{-\frac{1+\beta}{\rho-1}} f\left(\frac{|x|}{[c(1+t)]^{\frac{1+\beta}{2}}}\right), \tag{3.16}$$

with $c^{1+\beta}b_0(1+\beta)=1$ when $\rho<\rho_F(N)$. The decay rate is

$$\left\|w_0(t,\cdot)\right\|_{L^p} \leqslant C(1+t)^{-(1+\beta)(\frac{1}{\rho-1}-\frac{N}{2p})}, \tag{3.17}$$

which is same as (2.19)–(2.20). When $\rho>\rho_F(N)$, the exponent $\frac{(1+\beta)N}{2}(1-\frac{1}{p})$ of L^p-decay rate ($1\leqslant p\leqslant 2$) of u to (3.11)–(3.12) is bigger than $(1+\beta)(\frac{1}{\rho-1}-\frac{N}{2p})$. Therefore, $\rho_F(N)$ is exactly critical from the viewpoint of the diffusion phenomena or effective damping. Also, we note that, if the solution to (1.1) behaves as that to (3.11)–(3.12), then as $t\to\infty$,

$$\frac{1}{b(t)}\int_{\mathbb{R}^N} |u|^{\rho-1}u(t,x)\,dx = O\left(t^{\beta-\rho[\frac{(1+\beta)N}{2}(1-\frac{1}{\rho})]}\right) = O\left(t^{-1-\frac{(1+\beta)N}{2}(\rho-1-\frac{2}{N})}\right),$$

i.e.

$$\frac{1}{b(t)}\int_{\mathbb{R}^N} |u|^{\rho-1}u(t,x)\,dx \in L^1(0,\infty) \quad \text{if } \rho>\rho_F(N).$$

Hence we conjecture that, when $\rho>\rho_F(N)$, the asymptotic profile of the solution u to (1.1) is given by $\theta_0 G_B(t,x)$ (θ_0: some constant). In the critical exponent $\rho=\rho_F(N)$ we will have a slightly sharper decay rate than (2.19)–(2.20) thanks to the absorbing term, but this also remains open.

4. Summary

We summarize our results and future considerations on the problem (1.1).

The case $-1<\beta<1$ (*effective damping case*).

In Theorem 1.1 we obtained the decay rates of the solution to (1.1) with compactly supported data in $H^1\times L^2$,

$$\left\|u(t,\cdot)\right\|_{L^1} = O\left(t^{-(\frac{1}{\rho-1}-\frac{N}{2})(1+\beta)}\right), \qquad \left\|u(t,\cdot)\right\|_{L^2} = O\left(t^{-(\frac{1}{\rho-1}-\frac{N}{4})(1+\beta)}\right) \tag{4.1}$$

as $t\to\infty$ provided that $1<\rho<\frac{N+2}{[N-2]_+}$.

In the subcritical case $\rho<\rho_F(N)=1+\frac{2}{N}$, we have a self-similar solution to the corresponding parabolic equation $(3.13)_+$ in Proposition 3.1, whose decay rates are same as (4.1). Hence, our decay rates in (4.1) are optimal from the view point of the diffusion phenomena. Though the self-similar solution is expected to be an asymptotic profile of the solution to (1.1), it remains open. Even in the case $\beta=0$, we have the fact in [5] only when $\rho_F(N)-\varepsilon<\rho<\rho_F(N)$, $0<\varepsilon\ll 1$.

In the supercritical case $\rho>\rho_F(N)$, the solution of the linear problem (3.11)–(3.12) related to (1.1) is shown in [24] to decay with its rate

$$\left\|u(t)\right\|_{L^q} = O\left(t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})(1+\beta)}\right)$$

for the data in L^p, $\frac{1}{p}+\frac{1}{q}=1$, $p\in[1,2]$ with suitable regularity. Also, the solution of the corresponding linear parabolic problem (3.1)–(3.2) decays with

$$\left\|\phi(t,\cdot)\right\|_{L^1} = O(1), \qquad \left\|\phi(t,\cdot)\right\|_{L^2} = O\left(t^{-\frac{N}{4}(1+\beta)}\right), \tag{4.2}$$

as seen in (3.8), whose rates are sharper than (4.1) in the supercritical case. If u behaves as (4.2), then $\frac{1}{b(t)}|u|^{\rho-1}u\in L^1(0,\infty;L^1)$. Hence, even in the case of semilinear problem (1.1) we can expect the solution u to decay with the same rates in (4.2). More precisely, we expect that the solution $u(t,x)$ behaves as $\theta_0 G_B(t,x)$ for some constant θ_0, where G_B is given in (3.4).

In the critical case $\rho=\rho_F(N)$ the solution to Eq. (1.1) will decay with slightly faster rates than (4.1) or (4.2), thanks to the absorbing semilinear term. We also note that, though (4.1) is available in the critical and supercritical cases, the rates are less sharp than the expected ones.

The case $\beta\geqslant 1$.

As in [23], when $\beta>1$, the damping is not effective, and even for the semilinear problem (1.1) the solution will behave as that of the corresponding wave equation. The case $\beta=1$ is critical and the situation will be delicate as in [22].

The case $\beta \leqslant -1$.

We have no result in this case. As pointed in [24, Theorem 28], the solutions to the linear damped wave problem (3.11)–(3.12) converge to a function that is generally non-vanishing as t tends to infinity. See [24] for detail.

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