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Book Reviews

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Communicated by Andrei Martínez-Finkelshtein

Discrete Orthogonal Polynomials, Asymptotics and Applications. *By J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin and P. D. Miller*. Princeton University Press, Princeton and Oxford, 2007. Annals of Mathematics Studies 164. 184 pp. \$39.50 paper (ISBN 978-0-691-12734-7), \$79.50 cloth (ISBN 978-0-691-12733-0).

The method of the Matrix Riemann–Hilbert Problem (MR-HP) is one of the most successful and powerful modern techniques in the asymptotic analysis. In the reviewed monograph the method is developed and applied to the strong asymptotics of systems of polynomials orthogonal with respect to general discrete measures, i.e. measures supported on finite sets of nodes. If we recall the main features of the MR-HP method with regard to the asymptotics of polynomials P_n orthogonal with respect to continuous analytical weights, we see that the method is based on the reformulation of the orthogonal polynomials in terms of a 2×2 matrix Riemann–Hilbert problem (due to Fokas, Its and Kitaev) and the steepest descent analysis of this problem as n tends to infinity (due to Deift and Zhou). In its turn, for the steepest descent technique, this matrix Riemann–Hilbert problem has to be normalised at infinity using the equilibrium measure of the logarithmic potential with presence of an external field defined by the analytic weight of orthogonality. In case of the discrete orthogonality all these basic ingredients are essentially different. For discrete orthogonal polynomials the starting point is their reformulation in terms of a matrix interpolation problem for some rational matrix-valued functions with poles on the support of the discrete measure (this interpolation problem has been introduced by Borodin). Then this problem is transformed into a matrix Riemann–Hilbert boundary value problem turning these polar singularities into discontinuities along certain contours. The analysis of this matrix Riemann–Hilbert problem requires using equilibrium measures with constrains (these measures have been introduced by Rakhmanov to describe the weak asymptotics of discrete orthogonal polynomials).

The book under review contains the Preface, 7 Chapters and a couple of Appendices.

Chapter 1 is called Introduction. It discusses motivating applications, gives the definition of discrete orthogonal polynomials, states the equivalent matrix interpolation problem formulation

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for the discrete orthogonality, describes main steps of the method and gives an outline of the rest of the book.

The main results of this research monograph are rigorously stated in Chapters 2 and 3.

In Chapter 2 the detailed asymptotic behaviour of the discrete orthogonal polynomials in overlapping sets of the entire complex plane is presented. Then the general asymptotic results are applied to some special classical cases, such as the Krawtchouk polynomials and two classes of Hann polynomials.

Chapter 3 is devoted to the corresponding applications of the results on asymptotics of discrete orthogonal polynomials. There the notion of discrete orthogonal polynomials statistical ensembles is introduced and general results on the universality of these statistics are stated. Also the specific results implied by the general ones in context of the hexagon tiling problem are presented here. Another application is the continuum limit of the Toda lattice.

Chapters 4, 5 contain the detailed exposition of the method for proving the results stated in Chapters 2 and 3. In Chapter 4, the most of the new inventions in the asymptotic analysis of the discrete case are presented. Here a transformation from the matrix interpolation problem to a Riemann–Hilbert problem on a contour, double constrained equilibrium measure, reduction to a simpler limiting Riemann–Hilbert problem (global parametrix) are exploited and constructed. In Chapter 5, the authors find the solution of the global and local parametrix Riemann–Hilbert problems and carry out the error estimates.

Using the asymptotic analysis of the MR-HP developed in Chapters 4 and 5, in Chapter 6 the authors prove the theorems on the strong asymptotics of discrete orthogonal polynomials (stated in Chapter 2) and on universality of discrete orthogonal polynomials ensembles (in Chapter 7).

Appendix A contains proofs of some propositions for constructing the solution of the global parametrix problem based on the analysis on hyperelliptic Riemann surfaces and Riemann theta-functions. In Appendix B the equilibrium measure associated with Hahn polynomials (used in Chapter 2) is obtained. Also the monograph has a list of important symbols used throughout the book (Appendix C).

The monograph of Baik et al. is a comprehensive, detailed and impressive piece of work. Perhaps it is too technical, although it has an advantage that the reader after some work can follow from one line to the next one. The book should be useful for research mathematicians interested in modern methods of asymptotic analysis, orthogonal polynomials, integrable systems and random matrices. It should be also useful for students studying Matrix Riemann–Hilbert Problem method, as a complement to existing monographs.

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Experimental mathematics in action. *By David H. Bailey, Jonathan M. Borwein, Neil J. Calkin, Roland Girgensohn, D. Russell Luke, and Victor Moll.* A.K. Peters Ltd, 2007. xii +322 pp., $ 49.00 hardcover. ISBN 978-1-56881-271-7.

I like the term "experimental mathematics" because it reflects some aspects of the way I conceive and do research. Despite of the image of the purely deductive science that we partially encourage, mathematics has been always related with experiments, either "thought experiments", mechanical devises or coin tossing. But the authors refer to a very specific type of experiments made possible by computers. This book is meant to show that since the computers became widely available, mathematical activity will (and should) not be the same anymore:

> With the continued advance of computing power and accessibility, the view that "real mathematicians don't compute" no longer has any traction for a new generation of mathematicians that can really take advantage of computer aided research, especially given the modern computational packages such as Maple, Mathematica, and Matlab.

This statement is made loud and clear in the Preface, and we will find several types of arguments along the book to support it.

This monograph originated from a short course of the same name, consisting of 8 lectures, one for each of the 8 main chapters of the book. It is the third monograph in the series. The first two ("Mathematics by Experiment: Plausible Reasoning in the 21st Century" by Jonathan Borwein and David Bailey, and "Experimentation in Mathematics: Computational Paths to Discovery", again by Jonathan Borwein, David Bailey, and Roland Girgensohn) were also published by A. K. Peters in 2004. There is also a CD-ROM with hyperlinked, searchable PDF files of the above these two books.

Let us turn to the contents. Chapter 1 lays the philosophical foundations of the book, making a case for inductive methods, discussing forms of scientific experiment, and enumerating the eight roles of computation. Ten examples are presented to illustrate these roles and to give a flavor of what follows. The authors conclude the chapter saying that

> Above all, mathematics is primarily about *secure knowledge* not proof, and that while the aesthetic is central, we must put much more emphasis on notions of supporting evidence and attend more closely to the reliability of witnesses.

Chapters 2 and 3 present (sometimes in a very sketchy way) some of the algorithms used in experiments. We can learn about integer-relation detection (especially, the PSLQ algorithm), high-precision arithmetics, evaluation of integrals and series (both by closed expressions or by quadratures), polynomial root-finding, etc. My favorite: the Tanh-Sinh quadrature (Section 3.4), not widely known.

Chapter 4, having quite a different look and feel from the previous ones, is on inverse scattering, that is, on the problem of determining the model from an observation. It is meant to be an example of the experimental science the authors promote, combining both physical and mathematical experiments.

Chapter 5 is about exploring "strange" functions on the computer, starting with the famous Weierstrass' example of continuous and nowhere differentiable functions. Explorations are not merely reduced to function plots. The approach is quite pleasing for approximators, via systems of functional equations. A general question when a system of functional equations has a unique continuous solution is settled in a "traditional" way (i.e. proofs involving for instance the fixed point theorem), but along with this it is shown how graphical capabilities of the software can help us to understand the general trend and make the right conjectures. Taking advantage of the machinery of functional equations, infinite Bernoulli convolutions are introduced at the end of this chapter. The main focus here is on the analysis of the limiting measure, which can be either singular or absolutely continuous, depending on the parameter involved. Again, computer visualization allows to shorten the time spent on searching the right answer.

Chapter 6 discusses an experience of supervising undergraduate research, using as a case study the problem of factoring integers by a random vector approach. The well known Quadratic Sieve algorithm can be viewed as producing quasi-randomly distributed vectors, and its behaviour can be predicted by choosing an appropriate probabilistic model, which is done experimentally.

In Chapter 7 we get back to the problem of evaluation of definite integrals, but now the authors are interested in closed expressions and not in numerical approximations. Some of the integrals evaluated here are

$$\int_0^{\pi/2} (\log \cos x)^n \mathrm{d}x, \qquad \int_0^{\infty} \frac{\mathrm{d}x}{(x^2+1)^{m+1}}, \qquad \int_0^{\infty} \frac{\sinh(zx)}{\cosh(bx)}\,\mathrm{d}x,$$

and many more. The main tool used is *Mathematica*.

Chapter 8 is a "computational conclusion", emphasizing again the idea of usefulness of experiments in mathematics, but I found it much easier to read than Chapter 1. Either I was better prepared to assimilate the ideas, after going through the rest of the book, or simply the set of nice examples (visual computing, Hilberts inequality and Wittens Zeta Function, and further computational challenges) is more digestible.

The reader eager to test him/herself in the level of understanding of the main content of the book will enjoy Chapter 9, containing exercises (more than 10 per chapter, plus 36 additional problems).

This is a very enjoyable book, that should be read equipped with both a computer (loaded with an appropriate software) and a scratchpad. In fact, I think that one of the strengths of the book lies in a right balance between nice formal ("classical") proofs and exciting experiments. You will find plenty of both. Further content (examples, software, list of errata) is available at http://www.experimentalmath.info/.

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Fundamental Papers in Wavelet Theory. *By C. Heil and D. Walnut*. Princeton University Press, 2006. 912 pp., $49.50 paper, $85.00 cloth. ISBN: 978-0691127057.

The term *wavelet* has already been around for about 25 years. For many years, it has been impossible to follow the development of new results related to all facets of the topic. Therefore, new generations of researchers will have difficulties answering questions like

- Where did wavelets come from?
- What was the main motivation and idea behind the concept?
- Who did what?

The present book answers all questions of that type. The book is edited by Christopher Heil and David Walnut, who were Ph.D. students at the time wavelets were introduced. They have followed the topic from the beginning and have an excellent general view of the role of the first generation researchers.

The book collects 37 fundamental papers related to wavelets; all of these papers are published before and have played a key role in the development of wavelet analysis. The papers are divided into 7 groups, each of which are presented with a joint introduction by one of the leading researchers within that particular area. The sections are tied together by a common introduction written by John Benedetto, a foreword by Ingrid Daubechies, and a preface by the editors. Some of the papers were originally written in German or French, and appear here for the first time translated to the English language.

The preface by the editors explains the anticipated role of the book and the criteria used to select the papers. Ingrid Daubechies tells how wavelet analysis was born as a product of activities in many diverse fields of science, including mathematics, physics, electrical engineering, etc. She stresses the fact that all of these disciplines contributed (and still do) in an essential way.

The introduction by John Benedetto describes his own entrance to the wavelet area. It explains various ways of obtaining series expansions, e.g., the continuous way (via Calderón's reproducing formula), and the discrete way (wavelet frames, Gabor frames, and Fourier frames). It also relates the wavelet analysis in $L^2(\mathbb{R})$ to more general harmonic analysis in other function spaces. The text contains several interesting and funny anecdotes.

The first section, *Precursors in Signal Processing,* is introduced by Jelena Kovačević. Among the contributions in the section, the reader finds the paper by Burt and Adelson on efficient image coding based on pyramid coding, a paper by Esteban and Galand, where the idea of a quadrature mirror filter comes up, as well as papers by Smith and Barnwell, respectively, Mintzer, on how to obtain two-channel perfect reconstruction filter banks.

The section *Precursors in Physics: Affine Coherent States* is introduced by Jean-Pierre Antoine. He tells the story about how the idea of decomposing signals in terms of a wavelet system came up in a collaboration between Morlet and Grossmann: Morlet discovered that a "wavelet analysis" was useful in order to analyze seismic signals, and Grossmann found the explanation in terms of group theory. Antoine also discusses extensions of the continuous wavelet transform to higher dimensions. The section itself contains the paper by Aslaksen and Klauder from 1969 concerning what we today would call integral representations in terms of the continuous wavelet transform, as well as the seminal papers by Grossmann, Morlet, and Paul on integral expansions in terms of wavelets, respectively, general square integrable group representations. For many people, these papers mark the beginning of the wavelet era.

The section *Precursors in Mathematics: Early Wavelet Bases* is introduced by Hans Feichtinger. He explains the content of the papers in the section from the function theoretic point of view, and points out that many of the papers were motivated by purely mathematical reasons; still, the results lead to highly applicable constructions. This is a strong argument for the existence of mathematics as an independent research area! Interestingly, we learn that Haar presented his famous orthonormal basis for $L^2(0, 1)$ in the process of answering a theoretical question. The structure itself, which is nowadays the central issue, did not play the main role. Obviously, the paper by Haar, *On the theory of orthogonal function systems,* appears as the first paper in the section, in a translation from German to English. The subsequent paper by Franklin shows how to obtain an orthonormal basis for $L^2(0, 1)$ consisting of *continuous* functions. These papers are followed by Strömbergs paper on higher-order spline systems and their role as unconditional bases for the Hardy spaces. The papers by Lemarié and Meyer, e.g., with Meyer's famous construction of a wavelet for which the Fourier transform has compact support and is smooth, are also found here, for the first time in a translation from French to English.

The section *Precursors and Development in Mathematics: Atom and Frame Decompositions* is introduced by Yves Meyer. He explains the basic ideas by Morlet (as discussed above) from the point of view of Gabor analysis (a very interesting area of research that is developed more or less parallel to wavelet theory, but without having a section devoted to it in the present book). It also explains how the concept of a frame entered the scene. Frames were introduced by Duffin and Schaeffer already in 1951 (their paper appears as the first in the section), but apparently nobody were ready for the concept at that time. It was (again!) Grossmann who rediscovered the concept, and managed to connect it to wavelets. The famous paper by Daubechies, Grossmann and Meyer, where they construct tight frames for $L^2(\mathbb{R})$ having wavelet structure, respectively, Gabor structure, is also contained in the section; the same is the case for Daubechies's paper from 1990, where she, among many other things, relates the wavelet transform to time-frequency localization and discrete expansions in terms of general wavelet frames and Gabor frames.

Parallel to these concrete constructions in $L^2(\mathbb{R})$, other groups of researchers were working on more abstract and general constructions, e.g., by considering other function spaces than $L^2(\mathbb{R})$. The section contains a paper by Coifman and Weiss about extensions of the Hardy spaces, as well as a paper by Frazier and Jawerth on the φ-transform and the related decomposition of Besov spaces. Finally, one of the papers by Feichtinger and Gröchenig on decomposition of coorbit spaces in terms of square-integrable group representations is included. These papers have finally reached the level of popularity that they deserve; at the time they were written, they were certainly many years ahead of the development, and the community was not ready to understand the depth and importance yet.

The section *Multiresolution analysis* is introduced by Guido Weiss. He explains the concept of a classical multiresolution analysis in $L^2(\mathbb{R})$, and how it leads to a construction of a suitable wavelet. He also discusses some of the various extensions of the theory that have been obtained after the publication of the papers in this collection, e.g., the MRA-based tight frame constructions. Besides the references he mentions, I would like to add the unitary extension principle by Ron and Shen [4], as well as the extensions (with a higher number of vanishing moments) obtained simultaneously by two groups – see [1] and [2]. The section itself begins (of course!) with the papers by Mallat and Meyer, where the MRA-concept is introduced. These papers are followed by a paper by Cohen, where the characterization of the quadrature mirror filters associated with an MRA-analysis appears. The paper by Lawton with a construction of tight frames of compactly supported wavelets is also reprinted here. The section is (of course!) concluded by Daubechies' construction of MRA-based orthonormal bases of compactly supported wavelets, a paper that really opened for a completely new world.

The section *Multidimensional Wavelets* is also introduced by Guido Weiss. He describes the natural extension of the continuous wavelet transform to higher dimensions; an important by-product is an understanding of how to obtain discrete wavelet expansions. The section begins with a paper by Meyer that forms the basis for MRA-constructions in higher dimensions; it is followed by a paper by Gröchenig, where one of the one-dimensional algorithms in Meyer's paper is extended to higher dimensions as well. The paper by Kovačević and Vetterli, where *nonseparable* perfect reconstruction filter banks in higher dimensions are constructed, is also reprinted here. The same theme appears in the final paper in the section, where Gröchenig and Madych construct Haar-type orthonormal bases in $L^2(\mathbb{R}^n)$.

The final section, *Selected Applications,* is introduced by Mladen Victor Wickerhauser. He starts by explaining the need for sparse representations of operators and its relationship to multiresolution analysis. He also discusses the success of wavelets in the context of image compression, e.g., the use of wavelets to compress fingerprints.

It is well-known that wavelets have numerous applications, so it is clear that only a small fraction can be reprinted in the present book. The first contribution is a paper by Beylkin, Coifman and Rokhlin, which shows that the multipode algorithm by Rokhlin essentially is a multiresolution analysis. It is followed by a paper by DeVore, Jawerth, and Popov on nonlinear n-term approximation. The section also contains a paper by Donoho and Johnstone on (statistical) recovery of an unknown function based on noisy data, as well as a paper by Jaffard on estimates of the Hölder exponents of a function at a point based on the asymptotic decay of the wavelet coefficients. The section concludes with a paper by Shapiro, dealing with an efficient wavelet-based image compression algorithm.

Altogether, the book is a stunning collection of impressing articles. The editors took great care, and succeed, in the selection of very important articles that played a significant role for the development of wavelet theory. The book will be of great interest for anyone interested in wavelet analysis, including professional researchers, Ph.D. students, and people who shall teach a wavelet course.

Of course, every researcher in a wavelet-related area will have his own list of favorite publications, and will miss some of them in the book. From my own list, I just want to mention the paper *Continuous and discrete wavelet transforms* [3], written by the editors of the present book. It contains a survey of the stage of wavelet theory around 1989. The paper actually played an important role for many researchers and Ph.D. students who wanted to enter the wavelet arena: while the papers written by the established experts were frequently too complicated to understand, the paper by Heil and Walnut was written in a language that nonexperts could understand. After reading [3], it was possible to move to more advanced literature. The paper could very well have been included in the book; apparently, the editors were too modest to mention the importance of one of their own contributions!

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Numerical methods for roots of polynomials, Part I. *By J.M. McNamee*. Studies in Computational Mathematics 14, Elsevier, 2007. xx+333 pp., $160.00, ISBN 0 444 52729 X.

Some 15 years ago, in 1993, the author published *A bibliography on roots of polynomials* (see [1]) and supplements in 1997 [2], and 1999 [3], and an electronic update in 2002 [4] which are available via the Journal's website. Downloading it now results in a Microsoft Access table of 3003 items. The subjects of the papers are classified into 32 categories including “Bracketing methods”, “Newton's method”, “simultaneous root finding”, “matrix methods”, etc. Up to 6 categories can be assigned to an entry so that one could for example select all the papers related to Newton's method. Since 2003, the database is maintained on the author's website www.yorku.ca/mcnamee. Downloading it gave 3494 items, the last ones dated from 2006. The Access version seems to be not very recent since there are some problems when it is converted to more recent versions, a typical Microsoft feature.

The idea of the book is to guide the reader through this huge amount of information. Some items are not dated, the oldest dated items are from 625 and 650 AD, the last ones in that list are from 2003. In his introduction to this book, the author mentions a list of about 8000 items of which about 50 are from 2005. Unless it is due to the conversion problem, this is not what is available on his website.

The present book is part I of a two-part book-set on this topic. In this first volume, the text does not tell in detail what volume two will contain. But some chapters are mentioned in the introduction. The rough idea is to go over the different categories that were used to classify the earlier bibliography. Not all the 32 categories are treated separately. They are grouped in larger chapters. The references are not given in a long list at the end of the book, but the relevant ones are listed ordered alphabetically by author after each chapter. So it is difficult to find out whether all the references from the 2003 list are included, or if all the 8000 references appear, or up to which year the literature is covered.

It is the author's intension to write a "Handbook of methods for polynomial root-solving" accessible for readers with a knowledge of undergraduate calculus and linear algebra. So in his introduction he mentions very briefly some history, some applications and the basic techniques on which many variants are grafted. More details are to follow in subsequent chapters, so that this forms at the same time a summary of the two books. These primal methods are: bisection (chapter 7), Newton (chapter 5), simultaneous methods (chapter 4), Graeffe (chapter 8), Laguerre (chapter 9), Jenkins-Traub (chapter 12), interpolation methods (chapter 7), matrix methods (chapter 6), Routh-Hurwitz techniques for root location (chapter 15). Only chapters 1–6 are found in this first volume.

So, let us briefly survey the actual contents of the different chapters. The first chapter deals with generalities discussing the efficient and accurate evaluation of polynomials and their derivatives, the order of convergence, the efficiency, and stopping criteria of iterative methods, and bounds for polynomial zeros. (Some 45 different formulas are listed from the literature that give bounds for complex zeros.)

The style used in this first chapter is maintained in the rest of the text. Some elementary and some of the most basic formulas are "derived" or actually proved, while other more advanced variants or less simple results, requiring more than just a few lines are quoted from the literature. Often a numerical example is included for which the general formulas are "checked" to see that they do indeed give the expected result. In the following chapters, references to software are also included whenever they exist.

Chapter 2 deals with Sturm sequences and greatest common divisors. In the current context these are important to find approximate locations of single roots, multiple roots, or clusters of roots. This is continued in chapter 3 with theorems by Descartes, Fourier, Vincent, Akritas, etc. and continued fraction methods to find the number of real zeros in a certain interval. Nothing really new has happened about these topics in the last decades.

That is somewhat different for the methods of chapter 4. These methods may compute several roots simultaneously, or clusters of roots of multiple roots. Although having older origins, they have been reactivated since the 1960's. Not only methods suitable for parallel implementation, but other techniques involving higher order derivatives, square roots, interval arithmetic, Gauss–Seidel and SOR techniques, or hybrid methods are covered. Their convergence, accuracy, efficiency, and implementation, are discussed.

The same topics return in chapter 5 collecting methods that are in one way or another related to Newton's method, the archetype of all iteration formulas. The longest chapter, reflecting the currently still active field of research, is the collection of methods surveyed in chapter 6. The methods of this chapter are based on matrix techniques, the simplest being the computation of the eigenvalues of the companion matrix. But many recent fast and superfast variants are described as well.

This book is an extensive guide through the jungle of many methods for a very old problem. The diversity of theoretical, numerical, computational, and complexity aspects, and the wide range of approaches makes the field quite difficult to survey in a well structured way. Where does a particular method belong? To give an example, I was a bit disappointed not to find a description of Bairstow's method, which I consider to be a classic in this context. I would expect in to be in chapter 5 because of its close connection with Newton's method. It is indeed mentioned there several times, but I could not find an explicit description. It may be in part 2 though, but I couldn't spot a cross-reference to where the details are to be found. That makes me wonder more generally what the actual strategy is behind the grouping of references in the chapters as they are now. For most of them it is a logical decision, but it is less obvious for others. So a good index is necessary (which for the moment is only referring to part 1). An hypertext version or a searchable index might even be better. That would include an index of authors, which is now lacking. So, a well elaborated wikipedia-like database with short descriptions of the methods, indexing of the papers, searchable and linked in all possible ways would be a valuable alternative.

One last remark. Being a bit of a LaTeX purist myself, I was a somewhat frustrated by the sloppy typesetting. Phrases like "…replace x by $-x$ in $p(x)$ and…" or "$z_j = \exp(\mathrm{i}\cos^{-1} x_j)$", which can be found throughout the book, or not having the proper punctuation after displayed formulas, and similar typesetting sins gives me the creeps.

But this remark does not change the fact that this book (and probably its sequel) in guiding you through a gigantic list of references that has been collected over many years: a sheer drudgery. It is indeed accessible for undergraduates, but it is probably more appreciated by a skilled researchers who wants to look up some idea or reference on a well focused item.

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Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday. *Edited by F. Gesztesy (Managing Editor), P. Deift, Ch. Galvez, P. Perry, and W. Schlag.* Proceedings of Symposia in Pure Mathematics 76. American Mathematical Society 2007. Hardcover. Volume I: 496 pp., ISBN 978-0-8218-4248-5. Volume II: 464 pp., ISBN 978-0-8218-4249-2.

This is a two-volume set of proceedings of a conference called SimonFest held at Caltech, March 27-31, 2006, to honor Barry Simon's 60th birthday. So why this review is not relegated to the final section where the reports on proceedings belong? Because it is not a proceedings volume in the usual sense. As the editors explain in the Preface,

> Barry requested that both his birthday conference and this Festschrift concentrate not so much on what the speaker or writer has done recently, but instead on reviews of the state of the art, with a focus on recent developments and open problems. (…) the contributions to this Festschrift contain a few additionally selected reviews. In the end, the bulk of the articles in this Festschrift are of this state of the art survey form with a few that instead review Barry's contributions to a particular area.

All papers have been divided in two parts (coinciding with the volumes of the set). Part 1 is entitled "Quantum field theory, statistical mechanics, and nonrelativistic quantum systems", and contains the following topics:

- Quantum field theory, statistical mechanics (4 papers),
- Nonrelativistic two-body and N-body quantum systems, resonances (12 papers),
- Electric and magnetic fields, semiclassical limit (4 papers).

Part II, entitled "Ergodic Schrödinger operators, singular spectrum, orthogonal polynomials, and inverse spectral theory", is split into two sections:

- Random and ergodic Schrödinger operators, singular continuous spectrum (8 papers),
- Orthogonal polynomials, inverse spectral theory (5 papers).

Although Part I contains many interesting contributions even from a "purely mathematical" point of view (see e.g. E.B. Davies' "Non-self-adjoint operators and pseudospectra"), clearly, Volume II is closer to an approximator's heart, so I restrict my brief and biased account to the last area.

P. Deift's article is a short and elegant survey on some applications of the Riemann–Hilbert (RH) methods in the theory of OPRL and OPUC (you should be acquainted with the abbreviations made popular by Barry: OP = "orthogonal polynomials", RL = "real line" and UC = "unit circle"). We usually identify the RH method with the steepest descent analysis of Deift and Zhou (briefly described here). However, it also allows us to prove identities: recurrence relations, differential equations, Pinter–Nevai formula for OPUC, formulas for Toeplitz and Hankel determinants.

Golinskii and Totik's paper is an introduction to the OP theory. The first part is an easy to read account on the general theory, both classical (moment problems, continued fractions, classical OP, ratio and weak asymptotics) and modern aspects (the RH theory). The second part specialises on OPUC. Finally, third part is an obligatory description of "Simon's earthquake" in this theory: analysis of the CMV matrices, clock theorems for zeros, and much more.

I usually like the titles of Khrushchev's papers, and this one didn't disappoint me either: "Orthogonal polynomials: the first minutes". It logically deals with the not very well known origins of the theory of OP, namely the role played by Wallis' infinite product and Brouncker's continued fraction. This is a fantastic historical research containing classical but highly nontrivial mathematics.

Spectra of Jacobi matrices are one of the Barry's favorite topics. The last paper of the book, by R. Killip, is about it: spectral theory of one-dimensional differential and finite-difference operators (Jacobi matrices, Krein systems, Schrödinger operators and CMV matrices). Emphasis is made on the role of the sum rules and trace formulas.

Both nicely edited volumes sum almost 1000 pages of interesting surveys and exhaustive bibliography that clearly benefits graduate students and researchers seeking for a quick introduction into areas mentioned above. Even if you are interested only in the topics covered in Part II, don't miss "A selection of Barry stories" at the beginning of Volume I. After having a chance to collaborate with Barry I felt totally identified with the stories, mostly humorous, but reflecting Simon's energy, passion and style of work.

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The Cauchy Transform. *By Joseph A. Cima, Alec L. Matheson, and William T. Ross*. American Mathematical Society, 2006. 272 pp., $75.00. ISBN: 978-0821838716.

Introduction

From book's Preface:

"The book is a survey of Cauchy transforms of measures on the unit circle. The study of these functions is quite old and quite vast: quite old in that it dates back to the mid 1800s with the classical Cauchy integral formula; quite vast in that even though we restrict our study to Cauchy transforms of measures supported on the circle and not in the plane, the subject still makes deep connections to complex analysis, functional analysis, distribution theory, perturbation theory, and mathematical physics".

So, the audience of this book splits into two circles. The first are experienced researches which can use the book for references. The second are graduate students interested in Function Theory and Complex Analysis. I think that this book may be extremely useful for a seminar on Cauchy integrals and Hardy spaces. It may happen that namely the later circumstance and of course volume restrictions made this book not self-contained:

"Unfortunately, this book is not self-contained. We present a review of the basic background material but leave the proofs to the references. The material on Cauchy transforms is self-contained and the results are presented with complete proofs".

In fact this book is a compromise between the interests of researches and graduate students. Being on the side of researches I can nevertheless complain that the book states too many interesting facts without proofs. Yes, the references are provided, but I do not think that it is a very good idea to revise the original papers in the library. I would prefer to read this book at home. Some guidance on the main ideas of the proofs standing behind the statements could be advantageous for the reader. From this point the authors missed the opportunity to include some questions into exercises with short hints and references. This would definitely saved necessary space. The exercises on the other hand could be very useful for graduate students. However, I must admit that many of the results mentioned are not so simple and require more detailed presentation.

There is another complain. It is quite obvious that the authors faced the problem of the volume. Therefore, it is a little bit strange to find on some pages such generously spaced formulas like for instance this one

$$\begin{aligned}
\|K(hdm)\|_p^p &= \left\| \sum_{n=0}^{N} \hat{h}(n) z^n \right\|_p^p \\
&\le \left\| \sum_{n=0}^{N} \hat{h}(n) z^n \right\|_2^p \\
&= \left(\sum_{n=0}^{N} |\hat{h}(n)|^2 \right)^{p/2} \\
&\le \|h\|_2^p \\
&\le \|h\|_\infty^p,
\end{aligned}$$

see p. 46. There are no reasons on my opinion to think that it is better for understanding than for example the following one

$$\begin{aligned}\|K(hdm)\|_p^p &= \left\|\sum_{n=0}^{N}\hat{h}(n)z^n\right\|_p^p \le \left\|\sum_{n=0}^{N}\hat{h}(n)z^n\right\|_2^p \\ &= \left(\sum_{n=0}^{N}|\hat{h}(n)|^2\right)^{p/2} \le \|h\|_2^p \le \|h\|_\infty^p,\end{aligned}$$

or even this formula

$$\|K(hdm)\|_p^p \le \left\|\sum_{n=0}^{N}\hat{h}(n)z^n\right\|_2^p = \left(\sum_{n=0}^{N}|\hat{h}(n)|^2\right)^{p/2} \le \|h\|_2^p \le \|h\|_\infty^p.$$

The problem is that there are many such places. So, when I saw them I had a natural thought that this space could be used to present some more material or make the book shorter.

The following theorem is a good illustration of the questions which are considered in this book.

Theorem (Boole, 1857) *Let*

$$g(x) = \sum_{k=1}^{n}\frac{c_k}{x-a_k},$$

where all c_k *are positive and* $a_1 < a_2 < \cdots < a_n$. *Then for every* $y > 0$ *the Lebesgue measures of both sets* $\{g > y\}$ *and* $\{g < -y\}$ *are equal to*

$$\frac{1}{y}\sum_{k=1}^{n}c_k.$$

Another example is the theorem obtained in 1929 by the great mathematician of St. Petersburg V.I. Smirnov in his paper [17] on Hardy spaces.

Theorem (Smirnov, 1929) *If* μ *is a positive Borel measure on the unit circle* $\mathbb{T}$, *then its Cauchy transform*

$$\mathcal{K}^\mu(z) = \int_{\mathbb{T}}\frac{\mathrm{d}\mu(\zeta)}{\zeta - z} \tag{1}$$

is in the Hardy class H^p *for every* $0 < p < 1$.

For those who do not know Hardy spaces: H^p is the class of all analytic functions in the unit disc $\mathbb{D}$ such that

$$\|f\|_p = \sup_{0<r<1}\left(\int_{\mathbb{T}}|f(r\zeta)|^p\mathrm{d}m\right)^{1/p} < \infty,$$

where m is the normalised Lebesgue measure on $\mathbb{T}$. With this notations the subject of the book is the space $\mathcal{K}$ of all Cauchy transforms $\mathcal{K}^\mu$ as analytic functions defined in $\mathbb{D}$, as well as their boundary values on $\mathbb{T}$. Smirnov's theorem claims that $\mathcal{K} \subset \bigcap_{p<1} H^p$. On the other hand $H^1 \subset \mathcal{K}$. Both inclusions are proper. So, the book under review is a good complement to famous books on Hardy spaces such as [4, 6, 15, 22].

Since the Banach space $M(\mathbb{T})$ of all finite complex Borel measures on $\mathbb{T}$ is the adjoint (dual) space to the Banach space $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$ with respect to the duality

$$(f, \mu) = \int_{\mathbb{T}} f(\mathrm{e}^{-\mathrm{i}t})\, \mathrm{d}\mu(\mathrm{e}^{\mathrm{i}t}) = \sum_{n \in \mathbb{Z}} \hat{f}(n)\hat{\mu}(n),$$

$\hat{f}(n)$ and $\hat{\mu}(n)$ being the Fourier coefficients of f and μ, it is not surprise that $\mathcal{K}$ is the dual space to the disk algebra A. The disk algebra A is a closed subalgebra of $C(\mathbb{T})$ made of functions admitting the analytic extension to $\mathbb{D}$. Equivalently $A = \{f \in C(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0\}$. The duality mentioned is the source for interesting connections especially in trigonometric series.

Although H^p is not locally convex if $0 < p < 1$, it has a nontrivial dual space. For instance, every functional $f \longmapsto f(z)$, $z \in \mathbb{D}$, is linear and bounded on every H^p. The conjugate space for H^p, $0 < p < 1$, was found in [5], see also [4, Theorem 7.5]. It is not my purpose to enter the details, but however some words should be said here. Let Λ_α be the set of all continuous functions on $\mathbb{T}$ satisfying

$$|g(t_1) - g(t_2)| \le C_f |t_1 - t_2|^\alpha, \quad t_1, t_2 \in \mathbb{T}, \tag{2}$$

where $0 < \alpha < 1$. Then the linear bounded functionals on H^p, $1/2 < p < 1$, are described by the formula

$$\phi(f) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(r\mathrm{e}^{\mathrm{i}\theta}) g(\mathrm{e}^{-\mathrm{i}\theta})\, \mathrm{d}\theta,$$

where $f \in H^p$ and $g \in \Lambda_\alpha$, $\alpha = 1/p - 1$.

In 1916 Privalov proved [16] that the harmonic conjugate $\tilde{f}$,

$$\tilde{f}(\mathrm{e}^{\mathrm{i}\theta}) \overset{def}{=} \lim_{\epsilon \to 0^+} \int_{|\theta - t| \ge \epsilon} \cot\left(\frac{\theta - t}{2}\right) f(\mathrm{e}^{\mathrm{i}t}) \frac{\mathrm{d}t}{2\pi}$$

of any $f \in \Lambda_\alpha$ is also in Λ_α. Equivalently, the Cauchy integral $\mathcal{K}^\mu$ of $\mu = f\mathrm{d}m$ extends to a continuous function g in the closed unit disc satisfying (2) on $\mathbb{T}$ and hence by the maximum principle on $\mathbb{D}$.

In 1925 Kolmogorov obtained the following result [14].

Theorem *For every* $f \in L^1(\mathbb{T})$

$$m\{|\tilde{f}| > y\} \le \frac{C}{y}. \tag{3}$$

It is clear that Boole's, Smirnov's and Kolmogorov's theorem are closely related. Moreover, as it is often happens in Mathematics Smirnov's and Kolmogorov's theorems can be derived from Boole's theorem. But it is also true that these theorems put a new light to the subject. Smirnov's proof is important for many questions of Hardy spaces. For instance a great deal of the theory of Muckenhoupt weights rests on Smirnov's ideas. On the other hand Kolmogorov's proof made a great contribution to trigonometric series and stochastic processes.

There is also another interesting topic related with $\mathcal{K}$ which is not mentioned in the book. And I think that it is a pity since the topic is related with the proof of Luzin's conjecture found in 1966 by Lennart Carleson [2]. Let us consider sequences $S = \{\mu_n\}_{n \ge 0}$ of finite Borel measures on $\mathbb{T}$ satisfying

$$\sum_{n=0}^{\infty} \|\mu_n\| < \infty.$$

Then the series of analytic functions $z^n\mathcal{K}^{\mu_n}$ converges uniformly and absolutely on compact subsets of $\mathbb{D}$ to an analytic function

$$\mathcal{K}^S(z) = \sum_{n=0}^{\infty} z^n \mathcal{K}^{\mu_n}(z). \tag{4}$$

The question is whether $\mathcal{K}^S \in \bigcap_{p<1} H^p$ or even may be $\mathcal{K}^S$ satisfies (3) with $\tilde{f} = \mathcal{K}^S$. If H^p were locally convex, then this question would be very simple. It is equally simple if we drop the multipliers z^n in (4). But it is very difficult as it is and was answered in positive by Vinogradov in [19], see also [20]. There is a simple motivation why the answer to this question must be positive. The power series of $z^n\mathcal{K}^{\mu_n}$ is the shift of that of $\mathcal{K}^{\mu_n}$. It follows that the space U of $f \in A$ which power series converge uniformly on the closed unit disc is the pre-dual space for the space of all distributions S of the type (4).

Let me recall that Luzin's conjecture claims that the trigonometric series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\mathrm{e}^{\mathrm{i}\theta}$$

of any $f \in L^2(\mathbb{T})$ converges to f almost everywhere with respect to the Lebesgue measure m. Vinogradov noticed that Carleson's theorem has a dual statement and that this statement if interpolated by Riesz–Torin implies his conjecture. The crucial observation is based on one Stein's theorem on convolution operators. It claims that if the kernel satisfies some conditions of "smoothness", then the convolution operator is of the weak-type (1, 1) provided it is bounded on L^2 (in the case of Vinogradov it was Carleson's theorem). Amazingly all conditions of this Stein's theorem are satisfied in Vinogradov's case. So the result follows. Vinogradov's theorem has important consequences for the study of the coefficients of uniformly convergent power series. For instance, as Kislyakov showed later in [12], these coefficient are very well spaced in the Fourier coefficients of functions in $L^2(\mathbb{T})$. They are so good spaced that for every sequence $\{a_n\}_{n\geq 0} \in l^2$ there is $f \in U$ such that $|\hat{f}(n)| \geq a_n$ for every $n \geq 0$. Vinogradov's theorem has also another important application to improvements of Menshov's Theorem [11, 13].

It is therefore clear that $\mathcal{K}$ lies at the very center of a number important theories including BMO, VMO, theory of trigonometric series, probability theory, theory of the disc algebra and others.

The contents

Chapter 1 contains preliminaries on Measure Theory, Lebesgue Integration and on basic facts of Functional Analysis and Operator Theory. At the end of the chapter the authors review the theory of Hardy spaces.

In Chapter 2 the Cauchy transform is treated as a function. Here Smirnov's theorem is proved. In Section 2 two important results are presented. The first is Ulyanov's theorem saying that $f = \mathcal{K}^\mu$, where $\mu \ll m$ is absolutely continuous with respect to m, is (A)-integrable on $\mathbb{T}$ and

$$f(z) = (A)\int_{\mathbb{T}} \frac{f(\zeta)}{1-\bar{\zeta}z}\mathrm{d}m(\zeta), \quad z \in \mathbb{D}.$$

The second result is a deep theorem of Alexandrov extending Ulyanov's theorem to every $f \in H_0^{1,\infty}$, where $H_0^{1,\infty}$ is the weak Hardy class corresponding to H^1. By Smirnov's Theorem $\mathcal{K}^\mu \in H_0^{1,\infty}$ if $\mu \ll m$. So this is clearly an extension. The rest of the chapter is devoted to Fatou's jump theorem, Plemelj's formula and the tangential boundary behaviour.

Chapter 3 is called: "The Cauchy transform as an operator". This part in fact can be found in many books on Harmonic Analysis and Hardy Spaces. Especially complete and clear presentation is given in [18] and [22]. In Section 3.4 Kolmogorov's Theorem is proved. A short section Section 3.5 is devoted to Muckenhoupt weights. The natural projector of L^p onto H^p is bounded for $1 < p < \infty$. This allows one to identify the dual space to H^p with H^q, where $q = p/(p-1)$. For H^1 the dual space is BMO. These results are the subject of Section 3.6. In Section 3.7 the best constant for the norms of different projections are stated. The last short section Section 3.8 is devoted to the Hilbert transform on the real line $\mathbb{R}$.

There are in fact three topologies in $\mathcal{K}$, the topology of the Banach space and two weak topologies. The first weak topology is induced by the duality $(A, \mathcal{K})$. It is called the weak-* topology. The second weak topology is induced on $\mathcal{K}$ by bounded linear functionals on $\mathcal{K}$. These topologies are considered in Section 4.1-3. At the end of Section 4.3 the authors show that $\mathcal{K}$ is weakly sequentially complete. They derive this result from Havin–Mooney's theorem [7]. In Section 4.4 Schauder bases are considered.

The title of Chapter 5 is the question: "Which functions are Cauchy integrals"? There are many answers to this question. The answer found by Havin is the topic of Section 5.1. In Section 5.3 Tumarkin's theorem is proved. A beautiful characterization of $\mathcal{K}$ in terms of Hardy spaces H^p was found by Alexandrov. The proof of Alexandrov's theorem is in Section 5.4.

In Section 5.5 the authors state another Havin's theorem as well as my results [10] on a possibility to place a nonzero Borel measure on a subset E of $\mathbb{T}$ so that $\mathcal{K}^\mu$ be in A or even in the space of infinitely differentiable functions in the closed unit disc. In the case of A this can be done if and only if $m(E) > 0$. In the case of infinitely differentiable functions this can be done if and only if E contains a closed subset F with $m(F) > 0$ such that

$$\sum_{n=1}^{\infty} m(I_n) \log m(I_n) > -\infty,$$

where $\{I_n\}_{n\geq 1}$ is a complete list of the complementary intervals of F.

Interesting geometric conditions sufficient for the inclusion $f \in \mathcal{K}$ can be found in Section 5.6. In particular it is proved that if $f(\mathbb{D})$ is contained in a region that omits two oppositely oriented half-lenes then $f \in \mathcal{K}$.

Chapter 6 is completely devoted to the multipliers of $\mathcal{K}$. An analytic function ϕ on $\mathbb{D}$ is called a *multiplier* of $\mathcal{K}$ if $f \in \mathcal{K} \Rightarrow \phi f \in \mathcal{K}$. This topic originates in [21]. It was proved in [21] that the space $\mathfrak{M}$ of all multipliers of $\mathcal{K}$ is contained in the Hardy algebra H^∞. However, the elements of $\mathfrak{M}$ have additional and very special properties. For instance the partial sums of the Maclaurin series of each multiplier $\phi \in \mathcal{K}$ are uniformly bounded in the closed unit disc. Also the angular limits of $\phi \in \mathfrak{M}$ exist at every point of $\mathbb{T}$. In the opposite direction there is a simple condition in order that $\phi \in \mathfrak{M}$:

$$\sum_{n=0}^{\infty} |\hat{\phi}(n)| \log(n+2) < \infty \Rightarrow \phi \in \mathfrak{M}.$$

A Blaschke product

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z},$$

is said to satisfy the uniform Frostman condition if

$$\sup_{\zeta\in\mathbb{T}}\sum_{n=1}^{\infty}\frac{1-|a_n|}{|\zeta-a_n|}<\infty. \tag{5}$$

The rest of the chapter is devotes to the proof of my joint theorem with Vinogradov.

Theorem *Let I be an inner function. Then $I\in\mathfrak{M}$ if and only if $I=B$ is a Blaschke product satisfying* (5).

This result is in a good agreement with Frostman's theorem on radial limits of Blaschke products B. By this theorem a Blaschke product B and all its sub-products have radial limits at $\zeta\in\mathbb{T}$ with unimodular values if and only if

$$\sum_{n=1}^{\infty}\frac{1-|a_n|}{|\zeta-a_n|}<\infty.$$

Blaschke products satisfying the uniform Frostman condition have curious properties [9]. The Stolz domain $\Omega_\alpha(\zeta)$ in $\mathbb{D}$ is defined as the convex hull of ζ and the circle $C_\alpha=\{z:|z|=\sin\alpha\}$. This C_α is seen from ζ under the angle 2α in the direction to the origin. Any $\Omega_\alpha(\zeta)$ is obtained from $\Omega_\alpha(1)$ by a simple rotation. For $z\in\Omega_\alpha(1)$ we have

$$|z|^2=|1-(1-z)|^2=1+|1-z|^2-2\mathfrak{Re}(1-z),$$

implying that

$$\frac{1-|z|}{|1-z|}\geq\frac{1}{2}\frac{1-|z|^2}{|1-z|}\geq\frac{1}{2}\left\{2\mathfrak{Re}\frac{1-z}{|1-z|}-|1-z|\right\}=\frac{2\cos\theta-|1-z|}{2},$$

where $\theta=\arg(1-z)$. The length of the cord through z and 1 is $2\cos\theta$. If $z\in\Omega_\alpha(1)$, then z cannot be on this cord between $\mathbb{T}$ and the intersection of the cord with C_α. Hence

$$\frac{1-|z|}{|1-z|}\geq\frac{1-\sin\alpha}{2},\qquad z\in\Omega_\alpha(1).$$

It follows that every Stolz domain may contain not more than $N(\alpha)$ points of the zero set of a Blaschke product satisfying the uniform Frostman condition. In turn this means that from every zero of such a B one can see in the scope of the angle 2α directed towards $\mathbb{T}$ only a finite number of points. Hence the limit set of $\{a_n\}$ is a closed nowhere dense subset of $\mathbb{T}$.

For an arbitrary closed and nowhere dense subset E of $\mathbb{T}$ let $\{I_n\}_{n\geq1}$ be the complete list of the complementary arcs of E. We denote by ξ_n the center of I_n and consider the sequence

$$a_n=\left(1-\frac{m(I_n)}{2^n}\right)\xi_n,\quad n=1,2,\ldots.$$

Then the Blaschke product B constructed by $\{a_n\}_{n\geq1}$ satisfies the uniform Frostman condition. Therefore, $B\in\mathfrak{M}$ and the limit set of its zeros is E. This Blaschke product has additional properties:

- the infinite product of B converges at every point of $\mathbb{T}$;
- the discontinuity set of B is E;
- the Fourier series of B converges everywhere on $\mathbb{T}$ and $\hat{B}(n)=O(1/n)$.

Chapter 7 deals with extensions of important Boole's theorem.

In Chapters 8 and 9 the theory of Ahern and Clark is sketched. By Beurling's theorem the subspaces $K_\theta = H^2 \ominus \theta H^2$, $\theta = BI$, B being a Blaschke product and I an inner function corresponding to a singular measure μ, are invariant subspaces of the backward shift on H^2. There are two important elements in K_θ:

$$K_0 = \frac{\theta(z) - \theta(0)}{z}, \quad k_0 = 1 - \overline{\theta(0)}\theta(z).$$

The *model* operator corresponding to θ is defined by

$$Sf = P_\theta z f, \quad f \in K_\theta,$$

P_θ being the orthogonal projection onto K_θ in H^2. It is clear that S is a contraction, i.e. $\|S\| \le 1$. It is easy to check that

$$\begin{aligned} Sf &= zf \quad &\Longleftrightarrow \quad f \perp K_0; \\ S^* f &= f z^{-1} \quad &\Longleftrightarrow \quad f \perp k_0. \end{aligned}$$

This shows that for every $\alpha \in \mathbb{T}$ the operator

$$U_\alpha f = \begin{cases} zf \text{ if } \perp K_0 \\ \alpha k_0 \text{ if } f = K_0 \end{cases}$$

is unitary. There is a simple formula for U_α:

$$U_\alpha f = z\left(f - (f, K_0)\frac{K_0}{\|K_0\|^2}\right) + \alpha(f, K_0)\frac{k_0}{\|K_0\|^2}. \tag{6}$$

Formula (6) lists all unitary one-dimensional perturbations of S. The idea of Clark's paper [3] is that using these unitary operators one can study the contraction S.

It is shown in [1] that the linear functional $f \longmapsto f(\zeta)$ is bounded on K_θ if and only if

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|^2} + \int_{\mathbb{T}} \frac{d\mu(\xi)}{|\zeta - \xi|^2} < \infty. \tag{7}$$

Equivalently the kernel function

$$k_\zeta(z) = \frac{1 - \overline{\vartheta(\zeta)}\vartheta(z)}{1 - \overline{\zeta} z}$$

is in K_θ if and only if (7) holds.

Theorem (Clark) *Suppose that $\theta(0) = 0$. A point ζ is an eigenvalue of U_α if and only if $|\theta'(\zeta)| < \infty$ and $\theta(\zeta) = \alpha$. In this case k_ζ is an eigenvector of U_α.*

Moreover, Clark constructs the spectral measures of the unitary operators U_α so that U_α is unitary equivalent to the multiplication operator z in $L^2(d\sigma)$. These measures are mutually singular and closely related with the inner function θ. These measures are called Clark measures and considered in Chapter 9. By the way Clark measures play an important role in the theory of Orthogonal Polynomials on the unit circle.

Chapters 10 develops the theory of the Cauchy transforms for functions $f \in L^1(d\mu)$, where μ is an arbitrary positive Borel measure on $\mathbb{T}$. The approach to this problem is based on an extension of Ahern and Clark theory.

A short Chapter 11 lists some operators defined with the help of Cauchy transforms.

An advise to a reader

I presented here some topics covered by this book. If you are interested in them but do not have any background in Hardy spaces just try to read first [8]. In spite of the fact that this book is old it is exactly what you need to understand the book under review. If necessary further references may be found in [4], [15] and [6]. So, as I said at the beginning, the book is an ideal guide for running a seminar for graduate students.

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Triangulations and Applications. *By by Øyvind Hjelle and Morten Dæhlen*. Springer Verlag, 2006. xii + 234 pp. Hardcover, $64.95. ISBN 978-3540332602.

Triangulation, or the ability to split a domain of $\mathbb{R}^2$ into triangles, originated in cartography. With the development of computers, it has become a key tool in computational geometry, computer aided design (CAD), scattered data approximation, finite elements method (FEM) and other branches of science and technology. Choosing an appropriate triangulation can make a big difference in the quality of approximation or visual properties of a triangle-based surface. The most well known triangulation is due to Delaunay. For a few triangles we can construct it almost by hand, with more triangles we can do it with a computer and some rudimentary programming, but for a typical application, where a huge amount of triangles is involved, a combination of the data structure and some fine mathematics is required.

This book, based on lecture notes for a graduate level course at the Department of Informatics at the University of Oslo, is a pedagogical account on computer-oriented aspects of triangulation and meshing: algorithms, data structures and software. The assumed mathematics background of the reader is quite basic: in Chapter 1 the authors define a triangle, make a quick survey of some elementary geometrical facts related with triangles, define the admissible triangulations and introduce the first simple triangulation algorithm based on edge-swapping.

Chapter 2 (Graphs and Data Structures) switches completely the topic and paves the way to serious applications. If you are interested only in "pure math", you can safely skip this part. But as long as you aim at implementation, this chapter is essential. Given a triangulation of a domain, which is the best way to store it in a computer, from the point of view both of size and manipulability? We learn here some basic facts from the graph theory and triangle-based data structures. At the end, a very special (but very important too) kind of triangulations (binary triangulations) is considered.

In Chapter 3 the authors deal with the most popular part of the theory: Delaunay triangulations and Voronoi diagrams. First, they resort to intuition to motivate the max–min criterion, and then introduce the Voronoi diagrams, the circle criterion, and the properties of the Delaunay triangulations. Algorithmic aspects are also taken into account in the discussion how to compute the circumcircle test. A curiosity: in Section 3.1 the authors say that "Delone is pronounced 'Delaunay' and now also commonly spelled that way in English". In fact, the spelling Delone is a straightforward transliteration from the Cyrillic alphabet, and it was apparently Delaunay himself who used the French transliteration of the name.

Delaunay triangulations are typically constructed by "divide-and-conquer" or incremental-type algorithms. Both are discussed in Chapter 4, skipping occasionally the most annoying situations when 4 or more vertices lie on a circle.

Once the basic theory is set, we move to more serious applications of Delaunay-based triangulations: data dependent (Chapter 5) and constrained triangulation (Chapter 6). This is the most advanced but also the most interesting part of the book. For instance, in Chapter 5 we are naturally lead to nontrivial combinatorial optimization problems, along with the method of simulated annealing. Chapter 7 deals with mesh generation, where the problem is quite different: instead of finding the best triangulation for a given set of nodes, we need now to introduce nodes to make the triangles smaller and smaller (refinement), but without loosing the Delaunay property. This is a very important procedure in CAD, visualization and FEM. Obviously, the data structure plays here an important role.

One of the advantages of having built a triangulation is that we can easily approximate scattered data on this domain using splines. The most widely used are linear splines, where the surface over each triangle is a plane (higher order splines, such as Clough-Tocher, are only mentioned in Chapter 1). The requirements we have imposed on our triangulations assure at least continuity of our approximant. With a few nodes we can interpolate, but when the amount of data is huge, the best option is the least squares approximation, as discussed in Chapter 8. Beside the simplest setting, more subtle aspects such as weighted or constrained least squares are also considered.

Finally, in Chapter 9 we deal again with implementation issues, developing a generic triangulation library that the authors call "triangulation template library" or TTL. It can be freely downloaded from the Internet site http://www.simula.no/ogl/ttl.

The book is profusely and very well illustrated, which in many cases helps to understand and visualise a theorem or an algorithm. Each chapter ends with a few (typically, no more than 10) exercises without solutions, of both theoretical and computational character. I have used part of this book teaching a graduate course of scattered data approximation for hydro-geological engineers, and found it very useful.

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Proceedings

Approximation Theory XII: San Antonio 2007, *Marian Neamtu, and Larry L. Schumaker, eds.* Nashboro Press (Brentwood TN), 2008, $100, ISBN 978-0-9728482-9-9.

These are the proceedings of the Twelfth International Conference "Approximation Theory", held on March 4–8, 2007 in San Antonio, Texas.

Algorithms for Approximation. Proceedings of the 5th International Conference, Chester, July 2005, *A. Iske, and J. Levesley, eds.* Springer Verlag 2007. 79.95 €. xiv + 389 pp. ISBN 978-3-540-33283-1.

This is a collection of 30 papers on relevant aspects of approximation theory and applications, bringing together modern methods from statistics, mathematical modelling and numerical simulation. The content is split into 6 parts: imaging and data mining, numerical simulation, statistical approximation methods, data fitting and modelling, differential and integral equations, and special functions and approximation on manifolds. Applications include pattern recognition, machine learning, multiscale modelling of fluid flow, metrology, geometric modelling, tomography, signal and image processing.

The book is nicely edited, as we usually expect it from Springer. This collection of papers is aimed at graduate students and researchers in science and engineering who wish to understand and develop numerical algorithms for the solution of their specific problems.

Foundations of Computational Mathematics, Santander 2005, *L.M. Pardo, A. Pinkus, E. Süli, and M.J. Todd, eds.* Cambridge University Press, 2006, ISBN: 9780521681612.

DESCRIPTION: This volume is a collection of articles based on the plenary talks presented at the 2005 meeting in Santander of the Society for the Foundations of Computational Mathematics. The talks were given by some of the foremost world authorities in computational mathematics. The topics covered reflect the breadth of research within the area as well as the richness and fertility of interactions between seemingly unrelated branches of pure and applied mathematics. As a result this volume will be of interest to researchers in the field of computational mathematics and also to nonexperts who wish to gain some insight into the state of the art in this active and significant field.

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