International Journal of Approximate Reasoning 49 (2008) 595–606

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Logics from Galois connections

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ARTICLE INFO

Article history:
Received 13 June 2007
Received in revised form 29 May 2008
Accepted 16 June 2008
Available online 27 June 2008

Keywords:
Rough sets
Fuzzy sets
Approximate reasoning
Knowledge representation
Modal logic

ABSTRACT

In this paper, Information Logic of Galois Connections (ILGC) suited for approximate reasoning about knowledge is introduced. In addition to the three classical propositional logic axioms and the inference rule of modus ponens, ILGC contains only two auxiliary rules of inference mimicking the performance of Galois connections of lattice theory, and this makes ILGC comfortable to use due to the flip-flop property of the modal connectives. Kripke-style semantics based on information relations is defined for ILGC. It is also shown that ILGC is equivalent to the minimal tense logic K_t, and decidability and completeness of ILGC follow from this observation. Additionally, relationship of ILGC to the so-called classical modal logics is studied. Namely, a certain composition of Galois connection mappings forms a lattice-theoretical interior operator, and this motivates us to axiomatize a logic of these compositions. It turns out that this logic satisfies the axioms of the non-normal logic EMT4. Hence, EMT4 can be viewed to be embedded in ILGC. EMT4 is complete with respect to the neighbourhood semantics. Here, we introduce an alternative semantics for EMT4. This is done by defining the so-called interior models, and completeness of EMT4 is proved with respect to the interior semantics.

1. Introduction

The theory of rough sets introduced by Pawlak [36] can be viewed as an extension of the classical set theory. Its fundamental idea is that our knowledge about objects of a given universe of discourse U may be inadequate or incomplete. The objects can then be observed only with the accuracy restricted by some indiscernibility relation. According to Pawlak's original definition, an indiscernibility relation E on U is an equivalence interpreted so that two elements of U are E-related if they cannot be distinguished by their properties. Since there is one-to-one correspondence between equivalences and partitions, each indiscernibility relation induces a partition on U. In this sense, our ability to distinguish objects can be understood to be blurred – we cannot distinguish individual objects, only their equivalence classes.

Each subset X of U can be approximated by two sets: the lower approximation $X^{\blacktriangledown}$ of X consists of E-equivalence classes that are included in X, and X's upper approximation $X^{\blacktriangle}$ contains E-classes intersecting with X. The lower approximation $X^{\blacktriangledown}$ can be viewed as a set of elements that are certainly in X and the upper approximation $X^{\blacktriangle}$ can be considered as a set of elements that possibly belong to X. Note also that approximations may be considered to be definable or exact in the sense that they are unions of classes of indistinguishable elements. This may be interpreted so that definable sets are describable as the disjunction of the properties of the objects they contain.

The literature, however, contains studies in which rough approximations are defined by relations that are not necessarily equivalences (see e.g. [16] for further details). In [15,19], we studied approximations in a more general setting of complete

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doi:10.1016/j.ijar.2008.06.003

atomic Boolean lattices, and definable sets determined by indiscernibility relations of different types were considered in [21,24].

To be as general as possible, in this paper R is allowed to be any arbitrary binary relation. We may also define for each subset X of the universe U the rough set approximations by the means of the inverse R^{-1} of R and these sets are denoted by $X^{\triangledown}$ and $X^{\vartriangle}$. Therefore, for every X we may attach two lower approximations, $X^{\blacktriangledown}$ and $X^{\triangledown}$, and two upper approximations, $X^{\blacktriangle}$ and $X^{\vartriangle}$. Note also that the studies appearing in the literature usually consider the pair of rough approximation mappings $^{\blacktriangle}$ and $^{\blacktriangledown}$ that are mutually dual. However, in this work we focus on the pair $({}^{\blacktriangle},{}^{\triangledown})$ forming a Galois connection. Obviously, even in this more general case, the set $X^{\triangledown}$ can be considered as the set of elements that necessarily are in X, because if an element y is R-related to some element $x \in X^{\triangledown}$, then y must be in X. Similarly, $X^{\blacktriangle}$ may be viewed as the set of elements possibly belonging to X, since if $x \in X^{\blacktriangle}$, then there exists an element y in X to which x is R-related.

In this work we also briefly consider Pawlak's information systems [35]. They consist of a set U of objects and a set of attributes A. Every attribute $a \in A$ attaches the value $a(x)$ of the attribute a to the object x. The key idea in Pawlak's information systems is that each subset $B \subseteq A$ of attributes determines an indiscernibility relation $\mathrm{ind}(B)$ which is defined so that two objects x and y of the universe U are B-indiscernible if their values for all attributes in the set B are equal, that is, $a(x) = a(y)$ for all $a \in B$. Orłowska and Pawlak introduced in [34] many-valued information systems as a generalization of Pawlak's original systems. In a many-valued information system each attribute attaches a set of values to objects. Therefore, in many-valued systems it is possible to define several types of information relations reflecting distinguishability or indistinguishability of objects of the system.

L-sets introduced by Goguen [13] determine relations reflecting knowledge about objects. The idea presented by Kortelainen [25,26] is that each L-set φ on U induces a binary relation $\lesssim$ such that $x \lesssim y$ holds, whenever y belongs to the set represented by φ at least to the same extent as x. Now the relation $\lesssim$, or its inverse $\gtrsim$ as well, can be used to determine the approximation mappings. The essential connections between modal-like operators, topologies and fuzzy sets are studied in [22].

In the literature there are several studies on logical foundations of rough sets. Usually these logics have a semantics similar to the one by Kripke [27]. In the paper [37], Pawlak formulated some notions of rough logics. Rasiowa and Skowron [39] have introduced first-order predicate logic suited for rough approximations and definability. Orłowska with her coauthors has extensively studied several logics for knowledge representation – see [12,33,34], for example. Vakarelov [41,42] has investigated modal logics for information relations of many-valued information systems. Many of these mentioned logics are examined for instance in survey papers [3,44]. Orłowska has also introduced Kripke models with relative accessibility relations in [32] – these are modifications of the ordinary Kripke structure such that 'accessibility relations' are determined by sets of parameters interpreted as a properties of objects. In addition to this, Demri and Goré [7] have defined cut-free display calculi for knowledge representation logics with relative accessibility relations, and in [9] Demri and Stepaniuk fully characterized the computational complexity of approximation multimodal logics.

It also should be mentioned that Mattila has considered the so-called modifier logics closely related to fuzzy logic in several works; see e.g. [29], for further details and references. For example, in [30] a modifier calculi together with relational frame semantics and some ideas for topological semantics is given. Finally, note that von Karger has developed in [43] several temporal logics from the theory of complete lattices, Galois connections, and fixed points.

This paper is presented as follows: In Section 2, we define Galois connections and recall some of their well-known properties. We also introduce generalized rough approximation operations based on information relations. We show how they induce Galois connections and give two examples of approximation operators determined by information relations of information systems and L-sets. Furthermore, the so-called rough fuzzy sets are briefly considered. Section 3 introduces Information Logic of Galois Connections (ILGC), which is just the standard propositional logic with two modal connectives $\triangledown$ and $\blacktriangle$. Concerning $\triangledown$ and $\blacktriangle$, we have only two additional rules of inference mimicking the behavior of Galois connection maps. In Section 4, we show that in fact the introduced logic ILGC is equivalent to the well-known minimal tense logic K_t with respect to provability of formulae. This is interesting since ILGC has only two additional deduction rules and no further axioms, as K_t has four additional axioms and two auxiliary rules of inference. Therefore, ILGC can be viewed as a very simple formulation of K_t. The equivalence of ILGC and K_t implies easily that ILGC is decidable, since K_t is known to be decidable. In addition, completeness of ILGC follows from a similar argument, because semantics defined for these logics may be easily identified. Finally, in Section 5 we study the relationship between ILGC and the non-normal modal logic EMT4. Namely, a certain composition of Galois connection mappings forms a lattice-theoretical interior operator, and we are formalizing a logic for such compositions. The presented logic is not normal in the sense that it does not satisfy (RN) nor (K). So, it is clear that we cannot define Kripke-style of semantics for our new connective $\Box$ by the means of a frame $\mathscr{F} = (U,R)$ of just one binary relation in a standard way. We show that the introduced logic of compositions satisfies the axioms of EMT4, and hence EMT4 can be extracted from ILGC. We also introduce the interior semantics for EMT4 and prove completeness of EMT4 with respect to this semantics. Finally, some concluding remarks are given.

2. Galois connections of information relations

We begin our study by recalling Galois connections and their basic properties; these can be found in [11], for example. For two ordered sets P and Q, a pair $({}^{\blacktriangleright},{}^{\triangleleft})$ of maps ${}^{\blacktriangleright}\colon P \to Q$ and ${}^{\triangleleft}\colon Q \to P$ is called a *Galois connection* between P and Q if for all $p \in P$ and $q \in Q$,

$$p^{\blacktriangleright} \leqslant q \iff p \leqslant q^{\triangleleft}.$$

The function $^{\blacktriangleright}$ is called a *residuated map* and the function $^{\triangleleft}$ is called a *residual map*. The next proposition gives some well-known properties of Galois connections.

Proposition 1. *Assume* $(^{\blacktriangleright}, ^{\triangleleft})$ *is a Galois connection between ordered sets P and Q. Let* $p, p_1, p_2 \in P$ *and* $q, q_1, q_2 \in Q$. *Then the following assertions hold:*

(i) $p_1 \leqslant p_2 \Rightarrow p_1^{\blacktriangleright} \leqslant p_2^{\blacktriangleright}$ *and* $q_1 \leqslant q_2 \Rightarrow q_1^{\triangleleft} \leqslant q_2^{\triangleleft}$.
(ii) $p \leqslant p^{\blacktriangleright\triangleleft}$ *and* $q^{\triangleleft\blacktriangleright} \leqslant q$.
(iii) $p^{\blacktriangleright} = p^{\blacktriangleright\triangleleft\blacktriangleright}$ *and* $q^{\triangleleft} = q^{\triangleleft\blacktriangleright\triangleleft}$.
(iv) $^{\blacktriangleright}$ *preserves all existing joins and* $^{\triangleleft}$ *preserves all existing meets.*
(v) *The composite* $^{\blacktriangleright\triangleleft}: P \to P$ *is a lattice-theoretical closure operator and the composite* $^{\triangleleft\blacktriangleright}: Q \to Q$ *is a lattice-theoretical interior operator.*

It is known that $(^{\blacktriangleright}, ^{\triangleleft})$ is a Galois connection between two ordered sets if and only if $^{\blacktriangleright}$ and $^{\triangleleft}$ satisfy (i) and (ii). Notice also that Galois connections were originally defined with functions that reverse order. We use the above form since it is more suitable for our purposes.

Proposition 1 implies that if P and Q are bounded lattices, then $^{\blacktriangleright}$ is a $\vee$-homomorphism and $^{\triangleleft}$ is a $\wedge$-homomorphism, that is, $(a \vee b)^{\blacktriangleright} = a^{\blacktriangleright} \vee b^{\blacktriangleright}$ and $(x \wedge y)^{\triangleleft} = x^{\triangleleft} \wedge y^{\triangleleft}$ for all $a, b \in P$ and $x, y \in Q$. Additionally, $^{\blacktriangleright}$ is $\bot$-preserving and $^{\triangleleft}$ is $\top$-preserving, that is, $\bot^{\blacktriangleright} = \bot$ and $\top^{\triangleleft} = \top$.

Next we consider generalized rough set approximations – see e.g. [23] for further details. Let U be a set, called the *universe of discourse* and let R be a binary relation on U. The *upper approximation* of a set $X \subseteq U$ is

$$X^{\blacktriangle} = \{x \in U \mid (\exists y \in U)\, x\, R\, y \,\&\, y \in X\}$$

and the *lower approximation* of X is

$$X^{\blacktriangledown} = \{x \in U \mid (\forall y \in U)\, x\, R\, y \Rightarrow y \in X\}.$$

Obviously, the maps are dual, that is, for any $X \subseteq U$,

$$X^{c\blacktriangle} = X^{\blacktriangledown c} \quad \text{and} \quad X^{c\blacktriangledown} = X^{\blacktriangle c},$$

where $X^c = \{x \in U \mid x \notin X\}$ is the complement of X in the universe U.

We may also define an analogous pair of mappings $\wp(U) \to \wp(U)$ by reversing the relation R. For any set $X \subseteq U$, let us define

$$X^{\vartriangle} = \{x \in U \mid (\exists y \in U)\, y\, R\, x \,\&\, y \in X\}$$

and

$$X^{\triangledown} = \{x \in U \mid (\forall y \in U)\, y\, R\, x \Rightarrow y \in X\}.$$

Trivially, $^{\vartriangle}$ and $^{\triangledown}$ also are dual. The next result is well known.

Proposition 2. *For any binary relation, the pairs* $(^{\blacktriangle}, ^{\triangledown})$ *and* $(^{\vartriangle}, ^{\blacktriangledown})$ *are Galois connections.*

We end this section by considering two more concrete examples of approximation operations.

Information relations. Many-valued information systems were introduced in [34], and different types of information relations considered here can be found in [8], for instance. A *many-valued information system* is a pair (U, A), where U is a set of objects and A is a set of attributes such that each attribute is a map $a: U \to \wp(V_a)$, where V_a is the *value set* of the attribute a. This means that attributes attach sets of values to objects. For example, if a is the attribute 'knowledge of languages' and a person denoted by x knows English and Finnish, then $a(x) = \{\text{English}, \text{Finnish}\}$.

Objects of an information system may be related in different ways with respect to their values of attributes. We recall some *information relations* reflecting indistinguishability of objects of an information system (U, A). For any $B \subseteq A$, the following relations may be defined:

$$(x, y) \in \mathrm{ind}(B) \iff (\forall a \in B)\, a(x) = a(y)$$
$$(x, y) \in \mathrm{sim}(B) \iff (\forall a \in B)\, a(x) \cap a(y) \neq \emptyset$$
$$(x, y) \in \mathrm{inc}(B) \iff (\forall a \in B)\, a(x) \subseteq a(y)$$

These relations are referred to as *B-indiscernibility*, *B-similarity* and *B-inclusion*, respectively.

If a is again the attribute 'knowledge of languages' and R is the a-similarity relation, then two objects x and y are R-related if they have a common lanzguage. The similarity relation is obviously symmetric, which gives that $X^{\blacktriangle} = X^{\vartriangle}$ and $X^{\blacktriangledown} = X^{\triangledown}$. Obviously, $x \in X^{\blacktriangle}$ if there exists a person $y \in X$ who has a common language with x. Similarly, $x \in X^{\triangledown}$ if all persons having a common language with x are in X.

Fuzzy sets. Fuzzy sets were defined by Zadeh [45] as mappings from a non-empty set U into the unit interval $[0,1]$. Then, fuzzy sets were generalized to L-fuzzy sets by Goguen [13] in such a way that an *L-fuzzy set* φ on U is a mapping $\varphi\colon U \to L$, where L is equipped with some ordering structure. However, in this paper, we use the term 'L-set' instead of 'L-fuzzy set'.

Notice that in the literature L is usually assumed to be at least a complete lattice. The motivation for this is that in such a setting it is possible to consider many-valued logics in which some truth values are incomparable. The least element $\bot$ and the greatest element $\top$ of L may be viewed as the 'absolute' truth values *false* and *true*. In this work, L is always assumed to be a preordered set, that is, the set L is equipped with a reflexive and transitive binary relation $\leqslant$. Typically, L may consist of linguistic membership values such as 'good', 'excellent', 'poor' and 'adequate', and the preorder relation $w_1 \leqslant w_2$ holds between two values w_1 and w_2, if w_2 is 'stronger' than w_1. For instance, 'poor' $\leqslant$ 'excellent'. It is natural to assume that the relation $\leqslant$ is not antisymmetric: if $w_1, w_2 \in L$ are synonyms, that is, distinct words or expressions w_1 and w_2 that are used with same meaning, then it is the case that $w_1 \leqslant w_2$ and $w_1 \geqslant w_2$. Hence, w_1 and w_2 are in a sense equivalent, but not the same words. This kind of more general setting enables us to move towards the methodology called *computing with words* [46], in which the objects of computation are given by a natural language. Computing with words, in general, is inspired by the human capability to perform a wide variety of tasks without any measurements and any quantizations. Note also that in [17] the operations of union, intersection, and complement for preorder-based fuzzy sets were considered.

As noted in [25], each L-set $\varphi\colon U \to L$ determines a preorder $\lesssim$ on U by

$$x \lesssim y \iff \varphi(x) \leqslant \varphi(y).$$

Assume now that $\varphi\colon U \to L$ is an L-set describing the ability of persons in U to speak Japanese. Furthermore, we denote the inverse relation of $\lesssim$ by $\gtrsim$. Then, $x \gtrsim y$ is true if x can speak Japanese at least as well as y.

Let us consider the approximations defined by the relation $\gtrsim$, that is,

$$X^{\blacktriangle} = \{x \in U \mid (\exists y \in U)\, x \gtrsim y \,\&\, y \in X\}$$

and

$$X^{\triangledown} = \{x \in U \mid (\forall y \in U)\, y \gtrsim x \Rightarrow y \in X\}.$$

Now, $x \in X^{\blacktriangle}$ if and only if x can speak Japanese at least as well as some person in X. Furthermore, $x \in X^{\triangledown}$ if and only if $y \gtrsim x$ implies $y \in X$, that is, there cannot be a person outside X speaking Japanese at least as well as x. Thus, approximations have a nice interpretation also in case of fuzzy sets.

Notice that the other pair of approximation maps $(^{\vartriangle}, ^{\blacktriangledown})$ also forming a Galois connection is defined by $x \in X^{\vartriangle}$ if and only if there exists $y \in U$ such that $x \lesssim y$ and $y \in X$, and $x \in X^{\blacktriangledown}$ whenever for all $y \in U$, $y \lesssim x$ implies $y \in X$.

Rough fuzzy sets. Let us denote by $\mathbb{I}$ the unit interval $[0,1]$. For any non-empty set U, we denote by $\mathbb{I}^U$ the set of all *fuzzy sets* on U. Because the interval $\mathbb{I}$ may be ordered with its usual order, also the set $\mathbb{I}^U$ can be ordered *pointwise* by setting

$$\varphi \leqslant \psi \iff (\forall x \in U)\varphi(x) \leqslant \psi(x)$$

for all $\varphi, \psi \in \mathbb{I}^U$. It is easy to observe that with respect to the pointwise order, $\mathbb{I}^U$ is a distributive lattice such that for all $\varphi, \psi \in \mathbb{I}^U$ and $x \in U$,

$$(\varphi \vee \psi)(x) = \max\{\varphi(x), \psi(x)\}$$

and

$$(\varphi \wedge \psi)(x) = \min\{\varphi(x), \psi(x)\}.$$

The so-called *Gödel implication* (see e.g. [14]) is defined in $\mathbb{I}$ by

$$x \to y = \begin{cases} 1, & \text{if } x \leqslant y, \\ y, & \text{otherwise.} \end{cases}$$

We can now define implication in $\mathbb{I}^U$ pointwise:

$$(\varphi \to \psi)(x) = \varphi(x) \to \psi(x).$$

It is clear that in the lattice $\mathbb{I}^U$ the fuzzy set $\varphi \to \psi$ is the relative pseudocomplement of φ with respect to ψ in the sense of [38], that is, if for all $\mu \in \mathbb{I}^U$,

$$\varphi \wedge \mu \leqslant \psi \iff \mu \leqslant \varphi \to \psi.$$

Furthermore, a *fuzzy relation* on U is a mapping $U \times U \to [0,1]$.

Dubois and Prade [10] introduced in 1990 *rough fuzzy sets*. The idea is that the objects to be operated are fuzzy instead of classical sets. As a result we obtain certain kind of 'coarsened fuzzy sets'. Let $\varphi \in \mathbb{I}^U$ be a fuzzy set and let R be a fuzzy relation on U. Then we may define the fuzzy sets $\varphi^{\blacktriangle}$ and $\varphi^{\triangledown}$ on U by setting

$$\varphi^{\blacktriangle}(x) = \bigvee_{y \in U}\{R(x,y) \wedge \varphi(y)\}$$

$$\varphi^{\triangledown}(x) = \bigwedge_{y \in U}\{R(y,x) \to \varphi(y)\}$$

for all $x \in U$. It is easy to see that the pair $(^{\blacktriangle}, ^{\triangledown})$ is a Galois connection on $\mathbb{I}^U$.

Interestingly, these operators can be viewed as fuzzy set modifiers, or hedges in sense of Lakoff [28]. In [20], we introduced a many-valued logic for modifiers of fuzzy sets together with its axiomatization and semantics. Furthermore, we proved completeness of the logic there.

3. Syntax and semantics of ILGC

In this section, we introduce a simple propositional logic ILGC – an acronym for Information Logic of Galois Connections – with two additional connectives $\blacktriangle$ and $\triangledown$.

Let P be an enumerable set, whose elements are called *propositional variables*. The set of *connectives* consists of logical symbols $\rightarrow$, $\neg$, $\blacktriangle$, and $\triangledown$. A *formula* of ILGC is defined inductively as follows:

(i) Every propositional variable is a formula.
(ii) If A and B are formulae of ILGC, then so are $A \rightarrow B$, $\neg A$, $\blacktriangle A$, and $\triangledown A$.

Let us denote by Φ the set of all formulae of ILGC.

The logical system ILGC has the following three *axioms* of classical propositional logic:

(Ax1) $A \rightarrow (B \rightarrow A)$
(Ax2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(Ax3) $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

Furthermore, ILGC has the following three *rules of inference*:

$$(\mathrm{MP})\quad \frac{A \qquad A \rightarrow B}{B} \qquad (\mathrm{GC1})\quad \frac{A \rightarrow \triangledown B}{\blacktriangle A \rightarrow B} \qquad (\mathrm{GC2})\quad \frac{\blacktriangle A \rightarrow B}{A \rightarrow \triangledown B}$$

The first rule is the classical *modus ponens*, and (GC1) and (GC2) mimic the conditions appearing in the definition of Galois connections.

An ILGC-formula A is said to be *provable*, if there is a finite sequence $A_1, A_2, \ldots, A_n$ of ILGC-formulae such that $A = A_n$ and for every $1 \leqslant i \leqslant n$:

(i) Either A_i is an axiom of ILGC,
(ii) or A_i is the conclusion of some inference rules, whose premises are in the set $\{A_1, \ldots, A_{i-1}\}$.

That A is provable in ILGC is denoted by $\vdash A$.

We can show that $\vdash B \rightarrow (A \rightarrow A)$ for all formulae $A, B \in \Phi$. Therefore, we have that $A \rightarrow A$ and $B \rightarrow B$ are equivalent to each other. We may use this 'equivalence class' as the constant *true*. Formally, let p be some fixed propositional variable. Then, we set

$$\top := p \rightarrow p.$$

The constant *false* is defined by

$$\bot := \neg\top.$$

It is clear that $\vdash A \rightarrow \top$ and $\vdash \bot \rightarrow A$ for all $A \in \Phi$. Additionally, the following abbreviations for *disjunction*, *conjunction*, and *equivalence* are introduced:

$$A \vee B := (A \rightarrow B) \rightarrow B$$
$$A \wedge B := \neg(\neg A \vee \neg B)$$
$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

Our next proposition presents some provable formulae and additional inference rules of ILGC.

Proposition 3. *For all* ILGC-*formulae A and B, we have:*

(i) $\dfrac{A \rightarrow B}{\triangledown A \rightarrow \triangledown B}$ *and* $\dfrac{A \rightarrow B}{\blacktriangle A \rightarrow \blacktriangle B}$
(ii) $\vdash A \rightarrow \triangledown\blacktriangle A$ *and* $\vdash \blacktriangle\triangledown A \rightarrow A$
(iii) $\vdash \triangledown A \leftrightarrow \triangledown\blacktriangle\triangledown A$ *and* $\vdash \blacktriangle A \leftrightarrow \blacktriangle\triangledown\blacktriangle A$
(iv) $\vdash \triangledown\top \leftrightarrow \top$ *and* $\vdash \blacktriangle \bot \leftrightarrow \bot$
(v) $\dfrac{A}{\triangledown A}$

(vi) $\vdash \nabla(A \wedge B) \leftrightarrow \nabla A \wedge \nabla B$ *and* $\vdash \blacktriangle(A \vee B) \leftrightarrow \blacktriangle A \vee \blacktriangle B$
(vii) $\vdash \nabla(A \to B) \to (\nabla A \to \nabla B)$

Proof. Note that we prove only the first claims of (i)–(iv) and (vi), because their second parts can be proved in an analogous manner.

(i) Suppose that $\vdash A \to B$. Because $\vdash \nabla A \to \nabla A$ holds trivially, we obtain $\vdash \blacktriangle\nabla A \to A$ by (GC1). Hence $\vdash \blacktriangle\nabla A \to B$, which gives $\vdash \nabla A \to \nabla B$ by (GC2).
(ii) Because $\vdash \blacktriangle A \to \blacktriangle A$, we have $\vdash A \to \nabla\blacktriangle A$ by (GC2).
(iii) Obviously, by (ii), $\vdash \nabla A \to \nabla\blacktriangle\nabla A$. Furthermore, since $\vdash \blacktriangle\nabla A \to A$, we get $\vdash \nabla\blacktriangle\nabla A \to \nabla A$ by (i).
(iv) It is clear that $\vdash \nabla\top \to \top$. Conversely, $\vdash \blacktriangle\top \to \top$ implies $\vdash \top \to \nabla\top$ by (GC2).
(v) Assume $\vdash A$. This means $\vdash \top \to A$ and we get $\vdash \nabla\top \to \nabla A$ by (i). Because $\vdash \top \to \nabla\top$ by (iv), we obtain $\vdash \top \to \nabla A$. Thus, $\vdash \nabla A$.
(vi) Because $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, we have $\vdash \nabla(A \wedge B) \to \nabla A$ and $\vdash \nabla(A \wedge B) \to \nabla B$ by (i). Hence, $\vdash \nabla(A \wedge B) \to \nabla A \wedge \nabla B$. On the other hand, $\vdash \nabla A \wedge \nabla B \to \nabla A$ yields $\vdash \blacktriangle(\nabla A \wedge \nabla B) \to A$ by (GC2). Similarly, we may show $\vdash \blacktriangle(\nabla A \wedge \nabla B) \to B$. This gives that $\vdash \blacktriangle(\nabla A \wedge \nabla B) \to A \wedge B$ and $\vdash \nabla A \wedge \nabla B \to \nabla(A \wedge B)$ by (GC2).
(vii) Since $\vdash A \wedge (A \to B) \to B$, we have $\vdash \nabla(A \wedge (A \to B)) \to \nabla B$. Furthermore, by (vi), we obtain $\vdash \nabla A \wedge \nabla(A \to B) \to \nabla(A \wedge (A \to B))$. Thus, $\vdash \nabla A \wedge \nabla(A \to B) \to \nabla B$, which is equivalent to $\vdash \nabla(A \to B) \to (\nabla A \to \nabla B)$. □

We may also introduce another pair $\triangle$ and $\blacktriangledown$ of connectives. This is done by defining them as the *duals* of ∇ and $\blacktriangle$. Let us set

$$\triangle A := \neg\nabla\neg A \quad \text{and} \quad \blacktriangledown A := \neg\blacktriangle\neg A.$$

For the connectives $\triangle$ and $\blacktriangledown$, we have similar inference rules that we have for the original connectives $\blacktriangle$ and ∇.

Lemma 4. *For all* ILGC-*formulae A and B, we have*

$$\frac{A \to \blacktriangledown B}{\triangle A \to B} \quad \textit{and} \quad \frac{\triangle A \to B}{A \to \blacktriangledown B}.$$

Proof. We prove the first rule – the second can be proved in an analogous manner. Assume that $\vdash A \to \blacktriangledown B$. By (Ax3), $\vdash (A \to \blacktriangledown B) \to (\neg\blacktriangledown B \to \neg A)$. Therefore, $\vdash \neg\blacktriangledown B \to \neg A$ by (MP) and hence $\vdash \blacktriangle\neg B \to \neg A$. By applying (GC2), we obtain $\vdash \neg B \to 1\nabla\neg A$ and $\vdash \neg B \to \neg\triangle A$. This implies $\vdash \triangle A \to B$ by (Ax3) and (MP). □

Note that Lemma 4 means that the connectives $\blacktriangledown$ and $\triangle$ have all the properties listed in Proposition 3 for ∇ and $\blacktriangle$.

We introduce the semantics of the language ILGC. A relational structure $\mathscr{F} = (U, R)$, where U is a non-empty set and R is a binary relation on U, is called an ILGC-*frame*. Let v be a function $v: P \to \wp(U)$ assigning to each propositional variable p in P a subset $v(p)$ of U. Such functions are called *valuations* and the triple $\mathscr{M} = (U, R, v)$ is called an ILGC-*model*.

For any $x \in U$ and $A \in \Phi$, we define a *satisfiability relation* $\mathscr{M}, x \models A$ according the usual Kripke semantics of the formula A inductively by the following way:

$\mathscr{M}, x \models p$ iff $x \in v(p)$
$\mathscr{M}, x \models \neg A$ iff $\mathscr{M}, x \not\models A$
$\mathscr{M}, x \models A \to B$ iff $\mathscr{M}, x \models A$ implies $\mathscr{M}, x \models B$
$\mathscr{M}, x \models \blacktriangle A$ iff there exists $y \in U$ such that xRy and $\mathscr{M}, y \models A$
$\mathscr{M}, x \models \nabla A$ iff for all $y \in U$, yRx implies $\mathscr{M}, y \models A$

We may extend the valuation function v to all Φ-formulae by setting

$$v(A) = \{x \in U \mid \mathscr{M}, x \models A\}.$$

It is then easy to see that for all $A, B \in \Phi$:

(i) $v(\bot) = \emptyset$ and $v(\top) = U$
(ii) $v(A \vee B) = v(A) \cup v(B)$ and $v(A \wedge B) = v(A) \cap v(B)$
(iii) $v(\neg A) = v(A)^c$ and $v(A \to B) = v(A)^c \cup v(B)$
(iv) $v(\blacktriangle A) = v(A)^{\blacktriangle}$ and $v(\nabla A) = v(A)^{\nabla}$
(v) $v(\triangle A) = v(A)^{\triangle}$ and $v(\blacktriangledown A) = v(A)^{\blacktriangledown}$

An ILGC-formula A is said to be *true* in an ILGC-model $\mathscr{M} = (U, R, v)$, written $\mathscr{M} \models A$, if for all $x \in U$, $\mathscr{M}, x \models A$. The formula A is *valid* in an ILGC-frame $\mathscr{F} = (U, R)$, if A is true in all ILGC-models $\mathscr{M} = (U, R, v)$ based on $\mathscr{F}$. Furthermore, A is *valid*, if A is valid in all ILGC-frames.

Example 5. In classical modal logic necessity and possibility are usually explained by reference to the notion of *possible worlds* in such a way that a valuation gives a truth value to each propositional variable for each of the *possible worlds*. Hence, the value assigned to a propositional variable p for world w may differ from the value assigned to p for another world w'. Similarly, in temporal logics, the same sentence may have different truth values in different times. The logic ILGC can be interpreted as an information logic in which formulae are viewed to represent properties that objects of a given restricted universe of discourse may have.

For example, let U be some set of human beings and let R be a relation reflecting similarity of people with respect to some suitable attributes – what those properties might be is irrelevant for this consideration. Then, the pair $\mathscr{F} = (U, R)$ is clearly an ILGC-frame. Let $\mathscr{M} = (U, R, v)$ be a model based on the frame $\mathscr{F}$ and let A be an ILGC-formula such that $v(A)$ consists of 'good teachers'. Then, $\mathscr{M}, x \models A$ can be interpreted as a sentence 'x is a good teacher', and $\mathscr{M}, x \models \blacktriangle A$ holds if there exists $y \in U$ such that $x R y$ and $\mathscr{M}, y \models A$, that is, there is a good teacher y to whom x is similar. Analogously, $\mathscr{M}, x \models \bigtriangledown A$ means that $y R x$ implies $\mathscr{M}, y \models A$, that is, all people similar to x are good teachers.

In case of fuzzy sets, we may consider a situation in which an L-set $\varphi\colon U \to L$ represents how an expert evaluates the suitability of the persons in U to act as a teacher by using some expressions and attributes L of his own language. Let us now consider the relation $\gtrsim$ on U. Then $x \gtrsim y$ means simply that the expert has the opinion that x is at least as good a teacher as y. Let B now be an ILGC-formula such that people in $v(B)$ are currently acting as teachers. Then, $\mathscr{M}, x \models \blacktriangle B$ holds if there exists $y \in U$ such that $x \gtrsim y$ and $\mathscr{M}, y \models B$, that is, x is at least as good as one acting teacher, and $\mathscr{M}, x \models \bigtriangledown B$ if $y \gtrsim x$ implies $\mathscr{M}, y \models B$, which may be interpreted so that all persons who have at least as good teaching abilities as x are all acting as teachers.

Note also that being a valid formula has the interpretation that all objects in the universe of discourse U have the property the formula represents.

4. ILGC is K_t

In this section we study how our logic relates to the well-known minimal tense logic K_t. As before, let P be an enumerable set of propositional variables. Now the set of connectives consists of logical symbols $\to$, $\neg$, **G** and **H**. K_t-formulae are defined inductively as ILGC-formulae, and the set of all K_t-formulae is denoted by Ψ. In distinction, recall that the set of ILGC-formulae is denoted by Φ.

A formula **G**A is interpreted as 'it will always be the case that A' and **H**A has the meaning 'it has always been the case that A'. Furthermore, their dual connectives **P** and **F** are defined by

$$\mathbf{F}A := \neg\mathbf{G}\neg A \quad \text{and} \quad \mathbf{P}A := \neg\mathbf{H}\neg A.$$

The logic K_t has the following seven axioms:

(Ax1) $A \to (B \to A)$
(Ax2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
(Ax3) $(\neg A \to \neg B) \to (B \to A)$
(Ax4) $A \to \mathbf{HF}A$
(Ax5) $A \to \mathbf{GP}A$
(Ax6) $\mathbf{H}(A \to B) \to (\mathbf{H}A \to \mathbf{H}B)$
(Ax7) $\mathbf{G}(A \to B) \to (\mathbf{G}A \to \mathbf{G}B)$

Furthermore, K_t has three rules of inference:

$$\text{(MP)} \quad \frac{A \qquad A \to B}{B} \qquad \text{(RH)} \quad \frac{A}{\mathbf{H}A} \qquad \text{(RG)} \quad \frac{A}{\mathbf{G}A}$$

That a K_t-formula A is provable is defined as in case of ILGC.

Our purpose is to show that ILGC and K_t are equivalent with respect to provability. Indeed, ILGC appears much simpler than K_t, since ILGC has only three axioms (Ax1)–(Ax3) and three rules of inference. Therefore, ILGC can also viewed as a very simple formulation of K_t.

At the first glance the language of our logic ILGC is different from the one of K_t. However, if we replace for an ILGC-formula $A \in \Phi$ every symbol $\blacktriangle$ by **F** and every $\bigtriangledown$ by **H**, we will obtain a K_t-formula $A^{\psi} \in \Psi$. Similarly, any K_t-formula $B \in \Psi$ can be transformed to an ILGC-formula B^{ϕ} by replacing the occurrences of **F**, **G**, **P**, and **H** by $\blacktriangle$, $\blacktriangledown$, $\triangle$, and $\bigtriangledown$, respectively. Therefore, the languages of these may be considered to be exactly the same.

It is straightforward to prove the next lemma stating that each provable ILGC-formula A can be translated to a provable K_t-formula A^{ψ}.

Lemma 6. *If an* ILGC*-formula $A \in \Phi$ is* ILGC*-provable, then the corresponding* K_t*-formula $A^{\psi} \in \Psi$ is* K_t*-provable.*

Proof. Assume $A \in \Phi$ is a provable ILGC-formula. We prove the claim by induction. If A is an ILGC-axiom, then the assertion holds trivially because the axioms of ILGC are included in the axioms of K_t.

If A is deduced from B and $B \to A$ by (MP), then, by the induction hypothesis B^{ψ} and $(B \to A)^{\psi}$ are provable K_t-formulae. Since $(B \to A)^{\psi}$ is $B^{\psi} \to A^{\psi}$, A^{ψ} is a provable K_t-formula by (MP).

Assume A is equal to $B \to \nabla C$ for some $B, C \in \Phi$, and $B \to \nabla C$ is deduced from $\blacktriangle B \to C$ by (GC2). Because $\mathbf{F}B^{\psi} \to C^{\psi}$ is a provable K_t-formula by the induction hypothesis, we have that $\mathbf{HF}B^{\psi} \to \mathbf{H}C^{\psi}$ is K_t-provable by (RH). Additionally, $B^{\psi} \to \mathbf{HF}B^{\psi}$ is K_t-provable by (Ax4). Thus, we obtain that $B^{\psi} \to \mathbf{H}C^{\psi}$ is a provable K_t-formula. This gives that $B^{\psi} \to (\nabla C)^{\psi}$ and $(B \to \nabla C)^{\psi}$ are K_t-provable.

The case involving (GC1) can be proved in an analogous way. □

Our next lemma states that also the converse statement holds.

Lemma 7. *If a* K_t*-formula* $A \in \Psi$ *is* K_t*-provable, then the* ILGC*-formula* $A^{\phi} \in \Phi$ *is* ILGC*-provable.*

Proof. The proof is clear by Proposition 3 and Lemma 4. □

Lemmas 6 and 7 imply that ILGC and K_t are equivalent with respect to provability. It is well-known that K_t is *decidable*, that is, there exists an algorithm which for every K_t-formula is capable of deciding in finitely many steps whether the formula is provable in the system or not. Therefore, we may give the following theorem.

Theorem 8 (Decidability theorem). *ILGC is decidable.*

Next we recall the standard model-theoretic semantics of tense logic K_t. A *temporal frame* $(T, <)$ consists of a set T of entities called *times* together with a binary relation $<$ on T. This defines the 'flow of time' over which the meanings of the tense operators are to be defined. An interpretation of the tense-logical language gives a truth value to each atomic formula at each time $t \in T$ in the temporal frame. The meanings of **G** and **H** can be defined by:

$\mathbf{H}A$ is true at t iff A is true at all times t' such that $t' < t$
$\mathbf{G}A$ is true at t iff A is true at all times t' such that $t < t'$

It is now clear that if we identify $<$ and R as well as T and U, then $\mathbf{H}A$ is true exactly when ∇A is true, and $\mathbf{G}A$ is true in case $\blacktriangledown A$ is. As a corollary of this observation, we get completeness of ILGC.

Theorem 9 (Completeness theorem). *An* ILGC*-formula is valid if and only if it is provable.*

5. EMT4 from ILGC

Here, we study the relationship between ILGC and the well-known non-normal modal logic EMT4. Most of the so-called 'normal modal logics' include the *necessitation rule*:

$$\text{(RN)} \quad \frac{A}{\Box A}.$$

Furthermore, the *distribution axiom*

$$\text{(K)} \quad \Box(A \to B) \to \Box A \to \Box B$$

is usually included. The weakest normal modal logic, named K in honor of Saul Kripke, is simply the propositional calculus added with an extra connective □, the rule (RN), and the axiom (K). Let us recall also the axioms (T) and (4):

$$\text{(T)} \quad \Box A \to A$$
$$\text{(4)} \quad \Box A \to \Box\Box A$$

The logic S4 is characterized by axioms (T), (4) and (K) together with the rule (RN).

If we now come back to ILGC, we may define an additional connective □ by setting for any ILGC-formula $A \in \Phi$,

$$\Box A := \blacktriangle \nabla A.$$

By Proposition 3, $\vdash \Box A \to A$ and $\vdash \Box A \to \Box\Box A$, that is, the (T) and (4) are provable in ILGC. Similarly, we may show that

$$\text{(RM)} \quad \frac{A \to B}{\Box A \to \Box B}$$

is a rule of inference. We can also define the modal connective of possibility ◊ in a standard way as the dual of □ by setting for all $A \in \Phi$,

$$\Diamond A := \neg\Box\neg A.$$

Interestingly, this means that $\Diamond A = \neg\blacktriangle\nabla\neg A = \blacktriangledown\triangle A$. Unfortunately, our logic is not normal in the sense that it does not satisfy (RN) nor (K).

The so-called *classical modal logics* (see, e.g. [6]) do not validate the rule (RN), nor the axiom (K). Therefore, they resemble much our logic. The only rule that is common to all classical modal logics is

(RE) $\dfrac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$.

The smallest classical modal logic satisfying (RE) is denoted by E. The logic EM extends E by adding the axiom

(M) $\Box(A \wedge B) \to (\Box A \wedge \Box B)$.

It is well known that the logic EM is equal to the logic E added by the rule (RM). If we add the axioms (T) and (4) to the logic EM, we obtain the logic EMT4.

Now it is clear that we can embed EMT4 to our logic ILGC by setting $\Box A := \blacktriangle \triangledown A$. This means that EMT4 may be considered as a 'sublogic' of ILGC, and each provable EMT4-formula can be viewed as a provable ILGC-formula.

Neighbourhood semantics is a generalization of Kripke semantics invented independently by Scott [40] and Montague [31]. A *neighbourhood model* is a triple $\mathscr{M} = (U, N, v)$, where U is a non-empty set, v is a valuation, and $N : U \to \wp(\wp(U))$ is a map associating for any point $x \in U$ a collection $N(x)$ of subsets of U called the *neighbourhood of* x. Satisfiability for $\Box A$ is defined by

$$\mathscr{M}, x \models \Box A \text{ iff } \{y \in U \mid \mathscr{M}, y \models A\} \in N(x).$$

EMT4 is known (see, e.g. [6]) to be complete with respect to neighbourhood frames (U, N) such that

(i) $(\forall x \in U)\, X \subseteq Y$ and $X \in N(x)$ imply $Y \in N(x)$,
(ii) $(\forall x \in U)\, X \in N(x)$ implies $x \in X$,
(iii) $(\forall x \in U)\, X \in N(x)$ implies $\{y \in U \mid X \in N(y)\} \in N(x)$.

Conditions (i)–(iii) correspond the axioms (M), (T), and (4), respectively.

Next we present an alternative semantics for EMT4 motivated by the fact that each Galois connection between two power sets determines a lattice-theoretical interior operator. Our approach resembles the topological interpretation of modal logic initiated by Tarski (see, e.g. [1], where further references can be found), in which each propositional variable represents a region of the topological space, and so does every formula. The connectives $\neg$, $\vee$ and $\wedge$ are interpreted as complement, union and intersection, respectively. The modal connective $\Box$ becomes the topological interior operator, and if we define the connective $\Diamond$ as the dual $\Diamond A := \neg\Box\neg A$, then $\Diamond$ is the corresponding topological closure operator. Topological models $\mathscr{M} = (U, \mathscr{T}, v)$ are topological spaces $(U, \mathscr{T})$ equipped with a valuation function $v: P \to \wp(U)$.

Here we may proceed in an analogous manner. A map $^{\Box}: \wp(U) \to \wp(U)$ is called a *lattice-theoretical interior operator*, if for all $X, Y \subseteq U$,

(Int1) $X^{\Box} \subseteq X$,
(Int2) $X \subseteq Y$ implies $X^{\Box} \subseteq Y^{\Box}$, and
(Int3) $X^{\Box\Box} = X^{\Box}$

For any family $\mathscr{I} \subseteq \wp(U)$, the pair $(U, \mathscr{I})$ is called an *interior system* if $\bigcup \mathscr{S} \in \mathscr{I}$ for all $\mathscr{S} \subseteq \mathscr{I}$, that is, $\mathscr{I}$ is closed under arbitrary unions of its elements.

Lattice-theoretical interior operators and systems are closely connected. Namely, if $(U, \mathscr{I})$ is an interior system, then the map $X \mapsto \bigcup\{Y \in \mathscr{I} \mid Y \subseteq X\}$ is a lattice-theoretical interior operator. Similarly, if $X \mapsto X^{\Box}$ is a lattice theoretical interior operator, then $\mathscr{I} = \{X^{\Box} \mid X \subseteq U\}$ is closed under unions, and the pair $(U, \mathscr{I})$ is an interior system. This correspondence between interior systems and lattice-theoretical interior operators is bijective. This is analogous to the correspondence between lattice-theoretical closure operators and closure systems; see [5], for example.

Each interior system $(U, \mathscr{I})$ may also be viewed as a complete lattice $(\mathscr{I}, \subseteq)$, where

$$\bigvee \mathscr{S} = \bigcup \mathscr{S} \quad \text{and} \quad \bigwedge \mathscr{S} = \left(\bigcap \mathscr{S}\right)^{\Box}$$

for every $\mathscr{S} \subseteq \mathscr{I}$, where $^{\Box}: \wp(U) \to \wp(U)$ is the interior operator corresponding $(U, \mathscr{I})$. The least element of $\mathscr{I}$ is $\emptyset$ and the greatest element of $\mathscr{I}$ is $U^{\Box}$. Note that possibly $U^{\Box} \neq U$, that is, $U \notin \mathscr{I}$.

It is also well known that if $^{\Box}: \wp(U) \to \wp(U)$ is a lattice-theoretical interior operator, then its dual $^{\Diamond}: \wp(U) \to \wp(U)$ defined by $X^{\Diamond} := X^{c\Box c}$ is a lattice-theoretical closure operator, that is, for all $X, Y \subseteq U$: (i) $X \subseteq X^{\Diamond}$, (ii) $X \subseteq Y$ implies $X^{\Diamond} \subseteq Y^{\Diamond}$, and (iii) $X^{\Diamond\Diamond} = X^{\Diamond}$. The dual notion of interior systems is closure systems, which are families of sets closed under intersections.

We introduce interior models to define semantics for EMT4-formulae. For an interior system $(U, \mathscr{I})$, an *interior model* is a triple $\mathscr{M} = (U, \mathscr{I}, v)$, where $v: P \to \wp(U)$ is a valuation function. Validity of formulae can be defined inductively as in Section 3, except that

$$\mathscr{M}, x \models \Box A \text{ iff } (\exists X \in \mathscr{I})\, x \in X \text{ and } \mathscr{M}, y \models A \text{ for all } y \in X.$$

Lemma 10. *For any formula A,* $v(\Box A) = v(A)^{\Box}$.

Proof. ($\subseteq$) Suppose that $x \in v(\Box A)$. Then $\mathscr{M}, x \models \Box A$, which means that there exists $X \in \mathscr{I}$ such that $x \in X$ and $\mathscr{M}, y \models A$ for all $y \in X$. Thus, $y \in v(A)$ for all $y \in X$, that is, $X \subseteq v(A)$. This implies $x \in X = X^{\Box} \subseteq v(A)^{\Box}$.

($\supseteq$) If $x \in v(A)^{\Box} (\in \mathscr{I})$, then $v(A)^{\Box} \subseteq v(A)$ implies that for all $y \in v(A)^{\Box}$, $y \in v(A)$ and $\mathscr{M}, y \models A$. Thus, $\mathscr{M}, x \models \Box A$ and $x \in v(\Box A)$. $\Box$

An EMT4-formula A is said to be *true* in an interior model $\mathscr{M} = (U, \mathscr{I}, v)$, written $\mathscr{M} \models A$, if for all $x \in U$, $\mathscr{M}, x \models A$. The formula A is *valid* in an interior system $(U, \mathscr{I})$ if it is true in all interior models based on $(U, \mathscr{I})$. Finally, A is *valid* if it is valid in all interior systems.

Theorem 11 (Soundness theorem). *Each provable* EMT4-*formula is valid.*

Proof. We show that axioms (T) and (4) are valid, and that rule (RM) preserves validity. That (MP) preserves validity is trivial.

(T) $v(\Box A \to A) = v(\Box A)^c \cup v(A) = (v(A)^{\Box})^c \cup v(A) \supseteq v(A)^c \cup v(A) = U$.

(4) $v(\Box A \to \Box\Box A) = v(\Box A)^c \cup v(\Box\Box A) = (v(A)^{\Box})^c \cup (v(A)^{\Box})^{\Box} = (v(A)^{\Box})^c \cup v(A)^{\Box} = U$.

(RM) Assume that $A \to B$ is valid. Then $v(A) \subseteq v(B)$. This implies $v(\Box A) = v(A)^{\Box} \subseteq v(B)^{\Box} = v(\Box B)$. Thus, also $\Box A \to \Box B$ is valid. $\Box$

Next we shall show the converse, that is, every valid EMT4-formula is provable. We first recall some notions that will be needed for the proof. A subset Γ of formulae is called *inconsistent* if there are formulae $A_1, \ldots, A_n \in \Gamma$ such that $\vdash \neg(A_1 \wedge \cdots \wedge A_n)$; otherwise Γ is *consistent*. Additionally, Γ is a *maximal consistent set* if Γ is consistent, and any set of formulae properly containing Γ is inconsistent.

The next two lemmas present some well-known general properties of maximal consistent sets. These results can be found in [4], for example.

Lemma 12. *Let Γ be a maximal consistent set of formulae. Then for all formulae A and B:*

(i) If A provable, then $A \in \Gamma$.
(ii) $A \in \Gamma \iff \neg A \notin \Gamma$.
(iii) Γ is closed under modus ponens, that is, if A and $A \to B$ are in Γ, then also B is in Γ.
(iv) $A \wedge B \in \Gamma \iff A \in \Gamma$ and $B \in \Gamma$.
(v) $A \vee B \in \Gamma \iff A \in \Gamma$ or $B \in \Gamma$.

Lemma 13 (Lindenbaum's lemma). *Let Γ be a consistent set of formulae. Then there exists a maximal consistent set of formulae Γ^+ such that $\Gamma \subseteq \Gamma^+$.*

Next we construct the canonical interior system and the corresponding canonical model. For that, we denote by U^* the family of all maximal consistent sets of formulae. In addition, for any formula A, we define

$$\widehat{A} = \{\Gamma \in U^* \mid A \in \Gamma\}.$$

The *canonical interior system* is a pair $(U^*, \mathscr{I}^*)$ such that $\mathscr{I}^*$ is a subfamily of $\wp(U^*)$ generated by the all unions of the *basic sets*

$$\{\widehat{\Box A} \mid A \text{ is a formula}\}.$$

It is easy to see that $(U^*, \mathscr{I}^*)$ is really an interior system. Namely, if $\mathscr{S} \subseteq \mathscr{I}^*$, then each set in $\mathscr{S}$ is a union of some basic sets. This means that also $\bigcup \mathscr{S}$ must be a union of some basic sets. Hence, $\bigcup \mathscr{S} \in \mathscr{I}^*$.

The *canonical interior model* is a triple $\mathscr{M}^* = (U^*, \mathscr{I}^*, v^*)$, where

(i) $(U^*, \mathscr{I}^*)$ is the canonical interior system.
(ii) $v^*: P \to \wp(U^*)$ is the *canonical valuation* defined by

$$v^*(p) = \{\Gamma \in U^* \mid p \in \Gamma\}.$$

Note that $v^*(p) = \widehat{p}$ for all variables $p \in P$. It is clear that for any maximal consistent set $x \in U^*$ and formula A,

$$x \in \widehat{A} \iff A \in x.$$

We may now present the Truth Lemma for canonical interior models. The proof is similar to the one for the corresponding result for canonical topological models presented in [1,2], for instance.

Lemma 14 (Truth lemma). *Let $\mathscr{M}^* = (U^*, \mathscr{I}^*, v^*)$ be the canonical interior model. Then for any maximal consistent set $x \in U^*$ and formula A,*

$$\mathscr{M}^*, x \models A \textit{ iff } A \in x.$$

Proof. It suffices to consider the interesting case of the modal operator $\Box$. We show the directions separately.

($\Leftarrow$) Suppose $\Box A \in x$, that is, $x \in \widehat{\Box A}$. By definition, $\widehat{\Box A}$ is a basic set and hence $\widehat{\Box A} \in \mathscr{I}^*$. Furthermore, axiom (T) implies $\widehat{\Box A} \subseteq \widehat{A}$. This means that there exists $X = \widehat{\Box A}$ such that $x \in X \in \mathscr{I}^*$ and for all $y \in X$, $y \in \widehat{A}$. Thus, for all $y \in X$, $A \in y$, and so by the induction hypothesis $\mathscr{M}^*, y \models A$. Thus, $\mathscr{M}^*, x \models \Box A$.

($\Rightarrow$) Assume that $\mathscr{M}^*, x \models \Box A$. Then there exists $X \in \mathscr{I}^*$ such that $x \in X$ and $\mathscr{M}^*, y \models A$ for all $y \in X$. Since X is a union of some basic sets, we have that there is a basic set $\widehat{\Box B}$ for some formula B such that $x \in \widehat{\Box B}$ and for all $y \in \widehat{\Box B}$ ($\subseteq X$), $\mathscr{M}^*, y \models A$, that is, $A \in y$ and $y \in \widehat{A}$ by the induction hypothesis. This means that $\widehat{\Box B} \subseteq \widehat{A}$. But this implies that we can prove the implication $\Box B \to A$; namely, if not, then there would be some maximal consistent set containing $\Box B$ and $\neg A$, and this would give $\widehat{\Box B} \not\subseteq \widehat{A}$. By rule (RM), we can prove also the implication $\Box\Box B \to \Box A$. Therefore, by using axiom (4), we have $\Box B \to \Box A$. This implies $x \in \widehat{\Box B} \subseteq \widehat{\Box A}$, that is, $\Box A \in x$. $\Box$

Completeness is now obvious.

Theorem 15 (Completeness theorem). *An* EMT4*-formula is valid if and only if it is provable.*

Notice that we may also easily include the axiom

(N) $\Box\top$

to our axiom system – we have to only assume that $U \in \mathscr{I}$ must hold, which means that $(U, \mathscr{I})$ is the so-called *topped interior system*.

This modified logic is sound, because $v(\Box\top) = v(\top)^{\Box} = U^{\Box} = U$, that is, the axiom (N) is also valid. Furthermore, the canonical interior system $(U^*, \mathscr{I}^*)$ is now a topped interior system, because (N) implies $\Box\top \in \Gamma$ for all $\Gamma \in U^*$ and

$$\widehat{\Box\top} = \{\Gamma \in U^* \mid \Box\top \in \Gamma\} = U^*,$$

which gives directly $U^* \in \mathscr{I}^*$.

Let us now conclude this section by returning to our starting point, that is, we consider again rough set approximation operators. Let R be any binary relation on U and let the maps $^{\triangledown}\colon \wp(U) \to \wp(U)$ and $^{\blacktriangle}\colon \wp(U) \to \wp(U)$ be defined as in Section 2. We may now define a mapping $^{\Box}\colon \wp(U) \to \wp(U)$ by setting

$$X^{\Box} := X^{\triangledown\blacktriangle} = \{x \in U \mid (\exists y \in U)(\forall z \in U) x R y \;\&\; (z R y \Rightarrow z \in X)\}.$$

Similarly, its dual $^{\Diamond}\colon \wp(U) \to \wp(U)$ is defined as

$$X^{\Diamond} := X^{\triangle\blacktriangledown} = \{x \in U \mid (\forall y \in U)(\exists z \in U) x R y \Rightarrow (z R y \;\&\; z \in X)\}.$$

If we put $\mathscr{I} = \{X^{\Box} \mid X \subseteq U\}$, then the elements in $\mathscr{I}$ are such that $X^{\blacktriangledown\triangle} = X$. Hence, each $X \in \mathscr{I}$ may be interpreted in such a way that X consists exactly of elements that are 'possibly certainly' in X. Note that if the relation R is *serial*, that is, for all $x \in U$, there exists $y \in U$ such that xRy, then $U^{\Box} = U$. This means that $U \in \mathscr{I}$ and hence $(U, \mathscr{I})$ becomes a topped interior system, and the axiom (N) is valid. Note also that the assumption of seriality is quite natural – it means simply that each element of the universe is 'comparable' at least with one element. Trivially, reflexivity implies seriality.

Moreover, if R is a preorder, then $X^{\Box} = X^{\triangledown}$ and $X^{\Diamond} = X^{\triangle}$ are topological interior and closure operators, and the family $\mathscr{I}$ is a topology closed also under arbitrary intersections – that is, a so-called *Alexandrov topology*. The corresponding logic is S4.

6. Concluding remarks

In this paper, we introduced the logic ILGC. The logic has only two additional inference rules. Since we showed that ILGC is equivalent to K_t, ILGC can be seen as a very simple formulation of K_t. However, ILGC is valuable as such, because it is an information logic for generalized rough set operations. What is also interesting is that the additional inference rules (GC1) and (GC2) do not involve negations. This means that we can easily append these rules to several non-classical logics, also. For instance, in [18] we introduced a many-valued negationless logic of Galois connections suitable for rough L-sets.

We also showed how ILGC – and hence K_t – embeds in itself the non-normal modal logic EMT4. EMT4 is complete with respect to the neighbourhood semantics. Here, we introduced an alternative semantics based on interior systems for EMT4. Completeness of EMT4 with respect to interior semantics was proved. It is now clear that if A is a valid EMT4-formula with respect to neighbourhood semantics, A must be valid also with respect to interior semantics, and vice versa.

Acknowledgements

The authors thank the anonymous referees for their comments and several improvements. Jouni Järvinen wants also to thank Wojciech Dzik for fruitful discussions.

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