J. Differential Equations 252 (2012) 2339–2369

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Periodic Riemannian manifold with preassigned gaps in spectrum of Laplace–Beltrami operator

Andrii Khrabustovskyi

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, Lenin avenue 47, Kharkiv 61103, Ukraine

ARTICLE INFO

Article history:
Received 12 May 2011
Revised 28 September 2011
Available online 24 October 2011

Keywords:
Periodic manifold
Laplace–Beltrami operator
Spectrum
Gaps
Homogenization

ABSTRACT

It is known (E.L. Green (1997), O. Post (2003)) that for an arbitrary $m \in \mathbb{N}$ one can construct a periodic non-compact Riemannian manifold M with at least m gaps in the spectrum of the corresponding Laplace–Beltrami operator $-\Delta_M$. In this work we want not only to produce a new type of periodic manifolds with spectral gaps but also to control the edges of these gaps. The main result of the paper is as follows: for arbitrary pairwise disjoint intervals $(\alpha_j, \beta_j) \subset [0, \infty)$, $j = 1, \dots, m$ $(m \in \mathbb{N})$, for an arbitrarily small $\delta > 0$ and for an arbitrarily large $L > 0$ we construct a periodic non-compact Riemannian manifold M with at least m gaps in the spectrum of the operator $-\Delta_M$, moreover the edges of the first m gaps belong to δ-neighbourhoods of the edges of the intervals (α_j, β_j), while the remaining gaps (if any) are located outside the interval $[0, L]$.

0. Introduction

In this paper we deal with non-compact periodic manifolds. The n-dimensional Riemannian manifold M is called *periodic* if there is a discrete finitely generated abelian group Γ acting isometrically, properly discontinuously and co-compactly on M. Roughly speaking M is glued from countably many copies of some compact manifold $\mathbf{M}$ (period cell) and each $\gamma \in \Gamma$ maps $\mathbf{M}$ to one of these copies.

Let M be an n-dimensional periodic Riemannian manifold. We denote by $-\Delta_M$ the Laplace–Beltrami operator on M. It is known (see e.g. [23]) that the spectrum $\sigma(-\Delta_M)$ of the operator $-\Delta_M$ has band-gap structure, that is

E-mail address: andry9@ukr.net.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.10.011

$$\sigma(-\Delta_M) = \bigcup_{k=1}^{\infty} \mathcal{J}_k(M), \tag{0.1}$$

where $\mathcal{J}_k(M) = [a_k, b_k] \subset [0, \infty)$ are compact intervals called *bands*, $a_k, b_k \underset{k\to\infty}{\nearrow} \infty$, $a_1 = 0$. In general the bands may overlap. The open interval (α, β) is called a *gap* if $(\alpha, \beta) \cap \sigma(-\Delta_M) = \varnothing$ and $\alpha, \beta \in \sigma(-\Delta_M)$.

The existence of gaps in the spectrum is not guaranteed: for instance the spectrum of the operator $-\Delta_{\mathbb{R}^n} = -\sum_{j=1}^n \partial^2/\partial x_j^2$ in $\mathbb{R}^n$ coincides with $[0, \infty)$. It is easy to see (cf. [6]) that in 1-dimensional case any periodic Laplace–Beltrami operator has no gaps. However in the case $n \geqslant 2$ we have essentially another situation. Namely, E.B. Davies and E.M. Harrell II [6] considered the manifold $M = \mathbb{R}^n$ $(n \geqslant 2)$ with a periodic conformally flat metric $g_{ij} = a\delta_{ij}$, where $a = a(x)$ is a periodic strictly positive smooth function. The authors proved that $a(x)$ can be chosen in such a way that at least one gap in the spectrum of the operator $-\Delta_M$ exists.

Further, E.L. Green [12] for any $m \in \mathbb{N}$ constructed a periodic conformally flat metric in $\mathbb{R}^2$ such that the corresponding Laplace–Beltrami operator has at least m gaps in the spectrum.

Manifolds of another type were studied by O. Post in [24], where the author considered two different constructions: first, he constructed a periodic manifold M^ε ($\varepsilon > 0$ is a small parameter) starting from countably many copies of a fixed compact manifold connected by small cylinders (the parameter ε characterizes a size of the cylinders), in the second construction he started from a periodic manifold which further is conformally deformed (the parameter ε characterizes sizes of domains where the metric is deformed). For any $m \in \mathbb{N}$ the existence of m gaps is proved for ε small enough. These results were generalized by F. Lledo and O. Post [21] to the case of periodic manifolds with non-abelian group Γ.

Also P. Exner and O. Post [7] proved the existence of gaps for some graph-like manifolds, i.e. the manifolds which shrink with respect to an appropriate parameter to a graph.

We remark that a similar problem (i.e. the existence of gaps in the spectrum) was studied in [8,11, 14,30] for periodic divergence type elliptic operators in $\mathbb{R}^n$, in [13] for periodic magnetic Schrödinger operator, and in [9,10] for periodic Maxwell operator. In these works the gaps in the spectrum are the consequence of a high contrast in the coefficients. We refer to the overview [15] where these and other related questions are discussed in detail.

In the present work we want not only to construct a new type of periodic Riemannian manifolds with gaps in the spectrum of the Laplace–Beltrami operator but also be able *to control the edges of these gaps*. Namely the goal of the work is to solve the following problem: for an arbitrary finite set of pairwise disjoint finite intervals on the positive semi-axis to construct a periodic Riemannian manifolds M with at least m gaps in the spectrum of $-\Delta_M$ (here m is the number of the preassigned intervals), moreover the first m gaps have to be "close" to the preassigned intervals, and the remaining gaps (if any) have to be "close" to infinity.

Let us formulate the main result of the paper.

Theorem 0.1 *(Main theorem). Let $(\alpha_j, \beta_j) \subset [0, \infty)$ $(j = 1, \ldots, m, m \in \mathbb{N})$ be arbitrary pairwise disjoint finite intervals. Let $\delta > 0$ be an arbitrarily small number, $L > 0$ be an arbitrarily large number. Let $n \in \mathbb{N} \setminus \{1\}$.*

Then there exists an n-dimensional periodic Riemannian manifold M, which can be constructed in the explicit form, such that

$$\sigma(-\Delta_M) = [0, \infty) \setminus \left(\bigcup_{j=1}^{m'} \left(\alpha_j^\delta, \beta_j^\delta\right) \right), \quad m \leqslant m' \leqslant \infty, \tag{0.2}$$

where $(\alpha_j^\delta, \beta_j^\delta) \subset [0, \infty)$ are pairwise disjoint finite intervals satisfying

$$\begin{aligned} &\left|\alpha_j^\delta - \alpha_j\right| + \left|\beta_j^\delta - \beta_j\right| < \delta, \quad j = 1, \ldots, m, \\ &\left(\alpha_j^\delta, \beta_j^\delta\right) \subset (L, \infty), \quad j = m+1, \ldots, m'. \end{aligned} \tag{0.3}$$

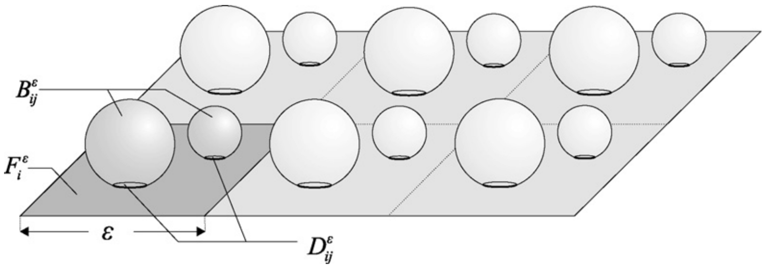

Fig. 1. The manifold M^ε $(m = 2)$. The period cell $\mathbf{M}^\varepsilon_i$ is tinted in more dark colour.

Remark 0.1. In 1987 Y. Colin de Verdière obtained the following remarkable result [4]: for arbitrary numbers $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_m$ $(m \in \mathbb{N})$ and $n \in \mathbb{N} \setminus \{1\}$ there exists an n-dimensional *compact* Riemannian manifold M such that the first m eigenvalues of the corresponding Laplace–Beltrami operator $-\Delta_M$ are exactly $\{\lambda_j\}_{j=1}^m$. Our main theorem can be regarded as an analogue of this result for the case of *non-compact periodic* Riemannian manifolds.

Remark 0.2. Obviously it is sufficient to prove Theorem 0.1 only for such intervals (α_j, β_j) that are *nonvoid* and their *closures* are pairwise disjoint and belong to $(0, \infty)$. For definiteness we renumber the intervals in the increasing order, i.e.

$$0 < \alpha_1, \qquad \alpha_j < \beta_j < \alpha_{j+1}, \quad j = \overline{1, m-1}, \qquad \alpha_m < \beta_m < \infty. \tag{0.4}$$

Proving Theorem 0.1 we suppose that the intervals (α_j, β_j) satisfy (0.4).

The idea how to construct the manifold M comes from one of the directions in the theory of homogenization of PDE's (for classical problems of the homogenization theory we refer e.g. to the monographs [22,27,29]). This direction deals with problems of the following type. Let M^ε be a Riemannian manifold depending on a small parameter ε: it consists of one or several copies of some fixed manifold (we call it "basic manifold") with many attached small surfaces whose number tends to infinity as $\varepsilon \to 0$. On M^ε some PDE (heat equation, wave equation, Maxwell equations etc.) is considered. The problem is to describe the behaviour of its solutions as $\varepsilon \to 0$. More exactly the problem is to find the equation on the basic manifold (so-called "homogenized equation") whose solutions approximate the solutions of the pre-limit equation as $\varepsilon \to 0$.

Firstly the problem of this type was studied by L. Boutet de Monvel and E.Ya. Khruslov in [2] where the behaviour of the diffusion equation was investigated. The asymptotic behaviour of the spectrum of the Laplace–Beltrami operator was studied in [5,17–20], in these works only compact manifolds were considered.

Let us describe briefly the construction of the manifold M solving our main problem. We denote by Ω^ε (ε is a small parameter) a non-compact domain which is obtained by removing from $\mathbb{R}^n$ a countable set of pairwise disjoint balls D^ε_{ij} $(i \in \mathbb{Z}^n,\ j = 1, \dots, m)$. It is supposed that $D^\varepsilon_{ij} = D^\varepsilon_{0j} + \varepsilon i$ and $D^\varepsilon_{0j} \subset \square^\varepsilon_0 = \{x \in \mathbb{R}^n\colon\ 0 \leqslant x_\alpha \leqslant \varepsilon,\ \forall\alpha\}$. We denote by d^ε_j the radius of the ball D^ε_{ij}. Let B^ε_{ij} $(i \in \mathbb{Z}^n$, $j = 1, \dots, m)$ be an n-dimensional surface (we call it "bubble") obtained by removing a small segment from the n-dimensional sphere of the radius b^ε_j. Identifying the points of $\partial D^\varepsilon_{ij}$ and $\partial B^\varepsilon_{ij}$ we glue the bubbles B^ε_{ij} $(i \in \mathbb{Z}^n,\ j = 1, \dots, m)$ to the domain Ω^ε and obtain the n-dimensional manifold M^ε:

$$M^\varepsilon = \Omega^\varepsilon \cup \left(\bigcup_{i \in \mathbb{Z}^n} \bigcup_{j=1}^m B^\varepsilon_{ij} \right).$$

The manifold M^ε (for $m = 2$) is presented on Fig. 1.

We equip M^ε with the Riemannian metric g^ε which coincides with the flat Euclidean metric in Ω^ε and coincides with the spherical metric on the bubbles B_{ij}^ε.

The manifold M^ε is periodic, the set

$$\mathbf{M}_i^\varepsilon = F_i^\varepsilon \cup \left(\bigcup_{j=1}^m B_{ij}^\varepsilon\right), \quad \text{where } F_i^\varepsilon = \square_0^\varepsilon \setminus \left(\bigcup_{j=1}^m D_{0j}^\varepsilon\right) + \varepsilon i$$

is a period cell (for any $i \in \mathbb{Z}^n$).

We set $d_j^\varepsilon = d_j \varepsilon^{\frac{n}{n-2}}$ if $n > 2$ and $d_j^\varepsilon = \exp(-\frac{1}{d_j \varepsilon^2})$ if $n = 2$, $b_j^\varepsilon = b_j \varepsilon$. Here d_j, b_j $(j = 1, \ldots, m)$ are some positive constants which will be chosen later.

We prove (see Theorem 2.1) that the spectrum $\sigma(-\Delta_{M^\varepsilon})$ of the operator $-\Delta_{M^\varepsilon}$ has at least m gaps when ε is small enough (i.e. when ε is less than some ε_0). We denote by $(\sigma_j^\varepsilon, \mu_j^\varepsilon)$ $(j = 1, \ldots, m)$ the first m gaps, by $\mathcal{J}^\varepsilon$ we denote the union of the remaining gaps (if any):

$$\sigma(-\Delta_{M^\varepsilon}) = [0, \infty) \setminus \left[\left(\bigcup_{j=1}^m (\sigma_j^\varepsilon, \mu_j^\varepsilon)\right) \cup \mathcal{J}^\varepsilon\right]. \tag{0.5}$$

Then

$$\forall j = 1, \ldots, m: \quad \lim_{\varepsilon \to 0} \sigma_j^\varepsilon = \sigma_j, \qquad \lim_{\varepsilon \to 0} \mu_j^\varepsilon = \mu_j, \tag{0.6}$$

$$\lim_{\varepsilon \to 0} \inf \mathcal{J}^\varepsilon = \infty, \tag{0.7}$$

where the numbers σ_j, μ_j depend in a special way on d_j, b_j and satisfy the conditions

$$0 < \sigma_1, \qquad \sigma_j < \mu_j < \sigma_{j+1}, \quad j = \overline{1, m-1}, \qquad \sigma_m < \mu_m < \infty.$$

The set $[0, \infty) \setminus (\bigcup_{j=1}^m (\sigma_j, \mu_j))$ coincides with the spectrum of some operator $\mathcal{A}$ acting in the Hilbert space $H = L_2(\mathbb{R}^n) \bigoplus_{j=\overline{1,m}} L_2(\mathbb{R}^n, \rho_j\, dx)$, where ρ_j $(j = 1, \ldots, m)$ are some positive constant weights, by dx we denote the density of the Lebesgue measure.

Remark 0.3. In the case when Ω^ε is obtained by removing a system of balls from some *compact* domain Ω and $m = 1$ (i.e. the removed balls are equivalent, the attached bubbles are also equivalent) the behaviour of the spectrum of the Laplace–Beltrami operator with Dirichlet boundary conditions on $\partial M^\varepsilon = \partial\Omega$ was studied in [17], also it was studied in [19] for another size of the removed balls, namely $\varepsilon^{\frac{n}{n-2}} \ll d_j^\varepsilon \ll \varepsilon$ if $n > 2$ and $\exp(-\frac{1}{a\varepsilon^2}) \ll d_j^\varepsilon \ll \varepsilon$ $(\forall a > 0)$ if $n = 2$. The same manifolds were also considered in [3] where the behaviour of attractors for semi-linear parabolic equations was investigated.

It was proved in [17] that the spectrum of the operator $-\Delta_{M^\varepsilon}^D$ (here ^{D} means the Dirichlet boundary conditions) converges in the Hausdorff sense (see the definition at the beginning of Section 3) to the spectrum of some self-adjoint operator $\mathcal{A}^D$ acting in the space $L_2(\Omega) \oplus L_2(\Omega, \rho\, dx)$, where $\rho > 0$ is some constant weight. The spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}^D$ has the form

$$\sigma(\mathcal{A}^D) = \{\sigma\} \cup \{\lambda_k^{D,-}: k = 1, 2, 3, \ldots\} \cup \{\lambda_k^{D,+}: k = 1, 2, 3, \ldots\},$$

where $\sigma > 0$ is a point of the essential spectrum, the nondecreasing sequences $\lambda_k^{D,-}$, $\lambda_k^{D,+}$ belong to the discrete spectrum, moreover $\lim_{k\to\infty} \lambda_k^{D,-} = \sigma$, $\lim_{k\to\infty} \lambda_k^{D,+} = \infty$ and $\lambda_1^{D,+} > \mu$, where $\mu = \sigma + \sigma\rho$. Thus, $(\sigma, \mu) \cap \sigma(\mathcal{A}^D) = \varnothing$, and, therefore, for an arbitrarily small $\delta > 0$ the interval $(\sigma + \delta, \mu - \delta)$

does not intersect with the spectrum of the operator $-\Delta^D_{M^\varepsilon}$ when $\varepsilon=\varepsilon(\delta)$ is small enough. A similar result is valid for the Neumann Laplacian $-\Delta^N_{M^\varepsilon}$: the spectrum of the corresponding limit operator $\mathcal{A}^N$ consists of the point σ and two nondecreasing sequences $\lambda_k^{N,-}$, $\lambda_k^{N,+}$ such that $\lim_{k\to\infty}\lambda_k^{N,-}=\sigma$, $\lim_{k\to\infty}\lambda_k^{N,+}=\infty$. Moreover $\lambda_1^{N,+}=\mu$. It is important that σ, ρ are independent of the shape of the domain Ω and the type of the boundary conditions. These facts suggest that in the case $\Omega=\mathbb{R}^n$ the spectrum $\sigma(-\Delta_{M^\varepsilon})$ has a gap when ε is small enough, and this gap is close to the interval (σ,μ).

The proof of Theorem 2.1 consists of three steps. Firstly we prove that the set $[0,\infty)\setminus(\bigcup_{j=1}^m(\sigma_j,\mu_j))$ coincides with the spectrum $\sigma(\mathcal{A})$ of the operator $\mathcal{A}$. Then we make the main step: we show that for an arbitrary $L\notin\bigcup_{j=1}^m\{\mu_j\}$ the set $\sigma(-\Delta_{M^\varepsilon})\cap[0,L]$ converges in the Hausdorff sense to the set $\sigma(\mathcal{A})\cap[0,L]$ as $\varepsilon\to0$. Finally, we prove that within an arbitrary finite interval $[0,L]$ the spectrum $\sigma(-\Delta_{M^\varepsilon})$ has at most m gaps when ε is small enough. Together with the Hausdorff convergence this fact will imply the properties (0.5)–(0.7) (see Proposition 3.1 at the beginning of Section 3).

We note that the metric g^ε is continuous but piecewise-smooth. However one can approximate it by a smooth metric $g^{\varepsilon\rho}$ that differs from g^ε only in a small ρ-neighbourhoods of $\partial B^\varepsilon_{ij}$. Moreover when $\rho=\rho(\varepsilon)$ is sufficiently small then the spectra of the operator $-\Delta_{(M^\varepsilon,g^{\varepsilon\rho})}$ and the operator $-\Delta_{M^\varepsilon}$ have the same limit as $\varepsilon\to0$ (here $-\Delta_{(M^\varepsilon,g^{\varepsilon\rho})}$ is the Laplace–Beltrami operator on M^ε equipped with the metric $g^{\varepsilon\rho}$). For precise statement see Remark 4.2 at the end of the paper.

In order to omit cumbersome calculations further we will work with the metric g^ε.

Now, let $\delta>0$ be arbitrarily small number, $L>0$ be arbitrarily large number. It follows from Theorem 2.1 that there is such small $\varepsilon=\varepsilon(\delta,L)$ that the structure of the spectrum $\sigma(-\Delta_{M^\varepsilon})$ is as follows: $\sigma(-\Delta_{M^\varepsilon})$ has m gaps whose edges are located in δ-neighbourhoods of the edges of some fixed intervals (σ_j,μ_j) $(j=1,\dots,m)$ while the remaining gaps (if any) belong to (L,∞). So we set $M=M^\varepsilon$, $\varepsilon=\varepsilon(\delta,L)$. In order to continue the proof of Theorem 0.1 we have to prove that for arbitrary preassigned intervals (α_j,β_j) satisfying (0.4) it is possible to choose such d_j, b_j that

$$\sigma_j=\alpha_j,\qquad \mu_j=\beta_j,\quad j=\overline{1,m}. \tag{0.8}$$

We will prove this fact and present the exact formulae for the constants d_j, b_j (see Theorem 4.1).

The paper is organized as follows. In Section 1 we recall some definitions and facts from the spectral theory for the Laplace–Beltrami operator. In Section 2 we construct the manifold M^ε and formulate Theorem 2.1 describing the behaviour of $\sigma(-\Delta_{M^\varepsilon})$ as $\varepsilon\to0$. Theorem 2.1 is proved in Section 3. And, finally, in Section 4 we present the formulae for the parameter d_j, b_j.

1. Theoretical background

In this section we present the definitions and some well-known results related to the Laplace–Beltrami operator and periodic manifolds. For more details on the Laplace–Beltrami operator see e.g. [28], for more details on periodic manifolds we refer to [23].

Let M be an n-dimensional Riemannian manifold with the metric g. By $g_{\alpha\beta}$ we denote the components of g in local coordinates $(x_1,\dots,x_n)$.

As usual we denote by $L_2(M)$ the Hilbert space of square integrable (with respect to Riemannian measure) functions on M. The scalar product and norm are defined by

$$(u,v)_{L_2(M)}=\int\limits_M u\bar v\,dV,\qquad \|u\|_{L_2(M)}=\sqrt{(u,u)_{L_2(M)}},$$

where $dV=\sqrt{\det g}\,dx_1\dots dx_n$ is the density of the Riemannian measure on M.

By $C^\infty(M)$ (resp. $C_0^\infty(M)$) we denote the space of smooth (resp. smooth and compactly supported) functions on M.

If the manifold M (possibly non-compact) has an empty boundary then we define the *Laplace–Beltrami operator* $-\Delta_M$ on M in the following way. By $\bar{\eta}_M[u,v]$ we denote the closure of the sesquilinear form $\eta_M[u,v]$ defined by the formula:

$$\eta_M[u,v] = (\nabla u, \nabla v)_{L_2(M)} \equiv \int_M (\nabla u, \nabla \bar{v})\, dV \tag{1.1}$$

with $\mathrm{dom}(\eta_M) = C_0^\infty(M)$. Here $(\nabla u, \nabla \bar{v})$ is the scalar product of the vectors ∇u and $\nabla \bar{v}$ with respect to the metric g: in local coordinates $(\nabla u, \nabla \bar{v}) = \sum_{\alpha,\beta=1}^n g^{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial \bar{v}}{\partial x_\beta}$, where $g^{\alpha\beta}$ are the components of the tensor inverse to $g_{\alpha\beta}$. The form $\bar{\eta}$ is densely defined, closed and positive (by the way $\mathrm{dom}(\bar{\eta}_M) = H^1(M) \equiv \{u \in L_2(M)\colon\ \nabla u \in L_2(M)\}$). Then there exists the unique self-adjoint and positive operator $-\Delta_M$ associated with the form $\bar{\eta}_M[u,v]$, i.e.

$$(-\Delta_M u, v)_{L_2(M)} = \bar{\eta}_M[u,v] \quad \text{for all } u \in \mathrm{dom}(\Delta_M),\ v \in \mathrm{dom}(\bar{\eta}_M).$$

For a smooth function u the Laplace–Beltrami operator is given in local coordinates by the formula

$$-\Delta_M u = -\sum_{\alpha,\beta=1}^n \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_\alpha} \left(g^{\alpha\beta} \sqrt{\det g} \frac{\partial u}{\partial x_\beta} \right). \tag{1.2}$$

If M is a compact manifold with a piecewise smooth boundary ∂M we define the *Laplace–Beltrami operator with Neumann* (resp. *Dirichlet*) *boundary conditions* $-\Delta_M^N$ (resp. $-\Delta_M^D$) as the operator associated with the sesquilinear form $\bar{\eta}_M^N$ (resp. $\bar{\eta}_M^D$) which is the closure of the form η_M^N (resp. η_M^D) defined by formula (1.1) and by the definitional domain $\mathrm{dom}(\eta_M^N) = C^\infty(M)$ (resp. $\mathrm{dom}(\eta_M^D) = C_0^\infty(M)$).

The spectra of the operators $-\Delta_M^N$ and $-\Delta_M^D$ are purely discrete. We denote by $\{\lambda_k^N(\mathbf{M})\}_{k\in\mathbb{N}}$ (resp. $\{\lambda_k^D(\mathbf{M})\}_{k\in\mathbb{N}}$) the sequence of eigenvalues of $-\Delta_M^N$ (resp. $-\Delta_M^D$) written in the increasing order and repeated according to their multiplicity.

Now we present the concept of periodic Riemannian manifolds.

We say that *the group Γ acts on the manifold M* if there is a map $\Gamma \times M \to M$ (denoted $(\gamma, x) \mapsto \gamma \cdot x$) such that $\forall \gamma_1, \gamma_2 \in \Gamma$, $\forall x \in M$ one has $(\gamma_1 * \gamma_2) \cdot x = \gamma_1 \cdot (\gamma_2 \cdot x)$, where $*$ is the group operation, and $\forall x \in M$ one has $id \cdot x = x$, where id is the identity element of Γ.

The Riemannian manifold M is called *periodic* (or more precisely *Γ-periodic*) if a discrete finitely generated abelian group Γ acts on M, moreover

- Γ acts *isometrically* on M, i.e. $\forall \gamma \in \Gamma$: $\gamma\cdot$ is the isometrical map,
- Γ acts *properly discontinuously* on M, i.e. for each $x \in M$ there exists a neighbourhood U_x such that the sets $\gamma \cdot U_x$ $(\gamma \in \Gamma)$ are pairwise disjoint,
- Γ acts *co-compactly* on M, i.e. the quotient space M/Γ is compact.

A compact subset $\mathbf{M} \subset M$ is called a *period sell* if $\bigcup_{\gamma\in\Gamma} \gamma \cdot \mathbf{M} = M$ and $\mathbf{M}$ is a closure of an open connected domain $\mathbf{D}$ such that $\forall \gamma \in \Gamma$, $\gamma \neq id$: $\mathbf{D} \cap \gamma \cdot \mathbf{D} = \varnothing$.

For convenience throughout our work we will use the same notation γ for the element $\gamma \in \Gamma$ and the corresponding map $\gamma\cdot : M \to M$.

By $\hat{\Gamma}$ we denote the dual group of Γ, i.e. the group of homomorphism from Γ into $\mathbb{S}^1$. We remark that if Γ is isomorphic to $\mathbb{Z}^n$ (as for the manifold M^ε, which will be considered in the next section) then $\hat{\Gamma}$ is isomorphic to the n-dimensional torus $\mathbb{T}^n = \{\theta = (\theta_1, \ldots, \theta_r) \in \mathbb{C}^n\colon\ \forall \alpha,\ |\theta_\alpha| = 1\}$.

Let $\theta \in \hat{\Gamma}$. We define *the Laplace–Beltrami operator with θ-periodic boundary conditions* $-\Delta_{\mathbf{M}}^\theta$ in the following way. By $C_\theta^\infty(\mathbf{M})$ we denote the space of functions $u \in C^\infty(\mathbf{M})$ satisfying

$$u(\gamma x) = \overline{\theta(\gamma)} u(x)$$

for each $x \in \partial\mathbf{M}$ and for each $\gamma \in \Gamma$ such that $\gamma x \in \partial\mathbf{M}$. Then we define the operator $-\Delta^\theta_{\mathbf{M}}$ as the operator associated with the form $\bar\eta^\theta_{\mathbf{M}}$ which is the closure of the form $\eta^\theta_{\mathbf{M}}$ defined by formula (1.1) (with $\mathbf{M}$ instead of M) and by the definitional domain $\operatorname{dom}(\eta^\theta_{\mathbf{M}}) = C^\infty_\theta(\mathbf{M})$.

The operator $-\Delta^\theta_{\mathbf{M}}$ has purely discrete spectrum. We denote by $\{\lambda^\theta_k(\mathbf{M})\}_{k\in\mathbb{N}}$ the sequence of eigenvalues of $-\Delta^\theta_{\mathbf{M}}$ written in the increasing order and repeated according to their multiplicity.

For any $\theta \in \hat\Gamma$ the following inequality holds:

$$\lambda^N_k(\mathbf{M}) \leqslant \lambda^\theta_k(\mathbf{M}) \leqslant \lambda^D_k(\mathbf{M}). \tag{1.3}$$

It turns out that analysis of the spectrum $\sigma(-\Delta_M)$ of the operator $-\Delta_M$ on the periodic manifold M can be reduced to analysis of the spectra $\sigma(-\Delta^\theta_{\mathbf{M}})$ of the operators $\sigma(-\Delta^\theta_{\mathbf{M}})$, $\theta \in \hat\Gamma$. Namely one has the following fundamental result.

Theorem. *Let M be Γ-periodic manifold with a period cell $\mathbf{M}$. Then*

$$\sigma(-\Delta_M) = \bigcup_{k\in\mathbb{N}} \mathcal{J}_k(M), \tag{1.4}$$

where $\mathcal{J}_k(M) = \{\lambda^\theta_k(\mathbf{M})\colon \theta \in \hat\Gamma\}$, $k \in \mathbb{N}$ are compact intervals.

2. Construction of the manifold

In this section we construct the manifold M^ε and describe the behaviour of the spectrum $\sigma(-\Delta_{M^\varepsilon})$ of the Laplace–Beltrami operator $-\Delta_{M^\varepsilon}$ as $\varepsilon \to 0$.

Let $\{D^\varepsilon_{ij}\colon i \in \mathbb{Z}^n,\ j = 1,\dots,m\}$ be the system of pairwise disjoint balls in $\mathbb{R}^n$ ($n \geqslant 2$) depending on small parameter $\varepsilon > 0$. We suppose that:

(1) the balls D^ε_{0j}, $j = 1,\dots,m$ belong to the cube $\square^\varepsilon_0 = \{x \in \mathbb{R}^n\colon\ 0 \leqslant x_\alpha \leqslant \varepsilon,\ \forall\alpha\}$;
(2) $\forall j = 1,\dots,m$: $\kappa\varepsilon \leqslant \operatorname{dist}(D^\varepsilon_{0j}, \partial\square^\varepsilon_0 \cup (\bigcup_{l\neq j} D^\varepsilon_l))$, where the constant $\kappa > 0$ is independent of ε;
(3) $\forall i \in \mathbb{Z}^n$, $\forall j = 1,\dots,m$: $D^\varepsilon_{ij} = D^\varepsilon_{0j} + \varepsilon i$.

By x^ε_{ij} we denote the centre of D^ε_{ij}, by d^ε_j we denote the radius of D^ε_{ij} (the third condition above implies that the radius of D^ε_{ij} depends only on the index j).

We denote by B^ε_{ij} the truncated n-dimensional sphere (we call it "bubble") of the radius b^ε_j:

$$B^\varepsilon_{ij} = \bigl\{(\theta_1,\theta_2,\dots,\theta_n)\colon\ \theta_1 \in [0,2\pi),\ \theta_k \in [0,\pi),\ k = 2,\dots,n-1,\ \theta_n \in \bigl[\Theta^\varepsilon_j,\pi\bigr]\bigr\}.$$

Here $\Theta^\varepsilon_j = \arcsin(\frac{d^\varepsilon_j}{b^\varepsilon_j})$, where b^ε_j ($j = 1,\dots,m$) are positive numbers satisfying $b^\varepsilon_j > d^\varepsilon_j$.

Let us introduce in Ω^ε the spherical coordinates $(\theta_1,\dots,\theta_n,r)$ with the origin at x^ε_{ij}. Here r is the distance to x^ε_{ij}. Identifying the points $(\theta_1,\dots,\theta_{n-1},d^\varepsilon_j) \in \partial D^\varepsilon_{ij}$ and $(\theta_1,\dots,\theta_{n-1},\Theta^\varepsilon_j) \in \partial B^\varepsilon_{ij}$ we glue the bubbles B^ε_{ij} to the perforated domain Ω^ε and obtain an n-dimensional manifold M^ε:

$$M^\varepsilon = \Omega^\varepsilon \cup \left(\bigcup_{i\in\mathbb{R}^n}\bigcup_{j=1}^m B^\varepsilon_{ij}\right). \tag{2.1}$$

The manifold M^ε is presented on Fig. 1. By $\tilde x$ we denote the points of M^ε. If the point $\tilde x$ belongs to Ω^ε sometimes we will write x instead of $\tilde x$ having in mind a corresponding point in $\mathbb{R}^n$.

Clearly M^ε can be covered by a system of charts and suitable local coordinates $(x_1,\dots,x_n)\mapsto \tilde{x}\in M^\varepsilon$ can be introduced. In particular in a small neighbourhood of $\partial B_{ij}^\varepsilon$ we introduce them in the following way (below by U_{ij}^ε we denote this neighbourhood):

$$x_k=\theta_k,\quad k=1,\dots,n-1,$$
$$x_n=\begin{cases} r-d_j^\varepsilon, & \tilde{x}=(\theta_1,\dots,\theta_{n-1},r)\in\Omega^\varepsilon\cap U_{ij}^\varepsilon,\\ -b_j^\varepsilon(\theta_n-\Theta_j^\varepsilon), & \tilde{x}=(\theta_1,\dots,\theta_{n-1},\theta_n)\in B_{ij}^\varepsilon\cap U_{ij}^\varepsilon\end{cases} \tag{2.2}$$

(that is $\partial B_{ij}^\varepsilon=\{(x_1,\dots,x_n)\colon\ x_n=0\}$).

We equip M^ε with the Riemannian metric g^ε that coincides with the flat Euclidean metric on Ω^ε and coincides with the spherical metric on the bubbles B_{ij}^ε. This last means that in the spherical coordinates $(\theta_1,\dots,\theta_n)$ the components $g_{\alpha\beta}^\varepsilon$ of the metric g^ε have the form

$$g_{\alpha\beta}^\varepsilon=\delta_{\alpha\beta}\big(b_j^\varepsilon\big)^2\prod_{k=\alpha+1}^{n}\sin^2\theta_k,\quad \alpha,\beta=\overline{1,n}$$

(for $\alpha=n$ we set $\prod_{k=\alpha+1}^{n}\sin^2\theta_k:=1$). Here $\delta_{\alpha\beta}$ is the Kronecker delta.

The metric g^ε is continuous and piecewise smooth: in the coordinates $(x_1,\dots,x_n)$, which are introduced above in the neighbourhood of $\partial B_{ij}^\varepsilon$ by formulae (2.2), the components $g_{\alpha\beta}^\varepsilon=g_{\alpha\beta}^\varepsilon(x_1,\dots,x_n)$ of the metric g^ε have the form:

$$g_{\alpha\beta}^\varepsilon=\begin{cases} g_{+\alpha\beta}^\varepsilon, & x_n\geqslant 0,\\ g_{-\alpha\beta}^\varepsilon, & x_n<0,\end{cases}\quad \alpha,\beta=\overline{1,n-1},\qquad g_{n\beta}=\delta_{n\beta} \tag{2.3}$$

where

$$g_{+\alpha\beta}^\varepsilon=\delta_{\alpha\beta}\big(x_n+d_j^\varepsilon\big)^2\prod_{k=\alpha+1}^{n-1}\sin^2\theta_k,$$
$$g_{-\alpha\beta}^\varepsilon=\delta_{\alpha\beta}\big(b_j^\varepsilon\big)^2\sin^2\left(\frac{|x_n|}{b_j^\varepsilon}+\Theta_j^\varepsilon\right)\prod_{k=\alpha+1}^{n-1}\sin^2\theta_k. \tag{2.4}$$

It is clear that as $x_n=0$ (i.e. on $\partial B_{ij}^\varepsilon$) the coefficients $g_{\alpha\beta}^\varepsilon$ lose smoothness.

Remark that g^ε can be approximated by a smooth metric $g^{\varepsilon\rho}$ that differs from g^ε only in a small ρ-neighbourhood of $\partial B_{ij}^\varepsilon$, moreover when $\rho=\rho(\varepsilon)$ is sufficiently small then the spectra $\sigma(-\Delta_{M^\varepsilon})$ and $\sigma(-\Delta_{(M^\varepsilon,g^{\varepsilon\rho})})$ have the same limit as $\varepsilon\to 0$ (for more precise statement see Remark 4.2). However in order to omit cumbersome calculations further we will work with the metric g^ε.

Remark 2.1. It is easy to see that the manifold M^ε can be immersed into the space $\mathbb{R}^{n+1}$ via the following map $\widehat{F}^\varepsilon: M^\varepsilon\to\widehat{M}^\varepsilon\subset\mathbb{R}^{n+1}$ (below $x\in\mathbb{R}^n$, $z\in\mathbb{R}$, $(x,z)\in\mathbb{R}^{n+1}$):

- if $\tilde{x}=x\in\Omega^\varepsilon$ then $\widehat{F}^\varepsilon(\tilde{x})=(x,0)$,
- if $\tilde{x}=(\theta_1,\dots,\theta_n)\in B_{ij}^\varepsilon$ then $\widehat{F}^\varepsilon(\tilde{x})=(x_1,\dots,x_n,z)$, where

$$x_1=\big(x_{ij}^\varepsilon\big)_1+b_j^\varepsilon\prod_{l=1}^{n}\sin\theta_l,\qquad x_k=\big(x_{ij}^\varepsilon\big)_k+b_j^\varepsilon\cos\theta_{k-1}\prod_{l=k}^{n}\sin\theta_l\quad (k=\overline{2,n}),$$
$$z=b_j^\varepsilon\big(\cos\Theta_j^\varepsilon-\cos\theta_n\big).$$

Note: one should not confuse $(x_1,\dots,x_n)$ with the local coordinates introduced above in a neighbourhood of $\partial B^\varepsilon_{ij}$.

Thus, $\widehat{F}^\varepsilon$ maps B^ε_{ij} onto the surface $\widehat{B}^\varepsilon_{ij}$ which is obtained by removing from the sphere $\widehat{\mathcal{B}}^\varepsilon_{ij} = \{(x,z)\in\mathbb{R}^{n+1}\colon\ |x-x^\varepsilon_{ij}|^2+(z-b^\varepsilon_j\cos\Theta^\varepsilon_j)^2=(b^\varepsilon_j)^2\}$ the segment $\{(x,z)\in\widehat{\mathcal{B}}^\varepsilon_{ij}\colon\ z<0\}$.

The map $\widehat{F}^\varepsilon$ is a local homeomorphism, i.e. for any $\tilde{x}\in M^\varepsilon$ there is a neighbourhood $U(\tilde{x})\subset M^\varepsilon$ such that $\widehat{F}^\varepsilon|_{U(\tilde{x})}$ is a homeomorphism (and even diffeomorphism if $\tilde{x}\notin\bigcup_{i,j}\partial B^\varepsilon_{ij}$). If the surfaces $\widehat{B}^\varepsilon_{ij}$ ($i\in\mathbb{Z}^n$, $j=1,\dots,m$) are pairwise disjoint (e.g. if $b^\varepsilon_j<d^\varepsilon_j+\kappa\varepsilon/2$) then $\widehat{F}^\varepsilon$ is a global homeomorphism. Furthermore $\widehat{F}^\varepsilon$ is an isometric map: if $\hat{g}^\varepsilon$ is a metric on $\widehat{M}^\varepsilon$ which is generated by the Euclidean metric in $\mathbb{R}^{n+1}$ then g^ε coincides with the pull-back $(\widehat{F}^\varepsilon)^*\hat{g}^\varepsilon$.

Let the group $\Gamma^\varepsilon\cong\mathbb{Z}^n$ act on M^ε by the following rule (below by γ^ε_k, $k\in\mathbb{Z}^n$ we denote the elements of Γ^ε):

- if $\tilde{x}=x\in\Omega^\varepsilon$ then γ^ε_k maps $\tilde{x}$ into the point $\gamma_k\tilde{x}=x+k\varepsilon\in\Omega^\varepsilon$,
- if $\tilde{x}=(\theta_1,\dots,\theta_n)\in B^\varepsilon_{ij}$ then γ^ε_k maps $\tilde{x}$ into the point $\gamma^\varepsilon_k\tilde{x}\in B_{i+k,j}$ with the same angle coordinates $(\theta_1,\dots,\theta_n)$.

Obviously M^ε is Γ^ε-periodic Riemannian manifold. For an arbitrary $i\in\mathbb{Z}^n$ the set

$$\mathbf{M}^\varepsilon_i = F^\varepsilon_i\cup\left(\bigcup_{j=1}^m B^\varepsilon_{ij}\right),\quad \text{where } F^\varepsilon_i=\left\{\tilde{x}\in\Omega^\varepsilon\colon\ x-\varepsilon i\in\square^\varepsilon_0\setminus\left(\bigcup_{j=1}^m D^\varepsilon_{0j}\right)\right\} \tag{2.5}$$

is a period cell.

We assume that the radii of the holes and bubbles are the following:

$$d^\varepsilon_j=\begin{cases} d_j\varepsilon^{\frac{n}{n-2}}, & n>2,\\ \exp(-\frac{1}{d_j\varepsilon^2}), & n=2,\end{cases} \tag{2.6}$$

$$b^\varepsilon_j=b_j\varepsilon \tag{2.7}$$

where d_j, b_j ($j=1,\dots,m$) are some positive constants (we choose them later in Section 4).

We will use the following notations:

$$R^\varepsilon_{ij}=\left\{\tilde{x}\in\Omega^\varepsilon\colon\ d^\varepsilon_j\leqslant\left|x-x^\varepsilon_{ij}\right|<d^\varepsilon_j+\frac{\kappa\varepsilon}{2}\right\},$$
$$G^\varepsilon_{ij}=R^\varepsilon_{ij}\cup B^\varepsilon_{ij},$$
$$S^\varepsilon_{ij}=\left\{\tilde{x}\in\Omega^\varepsilon\colon\ \left|x-x^\varepsilon_{ij}\right|=d^\varepsilon_j+\frac{\kappa\varepsilon}{2}\right\}\equiv\partial G^\varepsilon_{ij},$$

ω_n is the volume of n-dimensional unit sphere.

According to the notations introduced above in Section 1 we denote by $\{\lambda^D_k(G^\varepsilon_{ij})\}_{k\in\mathbb{N}}$ the sequence of the eigenvalues of the operator $-\Delta^D_{G^\varepsilon_{ij}}$ which is the Laplace–Beltrami operator in G^ε_{ij} with Dirichlet boundary conditions on S^ε_{ij}. It is clear that $\{\lambda^D_k(G^\varepsilon_{ij})\}_{k\in\mathbb{N}}$ depends only on the index j.

One can prove (see Lemma 3.2 below) that

$$\forall j=1,\dots,m:\quad \lim_{\varepsilon\to0}\lambda_1\left(G^\varepsilon_{ij}\right)=\sigma_j$$

where

$$\sigma_j = \begin{cases} \frac{d_j}{4b_j^2}, & n=2, \\ \frac{n-2}{2} \cdot \frac{d_j^{n-2}\omega_{n-1}}{b_j^n \omega_n}, & n>2. \end{cases} \tag{2.8}$$

Note that in spite of the fact that the diameter of G_{ij}^ε converges to zero as $\varepsilon \to 0$, $\lambda_1(G_{ij}^\varepsilon)$ does not blow up as $\varepsilon \to 0$. This is due to a weak connection between B_{ij}^ε and R_{ij}^ε.

We assume that the coefficients d_j and b_j are such that $\sigma_i \neq \sigma_j$ if $i \neq j$. For definiteness we suppose that $\sigma_j < \sigma_{j+1}$, $j = 1, \ldots, n-1$.

We introduce the Hilbert space

$$H = L_2(\mathbb{R}^n) \bigoplus_{j=\overline{1,m}} L_2(\mathbb{R}^n, \rho_j\,dx)$$

where by dx we denote the density of the Lebesgue measure, the constant weights ρ_j, $j = 1, \ldots, m$ are defined by the formula

$$\rho_j = (b_j)^n \omega_n. \tag{2.9}$$

Since $\lim_{\varepsilon\to 0}(d_j^\varepsilon / b_j^\varepsilon) = 0$, then $\rho_j = \lim_{\varepsilon \to 0} \varepsilon^{-n} |B_{ij}^\varepsilon|$ (here by $|\cdot|$ we denote the Riemannian volume).

And, finally, let us consider the following equation (with unknown $\lambda \in \mathbb{R}$):

$$\mathcal{F}(\lambda) \equiv 1 + \sum_{j=1}^m \frac{\sigma_j \rho_j}{\sigma_j - \lambda} = 0. \tag{2.10}$$

It is easy to obtain (see the proof of Theorem 2.1) that this equation has exactly m roots μ_j ($j = 1, \ldots, m$), moreover one can renumber them in such a way that

$$\sigma_j < \mu_j < \sigma_{j+1}, \quad j = \overline{1, m-1}, \qquad \sigma_m < \mu_m < \infty.$$

By the way if $m = 1$ then $\mu_1 = \sigma_1 + \sigma_1 \rho_1$ (cf. Remark 0.3).

Now we are able to formulate the theorem describing the behaviour of $\sigma(-\Delta_{M^\varepsilon})$.

Theorem 2.1. *The spectrum $\sigma(-\Delta_{M^\varepsilon})$ of the operator $-\Delta_{M^\varepsilon}$ has the following structure when ε is small enough (i.e. when $\varepsilon < \varepsilon_0$):*

$$\sigma(-\Delta_{M^\varepsilon}) = [0, \infty) \setminus \left[\left(\bigcup_{j=1}^m (\sigma_j^\varepsilon, \mu_j^\varepsilon)\right) \cup \mathcal{J}^\varepsilon\right]. \tag{2.11}$$

Here $\mathcal{J}^\varepsilon$ is a union of some open finite intervals (possibly $\mathcal{J}^\varepsilon = \varnothing$) and

$$0 < \sigma_1^\varepsilon, \qquad \sigma_j^\varepsilon < \mu_j^\varepsilon < \sigma_{j+1}^\varepsilon, \quad j = \overline{1, m-1}, \qquad \sigma_m^\varepsilon < \mu_m^\varepsilon < \inf \mathcal{J}^\varepsilon.$$

Moreover

$$\forall j = 1, \ldots, m: \quad \lim_{\varepsilon \to 0} \sigma_j^\varepsilon = \sigma_j, \qquad \lim_{\varepsilon \to 0} \mu_j^\varepsilon = \mu_j, \tag{2.12}$$

$$\lim_{\varepsilon \to 0} \inf \mathcal{J}^\varepsilon = \infty. \tag{2.13}$$

The set $[0,\infty)\setminus(\bigcup_{j=1}^m(\sigma_j,\mu_j))$ *coincides with the spectrum* $\sigma(\mathcal{A})$ *of the self-adjoint operator* $\mathcal{A}$ *which acts in* H *and is defined by the formulae*

$$\mathcal{A}U=\begin{pmatrix}-\Delta_{\mathbb{R}^n}u+\sum_{j=1}^m\sigma_j\rho_j(u-u_j)\\ \sigma_1(u_1-u)\\ \sigma_2(u_2-u)\\ \cdots\\ \sigma_m(u_m-u)\end{pmatrix},\qquad U=\begin{pmatrix}u\\ u_1\\ u_2\\ \cdots\\ u_m\end{pmatrix}\in\operatorname{dom}(\mathcal{A}),\tag{2.14}$$

$$\operatorname{dom}(\mathcal{A})=\operatorname{dom}(\Delta_{\mathbb{R}^n})\bigoplus_{j=\overline{1,m}}L_2\big(\mathbb{R}^n,\rho_j\,dx\big).\tag{2.15}$$

We prove this theorem in the next section. In the last section we present the formulae for d_j, b_j which will ensure the fulfilment of the equalities (0.8).

3. Proof of Theorem 2.1

Before we prove the result in full detail we will sketch the main ideas of the proof.

At first (Section 3.1) we prove the equality

$$\sigma(\mathcal{A})=[0,\infty)\setminus\left(\bigcup_{j=1}^m(\sigma_j,\mu_j)\right).\tag{3.1}$$

In the main part of the proof (Sections 3.2–3.3) we show that

for an arbitrary $L>0$, $L\notin\bigcup_{j=1}^m\{\mu_j\}$ *the set* $\sigma(-\Delta_{M^\varepsilon})\cap[0,L]$ *converges in the Hausdorff sense to the set* $\sigma(\mathcal{A})\cap[0,L]$ *as* $\varepsilon\to 0$.

Let us recall the definition of Hausdorff convergence.

Definition 3.1. The set $\mathcal{B}^\varepsilon\subset\mathbb{R}$ converges in the Hausdorff sense to the set $\mathcal{B}\subset\mathbb{R}$ as $\varepsilon\to 0$ if the following conditions (A) and (B) hold:

$$\text{if } \lambda^\varepsilon\in\mathcal{B}^\varepsilon \text{ and } \lim_{\varepsilon\to 0}\lambda^\varepsilon=\lambda \text{ then } \lambda\in\mathcal{B},\tag{A}$$

$$\text{for any } \lambda\in\mathcal{B} \text{ there exists } \lambda^\varepsilon\in\mathcal{B}^\varepsilon \text{ such that } \lim_{\varepsilon\to 0}\lambda^\varepsilon=\lambda.\tag{B}$$

Property (A) is verified in Section 3.2, property (B) is verified in Section 3.3.

In the last part of the proof (Section 3.4) we show that within an arbitrary finite interval $[0,L]$ the spectrum $\sigma(-\Delta_{M^\varepsilon})$ has at most m gaps when ε is small enough (i.e. when $\varepsilon<\varepsilon_0$). This fact and the Hausdorff convergence of $\sigma(-\Delta_{M^\varepsilon})\cap[0,L]$ to $\sigma(\mathcal{A})\cap[0,L]=[0,L]\setminus(\bigcup_{j=1}^m(\sigma_j,\mu_j))$ imply the properties (2.11)–(2.13). Indeed one can easily prove the following simple proposition.

Proposition 3.1. *Let* $\mathcal{B}^\varepsilon=[0,L]\setminus(\bigcup_{j=1}^{m^\varepsilon}(\alpha_j^\varepsilon,\beta_j^\varepsilon))$, $\mathcal{B}=[0,L]\setminus(\bigcup_{j=1}^m(\alpha_j,\beta_j))$, *where* $L<\infty$ *and*

$$0\leqslant\alpha_1^\varepsilon,\qquad \alpha_j^\varepsilon<\beta_j^\varepsilon\leqslant\alpha_{j+1}^\varepsilon,\quad j=\overline{1,m^\varepsilon-1},\qquad \alpha_{m^\varepsilon}^\varepsilon\leqslant L,$$
$$0<\alpha_1,\qquad \alpha_j<\beta_j<\alpha_{j+1},\quad j=\overline{1,m-1},\qquad \alpha_m<L,$$
$$m^\varepsilon\leqslant m.$$

Suppose that the set $\mathcal{B}^\varepsilon$ *converges to the set* $\mathcal{B}$ *in the Hausdorff sense as* $\varepsilon\to 0$.

Then $m^\varepsilon = m$ when ε becomes small (i.e. when ε is less than some ε_0) and

$$\forall j = 1, \ldots, m: \quad \lim_{\varepsilon \to 0} \alpha_j^\varepsilon = \alpha_j, \qquad \lim_{\varepsilon \to 0} \beta_j^\varepsilon = \beta_j.$$

3.1. Structure of $\sigma(\mathcal{A})$

Let $\lambda \in \mathbb{C} \setminus \bigcup_{j=1}^m \{\sigma_j\}$. Let

$$F = \begin{pmatrix} f \\ f_1 \\ \cdots \\ f_m \end{pmatrix} \in \operatorname{im}(\mathcal{A} - \lambda \mathrm{I}),$$

i.e. there is

$$U = \begin{pmatrix} u \\ u_1 \\ \cdots \\ u_m \end{pmatrix} \in \operatorname{dom}(\mathcal{A})$$

satisfying $\mathcal{A}U - \lambda U = F$. Then $u_j = \frac{\sigma_j u + f_j}{\sigma_j - \lambda}$ and

$$-\Delta_{\mathbb{R}^n} u - \lambda \mathcal{F}(\lambda) u = f + \sum_{j=1}^m \frac{\sigma_j \rho_j f_j}{\sigma_i - \lambda} \tag{3.2}$$

where $\mathcal{F}(\lambda)$ is defined by (2.10). Equality (3.2) implies that

$$\lambda \in \sigma(\mathcal{A}) \setminus \bigcup_{j=1}^m \{\sigma_j\} \quad \Longleftrightarrow \quad \lambda \mathcal{F}(\lambda) \in \sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty). \tag{3.3}$$

At first we study the function $\lambda \mathcal{F}(\lambda)$ on the real line. It is easy to see that $\lambda \mathcal{F}(\lambda)$ is a strictly increasing function on the intervals $(-\infty, \sigma_1)$, (σ_m, ∞), (σ_j, σ_{j+1}), $j = 1, \ldots, m-1$, $\lim_{\lambda \to \pm\infty} \lambda \mathcal{F}(\lambda) = \pm\infty$, $\lim_{\lambda \to \sigma_j \pm 0} \lambda \mathcal{F}(\lambda) = \mp\infty$, furthermore there are the points μ_j, $j = 1, \ldots, m$, such that

$$\mathcal{F}(\mu_j) = 0, \quad j = 1, \ldots, m-1,$$

$$\sigma_j < \mu_j < \sigma_{j+1}, \quad j = 1, \ldots, m, \qquad \sigma_m < \mu_m < \infty,$$

$$\left\{ \lambda \in \mathbb{R} \setminus \bigcup_{j=1}^m \{\sigma_j\} \colon \lambda \mathcal{F}(\lambda) \geqslant 0 \right\} = [0, \sigma_1) \cup \left(\bigcup_{j=1}^{m-1} [\mu_j, \sigma_{j+1}) \right) \cup [\mu_m, \infty).$$

Let us consider the equation $\lambda \mathcal{F}(\lambda) = a$, where $a \in [0, \infty)$. One the one hand it is equivalent to the equation $(\prod_{j=1}^m (\sigma_j - \lambda))^{-1} P_{m+1}(\lambda) = 0$, where P_{m+1} is a polynomial of the degree $m+1$, and, therefore, in $\mathbb{C}$ this equation has at most $m+1$ roots. On the other hand it is easy to see that on $[0, \infty)$ the equation $\lambda \mathcal{F}(\lambda) = a$ has $m+1$ roots (if $a = 0$ then these roots are $0, \mu_1, \ldots, \mu_m$). Hence we obtain that the set $\{\lambda \in \mathbb{C} \colon \lambda \mathcal{F}(\lambda) \geqslant 0\}$ belongs to $[0, \infty)$.

The graph of the function $\lambda \mathcal{F}(\lambda)$, $\lambda \in \mathbb{R}$ is presented on Fig. 2.

Thus, we conclude that $\lambda \in \sigma(\mathcal{A}) \setminus \bigcup_{j=1}^m \{\sigma_j\}$ iff $\lambda \in [0, \sigma_1) \cup (\bigcup_{j=1}^{m-1} [\mu_j, \sigma_{j+1})) \cup [\mu_m, \infty)$. Since the spectrum $\sigma(\mathcal{A})$ is a closed set, then the points σ_j ($j = 1, \ldots, m$) also belong to $\sigma(\mathcal{A})$. Equality (3.1) is proved.

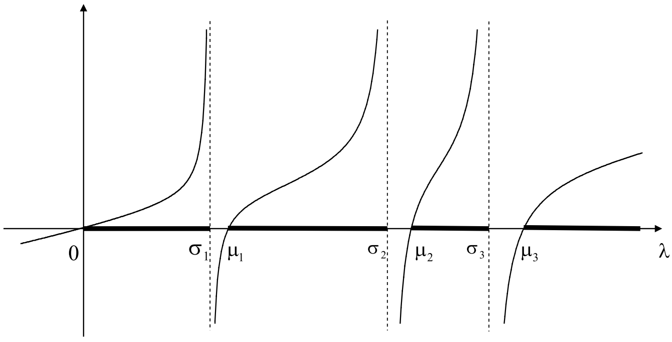

Fig. 2. The graph of the function $\lambda\mathcal{F}(\lambda)$ (for $m=3$). The bold intervals are the components of $\sigma(\mathcal{A})$.

3.2. Property (A) of Hausdorff convergence

We present the proof for the case $n \geqslant 3$ only. For the case $n=2$ the proof is repeated word-by-word with small modifications in some estimates.

Let $\lambda^\varepsilon \in \sigma(-\Delta_{M^\varepsilon}) \cap [0,L]$ and $\lim_{\varepsilon\to 0}\lambda^\varepsilon = \lambda$. Obviously $\lambda \in [0,L]$, thus, we have to prove that λ belongs to $\sigma(\mathcal{A})$. If $\lambda \in \bigcup_{j=1}^m \{\sigma_j\}$ then this statement follows from (3.1). Therefore, we can focus on the case $\lambda \notin \bigcup_{j=1}^m \{\sigma_j\}$.

Let us consider the sequence $\varepsilon_N \subset \varepsilon$, where $\varepsilon_N = \frac{1}{N}$, $N = 1,2,3,\ldots$. For convenience we preserve the same notation ε having in mind the sequence ε_N.

We introduce the following cubes in $\mathbb{R}^n$:

$$\square = \left\{x \in \mathbb{R}^n \colon\ 0 \leqslant x_\alpha \leqslant 1,\ \forall\alpha\right\},$$
$$\square_i^\varepsilon = \left\{x \in \mathbb{R}^n \colon\ \varepsilon i_\alpha \leqslant x_\alpha \leqslant \varepsilon(i_\alpha+1),\ \forall\alpha\right\}, \quad i = (i_1,\ldots,i_n) \in \mathbb{Z}^n.$$

Since $\varepsilon^{-1} \in \mathbb{N}$, then $\square = \bigcup_{i\in\mathcal{I}^\varepsilon} \square_i^\varepsilon$, where

$$\mathcal{I}^\varepsilon = \left\{i \in \mathbb{Z}^n \colon\ 0 \leqslant i_\alpha \leqslant \left(\varepsilon^{-1}-1\right),\ \forall\alpha\right\}.$$

Also we introduce the following set in M^ε:

$$\mathbf{M}^\varepsilon = \bigcup_{i\in\mathcal{I}^\varepsilon} \mathbf{M}_i^\varepsilon$$

where $\mathbf{M}_i^\varepsilon$ is defined by formulae (2.5).

In Section 2 we concluded that M^ε is Γ^ε-periodic manifold, the set $\mathbf{M}_i^\varepsilon$ is a corresponding periodic cell. On the other hand since $\varepsilon^{-1} \in \mathbb{N}$, then M^ε is also Γ-periodic manifold on which the group $\Gamma \cong \mathbb{Z}^n$ acts by the following rule (below by γ_k, $k \in \mathbb{Z}^n$ we denote the elements of Γ):

- if $\tilde{x} = x \in \Omega^\varepsilon$ then γ_k maps $\tilde{x}$ into the point $\gamma_k\tilde{x} = x + k \in \Omega^\varepsilon$,
- if $\tilde{x} = (\theta_1,\ldots,\theta_n) \in B_{ij}^\varepsilon$ then γ_k maps $\tilde{x}$ into the point $\gamma_k\tilde{x} \in B_{i+k\varepsilon^{-1},j}$ with the same angle coordinates $(\theta_1,\ldots,\theta_n)$.

The set $\mathbf{M}^\varepsilon$ is a period cell. The boundary of $\mathbf{M}^\varepsilon$ is independent of ε: $\partial\mathbf{M}^\varepsilon = \{\tilde{x} \in \Omega^\varepsilon \colon\ x \in \partial\square\}$.

Roughly speaking if $\varepsilon^{-1} \in \mathbb{N}$ then M^ε is not only "ε-periodic" manifold but also "1-periodic" manifold. To prove property (A) of the Hausdorff convergence it is more convenient to look at M^ε as

Γ-periodic manifold (and to work with period cell $\mathbf{M}^\varepsilon$) since in this case we are able to utilize some ideas and methods developed in [2,3,17,19,20].

By $\mathbf{M}_\alpha$ ($\alpha = 1,\dots,2n$) we denote the components of $\partial\mathbf{M}^\varepsilon$:

$$\mathbf{M}_\alpha = \left\{\tilde{x} \in \Omega^\varepsilon\colon\ x_\alpha = 0 \text{ and } 0 \leqslant x_\beta \leqslant 1,\ \forall\beta \neq \alpha\right\} \quad \text{if } \alpha = 1,\dots,n,$$
$$\mathbf{M}_\alpha = \left\{\tilde{x} \in \Omega^\varepsilon\colon\ x_{\alpha-n} = 1 \text{ and } 0 \leqslant x_\beta \leqslant 1,\ \forall\beta \neq \alpha - n\right\} \quad \text{if } \alpha = n+1,\dots,2n.$$

The faces $\mathbf{M}_\alpha$ and $\mathbf{M}_{\alpha+n}$ ($\alpha = 1,\dots,n$) are parallel to each other and

$$\gamma_{e_\alpha}\mathbf{M}_\alpha = \mathbf{M}_{\alpha+n}, \quad \alpha = 1,\dots,n, \text{ where } e_\alpha = (0,0,\dots,\underset{\alpha\text{-th place}}{\underset{\uparrow}{1}},\dots,0). \tag{3.4}$$

Also we denote by $\mathbf{M}_\alpha$ the corresponding faces of $\partial\Box$.

Since $\lambda^\varepsilon \in \sigma(-\Delta_{M^\varepsilon})$, then there exists $\theta^\varepsilon \in \hat{\Gamma}$ such that $\lambda^\varepsilon \in \sigma(-\Delta^{\theta^\varepsilon}_{\mathbf{M}^\varepsilon})$. Since Γ is isomorphic to $\mathbb{Z}^n$, then the dual group $\hat{\Gamma}$ is isomorphic to $\mathbb{T}^n = \{\theta = (\theta_1,\dots,\theta_r) \in \mathbb{C}^r\colon\ \forall\alpha,\ |\theta_\alpha| = 1\}$. For convenience hereafter by θ^ε we will understand a corresponding element $(\theta^\varepsilon_1,\dots,\theta^\varepsilon_n) \in \mathbb{T}^n$.

We extract a subsequence (still denoted by ε) such that

$$\theta^\varepsilon \underset{\varepsilon\to 0}{\to} \theta = (\theta_1,\dots,\theta_n) \in \mathbb{T}^n.$$

Let $u^\varepsilon \in \operatorname{dom}(\Delta^{\theta^\varepsilon}_{\mathbf{M}^\varepsilon})$ be the eigenfunction corresponding to λ^ε, i.e. $-\Delta^{\theta^\varepsilon}_{\mathbf{M}^\varepsilon} u^\varepsilon = \lambda^\varepsilon u^\varepsilon$, $u^\varepsilon \neq 0$. We normalize u^ε by the condition $\|u^\varepsilon\|_{L_2(\mathbf{M}^\varepsilon)} = 1$, then $\|\nabla u^\varepsilon\|^2_{L_2(\mathbf{M}^\varepsilon)} = \lambda^\varepsilon$.

In order to describe the behaviour of u^ε as $\varepsilon \to 0$ we need some special operators. From now on by C we denote a generic constant independent of ε.

We denote

$$\Omega^\varepsilon_\Box = \left\{\tilde{x} \in \Omega^\varepsilon\colon\ x \in \Box \setminus \left(\bigcup_{i\in\mathcal{I}^\varepsilon}\bigcup_{j=1}^m D^\varepsilon_{ij}\right)\right\}$$

and introduce an extension operator $\Pi^\varepsilon : H^1(\mathbf{M}^\varepsilon) \to H^1(\Box)$ such that for each $u \in H^1(\mathbf{M}^\varepsilon)$:

$$\Pi^\varepsilon u(x) = u(\tilde{x}) \quad \text{for } \tilde{x} \in \Omega^\varepsilon_\Box, \tag{3.5}$$

$$\left\|\Pi^\varepsilon u\right\|_{H^1(\Box)} \leqslant C\|u\|_{H^1(\Omega^\varepsilon_\Box)}. \tag{3.6}$$

It is known (see e.g. [22, Chapter 4]) that such an operator exists.

By $\langle u\rangle_B$ we denote the average value of the function u over the domain $B \subset M^\varepsilon$ ($|B| \neq 0$), i.e. $\langle u\rangle_B = \frac{1}{|B|}\int_B u\,dV^\varepsilon$, where dV^ε is the density of the Riemannian measure on M^ε. The same notation remains for $B \subset \mathbb{R}^n$.

If $\Sigma \subset M^\varepsilon$ is a $(n-1)$-dimensional submanifold then g^ε induces on Σ the Riemannian metric and measure. We denote by dS^ε the density of this measure. Again by $\langle u\rangle_\Sigma$ we denote the average value of the function u over Σ, i.e. $\langle u\rangle_B = \frac{1}{|\Sigma|}\int_\Sigma u\,dS^\varepsilon$ (here $|\Sigma| = \int_\Sigma dS^\varepsilon$).

We introduce the operators $\Pi^\varepsilon_j : L_2(\mathbf{M}^\varepsilon) \to L_2(\Box)$ ($j = 1,\dots,m$) by the formula:

$$i \in \mathcal{I}^\varepsilon,\ x \in \Box^\varepsilon_i\colon \quad \Pi^\varepsilon_j u(x) = \left\langle u^\varepsilon\right\rangle_{B^\varepsilon_{ij}}.$$

Recall that $\square = \bigcup_{i\in\mathcal{I}^\varepsilon} \square_i^\varepsilon$. Using the Cauchy inequality and (2.7) we obtain

$$\left\| \Pi_j^\varepsilon u \right\|_{L_2(\square)} \leqslant C \|u\|_{L_2(\bigcup_{i\in\mathcal{I}^\varepsilon} \bigcup_{j=1}^m B_{ij}^\varepsilon)}. \tag{3.7}$$

In view of (3.6), (3.7) the norms $\|\Pi^\varepsilon u^\varepsilon\|_{H^1(\square)}$, $\|\Pi_j^\varepsilon u^\varepsilon\|_{L_2(\square)}$ $(j = 1,\dots,m)$ are bounded uniformly in ε. Using the embedding theorem (see e.g. [28, Chapter 4]) we obtain that the sub-sequence (still denoted by ε), the functions $u \in H^1(\square)$, $u_j \in L_2(\square)$, $j = 1,\dots,m$ exist such that

$$\Pi^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u \text{ weakly in } H^1(\square) \text{ and strongly in } L_2(\square), \qquad \Pi_j^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u_j \text{ weakly in } L_2(\square).$$

Moreover due to the trace theorem (see e.g. [28, Chapter 4]) $\Pi^\varepsilon u^\varepsilon$, $u \in L_2(\partial\square)$ and

$$\Pi^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u \text{ strongly in } L_2(\partial\square). \tag{3.8}$$

Since $u^\varepsilon \in \mathrm{dom}(\Delta_{\mathbf{M}^\varepsilon}^{\theta^\varepsilon})$, then in view of (3.4)

$$u^\varepsilon(x+e_\alpha) = \overline{\theta_\alpha^\varepsilon} u^\varepsilon(x), \quad \tilde{x} \in \mathbf{M}_\alpha,\ \alpha = 1,\dots,n.$$

Therefore,

$$u(x+e_\alpha) = \overline{\theta_\alpha} u(x), \quad x \in \mathbf{M}_\alpha,\ \alpha = 1,\dots,n.$$

Thus, $u \in \mathrm{dom}(\bar{\eta}_\square^\theta)$. Recall (see Section 1) that $\bar{\eta}_\square^\theta$ is the sesquilinear form which generates the operator $-\Delta_{\mathbf{M}^\varepsilon}^{\theta^\varepsilon}$.

We also need some auxiliary lemmas.

Lemma 3.1. *For any* $j = 1,\dots,m$:

$$\lim_{\varepsilon\to 0} \varepsilon^n \sum_{i\in\mathcal{I}^\varepsilon} \left|\langle u^\varepsilon\rangle_{S_{ij}^\varepsilon}\right|^2 = \|u\|_{L_2(\square)}^2. \tag{3.9}$$

Proof. We denote $\hat{R}_{ij}^\varepsilon = \{\tilde{x} \in \Omega^\varepsilon:\ d_j^\varepsilon + \frac{\kappa\varepsilon}{4} \leqslant |x - x_{ij}^\varepsilon| < d_j^\varepsilon + \frac{\kappa\varepsilon}{2}\}$. One has the inequalities:

$$0 \leqslant \left\|\Pi^\varepsilon u^\varepsilon\right\|_{L_2(\square_i^\varepsilon)}^2 - \varepsilon^n \left|\langle \Pi^\varepsilon u^\varepsilon\rangle_{\square_i^\varepsilon}\right|^2 \leqslant C\varepsilon^2 \left\|\nabla \Pi^\varepsilon u^\varepsilon\right\|_{L_2(\square_i^\varepsilon)}^2, \quad i \in \mathcal{I}^\varepsilon, \tag{3.10}$$

$$\left|\langle \Pi^\varepsilon u^\varepsilon\rangle_{\square_i^\varepsilon} - \langle u^\varepsilon\rangle_{\hat{R}_{ij}^\varepsilon}\right|^2 \leqslant C \left\|\nabla \Pi^\varepsilon u^\varepsilon\right\|_{L_2(\square_i^\varepsilon)}^2 \varepsilon^{2-n}, \quad i \in \mathcal{I}^\varepsilon, \tag{3.11}$$

$$\left|\langle u^\varepsilon\rangle_{S_{ij}^\varepsilon} - \langle u^\varepsilon\rangle_{\hat{R}_{ij}^\varepsilon}\right|^2 \leqslant C \left\|\nabla u^\varepsilon\right\|_{L_2(\hat{R}_{ij}^\varepsilon)}^2 \varepsilon^{2-n}, \quad i \in \mathcal{I}^\varepsilon, \tag{3.12}$$

which are valid for any $u^\varepsilon \in H^1(\Omega_\square^\varepsilon)$, $j = 1,\dots,m$. Inequality (3.10) is the Poincaré inequality, the inequality (3.11) follows directly from [19, Lemma 2.1], and the inequality (3.12) can be proved in the same way as inequality (2.2) from [20].[1]

Equality (3.9) follows directly from (3.10)–(3.12). The lemma is proved. □

[1] In [20] inequality (3.12) with $\partial\hat{R}_{ij}^\varepsilon \setminus S_{ij}^\varepsilon$ instead of S_{ij}^ε was proved. For S_{ij}^ε the proof is similar. We remark that in the case $n = 2$ inequality (3.12) is valid with $|\ln\varepsilon|$ instead of ε^{2-n}.

Lemma 3.2.[2] *For $j = 1, \dots, m$:*

$$\lim_{\varepsilon\to 0} \lambda_1^D\big(G_{ij}^\varepsilon\big) = \sigma_j$$

where σ_j is defined by formula (2.8)*.*

Proof. Let $v_{ij}^\varepsilon \in \operatorname{dom}(\Delta^D_{G_{ij}^\varepsilon})$ be the eigenfunction corresponding to $\lambda_1(G_{ij}^\varepsilon)$ such that $\langle v_{ij}^\varepsilon\rangle_{B_{ij}^\varepsilon} = 1$. Instead of calculating v_{ij}^ε in the exact form we construct a convenient approximation $\mathbf{v}_{ij}^\varepsilon$ for it.

We introduce the notations:

$$\hat{B}_{ij}^\varepsilon = \big\{\tilde{x} = (\theta_1, \dots, \theta_n) \in B_{ij}^\varepsilon\colon \theta_n \in \big[\Theta_j^\varepsilon, \pi/2\big]\big\},$$

$$\hat{G}_{ij}^\varepsilon = \hat{B}_{ij}^\varepsilon \cup R_{ij}^\varepsilon,$$

$$\hat{S}_{ij}^\varepsilon = \big\{\tilde{x} = (\theta_1, \dots, \theta_n) \in B_{ij}^\varepsilon\colon \theta_n = \pi/2\big\} = \partial \hat{B}_{ij}^\varepsilon \setminus \partial B_{ij}^\varepsilon.$$

Let the function $\hat{v}_{ij}^\varepsilon$ be the solution of the following boundary value problem:

$$-\Delta_{\hat{G}_{ij}^\varepsilon} \hat{v}_{ij}^\varepsilon = 0 \quad \text{in } \hat{G}_{ij}^\varepsilon, \tag{3.13}$$

$$\hat{v}_{ij}^\varepsilon\big|_{S_{ij}^\varepsilon} = 0, \qquad \hat{v}_{ij}^\varepsilon\big|_{\hat{S}_{ij}^\varepsilon} = 1. \tag{3.14}$$

Here by $-\Delta_{\hat{G}_{ij}^\varepsilon}$ we denote the operator which is defined by the operation (1.2) and the definitional domain $\operatorname{dom}(\Delta_{\hat{G}_{ij}^\varepsilon}) = \{u\colon\ u = v|_{\hat{G}_{ij}^\varepsilon},\ v \in \operatorname{dom}(\Delta_{M^\varepsilon})\}$. For convenience from now on we use the notation $-\Delta$ instead of $-\Delta_{\hat{G}_{ij}^\varepsilon}$. It is easy to see that the function $\hat{v}_{ij}^\varepsilon$ is smooth in R_{ij}^ε and B_{ij}^ε, the limiting values of $\hat{v}_{ij}^\varepsilon$ in the domains R_{ij}^ε and $\hat{B}_{ij}^\varepsilon$ coincide on $\partial B_{ij}^\varepsilon$, the normal derivatives satisfy the condition $\frac{\partial \hat{v}_{ij}^\varepsilon}{\partial r} + \frac{1}{b_j^\varepsilon}\frac{\partial \hat{v}_{ij}^\varepsilon}{\partial \theta_n} = 0$.

Due to the symmetry of $\hat{G}_{ij}^\varepsilon$ one can easily calculate $\hat{v}_{ij}^\varepsilon$ (recall that we consider the case $n \geqslant 3$):

$$\hat{v}_{ij}^\varepsilon(\tilde{x}) = \begin{cases} \mathrm{A}_j^\varepsilon |x - x_{ij}^\varepsilon|^{2-n} + \mathrm{B}_j^\varepsilon, & \tilde{x} \in R_{ij}^\varepsilon, \\ \mathrm{C}_j^\varepsilon F(\theta_n) + 1, & \tilde{x} = (\theta_1, \dots, \theta_n) \in \hat{B}_{ij}^\varepsilon \end{cases} \tag{3.15}$$

where $F(\theta_n) = \int_{\pi/2}^{\theta_n} (\sin^{1-n}\psi)\, d\psi$ and the constants A_j^ε, B_j^ε, C_j^ε are defined by the formulae

$$\mathrm{A}_j^\varepsilon = \frac{(d_j^\varepsilon)^{n-2}}{1 - (\frac{2d_j^\varepsilon}{\kappa\varepsilon})^{n-2} - (n-2)F(\Theta_j^\varepsilon)(\frac{d_j^\varepsilon}{b_j^\varepsilon})^{n-2}},$$

$$\mathrm{B}_j^\varepsilon = -\frac{\mathrm{A}_j^\varepsilon}{(\frac{\kappa\varepsilon}{2})^{n-2}}, \qquad \mathrm{C}_j^\varepsilon = (n-2)\frac{\mathrm{A}_j^\varepsilon}{(b_j^\varepsilon)^{n-2}}. \tag{3.16}$$

We redefine $\hat{v}_{ij}^\varepsilon$ by 1 in $B_{ij}^\varepsilon \setminus \hat{B}_{ij}^\varepsilon$ preserving the same notation.

[2] This result was given in [17] without a justification. In the current work we present a complete proof.

Direct calculations lead to the following asymptotics as $\varepsilon \to 0$:

$$\|\nabla \hat{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \sim \sigma_j \rho_j \varepsilon^n, \qquad \|\hat{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \sim \rho_j \varepsilon^n \tag{3.17}$$

where σ_j, ρ_j are defined by formulae (2.8), (2.9).

We define the function $\mathbf{v}_{ij} \in \operatorname{dom}(\Delta^D_{G_{ij}^\varepsilon})$ by the formula

$$\mathbf{v}_{ij}^\varepsilon(\tilde{x}) = \begin{cases} \hat{v}_{ij}^\varepsilon & \tilde{x} \in R_{ij}^\varepsilon, \\ 1 + (\hat{v}_{ij}^\varepsilon(\tilde{x}) - 1)\Phi(\theta_n), & \tilde{x} = (\theta_1, \ldots, \theta_n) \in \hat{B}_{ij}^\varepsilon, \\ 1, & \tilde{x} \in B_{ij}^\varepsilon \setminus \hat{B}_{ij}^\varepsilon. \end{cases} \tag{3.18}$$

Here $\Phi(\theta_n)$ is a twice continuously differentiable non-negative function on $[0, \infty)$ equal to 1 as $0 \leqslant \theta_n \leqslant \pi/4$ and equal to 0 as $\theta_n \geqslant \pi/2$. We have the following asymptotics as $\varepsilon \to 0$:

$$\|\nabla \mathbf{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \sim \|\nabla \hat{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)}, \qquad \|\mathbf{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \sim \|\hat{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)}, \qquad \|\Delta \mathbf{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} = O(\varepsilon^n), \tag{3.19}$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-n}\left(\|\mathbf{v}_{ij}^\varepsilon - 1\|^2_{L_2(B_{ij}^\varepsilon)} + \|\mathbf{v}_{ij}^\varepsilon\|^2_{L_2(R_{ij}^\varepsilon)}\right) = 0. \tag{3.20}$$

It follows from the min–max principle (see e.g. [26]) that

$$\lambda_1(G_{ij}^\varepsilon) = \frac{\|\nabla v_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)}}{\|v_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)}} \leqslant \frac{\|\nabla \mathbf{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)}}{\|\mathbf{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)}}. \tag{3.21}$$

Note, that this automatically gives the inequality $\lim_{\varepsilon \to 0} \lambda_1(G_{ij}^\varepsilon) \leqslant \sigma_j$.

We present the eigenfunction v_{ij}^ε in the form

$$v_{ij}^\varepsilon = \mathbf{v}_{ij}^\varepsilon + w_{ij}^\varepsilon. \tag{3.22}$$

Let us estimate the remainder w_{ij}^ε. One has the following estimates for the eigenfunction v_{ij}^ε (for the proof see [3, Lemma 4.2]):

$$\|v_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} = \|v_{ij}^\varepsilon\|^2_{L_2(B_{ij}^\varepsilon)} + O(\varepsilon^{n+2}) = |B_{ij}^\varepsilon| + O(\varepsilon^{n+2}), \tag{3.23}$$

$$\|v_{ij}^\varepsilon\|^2_{L_2(R_{ij}^\varepsilon)} \leqslant C\varepsilon^{n+2}. \tag{3.24}$$

Using (3.20), (3.23), (3.24) we obtain

$$\begin{aligned} \varepsilon^{-n}\|w_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \leqslant{} & 2\varepsilon^{-n}\left(\|\mathbf{v}_{ij}^\varepsilon\|^2_{L_2(R_{ij}^\varepsilon)} + \|v_{ij}^\varepsilon\|^2_{L_2(R_{ij}^\varepsilon)}\right. \\ & \left. + \|1 - \mathbf{v}_{ij}^\varepsilon\|^2_{L_2(B_{ij}^\varepsilon)} + \|v_{ij}^\varepsilon - 1\|^2_{L_2(B_{ij}^\varepsilon)}\right) \underset{\varepsilon \to 0}{\to} 0. \end{aligned} \tag{3.25}$$

Substituting (3.22) into (3.21) and integrating by parts we get

$$\|\nabla w_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \leqslant 2\left|(\Delta \mathbf{v}_{ij}^\varepsilon, w^\varepsilon)_{L_2(G_{ij}^\varepsilon)}\right| + \|\nabla \mathbf{v}_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)}\left(\frac{\|v_{ij}\|^2_{L_2(G_{ij}^\varepsilon)}}{\|\mathbf{v}_{ij}^\varepsilon\|_{L_2(G_{ij}^\varepsilon)}} - 1\right). \tag{3.26}$$

Taking into account (3.17), (3.19), (3.23), (3.25) we conclude that (3.26) implies

$$\varepsilon^{-n}\left\|\nabla w_{ij}^{\varepsilon}\right\|^2_{L_2(G_{ij}^{\varepsilon})} \underset{\varepsilon\to 0}{\to} 0. \tag{3.27}$$

It follows from (3.17), (3.19), (3.25), (3.27) that $\lim_{\varepsilon\to 0}\lambda_1^{\varepsilon}(G_{ij}^{\varepsilon}) = \sigma_j$. The lemma is proved. □

Lemma 3.3. *For* $j = 1,\dots,m$:

$$\lim_{\varepsilon\to 0}\lambda_2^D\left(G_{ij}^{\varepsilon}\right) = \infty.$$

Proof. Let $\mathbf{G}_j^{\varepsilon}$ be an n-dimensional surface embedded into $\mathbb{R}^{n+1}$ (below $x\in\mathbb{R}^n$, $z\in\mathbb{R}$):

$$\mathbf{G}_j^{\varepsilon} = \mathbf{R}_j^{\varepsilon}\cup\mathbf{B}_j^{\varepsilon}$$

where

$$\mathbf{R}_j^{\varepsilon} = \left\{(x,z)\in\mathbb{R}^{n+1}\colon\ \varepsilon^{-1}d_j^{\varepsilon}\leqslant |x| < \kappa/2,\ z=0\right\},$$
$$\mathbf{B}_j^{\varepsilon} = \left\{(x,z)\in\mathbb{R}^{n+1}\colon\ |x|^2 + \left(z - b_j\cos\Theta_j^{\varepsilon}\right)^2 = (b_j)^2,\ z\geqslant 0\right\}.$$

We equip $\mathbf{G}_j^{\varepsilon}$ with the Riemannian metric induced by the Euclidean metric in $\mathbb{R}^{n+1}$. By dV we denote the density of the Riemannian measure on $\mathbf{G}_j^{\varepsilon}$. Thus, $\mathbf{G}_j^{\varepsilon}$ is the ε^{-1}-homothetic image of G_{ij}^{ε}.

Evidently one has the following relation between the spectra of $-\Delta^D_{G_{ij}^{\varepsilon}}$ and $-\Delta^D_{\mathbf{G}_j^{\varepsilon}}$:

$$\forall k\in\mathbb{N}\colon\quad \lambda_k^D\left(G_{ij}^{\varepsilon}\right) = \varepsilon^{-2}\lambda_k^D\left(\mathbf{G}_{ij}^{\varepsilon}\right). \tag{3.28}$$

We denote

$$\mathbf{R} = \left\{(x,z)\in\mathbb{R}^{n+1}\colon\ |x| < \kappa/2,\ z=0\right\},\qquad \mathbf{B}_j = \left\{(x,z)\in\mathbb{R}^{n+1}\colon\ |x|^2 + z^2 = (b_j)^2\right\}.$$

Further we will prove that

$$\forall k\in\mathbb{N}\colon\quad \lambda_k^D\left(\mathbf{G}_j^{\varepsilon}\right) \underset{\varepsilon\to 0}{\to} \lambda_k \tag{3.29}$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of the operator $\mathcal{L}_j$ which acts in the space $L_2(\mathbf{R})\oplus L_2(\mathbf{B}_j)$ and is defined by the formula

$$\mathcal{L}_j = -\begin{pmatrix}\Delta_{\mathbf{R}}^D & 0\\ 0 & \Delta_{\mathbf{B}_j}\end{pmatrix}.$$

Here the eigenvalues are renumbered in the increasing order and with account of their multiplicity.

One has $\lambda_1 = \lambda_1(\mathbf{B}_j) = 0$, $\lambda_2 = \min\{\lambda_1^D(\mathbf{R}), \lambda_2(\mathbf{B}_j)\} > 0$. Therefore, in view of (3.28)–(3.29) $\lim_{\varepsilon\to 0}\lambda_2^D(G_{ij}^{\varepsilon}) = \infty$. Thus, to complete the proof of the lemma we have to prove (3.29). For that we use the abstract scheme proposed in the work [16].

Theorem. *(See [16].) Let $\mathcal{H}^{\varepsilon}$, $\mathcal{H}^0$ be separable Hilbert spaces, let $\mathcal{A}^{\varepsilon}:\mathcal{H}^{\varepsilon}\to\mathcal{H}^{\varepsilon}$, $\mathcal{A}^0:\mathcal{H}^0\to\mathcal{H}^0$ be linear continuous operators, $\operatorname{im}\mathcal{A}^0\subset\mathcal{V}\subset\mathcal{H}^0$, where $\mathcal{V}$ is a subspace in $\mathcal{H}^0$.*

Suppose that the following conditions C_1–C_4 hold:

C_1. *The linear bounded operators $R^\varepsilon : \mathcal{H}^0 \to \mathcal{H}^\varepsilon$ exist such that $\|R^\varepsilon f\|^2_{\mathcal{H}^\varepsilon} \underset{\varepsilon\to 0}{\to} \gamma \|f\|^2_{\mathcal{H}^0}$ for any $f \in \mathcal{V}$. Here $\gamma > 0$ is a constant.*

C_2. *Operators $\mathcal{A}^\varepsilon$, $\mathcal{A}^0$ are positive, compact and self-adjoint. The norms $\|\mathcal{A}^\varepsilon\|_{\mathcal{L}(\mathcal{H}^\varepsilon)}$ are bounded uniformly in ε.*

C_3. *For any $f \in \mathcal{V}$: $\|\mathcal{A}^\varepsilon R^\varepsilon f - R^\varepsilon \mathcal{A}^0 f\|_{\mathcal{H}^\varepsilon} \underset{\varepsilon\to 0}{\to} 0$.*

C_4. *For any sequence $f^\varepsilon \in \mathcal{H}^\varepsilon$ such that $\sup_\varepsilon \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$ the subsequence $\varepsilon' \subset \varepsilon$ and $w \in \mathcal{V}$ exist such that $\|\mathcal{A}^\varepsilon f^\varepsilon - R^\varepsilon w\|_{\mathcal{H}^\varepsilon} \underset{\varepsilon=\varepsilon'\to 0}{\longrightarrow} 0$.*

Then for any $k \in \mathbb{N}$

$$\mu_k^\varepsilon \underset{\varepsilon\to 0}{\to} \mu_k$$

where $\{\mu_k^\varepsilon\}_{k=1}^\infty$ and $\{\mu_k\}_{k=1}^\infty$ are the eigenvalues of the operators $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$, which are renumbered in the increasing order and with account of their multiplicity.

Let us apply this theorem. We set $\mathcal{H}^\varepsilon = L_2(\mathbf{G}_j^\varepsilon)$, $\mathcal{H}^0 = L_2(\mathbf{R}) \oplus L_2(\mathbf{B}_j)$, $\mathcal{A}^\varepsilon = (-\Delta^D_{\mathbf{G}_j^\varepsilon} + \mathrm{I})^{-1}$, $\mathcal{A}^0 = (\mathcal{L}_j + \mathrm{I})^{-1}$, $\mathcal{V} = \mathcal{H}^0$. We introduce the operator $R^\varepsilon : \mathcal{H}^0 \to \mathcal{H}^\varepsilon$ by the formula:

$$\big[R^\varepsilon f\big](x,z) = \begin{cases} f^R(x), & (x,0) \in \mathbf{R}_j^\varepsilon, \\ f^B(x, z - b_j^\varepsilon \cos\Theta_j^\varepsilon), & (x,z) \in \mathbf{B}_j^\varepsilon, \end{cases} \qquad f = \big(f^R, f^B\big) \in \mathcal{H}^0 = L_2(\mathbf{R}) \oplus L_2(\mathbf{B}_j).$$

We also denote $H_0^1(\mathbf{R}) = \{u \in H^1(\mathbf{R}):\ u|_{\partial\mathbf{R}} = 0\}$, $H_0^1(\mathbf{G}_j^\varepsilon) = \{u \in H^1(\mathbf{G}_j^\varepsilon):\ u|_{\partial\mathbf{G}_j^\varepsilon} = 0\}$, $\mathcal{H}^1 = H_0^1(\mathbf{R}) \oplus H^1(\mathbf{B}_j) \subset \mathcal{H}^0$ and introduce the operator $Q^\varepsilon : H_0^1(\mathbf{G}_j^\varepsilon) \to \mathcal{H}^1$ satisfying the properties that are similar to those of the operator Π^ε (see above):

$$\forall\varepsilon > 0,\ \forall v \in H_0^1\big(\mathbf{G}_j^\varepsilon\big):\quad R^\varepsilon Q^\varepsilon v = v, \qquad \big\|Q^\varepsilon v\big\|_{\mathcal{H}^1} \leqslant C\|v\|_{H_0^1(\mathbf{G}_j^\varepsilon)}. \tag{3.30}$$

Evidently conditions C_1 (with $\gamma = 1$) and C_2 hold. We verify condition C_3. Let $f \in \mathcal{H}^\varepsilon$. We set $f^\varepsilon = R^\varepsilon f$, $v^\varepsilon = \mathcal{A}^\varepsilon f^\varepsilon$, $\hat{v}^\varepsilon = Q^\varepsilon v^\varepsilon$. One has

$$\int\limits_{\mathbf{G}_j^\varepsilon} \big(\big(\nabla v^\varepsilon, \nabla w^\varepsilon\big) + u^\varepsilon w^\varepsilon - f^\varepsilon w^\varepsilon\big)\, dV = 0, \quad \forall w^\varepsilon \in H_0^1\big(\mathbf{G}_j^\varepsilon\big). \tag{3.31}$$

Clearly the norms $\|v^\varepsilon\|^2_{H_0^1(\mathbf{G}_j^\varepsilon)}$ are bounded uniformly in ε. Taking into account (3.30) we conclude that the subsequence (still denoted by ε) and $v = (v^R, v^B) \in \mathcal{H}^1$ exist such that

$$\hat{v}^\varepsilon = \big(\hat{v}^{\varepsilon R}, \hat{v}^{\varepsilon B}\big) \underset{\varepsilon\to 0}{\to} v \text{ weakly in } \mathcal{H}^1 \text{ and strongly in } \mathcal{H}^0.$$

Let $w \in \widehat{\mathcal{H}}^1 = \{w = (w^R, w^B) \in \mathcal{H}^1:\ \operatorname{supp} f^R \subset \mathbf{R} \setminus \{(0,0)\},\ \operatorname{supp} f^B \subset \mathbf{B}_j \setminus \{(0,-b_j)\}\}$, i.e. $w^R = 0$ in a neighbourhood of $\{(0,0)\}$, $w^B = 0$ in a neighbourhood of $\{(0,-b_j)\}$. We set $w^\varepsilon = R^\varepsilon w$. Then, when ε is small enough, $w^\varepsilon = 0$ in some neighbourhood of $\partial\mathbf{B}_j^\varepsilon$ and $w^\varepsilon \in H_0^1(\mathbf{G}_j^\varepsilon)$. Substituting w^ε into (3.31) we obtain (ε is small enough):

$$\int\limits_{\mathbf{R}} \left(\left(\nabla \hat{v}^{\varepsilon R}, \nabla w^R \right) + \hat{v}^{\varepsilon R} w^R - f^R w^R \right) dx + \int\limits_{\mathbf{B}_j} \left(\left(\nabla \hat{v}^{\varepsilon B}, \nabla w^B \right) + \hat{v}^{\varepsilon B} w^B - f^B w^B \right) dV = 0. \tag{3.32}$$

Passing to the limit in (3.32) as $\varepsilon \to 0$ and taking into account that the space $\widehat{\mathcal{H}}^1$ is dense in $\mathcal{H}^1$ (see e.g. [25]), we obtain the equality $\mathcal{A}^0 f = v$ that obviously implies the fulfilment of C_3.

Finally, condition C_4 follows from the fact that if $\sup_\varepsilon \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$ then the norms $\|Q^\varepsilon \mathcal{A}^\varepsilon f^\varepsilon\|_{\mathcal{H}^1}$ are bounded uniformly in ε and, therefore, the subsequence $\varepsilon' \subset \varepsilon$ and $w \in \mathcal{H}^1$ exist such that

$$Q^\varepsilon \mathcal{A}^\varepsilon f^\varepsilon \underset{\varepsilon=\varepsilon'\to 0}{\longrightarrow} w \text{ weakly in } \mathcal{H}^1 \text{ and strongly in } \mathcal{H}^0.$$

Thus, the eigenvalues μ_k^ε of the operator $\mathcal{A}^\varepsilon$ converge to the eigenvalues μ_k of the operator $\mathcal{A}^0$ as $\varepsilon \to 0$. But $\lambda_k^D(\mathbf{G}_{ij}^\varepsilon) = (\mu_k^\varepsilon)^{-1} - 1$, $\lambda_k = (\mu_k)^{-1} - 1$ that implies (3.29). The lemma is proved. □

Lemma 3.4. *For $j = 1, \ldots, m$:*

$$\lim_{\varepsilon\to 0} \sum_{i\in\mathcal{I}^\varepsilon} \|u^\varepsilon\|^2_{L_2(B_{ij}^\varepsilon)} = \rho_j \left(\frac{\sigma_j}{\sigma_j - \lambda} \right)^2 \|u\|^2_{L_2(\square)}. \tag{3.33}$$

Proof. For $\tilde{x} \in G_{ij}^\varepsilon$ we denote $l^\varepsilon(\tilde{x}) = \operatorname{dist}_{g^\varepsilon}(\tilde{x}, S_{ij}^\varepsilon)$, where by $\operatorname{dist}_{g^\varepsilon}(\cdot,\cdot)$ we denote the distance with respect to the metric g^ε. We introduce the set

$$S_{ij}^\varepsilon[\tilde{x}] = \left\{ \tilde{y} \in G_{ij}^\varepsilon \colon l^\varepsilon(\tilde{y}) = l^\varepsilon(\tilde{x}) \right\}.$$

Obviously $S_{ij}^\varepsilon[\tilde{x}]$ is a $(n-1)$-dimensional sphere (in particular if $\tilde{x} \in \partial B_{ij}^\varepsilon$ then $l^\varepsilon(\tilde{x}) = \kappa\varepsilon/2$ and $S_{ij}^\varepsilon[\tilde{x}] = \partial B_{ij}^\varepsilon$).

We define the function $u_{ij}^\varepsilon(\tilde{x})$ by the formula:

$$u_{ij}^\varepsilon(\tilde{x}) = \langle u^\varepsilon \rangle_{S_{ij}^\varepsilon[\tilde{x}]}, \quad \tilde{x} \in G_{ij}^\varepsilon.$$

Using the Poincaré inequality (for the spheres $S_{ij}[\tilde{x}]$) we get

$$\sum_{i\in\mathcal{I}^\varepsilon} \|u^\varepsilon - u_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \leqslant C \sum_{i\in\mathcal{I}^\varepsilon} \max_{\tilde{x}\in G_{ij}^\varepsilon} \left(\operatorname{diam} S_{ij}^\varepsilon[\tilde{x}] \right)^2 \|\nabla u^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \leqslant C\varepsilon^2 \|\nabla u^\varepsilon\|^2_{L_2(\mathbf{M}^\varepsilon)}. \tag{3.34}$$

We denote $\mathbf{u}_{ij}^\varepsilon = u_{ij}^\varepsilon - \langle u^\varepsilon \rangle_{S_{ij}^\varepsilon}$. Clearly $\mathbf{u}_{ij}^\varepsilon \in \operatorname{dom}(\Delta^D_{G_{ij}^\varepsilon})$ and

$$-\Delta^D_{G_{ij}^\varepsilon} \mathbf{u}_{ij}^\varepsilon - \lambda^\varepsilon \mathbf{u}_{ij}^\varepsilon = \lambda^\varepsilon \langle u^\varepsilon \rangle_{S_{ij}^\varepsilon}.$$

In view of Lemmas 3.2, 3.3 and since $\lambda \notin \bigcup_{j=1}^m \{\sigma_j\}$, $\lambda^\varepsilon \notin \sigma(-\Delta^D_{G_{ij}^\varepsilon})$ when ε is small enough. Therefore, the following expansion is valid:

$$\mathbf{u}_{ij}^\varepsilon = \sum_{k=1}^\infty I_{ij}^k(\varepsilon), \quad \text{where } I_{ij}^k(\varepsilon) = \frac{v_k^D(G_{ij}^\varepsilon)}{\|v_k^D(G_{ij}^\varepsilon)\|^2_{L_2(G_{ij}^\varepsilon)}} \cdot \frac{(f_{ij}^\varepsilon, v_k^D(G_{ij}^\varepsilon))_{L_2(G_{ij}^\varepsilon)}}{(\lambda_k^D(G_{ij}^\varepsilon) - \lambda^\varepsilon)}. \tag{3.35}$$

Here $f_{ij}^\varepsilon = \lambda^\varepsilon \langle u^\varepsilon\rangle_{S_{ij}^\varepsilon}$, $\{v_k^D(G_{ij}^\varepsilon)\}_{k=1}^m$ is a system of the eigenfunctions of $-\Delta_{G_{ij}^\varepsilon}^D$ corresponding to $\{\lambda_k^D(G_{ij}^\varepsilon)\}_{k=1}^m$ and such that $(v_k^D(G_{ij}^\varepsilon), v_l^D(G_{ij}^\varepsilon))_{L_2(G_{ij}^\varepsilon)} = 0$ if $k \neq l$.

We denote $\Lambda^\varepsilon = \max_{j=\overline{1,m}} \max_{k=\overline{2,\infty}} |\lambda^\varepsilon - \lambda_k^D(G_{ij}^\varepsilon)|^{-2}$. Thus, it follows from Lemma 3.3 that $\lim_{\varepsilon\to 0} \Lambda^\varepsilon = 0$. Therefore, taking into account (2.7) and using Lemma 3.1 we obtain

$$\sum_{i\in\mathcal{I}^\varepsilon} \left\| \sum_{k=2}^\infty I_{ij}^k(\varepsilon)\right\|^2_{L_2(B_{ij}^\varepsilon)} \leqslant \Lambda^\varepsilon \sum_{i\in\mathcal{I}^\varepsilon} \| f_{ij}^\varepsilon\|^2_{L_2(G_{ij}^\varepsilon)} \leqslant C\left(\lambda^\varepsilon\right)^2 \Lambda^\varepsilon \sum_{i\in\mathcal{I}^\varepsilon} \left|\langle u^\varepsilon\rangle_{S_{ij}^\varepsilon}\right|^2 \varepsilon^n \underset{\varepsilon\to 0}{\to} 0. \tag{3.36}$$

As in Lemma 3.2 we denote $v_{ij}^\varepsilon = v_1^D(G_{ij}^\varepsilon)$. We normalize v_{ij}^ε by the condition $\langle v_{ij}^\varepsilon\rangle_{B_{ij}^\varepsilon} = 1$. Using the estimates (3.23), (3.24) and Lemma 3.2 we obtain that

$$\sum_{i\in\mathcal{I}^\varepsilon} \| I_{ij}^1(\varepsilon)\|^2_{L_2(B_{ij}^\varepsilon)} \sim \sum_{i\in\mathcal{I}^\varepsilon} \frac{\lambda^2\rho_j\varepsilon^n |\langle u^\varepsilon\rangle_{S_{ij}^\varepsilon}|^2}{(\sigma_j-\lambda)^2} \sim \frac{\lambda^2\rho_j\|u\|^2_{L_2(\square)}}{(\sigma_j-\lambda)^2} \tag{3.37}$$

as $\varepsilon \to 0$. Thus, it follows from (3.35)–(3.37) that

$$\lim_{\varepsilon\to 0} \sum_{i\in\mathcal{I}^\varepsilon} \|\mathbf{u}_{ij}^\varepsilon\|^2_{L_2(B_{ij}^\varepsilon)} = \frac{\lambda^2\rho_j\|u\|^2_{L_2(\square)}}{(\sigma_j-\lambda)^2}. \tag{3.38}$$

Finally, using (3.35), (3.36), (3.38) and Lemma 3.1 we get

$$\sum_{i\in\mathcal{I}^\varepsilon} \|u_{ij}^\varepsilon\|^2_{L_2(B_{ij}^\varepsilon)} = \sum_{i\in\mathcal{I}^\varepsilon} \left(\|\mathbf{u}_{ij}^\varepsilon\|^2_{L_2(B_{ij}^\varepsilon)} + 2\langle u^\varepsilon\rangle_{S_{ij}^\varepsilon} \int_{B_{ij}^\varepsilon} \mathbf{u}_{ij}^\varepsilon(\tilde{x})\, dV^\varepsilon + \left|\langle u^\varepsilon\rangle_{S_{ij}^\varepsilon}\right|^2 \cdot \left|B_{ij}^\varepsilon\right| \right)$$

$$\underset{\varepsilon\to 0}{\to} \left[\frac{\lambda^2\rho_j}{(\sigma_j-\lambda)^2} + \frac{2\lambda\rho_j}{\sigma_j-\lambda} + \rho_j \right] \|u\|^2_{L_2(\square)} = \rho_j \left(\frac{\sigma_j}{\sigma_j-\lambda}\right)^2 \|u\|^2_{L_2(\square)}. \tag{3.39}$$

Then (3.33) follows from (3.34) and (3.39). The lemma is proved. □

Lemma 3.5. *For any $w \in C_\theta^\infty(\square)$ the function $\hat{w}^\varepsilon \in C^\infty(\square)$ exists such that*:

$$w + \hat{w}^\varepsilon \in C_{\theta^\varepsilon}^\infty(\square), \tag{3.40}$$

$$\max_{x\in\square} \left|\hat{w}^\varepsilon(x)\right| + \max_{x\in\square}\left|\nabla \hat{w}^\varepsilon(x)\right| \underset{\varepsilon\to 0}{\to} 0. \tag{3.41}$$

Proof. We define the function $\mathbf{1}^\varepsilon \in C^\infty(\mathbb{R}^n)$ by the following recurrent formulae:

$$\mathbf{1}^\varepsilon(x_1,\ldots,x_n) = A_n(x_1,\ldots,x_{n-1})x_n + B_n(x_1,\ldots,x_{n-1}),$$

$$\alpha = 2,\ldots,n: \quad \begin{cases} B_\alpha(x_1,\ldots,x_{\alpha-1}) = A_{\alpha-1}(x_1,\ldots,x_{\alpha-2})x_{\alpha-1} + B_{\alpha-1}(x_1,\ldots,x_{\alpha-2}), \\ A_\alpha(x_1,\ldots,x_{\alpha-1}) = \left(\overline{\theta_\alpha^\varepsilon}/\overline{\theta_\alpha} - 1\right) B_\alpha(x_1,\ldots,x_{\alpha-1}), \end{cases}$$

$$B_1 = 1, \qquad A_1 = \overline{\theta_1^\varepsilon}/\overline{\theta_1} - 1.$$

It is easy to see that $\max_{x\in\square}|\mathbf{1}^\varepsilon(x)-1|+\max_{x\in\square}|\nabla\mathbf{1}^\varepsilon(x)|\underset{\varepsilon\to 0}{\to}0$ and $\mathbf{1}^\varepsilon\in C^\infty_{\theta^\varepsilon/\theta}(\square)$, where $\theta^\varepsilon/\theta:=(\theta_1^\varepsilon/\theta_1,\dots,\theta_n^\varepsilon/\theta_n)$. Then we set

$$\hat{w}^\varepsilon=\big(\mathbf{1}^\varepsilon-1\big)w.$$

Obviously the function $\hat{w}^\varepsilon$ satisfies the conditions (3.40), (3.41). The lemma is proved. $\square$

We continue the proof of Theorem 2.1. For an arbitrary $w^\varepsilon\in\mathrm{dom}(\bar{\eta}_{\mathbf{M}^\varepsilon})$ we have

$$\int\limits_{\mathbf{M}^\varepsilon}\big(\big(\nabla u^\varepsilon,\nabla w^\varepsilon\big)_\varepsilon-\lambda^\varepsilon u^\varepsilon w^\varepsilon\big)\,dV^\varepsilon=0 \tag{3.42}$$

where $(\nabla u^\varepsilon,\nabla w^\varepsilon)_\varepsilon$ is the scalar product of the vectors ∇u^ε and ∇w^ε with respect to the metric g^ε.

We substitute into (3.42) the test function w^ε of a special type. Namely, let w be an arbitrary function from $C^\infty_\theta(\square)$, $\hat{w}^\varepsilon\in C^\infty(\square)$ be the function satisfying (3.40), (3.41). Let w_j, $j=1,\dots,m$ be arbitrary functions from $C^\infty(\square)$. Let $\Phi(r)$ be a twice continuously differentiable non-negative function equal to 1 as $0\leqslant r\leqslant 1/4$ and equal to 0 as $r\geqslant 1/2$. We set

$$\widehat{\Phi}_{ij}^\varepsilon=\Phi\left(\frac{|x-x_{ij}^\varepsilon|-d_j^\varepsilon}{d_j^\varepsilon}\right),\qquad \Phi_{ij}^\varepsilon=\Phi\left(\frac{|x-x_{ij}^\varepsilon|-d_j^\varepsilon}{\kappa\varepsilon}\right).$$

Then we set $w^\varepsilon=\mathbf{w}^\varepsilon+\delta^\varepsilon$, where

$$\mathbf{w}^\varepsilon(\tilde{x})=\begin{cases} w(x), & \tilde{x}\in\Omega_\square^\varepsilon\setminus(\bigcup_{i\in\mathcal{I}^\varepsilon}\bigcup_{j=1}^m R_{ij}^\varepsilon),\\ w(x)+(w(x_{ij}^\varepsilon)-w(x))\widehat{\Phi}_{ij}^\varepsilon(x) & \\ \quad+(w_j(x_{ij}^\varepsilon)-w(x_{ij}^\varepsilon))\mathbf{v}_{ij}^\varepsilon(x)\Phi_{ij}^\varepsilon(x), & \tilde{x}\in R_{ij}^\varepsilon,\\ w_j(x_{ij}^\varepsilon)+(w(x_{ij}^\varepsilon)-w_j(x_{ij}^\varepsilon))(1-\mathbf{v}_{ij}^\varepsilon(\tilde{x})), & \tilde{x}\in B_{ij},\end{cases}$$

$$\delta^\varepsilon(\tilde{x})=\begin{cases} \hat{w}(x), & \tilde{x}\in\Omega_\square^\varepsilon\setminus(\bigcup_{i\in\mathcal{I}^\varepsilon}\bigcup_{j=1}^m R_{ij}^\varepsilon),\\ \hat{w}^\varepsilon(x)+(\hat{w}^\varepsilon(x_{ij}^\varepsilon)-\hat{w}^\varepsilon(x))\widehat{\Phi}_{ij}^\varepsilon(x), & \tilde{x}\in R_{ij}^\varepsilon,\\ \hat{w}^\varepsilon(x_{ij}^\varepsilon), & \tilde{x}\in B_{ij}^\varepsilon.\end{cases} \tag{3.43}$$

Here the function $\mathbf{v}_{ij}^\varepsilon$ is defined by (3.18), (3.15), (3.16). It follows from (3.40) that $w^\varepsilon\in\mathrm{dom}(\bar{\eta}_{\mathbf{M}^\varepsilon})$.

Substituting this w^ε into (3.42) and integrating by parts we obtain

$$\int\limits_{\mathbf{M}^\varepsilon}\big(-u^\varepsilon\Delta\mathbf{w}^\varepsilon-\lambda^\varepsilon u^\varepsilon w^\varepsilon\big)\,dV^\varepsilon+\int\limits_{\partial\mathbf{M}^\varepsilon}\nu\big[\mathbf{w}^\varepsilon\big]u^\varepsilon\,dS^\varepsilon+\int\limits_{\mathbf{M}^\varepsilon}\big(\big(\nabla u^\varepsilon,\nabla\delta^\varepsilon\big)_\varepsilon-\lambda^\varepsilon u^\varepsilon\delta^\varepsilon\big)\,dV^\varepsilon=0, \tag{3.44}$$

where ν is the outward normal vector field on $\partial\mathbf{M}^\varepsilon$.

In view of (2.6)–(2.7) and the Cauchy inequality, the last term in (3.44) is estimated by $C\|u^\varepsilon\|_{H^1(\mathbf{M}^\varepsilon)}\sqrt{\max_{x\in\square}|\hat{w}(x)|^2+\max_{x\in\square}|\nabla\hat{w}(x)|^2}$ and tends to zero as $\varepsilon\to 0$ in view of (3.41).

In view of (3.8) the second term tends to $\int_{\partial\square}\nu[w]u\,ds$ as $\varepsilon\to 0$, where ν is the outward normal vector field on $\partial\square$, ds is the density of the Lebesgue measure on $\partial\square$.

Now let us investigate the first term. Firstly we study the integrals over $\Omega^{\varepsilon}_{\square}$. Integrating by parts we get

$$\left|\sum_{i\in\mathcal{I}^{\varepsilon}}\sum_{j=1}^{m}\int\limits_{R_{ij}^{\varepsilon}} -\Delta\big\{\big(w\big(x_{ij}^{\varepsilon}\big)-w(x)\big)\widehat{\Phi}_{ij}^{\varepsilon}(x)\big\}u^{\varepsilon}(x)\,dV^{\varepsilon}\right|$$

$$=\left|\sum_{i\in\mathcal{I}^{\varepsilon}}\sum_{j=1}^{m}\Bigg(\int\limits_{R_{ij}^{\varepsilon}\cup D_{ij}^{\varepsilon}}\big(\nabla\big\{\big(w\big(x_{ij}^{\varepsilon}\big)-w(x)\big)\widehat{\Phi}_{ij}^{\varepsilon}(x)\big\},\nabla^{\varepsilon}\Pi^{\varepsilon}u^{\varepsilon}(x)\big)\,dx-\int\limits_{D_{ij}^{\varepsilon}}\Delta w\Pi^{\varepsilon}u^{\varepsilon}\,dx\Bigg)\right|$$

$$\leqslant C(w)\cdot\big\|\Pi^{\varepsilon}u^{\varepsilon}\big\|_{H^1(\square)}\cdot\sqrt{\sum_{i\in\mathcal{I}^{\varepsilon}}\sum_{j=1}^{m}\big|D_{ij}^{\varepsilon}\cup\operatorname{supp}\big[\nabla\widehat{\Phi}_{ij}^{\varepsilon}\big]\big|}\underset{\varepsilon\to 0}{\to}0. \tag{3.45}$$

Hereafter by $C(w)$ we denote a constant depending only on w.

Let us prove that the function $\xi^{\varepsilon}\in L_2(\square)$,

$$\xi^{\varepsilon}(x)=\begin{cases}\sum_{i\in\mathcal{I}^{\varepsilon}}\sum_{j=1}^{m}-\Delta\{(w_j^{\varepsilon}(x_{ij}^{\varepsilon})-w(x_{ij}^{\varepsilon}))\mathbf{v}_{ij}^{\varepsilon}(x)\Phi_{ij}^{\varepsilon}(x)\}, & x\in R_{ij}^{\varepsilon},\\ 0, & x\in\square\setminus\bigcup_{i\in\mathcal{I}^{\varepsilon}}\bigcup_{j=1}^{m}R_{ij}^{\varepsilon}\end{cases}$$

converges weakly in $L_2(\square)$ to the function $\sum_{j=1}^{m}\sigma_j\rho_j(w-w_j)$. Indeed using the properties of $\mathbf{v}_{ij}^{\varepsilon}$

$$x\in R_{ij}^{\varepsilon}:\quad \Delta\mathbf{v}_{ij}^{\varepsilon}(x)=0,\qquad \big|D^{\alpha}\mathbf{v}_{ij}^{\varepsilon}(x)\big|\leqslant C\varepsilon^{n}\big|x-x_{ij}^{\varepsilon}\big|^{2-n-|\alpha|},\quad \alpha=0,1$$

and the enclosure $\operatorname{supp}(D^{\alpha}\Phi_j^{\varepsilon})\subset\{x\in\Omega^{\varepsilon}_{\square}\colon \kappa\varepsilon/4\leqslant|x-x_{ij}^{\varepsilon}|\leqslant\kappa\varepsilon/2\}$ $(\alpha\neq 0)$ we obtain

$$\int\limits_{R_{ij}^{\varepsilon}}\big|-\Delta\big\{\big(w_j\big(x_{ij}^{\varepsilon}\big)-w\big(x_{ij}^{\varepsilon}\big)\big)\mathbf{v}_{ij}^{\varepsilon}(x)\Phi_{ij}^{\varepsilon}(x)\big\}\big|^2\,dx<C(w)\varepsilon^{n}. \tag{3.46}$$

Hence the norms $\|\xi^{\varepsilon}\|_{L_2(\square)}$ are bounded uniformly in ε. Taking into account (3.46) we obtain for an arbitrary $f\in C^{\infty}(\square)$ (below ν^{ε} is the normal vector field on $\partial D_{ij}^{\varepsilon}$ directed outward R_{ij}^{ε}):

$$\sum_{i\in\mathcal{I}^{\varepsilon}}\sum_{j=1}^{m}\int\limits_{R_{ij}^{\varepsilon}}-\Delta\big\{\big(w_j\big(x_{ij}^{\varepsilon}\big)-w\big(x_{ij}^{\varepsilon}\big)\big)\mathbf{v}_{ij}^{\varepsilon}(x)\Phi_{ij}^{\varepsilon}(x)\big\}f(x)\,dV^{\varepsilon}$$

$$=\sum_{i\in\mathcal{I}^{\varepsilon}}\sum_{j=1}^{m}f\big(x_{ij}^{\varepsilon}\big)\big(w\big(x_{ij}^{\varepsilon}\big)-w_j\big(x_{ij}^{\varepsilon}\big)\big)\int\limits_{\partial D_{ij}^{\varepsilon}}\nu^{\varepsilon}\big[\mathbf{v}_{ij}^{\varepsilon}\big]\,dS^{\varepsilon}+\bar{o}(1)$$

$$=\sum_{i\in\mathcal{I}^{\varepsilon}}\sum_{j=1}^{m}f\big(x_{ij}^{\varepsilon}\big)\big(w\big(x_{ij}^{\varepsilon}\big)-w_j\big(x_{ij}^{\varepsilon}\big)\big)\sigma_j\rho_j\varepsilon^{n}$$

$$+\bar{o}(1)\underset{\varepsilon\to 0}{\to}\sum_{j=1}^{m}\sigma_j\rho_j\int\limits_{\square}f(x)\big(w(x)-w_j(x)\big)\,dx. \tag{3.47}$$

Here we have used the following computations (below $r = |x - x_{ij}^\varepsilon|$):

$$\int\limits_{\partial D_{ij}^\varepsilon} \nu^\varepsilon\big[\mathbf{v}_{ij}^\varepsilon\big]\,dS^\varepsilon = -\frac{\partial \mathbf{v}_{ij}^\varepsilon}{\partial r}\bigg|_{r=d_j^\varepsilon}\big(d_j^\varepsilon\big)^{n-1}\omega_{n-1} \sim \frac{1}{2}\omega_{n-1}(n-2)\,d_j^{n-2}\varepsilon^n = \sigma_j\rho_j\varepsilon^n, \quad \varepsilon\to 0. \tag{3.48}$$

Since $\overline{C^\infty(\square)} = L_2(\square)$, then ξ^ε converges weakly in $L_2(\square)$ to $\sum_{j=1}^m \sigma_j\rho_j(w - w_j)$ as $\varepsilon\to 0$.

Using this, (3.5), (3.6) and (3.45) we conclude that

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\square^\varepsilon} -\Delta w^\varepsilon u^\varepsilon\,dV^\varepsilon = \int\limits_\square\Bigg(-\Delta wu + \sum_{j=1}^m \sigma_j\rho_j(w - w_j)u\Bigg)\,dx. \tag{3.49}$$

In the same way (using the estimate (3.20)) one can prove that

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\square^\varepsilon} \lambda^\varepsilon w^\varepsilon u^\varepsilon\,dV^\varepsilon = \int\limits_\square \lambda wu\,dx. \tag{3.50}$$

Now, we investigate the behaviour of the integrals in (3.42) over $\bigcup_{i,j} B_{ij}^\varepsilon$. Using (3.19) (the last asymptotics), (3.48) and the Poincaré inequality we get

$$\begin{aligned}
&\sum_{i\in\mathcal{I}^\varepsilon}\sum_{j=1}^m \int\limits_{B_{ij}^\varepsilon} -\Delta\big[\big(w\big(x_{ij}^\varepsilon\big) - w_j\big(x_{ij}^\varepsilon\big)\big)\big(1 - \mathbf{v}_{ij}^\varepsilon(\tilde{x})\big)\big]u^\varepsilon(\tilde{x})\,dV^\varepsilon \\
&\quad = \sum_{i\in\mathcal{I}^\varepsilon}\sum_{j=1}^m \langle u^\varepsilon\rangle_{B_{ij}^\varepsilon}\big(w\big(x_{ij}^\varepsilon\big) - w_j\big(x_{ij}^\varepsilon\big)\big)\int\limits_{\partial D_{ij}^\varepsilon} -\nu^\varepsilon\big[\mathbf{v}_{ij}^\varepsilon\big]\,dS^\varepsilon + \bar{o}(1) \\
&\quad = \sum_{j=1}^m \sigma_j\rho_j\int\limits_\square [\widehat{w_j - w}](x)\Pi_j^\varepsilon u^\varepsilon(x)\,dx \\
&\qquad + \bar{o}(1) \underset{\varepsilon\to 0}{\to} \sum_{j=1}^m \sigma_j\rho_j\int\limits_\square \big(w_j(x) - w(x)\big)u_j(x)\,dx,
\end{aligned} \tag{3.51}$$

where $[\widehat{w_j - w}] \in L_2(\square)$ is a step function: $[\widehat{w_j - w}](x) = w_j(x_{ij}^\varepsilon) - w(x_{ij}^\varepsilon)$, $x\in\square_i^\varepsilon$, $i\in\mathcal{I}^\varepsilon$; it is clear that $[\widehat{w_j - w}]$ converges to $w_j - w$ strongly in $L_2(\square)$ as $\varepsilon\to 0$.

In a similar manner we obtain

$$\lim_{\varepsilon\to 0}\sum_{i\in\mathcal{I}^\varepsilon}\sum_{j=1}^m\int\limits_{B_{ij}^\varepsilon}\lambda^\varepsilon w^\varepsilon u^\varepsilon\,dV^\varepsilon = \lambda\sum_{j=1}^m\rho_j\int\limits_\square w_j u_j\,dx. \tag{3.52}$$

Thus, from (3.49)–(3.52) we obtain that the functions $u\in\operatorname{dom}(\bar{\eta}_\square^\theta)$, $u_j\in L_2(\square)$ $(j = 1,\dots,m)$ satisfy the equality:

$$\int\limits_{\square}\left[-\Delta w u+\sum_{j=1}^{m}\sigma_j\rho_j u(w-w_j)+\sum_{j=1}^{m}\sigma_j\rho_j u_j(w_j-w)\right]dx$$
$$+\int\limits_{\partial\square}\nu[w]u ds-\lambda\int\limits_{\square}\left[uw+\sum_{j=1}^{m}\rho_j u_j w_j\right]dx=0 \tag{3.53}$$

for arbitrary $w\in C_\theta^\infty(\square)$, $w_j\in C^\infty(\square)$ $(j=1,\dots,m)$.

Substituting $w\equiv 0$, $w_j\equiv 0$, $j\neq k$ into (3.53) we obtain

$$u_k=\frac{\sigma_k u}{\sigma_k-\lambda},\quad k=1,\dots,m. \tag{3.54}$$

Then substituting into (3.53) $w_j\equiv 0$ $(\forall j)$, integrating by parts and taking into account (3.54), we conclude that $u\in\mathrm{dom}(\bar\eta_\square^\theta)$ satisfies the equality

$$\int\limits_{\square}\big[(\nabla u,\nabla w)-\lambda\mathcal{F}(\lambda)uw\big]dx=0,\quad \forall w\in C_\theta^\infty(\square)$$

where $\mathcal{F}(\lambda)$ is defined by (2.10). Hence $u\in\mathrm{dom}(\Delta_\square^\theta)$ and

$$-\Delta_\square^\theta u=\lambda\mathcal{F}(\lambda)u.$$

In view of Lemma 3.4 $u\neq 0$. Then $\lambda\mathcal{F}(\lambda)\in\sigma(-\Delta_{\mathbb{R}^n})$ and, therefore, due to (3.3) $\lambda\in\sigma(\mathcal{A})\setminus\bigcup_{j=1}^m\{\sigma_j\}$.

The fulfilment of property (A) is proved.

3.3. *Property (B) of Hausdorff convergence*

Let $\lambda\in\sigma(\mathcal{A})\cap[0,L]$, $L\notin\bigcup_{j=1}^m\{\mu_j\}$. We have to prove that there exists $\lambda^\varepsilon\in\sigma(-\Delta_{M^\varepsilon})\cap[0,L]$ such that $\lambda^\varepsilon\underset{\varepsilon\to 0}{\to}\lambda$.

At first we prove property (B) for the case $\lambda<L$.

We assume the opposite: the subsequence (still denoted by ε) and $\delta>0$ exist such that

$$\mathrm{dist}\big(\lambda,\sigma(-\Delta_{M^\varepsilon})\big)>\delta. \tag{3.55}$$

Since $\lambda\in\sigma(\mathcal{A})$, then the function

$$F=\begin{pmatrix} f\\ f_1\\ \cdots\\ f_m\end{pmatrix}\in H$$

exists such that

$$F\notin\mathrm{im}(\mathcal{A}-\lambda\mathrm{I}),\quad \text{where I is the identity operator}. \tag{3.56}$$

Let $f^\varepsilon(\tilde x)\in L_2(M^\varepsilon)$ be defined by the formula

$$f^\varepsilon(\tilde x)=\begin{cases} f(x), & \tilde x\in\Omega^\varepsilon,\\ \langle f_j\rangle_{\square_i^\varepsilon}, & \tilde x\in B_{ij}^\varepsilon.\end{cases}$$

It follows from the Cauchy inequality and (2.7) that the norms $\|f^\varepsilon\|_{L_2(M^\varepsilon)}$ are bounded uniformly in ε.

Inequality (3.55) implies that $\lambda \in \mathbb{R} \setminus \sigma(-\Delta_{M^\varepsilon})$. Then $\mathrm{im}(-\Delta_{M^\varepsilon} - \lambda \mathrm{I}) = L_2(M^\varepsilon)$ and thus, the unique $u^\varepsilon \in \mathrm{dom}(\Delta_{M^\varepsilon})$ exists satisfying

$$-\Delta_{M^\varepsilon} u^\varepsilon - \lambda u^\varepsilon = f^\varepsilon. \tag{3.57}$$

In consequence of (3.55) u^ε satisfies the inequality

$$\|u^\varepsilon\|_{L_2(M^\varepsilon)} \leqslant \delta^{-1} \|f^\varepsilon\|_{L_2(M^\varepsilon)} \leqslant C.$$

Furthermore

$$\|\nabla u^\varepsilon\|^2_{L_2(M^\varepsilon)} \leqslant \|f^\varepsilon\|_{L_2(M^\varepsilon)} \cdot \|u^\varepsilon\|_{L_2(M^\varepsilon)} + |\lambda| \cdot \|u^\varepsilon\|^2_{L_2(M^\varepsilon)} \leqslant C.$$

Then there exists a subsequence (still denoted by ε) such that

$$\Pi^\varepsilon u^\varepsilon \to u \in H^1(\mathbb{R}^n) \text{ weakly in } H^1(\mathbb{R}^n) \text{ and strongly in } L_2(G) \text{ for any compact set } G \subset \mathbb{R}^n,$$
$$\Pi_j^\varepsilon u^\varepsilon \to u_j \in L_2(\mathbb{R}^n) \text{ weakly in } L_2(\mathbb{R}^n) \quad (j = 1, \dots, m)$$

where Π^ε, Π_j^ε $(j = 1, \dots, m)$ are the extension operators introduced in the previous subsection.

For an arbitrary function $\mathbf{w}^\varepsilon \in C_0^\infty(M^\varepsilon)$ we have

$$\int_{M^\varepsilon} \left(\left(\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon \mathbf{w}^\varepsilon\right)_\varepsilon - \lambda u^\varepsilon \mathbf{w}^\varepsilon - f^\varepsilon \mathbf{w}^\varepsilon \right) dV^\varepsilon = 0. \tag{3.58}$$

Let $w \in C_0^\infty(\mathbb{R}^n)$, $w_j \in C_0^\infty(\mathbb{R}^n)$ $(j = 1, \dots, m)$ be arbitrary functions. Using them we construct the test-function $\mathbf{w}^\varepsilon$ by formula (3.43) (but with $\mathbb{R}^n$ instead of $\Omega^\varepsilon_\square$ and with $\mathbb{Z}^n$ instead of $\mathcal{I}^\varepsilon$) and substitute it into (3.58). Performing the same calculations as in the previous subsection we obtain

$$\int_{\mathbb{R}^n} \left[(\nabla u, \nabla w) + \sum_{j=1}^m \sigma_j \rho_j u (w - w_j) + \sum_{j=1}^m \sigma_j \rho_j u_j (w_j - w) \right.$$
$$\left. - \lambda \left(u w + \sum_{j=1}^m \rho_j u_j w_j \right) - \left(f w + \sum_{j=1}^m \rho_j f_j w_j \right) \right] dx = 0 \tag{3.59}$$

for arbitrary $w \in C_0^\infty(\mathbb{R}^n)$, $w_j \in C_0^\infty(\mathbb{R}^n)$ $(j = 1, \dots, m)$. It follows from (3.59) that

$$U = \begin{pmatrix} u \\ u_1 \\ \dots \\ u_m \end{pmatrix} \in \mathrm{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}U - \lambda U = F.$$

We obtain a contradiction with (3.56). Then there is $\lambda^\varepsilon \in \sigma(-\Delta_{M^\varepsilon})$ such that $\lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda$. Since $\lambda < L$, then $\lambda^\varepsilon < L$ when ε is small enough.

Finally, we verify the fulfilment of property (B) for the case $\lambda = L$. Since $L \notin \bigcup_{j=1}^m \{\mu_j^\varepsilon\}$, then (3.1) implies that $(L - \delta, L - \delta/2) \subset \sigma(\mathcal{A})$ when δ is small enough. Let $\lambda_\delta \in (L - \delta, L - \delta/2)$. We have just proved that if $\varepsilon < \varepsilon(\delta)$ then $\lambda^\varepsilon \in \sigma(-\Delta_{M^\varepsilon})$ exists such that $|\lambda^\varepsilon - \lambda_\delta| < \delta/2$. Then $\lambda^\varepsilon \in (L - 3\delta/2, L)$ as $\varepsilon < \varepsilon(\delta)$ that obviously implies the fulfilment of property (B).

3.4. End of the proof

In the proof of the Hausdorff convergence we used the fact that M^ε is Γ-periodic manifold, $\mathbf{M}^\varepsilon$ is a period cell. Now let us recall that M^ε is also Γ^ε-periodic manifold, $\mathbf{M}_i^\varepsilon$ is a corresponding period cell (i is arbitrary, so from now on we consider $i=0$). Then

$$\sigma(-\Delta_{M^\varepsilon}) = \bigcup_{k=1}^{\infty} \big[a_k^\varepsilon, b_k^\varepsilon\big],$$

where $[a_k^\varepsilon, b_k^\varepsilon] = \{\lambda_k^\theta(\mathbf{M}_0^\varepsilon),\ \theta \in \mathbb{T}^n\}$.

Lemma 3.6. $\lim_{\varepsilon\to 0} b_{m+1}^\varepsilon = \infty$.

Proof. As usual by $\lambda_k^N(\mathbf{M}_0^\varepsilon)$ we denote the k-th eigenvalue of the operator $-\Delta^N_{\mathbf{M}_0^\varepsilon}$, which is the Laplace–Beltrami operator on $\mathbf{M}_0^\varepsilon$ with Neumann boundary conditions.

Using the same idea as in the proof of Lemma 3.3 (i.e. ε^{-1}-homothetic image of $\mathbf{M}_0^\varepsilon$), we get

$$\lim_{\varepsilon\to 0} \varepsilon^2 \lambda_k\big(\mathbf{M}_0^\varepsilon\big) = \lambda_k, \quad k = 1, 2, 3, \ldots, \tag{3.60}$$

where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the eigenvalues of the operator $\mathcal{L}$ which acts in the space $L_2(\Box)\bigoplus_{j=\overline{1,m}} L_2(\mathbf{B}_j)$ and is defined by the operation

$$\mathcal{L} = -\begin{pmatrix} \Delta_\Box^N & 0 & \cdots & 0 \\ 0 & \Delta_{\mathbf{B}_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_{\mathbf{B}_m} \end{pmatrix}.$$

Recall that $\Box$ is the unit cube in $\mathbb{R}^n$, $\mathbf{B}_j$ is the n-dimensional sphere of the radius b_j $(j = 1, \ldots, m)$.

One has $\lambda_j = \lambda_1(\mathbf{B}_j) = 0$, $j = 1, \ldots, m$, $\lambda_{m+1} = \lambda_1^N(\Box) = 0$, and

$$\lambda_{m+2} = \min\big\{\lambda_2^N(\Box), \lambda_2(\mathbf{B}_j),\ j = 1, \ldots, m\big\} > 0.$$

Thus, in view of (3.60) $\lim_{\varepsilon\to 0}\lambda_{m+2}^N(\mathbf{M}_0^\varepsilon) = \infty$. Due to inequality (1.3) $\lambda_{m+2}^N(\mathbf{M}_0^\varepsilon) \leqslant a_{m+2}^\varepsilon$. Thus, $\lim_{\varepsilon\to 0} a_{m+2}^\varepsilon = \infty$.

Suppose that there exists a subsequence (still denoted by ε) such that the numbers b_{m+1}^ε are bounded uniformly in ε. Let $L > \max_{j=\overline{1,m}} \mu_j$ and $L > b_{m+1}^\varepsilon$. Let $L_1 > L$. Since $a_{m+2}^\varepsilon \underset{\varepsilon\to 0}{\to} \infty$, then $a_{m+2}^\varepsilon > L_1$ when ε is small enough. Hence $\sigma(-\Delta_{M^\varepsilon}) \cap [L, L_1] = \varnothing$ when ε is small enough. But this contradicts to property (B) of the Hausdorff convergence. Hence $b_{m+1}^\varepsilon \underset{\varepsilon\to 0}{\to} \infty$. The lemma is proved. $\Box$

It follows from Lemma 3.6 that within an arbitrary finite interval $[0, L]$ the spectrum $\sigma(-\Delta_{M^\varepsilon})$ has at most m gaps when ε is small enough, i.e.

$$\sigma(-\Delta_{M^\varepsilon}) \cap [0, L] = [0, L] \setminus \bigcup_{j=1}^{m^\varepsilon} \big(\sigma_j^\varepsilon, \mu_j^\varepsilon\big) \tag{3.61}$$

where $(\sigma_j^\varepsilon, \mu_j^\varepsilon) \subset [0, L]$ are some pairwise disjoint intervals, $m^\varepsilon \leqslant m$. Here we renumber the intervals in the increasing order.

Let $L > \max_{j=\overline{1,m}} \mu_j$ be arbitrarily large number. We have just proved that as $\varepsilon \to 0$ the set $\sigma(-\Delta_{M^\varepsilon}) \cap [0, L]$ converges to the set $\sigma(\mathcal{A}) \cap [0, L] = [0, L] \setminus (\bigcup_{j=1}^{m}(\sigma_j, \mu_j))$ in the Hausdorff sense. Then by Proposition 3.1 $m^\varepsilon = m$ when ε is small enough and

$$\forall j = 1, \ldots, m: \quad \lim_{\varepsilon\to 0} \sigma_j^\varepsilon = \sigma_j, \qquad \lim_{\varepsilon\to 0} \mu_j^\varepsilon = \mu_j.$$

Finally, we denote by $\mathcal{J}^\varepsilon$ the union of the remaining gaps (if any). Since $b_{m+1}^\varepsilon \underset{\varepsilon\to 0}{\to} \infty$ and $b_{m+1}^\varepsilon \leqslant \inf \mathcal{J}^\varepsilon$, then

$$\inf \mathcal{J}^\varepsilon > L$$

when ε is small enough. This concludes the proof of Theorem 2.1.

Remark 3.1. Actually, we have proved a slightly strong result: $\lim_{\varepsilon\to 0} a_{k+1}^\varepsilon = \mu_k$, $\lim_{\varepsilon\to 0} b_k^\varepsilon = \sigma_k$, $k = 1, \ldots, m$, $\lim_{\varepsilon\to 0} b_{m+1}^\varepsilon = \infty$, i.e. the first m gaps of the spectrum $\sigma(-\Delta_{M^\varepsilon})$ (ε is small enough) are located exactly between the first $(m+1)$ bands.

4. End of the proof of Theorem 0.1: choice of the constants d_j, b_j and conclusive remarks

In order to complete the proof of Theorem 0.1, we have to choose the constants d_j, b_j in (2.6), (2.7) such that equalities (0.8) hold.

Theorem 4.1. *Let* (α_j, β_j) $(j = 1, \ldots, m,\ m \in \mathbb{N})$ *be arbitrary intervals satisfying* (0.4). *Let* M^ε $(\varepsilon > 0)$ *be an* n*-dimensional periodic Riemannian manifolds of the form* (2.1).

Then (0.8) *holds if we choose*

$$d_j = \begin{cases} \left[\frac{2(\beta_j-\alpha_j)}{\omega_{n-1}(n-2)} \prod_{i=\overline{1,m}|i\neq j} (\frac{\beta_i-\alpha_j}{\alpha_i-\alpha_j})\right]^{\frac{1}{n-2}}, & n > 2, \\ \frac{(\beta_j-\alpha_j)}{\pi} \prod_{i=\overline{1,m}|i\neq j} (\frac{\beta_i-\alpha_j}{\alpha_i-\alpha_j}), & n = 2, \end{cases} \tag{4.1}$$

$$b_j = \left[\frac{\beta_j - \alpha_j}{\omega_n \alpha_j} \prod_{i=\overline{1,m}|i\neq j} \left(\frac{\beta_i - \alpha_j}{\alpha_i - \alpha_j}\right)\right]^{\frac{1}{n}}. \tag{4.2}$$

Remark 4.1. Since the intervals (α_j, β_j) satisfy (0.4), then

$$\forall j: \quad \alpha_j < \beta_j, \qquad \forall i \neq j: \quad \operatorname{sign}(\beta_i - \alpha_j) = \operatorname{sign}(\alpha_i - \alpha_j) \neq 0.$$

Therefore, the expressions $(\beta_j - \alpha_j) \prod_{i=\overline{1,m}|i\neq j} (\frac{\beta_i-\alpha_j}{\alpha_i-\alpha_j})$, $j = 1, \ldots, m$ are positive and thus the choice of d_j and b_j is correct.

Proof of Theorem 4.1. Substituting d_j, b_j (4.1), (4.2) into (2.8) we get

$$\sigma_j = \alpha_j,$$

i.e. the first equality in (0.8) holds. Furthermore substituting b_j (4.2) into (2.9) we obtain

$$\rho_j = \frac{\beta_j - \alpha_j}{\alpha_j} \prod_{i=\overline{1,m}|i\neq j} \left(\frac{\beta_i - \alpha_j}{\alpha_i - \alpha_j}\right). \tag{4.3}$$

It remains to prove that $\mu_j = \beta_j$. Recall that the numbers μ_j $(j = 1, \ldots, m)$ are the roots of Eq. (2.10). Therefore, in order to prove the equality $\mu_j = \beta_j$, we have to show that

$$\forall k = 1, \ldots, m: \quad \sum_{j=1}^{m} \frac{\alpha_j \rho_j}{\beta_k - \alpha_j} = 1. \tag{4.4}$$

Let us consider (4.4) as the linear algebraic system of m equations with unknowns ρ_j $(j = 1, \ldots, m)$. In order to end the proof of theorem we have to prove the following

Lemma 4.1. *The system* (4.4) *has the unique solution* $\rho_1, \ldots, \rho_m$ *which is defined by* (4.3)*.*

Proof. We prove the lemma by induction. For $m = 1$ its validity is obvious. Suppose that we have proved it for $m = N - 1$ and let us prove it for $m = N$.

Multiplying the k-th equation in (4.4) $(k = 1, \ldots, N)$ by $\beta_k - \alpha_N$ and then subtracting the N-th equation from the first $N - 1$ equations we obtain

$$\forall k = 1, \ldots, N-1: \quad \sum_{j=1}^{N-1} \frac{\alpha_j \hat{\rho}_j}{\beta_k - \alpha_j} = 1$$

where the new variables $\hat{\rho}_j$, $j = 1, \ldots, N-1$ are expressed in terms of ρ_j by the formula

$$\hat{\rho}_j := \rho_j \frac{\alpha_N - \alpha_j}{\beta_N - \alpha_j}, \quad j = 1, \ldots, N-1. \tag{4.5}$$

Thus, the numbers $\hat{\rho}_j$ satisfy the system (4.4) with $m = N - 1$. Therefore, by the induction

$$\hat{\rho}_j = \frac{\beta_j - \alpha_j}{\alpha_j} \prod_{i=\overline{1,N-1} | i \neq j} \left(\frac{\beta_i - \alpha_j}{\alpha_i - \alpha_j} \right). \tag{4.6}$$

It follows from (4.5), (4.6) that ρ_j, $j = 1, \ldots, N-1$, satisfy formula (4.3). The validity of this formula for ρ_N follows from the symmetry of the system.

Lemma 4.1 and Theorem 4.1 are proved. This completes the proof of the main theorem. □

Remark 4.2. We noted above that the metric g^ε of the manifold M^ε is continuous but piecewise-smooth (see formulae (2.3)–(2.4)). However one can approximate g^ε by a smooth metric $g^{\varepsilon\rho}$ which differs from g^ε only in small ρ-neighbourhoods of $\partial B_{ij}^\varepsilon$ and moreover the corresponding Laplace–Beltrami operator has the same spectral properties as $\varepsilon \to 0$.

Namely, in a small neighbourhood U_{ij}^ε of $\partial B_{ij}^\varepsilon$ we introduce the local coordinates $(x_1, \ldots, x_n)$ by formulae (2.2) and define $g^{\varepsilon\rho}$ by the formula

$$g_{\alpha\beta}^{\varepsilon\rho}(x_1, \ldots, x_n) = g_{+\alpha\beta}^{\varepsilon}(x_1, \ldots, x_n)\varphi(x_n/\rho) + g_{-\alpha\beta}^{\varepsilon}(x_1, \ldots, x_n)\big(1 - \varphi(x_n/\rho)\big)$$

where $\varphi(r)$, $r \in \mathbb{R}$, is a smooth positive function equal to 1 as $r \geqslant 1$, equal to 0 as $r \leqslant -1$ and positive as $-1 < r < 1$, the coefficients $g_{\pm\alpha\beta}^{\varepsilon}$ are defined by (2.4). Outside $\bigcup_{i,j} U_{ij}^\varepsilon$ we set $g^{\varepsilon\rho} = g^\varepsilon$.

It is easy to see that $A^{\varepsilon\rho} g^\varepsilon \leqslant g^{\varepsilon\rho} \leqslant B^{\varepsilon\rho} g^\varepsilon$, where $A^{\varepsilon\rho}$, $B^{\varepsilon\rho}$ are positive constants depending on ε and ρ in such a way that for **fixed** ε

$$\lim_{\rho \to 0} A^{\varepsilon\rho} = \lim_{\rho \to 0} B^{\varepsilon\rho} = 1. \tag{4.7}$$

Using the min–max principle one can obtain that

$$\forall k \in \mathbb{N},\ \forall \theta \in \mathbb{T}^n:\quad \frac{(A^{\varepsilon\rho})^{n/2}}{(B^{\varepsilon\rho})^{1+n/2}}\lambda_k^\theta\big(\mathbf{M}_i^\varepsilon\big) \leqslant \lambda_k^\theta\big(\mathbf{M}_i^\varepsilon, g^{\varepsilon\rho}\big) \leqslant \frac{(B^{\varepsilon\rho})^{n/2}}{(A^{\varepsilon\rho})^{1+n/2}}\lambda_k^\theta\big(\mathbf{M}_i^\varepsilon\big). \tag{4.8}$$

Here $\lambda_k^\theta(\mathbf{M}_i^\varepsilon, g^{\varepsilon\rho})$ is the k-th eigenvalue of the Laplace–Beltrami operator with θ-periodic boundary conditions on the manifold $\mathbf{M}_i^\varepsilon$ equipped with the metric $g^{\varepsilon\rho}$. This inequality is proved in [1, Chapter A] for manifolds without a boundary, for our case the proof is completely analogous.

Let $\delta_1 > 0$, $L_1 > 0$. We have just proved (see Theorems 2.1, 4.1) that there are such $\varepsilon = \varepsilon(\delta_1, L_1)$ and such d_j, b_j that the manifold $M = M^\varepsilon$ satisfies (0.2)–(0.3) with $\delta = \delta_1$, $L = L_1$.

So let us fix $\varepsilon = \varepsilon(L_1, \delta_1)$. Then it follows from (4.7), (4.8) that

$$\forall \theta \in \mathbb{T}^n,\ \forall k \in \mathbb{N}:\quad \big|\lambda_k^\theta\big(\mathbf{M}_i^\varepsilon\big) - \lambda_k^\theta\big(\mathbf{M}_i^\varepsilon, g^{\varepsilon\rho}\big)\big| \underset{\rho\to 0}{\to} 0 \tag{4.9}$$

uniformly in (θ, k) from $\mathbb{T}^n \times \mathbb{G}$, where $\mathbb{G}$ is any compact subset of $\mathbb{N}$. Then using (1.4), (4.9) and taking into account Remark 3.1 we conclude: there is such $\rho = \rho(\varepsilon(\delta_1, L_1))$ that the manifold $(M^\varepsilon, g^{\varepsilon\rho})$ satisfies (0.2)–(0.3) with $\delta = 2\delta_1$, $L = L_1 - \delta_1$.

Now, let $\delta > 0$, $L > 0$. Setting $\delta_1 = \delta/2$, $L_1 = L + \delta/2$ we conclude that the manifold $M = (M^\varepsilon, g^{\varepsilon\rho})$, where $\varepsilon = \varepsilon(\delta_1, L_1)$, $\rho = \rho(\varepsilon(\delta_1, L_1))$, satisfies (0.2)–(0.3).

Acknowledgments

The author is grateful to Prof. E. Khruslov for the helpful discussion. The work is supported by the French–Ukrainian grant "PICS 2009-2011. Mathematical Physics: Methods and Applications".

References

[1] C. Anné, Exemples de convergence de valeurs propres sur des surfaces ayant une anse trés fine, Thèse, Université S. et M. Grenoble, 1984.

[2] L. Boutet de Monvel, E.Ya. Khruslov, Averaging of the diffusion equation on Riemannian manifolds of complex microstructure, Trans. Moscow Math. Soc. (1997) 137–161.

[3] L. Boutet de Monvel, I.D. Chueshov, E.Ya. Khruslov, Homogenization of attractors for semilinear parabolic equations on manifolds with complicated microstructure, Ann. Mat. Pura Appl. (4) 172 (1997) 297–322.

[4] Y. Colin de Verdière, Construction de laplaciens dont une partie finie du spectre est donnee, Ann. Sci. Éc. Norm. Supér. (4) 20 (4) (1987) 599–615.

[5] G. Dal Maso, R. Gulliver, U. Mosco, Asymptotic spectrum of manifolds of increasing topological type, Preprint S.I.S.S.A. 78/2001/M, Trieste, 2001.

[6] E.B. Davies, E.M. Harrell II, Conformally flat Riemannian metrics, Schrödinger operators, and semiclassical approximation, J. Differential Equations 66 (2) (1987) 165–188.

[7] P. Exner, O. Post, Convergence of spectra of graph-like thin manifolds, J. Geom. Phys. 54 (1) (2005) 77–115.

[8] A. Figotin, P. Kuchment, Band-gap structure of the spectrum of periodic dielectric and acoustic media. I. Scalar model, SIAM J. Appl. Math. 56 (1) (1996) 68–88.

[9] A. Figotin, P. Kuchment, Band-gap structure of the spectrum of periodic dielectric and acoustic media. II. Two-dimensional photonic crystals, SIAM J. Appl. Math. 56 (6) (1996) 1561–1620.

[10] N. Filonov, Gaps in the spectrum of the Maxwell operator with periodic coefficients, Comm. Math. Phys. 240 (1–2) (2003) 161–170.

[11] L. Friedlander, On the density of states of periodic media in large coupling limit, Comm. Partial Differential Equations 27 (1–2) (2002) 355–380.

[12] E.L. Green, Spectral theory of Laplace–Beltrami operators with periodic metrics, J. Differential Equations 133 (1997) 15–29.

[13] R. Hempel, I. Herbst, Strong magnetic fields, Dirichlet boundaries, and spectral gaps, Comm. Math. Phys. 169 (2) (1995) 237–259.

[14] R. Hempel, K. Lienau, Spectral properties of periodic media in the large coupling limit, Comm. Partial Differential Equations 25 (2000) 1445–1470.

[15] R. Hempel, O. Post, Spectral gaps for periodic elliptic operators with high contrast: an overview, in: Progress in Analysis, Proceedings of the 3rd International ISAAC Congress Berlin 2001, vol. 1, 2003, pp. 577–587.

[16] G.A. Iosifyan, O.A. Olejnik, A.S. Shamaev, On the limiting behaviour of the spectrum of a sequence of operators defined on different Hilbert spaces, Russian Math. Surveys 44 (3) (1989) 195–196.

[17] A. Khrabustovskyi, Asymptotic behaviour of spectrum of Laplace–Beltrami operator on Riemannian manifolds with complex microstructure, Appl. Anal. 87 (12) (2002) 1357–1372.
[18] A. Khrabustovskyi, On the spectrum of Riemannian manifolds with attached thin handles, J. Math. Phys. Anal. Geom. 5 (2) (2009) 145–169.
[19] A. Khrabustovskyi, Homogenization of eigenvalue problem for Laplace–Beltrami operator on Riemannian manifold with complicated 'bubble-like' microstructure, Math. Methods Appl. Sci. 32 (16) (2009) 2123–2137.
[20] A. Khrabustovskyi, Homogenization of spectral problem on Riemannian manifold consisting of two domains connected by many tubes, Proc. Roy. Soc. Edinburgh Sect. A Math., submitted for publication; arXiv:1011.3931.
[21] F. Lledo, O. Post, Existence of spectral gaps, covering manifolds and residually finite groups, Rev. Math. Phys. 20 (2) (2008) 199–231.
[22] V.A. Marchenko, E.Ya. Khruslov, Homogenization of Partial Differential Equations, Prog. Math. Phys., vol. 46, Birkhäuser, Boston, 2006.
[23] O. Post, Periodic manifolds, spectral gaps, and eigenvalues in gaps, PhD thesis, Technische Universit́at Braunschweig, 2000, http://www.mathematik.hu-berlin.de/~post/papers/promo-e.pdf.
[24] O. Post, Periodic manifolds with spectral gaps, J. Differential Equations 187 (1) (2003) 23–45.
[25] J. Rauch, M. Taylor, Potential and scattering theory on wildly perturbed domains, J. Funct. Anal. 18 (1975) 27–59.
[26] M. Reed, B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, 1978.
[27] L. Tartar, The General Theory of Homogenization. A Personalized Introduction, Springer, Berlin, 2009.
[28] M. Taylor, Partial Differential Equations I. Basic Course, Springer-Verlag, Heidelberg, 1996.
[29] V.V. Zhikov, S.M. Kozlov, O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer, New York, 1994.
[30] V. Zhikov, On spectrum gaps of some divergent elliptic operators with periodic coefficients, St. Petersburg Math. J. 16 (5) (2005) 773–790.