One special double starlike graph is determined by its Laplacian spectrum

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\textbf{A B S T R A C T}

A tree is called \textit{double starlike} if it has exactly two vertices of degree greater than 2. We denote by $H_n(p, p)$ ($n \geq 2$, $p \geq 1$) one special double starlike graph. In this work, graph $H_n(p, p)$ will be proved to be determined by its Laplacian spectrum.

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\textbf{1. Introduction}

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. All graphs considered here are simple and undirected. Let matrix $A(G)$ be the $(0, 1)$-adjacency matrix of $G$ and $d_k$ the degree of the vertex $v_k$. The matrix $L(G) = D(G) - A(G)$ is called the \textit{Laplacian matrix} of $G$, where $D(G)$ is the $n \times n$ diagonal matrix with \{d$_1$, d$_2$, \ldots, d$_n$\} as diagonal entries (and all other entries 0). Since $A(G)$ and $L(G)$ are real and symmetric, their eigenvalues are real numbers and called the adjacency and Laplacian eigenvalues of $G$, respectively. Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (= 0)$ are the adjacency and the Laplacian eigenvalues of $G$, respectively. The multiset of eigenvalues of $A(G)$($L(G)$) is called the \textit{adjacency} (\textit{Laplacian}) \textit{spectrum} of $G$. Two non-isomorphic graphs are said to be \textit{cospectral} with respect to adjacency (Laplacian) matrix if they share the same adjacency (Laplacian) spectrum. A graph is said to be determined by the adjacency (Laplacian) spectrum if there is no other non-isomorphic graph with the same spectrum of adjacency (Laplacian).

Up to now, numerous examples of cospectral but non-isomorphic graphs are reported (see for example Fig. 1, and more examples can be found in [1]). But, only few of the graphs have been proved to be determined by their spectra [3,7,8,10–12]. So, “which graphs are determined by their spectrum? [10]” seems to be a difficult problem in the theory of graph spectra.

A tree is called \textit{double starlike} if it has exactly two vertices of degree greater than two. We denote by $H_n(p, p)$ ($n \geq 2$, $p \geq 1$) one special double starlike graph shown in Fig. 2. By Fig. 1, we can conclude that $H_n(p, p)$ is not determined by its adjacency spectrum. In this work, graph $H_n(p, p)$ will be proved to be determined by its Laplacian spectrum.

\textbf{2. Preliminaries}

Some previously established results as regards the spectrum are summarized in this section. They will an play important role throughout the work.
Fig. 1. Two non-isomorphic graphs with the same adjacency spectrum.

Fig. 2. Double starlike graph $H_n(p, p)$.

Fig. 3. Double starlike graph $H_3(p, q)$ ($p \geq q \geq 1$).

Lemma 2.1 ([2]). Let $T$ be a tree with $n$ vertices and $L(T)$ be its line graph. Then for $i = 1, 2, \ldots, n$, $\mu_i(T) = \lambda_i(L(T)) + 2$.

Some results of [6, 10] are summarized in the following lemma.

Lemma 2.2. Let $G$ be a graph. For the adjacency matrix and the Laplacian matrix, the following can be deduced from the spectrum:

(i) The number of vertices.
(ii) The number of edges.
(iii) Whether $G$ is regular.
(iv) Whether $G$ is regular with any fixed girth.

For the adjacency matrix, the following follow from the spectrum:

(v) The number of closed walk of any length.
(vi) Whether $G$ is bipartite.

For the Laplacian matrix, the following follow from the spectrum:

(vii) The number of components.
(viii) The number of spanning trees.
(ix) The sum of the squares of degrees of vertices.

Lemma 2.3 ([9]). The double star graph is determined by its Laplacian spectrum.

Lemma 2.4 ([9]). Let $G$ be the graph shown in Fig. 3. Then, $G$ is determined by its Laplacian spectrum.

The following lemma can be found in [4, 5].

Lemma 2.5. Let $G$ be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then

$$\Delta(G) + 1 \leq \mu_1 \leq \max \left\{ \frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v}, uv \in E(G) \right\}$$

where $\Delta(G)$ denotes the maximum vertex degree of $G$, and $m_v$ the average of the degrees of the vertices adjacent to vertex $v$ in $G$.

3. $H_n(p, p)$ is determined by its Laplacian spectrum

Theorem 3.1. Let $G$ be the double starlike graph shown in Fig. 2. Then, $G$ is determined by its Laplacian spectrum.

Proof. For $n = 2, n = 3$, $G$ is the double star $S(p, p)$ and $H_3(p, p)$. By Lemmas 2.3 and 2.4, they are determined by their Laplacian spectra, respectively. For $p = 1$ and $p = 2$, $G$ is a path and graph $W_n$ which are respectively determined by their Laplacian spectra [10, 8]. We consider $n \geq 4, p \geq 3$. Suppose $G'$ is cospectral with $G$ with respect to the Laplacian matrix. By (i), (ii), (vii) and (viii) of Lemma 2.2, $G'$ is a connected tree with $n + 2p$ vertices and $n + 2p - 1$ edges. Lemma 2.5 implies that $p + 2 \leq \mu_1(G) \leq p + 2 + \frac{1}{p+2}$, so the largest degree of graph $G'$ is at most $p + 1$. Now, suppose that $G'$ has $n_i$ vertices
of degree $i$, for $i = 1, 2, \ldots, \Delta'$, where $\Delta' \leq p + 1$ is the maximum degree of $G'$. (i), (ii) and (ix) of Lemma 2.2 imply the following equations:

$$\begin{align*}
\sum_{i=1}^{\Delta'} n_i &= n + 2p \\
\sum_{i=1}^{\Delta'} i n_i &= 2(n + 2p - 1) \\
\sum_{i=1}^{\Delta'} i^2 n_i &= 2(p + 1)^2 + 2p + 4(n - 2)
\end{align*}$$

(3.1) (3.2) (3.3)

Then,

$$\sum_{i=1}^{\Delta'} (i^2 - 3i + 2)n_i = 2[(p + 1)^2 - 3(p + 1) + 2].$$

(3.4)

Lemma 2.1 implies that the line graph $L(G)$ and $L(G')$ are cospectral with respect to adjacency matrix, and by (v) of Lemma 2.2, they have the same number of triangles (six times the number of closed walks of length 3), i.e.,

$$\sum_{i=1}^{\Delta'} \left( \frac{i}{3} \right) n_i = \frac{2}{3} \left( \frac{p + 1}{3} \right).$$

Case 1.1. $n_{p+1} = 0$, i.e., $\Delta' < p + 1$. Then

$$2 \left( \frac{p + 1}{3} \right) = \sum_{i=1}^{\Delta'} \left( \frac{i}{3} \right) n_i < \frac{p + 1}{6} \sum_{i=1}^{\Delta'} (i - 1)(i - 2)n_i,$$

i.e.,

$$2[(p + 1)^2 - 3(p + 1) + 2] < \sum_{i=1}^{\Delta'} (i^2 - 3i + 2)n_i$$

which is contrary to (3.4).

Case 1.2. $n_{p+1} = 1$, i.e., only one vertex has the largest degree $\Delta' = p + 1$. Then

$$2 \left( \frac{p + 1}{3} \right) = \sum_{i=1}^{\Delta'} \left( \frac{i}{3} \right) n_i,$$

i.e.,

$$2 \frac{p + 1}{6} [(p + 1)^2 - 3(p + 1) + 2] = \frac{p + 1}{6} [(p + 1)^2 - 3(p + 1) + 2] + \sum_{i=1}^{p} \frac{i}{6} (i - 1)(i - 2)n_i,$$

i.e.,

$$\frac{p + 1}{6} [(p + 1)^2 - 3(p + 1) + 2] = \sum_{i=1}^{p} \frac{i}{6} (i - 1)(i - 2)n_i < \frac{p + 1}{6} \sum_{i=1}^{p} (i - 1)(i - 2)n_i,$$

i.e.,

$$[(p + 1)^2 - 3(p + 1) + 2] < \sum_{i=1}^{p} (i - 1)(i - 2)n_i,$$

i.e.,

$$2[(p + 1)^2 - 3(p + 1) + 2] < [((p + 1)^2 - 3(p + 1) + 2] + \sum_{i=1}^{p} (i - 1)(i - 2)n_i$$

$$= \sum_{i=1}^{p+1} (i - 1)(i - 2)n_i = \sum_{i=1}^{\Delta'} (i^2 - 3i + 2)n_i,$$

which is contrary to (3.4).
Therefore, \( n_{p+1} = 2 \). By (3.4), \( n_i = 0 \) for \( i = 3, 4, \ldots, p \). By (3.1) and (3.2), we have \( n_1 + n_2 + 2 = n + 2p \) and \( n_1 + 2n_2 + 2(p+1) = 2(n + 2p - 1) \), which yields that \( n_1 = 2p \) and \( n_2 = n - 2 \). Then, \( G' \) is a double starlike tree with two vertices of degree \( p + 1 \), \( n - 2 \) vertices of degree 2 and 2p vertices of degree 1.

Case 2.1. It is clear that \( G' \) is isomorphic to \( G \) if its line graph \( L(G') \) has no vertex of degree 1. Let \( n'_i \geq 1 \) be the number of vertices of degree 1 in \( L(G') \). Consider first the case when two vertices of degree \( p + 1 \) are adjacent in \( G' \). Then \( n'_{p+1} = 1 \). \( n'_{p+1} = n'_1 \) and \( n'_2 = 2p - n'_1 \). Since \( n'_{2p} + n'_{p+1} + n'_p + n'_1 + n'_2 = n + 2p - 1 \), we easily obtain \( n'_2 = n - 2 - n'_1 \), where \( n'_i \) is the number of vertices of degree \( i \) in \( L(G') \) for \( i = 1, 2, p, p + 1 \). Clearly, there exist two vertices of degree \( p + 1 \), 2p vertices of degree \( p \) and \( n - 3 \) of degree 2 in the line graph \( L(G) \). Since in any graph the number of closed walks of length 4 equals twice the number of edges plus four times the number of induced paths of length 2 plus eight times the number of 4-cycles, so by (v) of Lemma 2.2, \( L(G) \) and \( L(G') \) have the same number of induced paths of length 2. Therefore

\[
2 \left( \frac{p + 1}{2} \right) + 2p \left( \frac{p}{2} \right) + (n - 3) \left( \frac{2}{2} \right) = \left( \frac{2p}{2} \right) + n'_1 \left( \frac{p + 1}{2} \right) + (2p - n'_1) \left( \frac{p}{2} \right) + (n - 2 - n'_1) \left( \frac{2}{2} \right),
\]

i.e.,

\[
0 = (p - 1)^2 + (p - 1)n'_1.
\]

Clearly, for \( p \geq 2 \), \( (p - 1)^2 + (p - 1)n'_1 \neq 0 \), a contradiction.

Consider now the case when the two vertices of degree \( p + 1 \) are not adjacent in \( G' \). Then \( n'_{p+1} = 2 + n'_1 \), \( n'_p = 2p - n'_1 \), \( n'_2 = n - 3 - n'_1 \), where \( n'_i \) is the number of vertices of degree \( i \) in \( L(G') \) for \( i = 1, 2, p, p + 1 \). Then, (v) of Lemma 2.2 implies the following equation:

\[
2 \left( \frac{p + 1}{2} \right) + 2p \left( \frac{p}{2} \right) + (n - 3) \left( \frac{2}{2} \right) = (2 + n'_1) \left( \frac{p + 1}{2} \right) + (2p - n'_1) \left( \frac{p}{2} \right) + (n - 3 - n'_1) \left( \frac{2}{2} \right),
\]

which yields that \( (p - 1)n'_1 = 0 \). Thus, \( n'_1 \) must be zero which proves that \( G' \) is isomorphic to \( G \). \( \square \)

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement [4], so the complement of \( H_a(p, p) \) is determined by its Laplacian spectrum.

4. Conclusion

In this work, one special double starlike graph is proved to be determined by its Laplacian spectrum. For another special double starlike graph \( H_a(p, q) (n \geq 2, p > q \geq 1) \), the most difficult problem seems to show its degree sequence. But, are all the double starlike graphs determined by their Laplacian spectra? The answer is unknown. Some new methods should be found to prove this problem.

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