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On the subset theorem in dimension theory

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ABSTRACT

The most general subset theorem for the covering dimension for arbitrary topological spaces is obtained in the paper.

"Space", "map", "(para)compactum", "fo" will be used instead of "topological space", "continuous mapping between spaces", "(para)compact Hausdorff space", "functionally open", respectively. Only the covering dimension dim defined by means of finite fo covers is considered. Below $r = 0, 1, \dots, \infty$ and for a set $\mathcal{A}$, $\mathcal{A}^*$ and $\mathcal{A}^*_\emptyset$ denote the sets of all finite and, respectively, of all finite non-empty subsets of $\mathcal{A}$.

In [4] and [5], a subspace X of a space Y was called *d-right* (in Y) if, for any fo subset U of X, there exists a σ-locally finite and fo in X family ν such that $U = \bigcup \nu$ and any $V \in \nu$ has a fo piecewise extension $W = W(V)$ in Y, i.e., W is fo in Y and V is closed–open in $W \cap X$.

It was announced in [4] and proved [5] that

$$\dim X \leqslant \dim Y \qquad (*)$$

if X is d-right in Y.

In [1], the d-rightness and the cited result were generalized in the following way (note that, for a map $f : X \to [0, 1]$, $\operatorname{coz} f = f^{-1}(0, 1]$).

A subspace X of a space Y is called *countable accessible* if for every fo set G of X there exists a system $[f]$ of maps $f_{is} : X \to [0, 1]$, $i \in \mathbb{N}$, $s \in S$, such that each $f_{is}|_{\operatorname{coz} f_{is}}$ has a continuous extension over Y (to $[0, 1]$), $\{\operatorname{coz} f_{is}\colon\ s \in S\}$ is locally finite in X for each i, and G is open in the topology τ on X generated by $[f]$, i.e., all sets $f_{is}^{-1} O$, where O is open in $[0, 1]$, $i \in \mathbb{N}$, $s \in S$, is a subbase for τ.

It was shown in [1] that $(*)$ is true if X is a countable accessible subspace of Y.

In our paper this result will be generalized.

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1. Topology generated by a family of subspaces

Recall one definition and some assertions.

Let we have a space (Z,τ) and its cover $\mathcal{Q}$ by its subspaces.

A subset O of Z will be called *$\mathcal{Q}$-open* if for any $Q\in\mathcal{Q}$, the set $O\cap Q$ is open in Q. Evidently,
the family $\tau_{\mathcal{Q}}$ of all $\mathcal{Q}$-open sets is a topology on the set Z called the *topology generated by* $\mathcal{Q}$;
the identical mapping $\mathrm{id}_{\mathcal{Q}}$ of $(Z,\tau_{\mathcal{Q}})$ onto (Z,τ) is continuous;
for any $Q\in\mathcal{Q}$, the identical map $\mathrm{id}_{\mathcal{Q}Q}$ of $Q\subset(Z,\tau_{\mathcal{Q}})$ onto $Q\subset(Z,\tau)$ is a homeomorphism;
a set $F\subset Z$ is closed in $(Z,\tau_{\mathcal{Q}})$ iff $F\cap Q$ is closed in $Q\subset(Z,\tau)$ for any $Q\in\mathcal{Q}$; and
if (Z,τ) is Hausdorff then $(Z,\tau_{\mathcal{Q}})$ is also Hausdorff.

Since $\mathrm{id}_{\mathcal{Q}Q}$ is a homeomorphism for any $Q\in\mathcal{Q}$ we have the following lemma.

Lemma 1.1. *Let a map $f:X\to(Z,\tau)$ be such that for any $x\in X$, there exist a neighborhood Ox of x and $Q\in\mathcal{Q}$ with $fOx\subset Q$. Then the mapping $f_{\mathcal{Q}}:X\to(Z,\tau_{\mathcal{Q}})$ such that $f=\mathrm{id}_{\mathcal{Q}}\circ f_{\mathcal{Q}}$ is continuous.*

Proposition 1.2. *Let (Z,τ) be Hausdorff; all $Q\in\mathcal{Q}$ be perfectly normal and closed in (Z,τ); $\mathcal{Q}=\bigcup\{\mathcal{Q}_n:\ n=0,1,\ldots\}$; $Z_n=\bigcup\mathcal{Q}_n$ be closed in (Z,τ); $Z_n\subset Z_{n+1}$; $(*)$ for any $Q\in\mathcal{Q}_n$ and $k\leqslant n$, $Q\cap Z_k$ be contained in the union of finite many elements of $\mathcal{Q}_k$; the family $\langle\mathcal{Q}_0\rangle=\mathcal{Q}_0$ be disjoint and open in Z_0 and the families $\langle\mathcal{Q}_{n+1}\rangle=\{\langle Q\rangle=Q\setminus Z_n:\ Q\in\mathcal{Q}_{n+1}\}$ be disjoint and open in $Z_{n+1}\subset(Z,\tau)$, $n=0,1,\ldots.$*

Then all $Q\in\mathcal{Q}$ and all Z_n are closed in $(Z,\tau_{\mathcal{Q}})$; $(**)$ $F\subset Z_n$ is closed in $Z_n\subset(Z,\tau_{\mathcal{Q}})$ (and in $(Z,\tau_{\mathcal{Q}})$) iff $F\cap Q$ is closed in $Q\subset(Z,\tau)$ for any $Q\in\bigcup\{\mathcal{Q}_i:\ i=0,1,\ldots,n\}$; $(Z,\tau_{\mathcal{Q}})$ is perfectly normal.

If

$$(***)\quad \dim Q\leqslant r \text{ for any } Q\in\mathcal{Q}_0 \text{ and } \dim\langle Q\rangle\leqslant r \text{ for any } Q\in\mathcal{Q}_n,\ n=1,2,\ldots,$$

then $\dim(Z,\tau_{\mathcal{Q}})\leqslant r$, $r=0,1,\ldots.$

Proof. Since all $\mathrm{id}_{\mathcal{Q}Q}$ are homeomorphisms, all $Q\subset(Z,\tau_{\mathcal{Q}})$ are perfectly normal.

Since $\mathrm{id}_{\mathcal{Q}}$ is a condensation, all Q and Z_n are closed in $(Z,\tau_{\mathcal{Q}})$; the family $\langle\mathcal{Q}_0\rangle$ is disjoint and open (and so discrete) in $Z_0\subset(Z,\tau_{\mathcal{Q}})$; and the family $\langle\mathcal{Q}_{n+1}\rangle$ is disjoint and open in $Z_{n+1}\subset(Z,\tau_{\mathcal{Q}})$ (and so it is discrete in its own union as a subspace of $(Z,\tau_{\mathcal{Q}})$), $n=0,1,\ldots.$ Hence $\langle Z_0\rangle=Z_0\subset(Z,\tau_{\mathcal{Q}})$ and $\langle Z_n\rangle=\bigcup\langle\mathcal{Q}_n\rangle\subset(Z,\tau_{\mathcal{Q}})$, $n=1,2,\ldots,$ are perfectly normal.

Take $F\subset Z_n$. If F is closed in $Z_n\subset(Z,\tau_{\mathcal{Q}})$ and $Q\in\bigcup\{\mathcal{Q}_i:\ i=0,1,\ldots,n\}$ then $F\cap Q$ is closed in Q as a subspace of $(Z,\tau_{\mathcal{Q}})$. Since $\mathrm{id}_{\mathcal{Q}Q}$ is a homeomorphism, $F\cap Q$ is closed in Q as a subspace of (Z,τ). Let $F\cap Q$ be closed in Q as a subspace of (Z,τ) for any $Q\in\bigcup\{\mathcal{Q}_i:\ i=0,1,\ldots,n\}$. Take $Q\in\mathcal{Q}_k$ for $k>n$. Then there exist $Q_i\in\mathcal{Q}_n$, $i=1,\ldots,p$, such that $Q\cap Z_n\subset Q_1\cup\cdots\cup Q_p$. It follows from this that $F\cap Q=F\cap Q\cap Z_n=F\cap Q\cap(Q_1\cup\cdots\cup Q_p)=(F\cap Q_1\cap Q)\cup\cdots\cup(F\cap Q_p\cap Q)$. Since $(F\cap Q_i)\cap Q$, $i=1,\ldots,p$, are closed in (Z,τ), $F\cap Q$ is closed in $(Z,\tau_{\mathcal{Q}})$ and in $Z_n\subset(Z,\tau_{\mathcal{Q}})$.

Fix $n=1,2,\ldots.$ The set $\langle Q\rangle\in\langle\mathcal{Q}_n\rangle$ is open in the perfectly normal space Q. Hence it is the union of closed in Q (and so in $(Z,\tau_{\mathcal{Q}})$) sets F_{Qni}, $i\in\mathbb{N}$. Take $Q'\in\mathcal{Q}_k$, $k\leqslant n$. If $k<n$ then $F_{ni}\cap Q'=\emptyset$. If $k=n$ then $F_{ni}\cap Q'=F_{Q'ni}$. Hence, by $(**)$, F_{ni} is closed in $Z_n\subset(Z,\tau_{\mathcal{Q}})$ and so in $(Z,\tau_{\mathcal{Q}})$. Thus $\langle Z_n\rangle$ is the union of countably many closed in $(Z,\tau_{\mathcal{Q}})$ perfectly normal subspaces and so $(Z,\tau_{\mathcal{Q}})$ also is the union of countably many closed and perfectly normal subspaces. Hence every open in $(Z,\tau_{\mathcal{Q}})$ set is of type F_σ.

Let us prove that $(Z,\tau_{\mathcal{Q}})$ is normal.

Take a closed in $(Z,\tau_{\mathcal{Q}})$ set F and a map f of F to the unite segment $I=[0,1]$.

Since every $Q\in\mathcal{Q}_0$ is perfectly normal, there exists a continuous extension $f_Q:Q\to I$ of $f|_{F\cap Q}$. Let f_0' be equal to f_Q on every $Q\in\mathcal{Q}_0$. Then, f_0' is continuous on $Z_0\subset(Z,\tau_{\mathcal{Q}})$. Let f_0 be equal to f on F and to f_0' on Z_0. Since Z_0 is closed in $(Z,\tau_{\mathcal{Q}})$, f_0 is continuous on $F\cup Z_0\subset(Z,\tau_{\mathcal{Q}})$. Take $Q\in\mathcal{Q}_1$. Since $Q\cap(F\cup Z_0)$ is closed in Q and Q is perfectly normal, there exists a continuous extension $f_Q:Q\to I$ of $f_0|_{Q\cap(F\cup Z_0)}$. Let f_1' be equal to f_Q on every $Q\in\mathcal{Q}_1$. Evidently, f_1' is defined correctly. Since $Z_0\subset Z_1$, as above, f_1' is continuous on $Z_1\subset(Z,\tau_{\mathcal{Q}})$. If f_1 is equal to f on F and to f_1' on Z_1 then, also as above, f_1 is continuous on $F\cup Z_1\subset(Z,\tau_{\mathcal{Q}})$. Evidently, $f_1|_F=f$ and $f_1|_{Z_0}=f_0$. In the same way we can define maps f_n of $F\cup Z_n\subset(Z,\tau_{\mathcal{Q}})$ to I, $n=2,3,\ldots,$ such that $f_n|_F=f$, and $f_n|_{Z_{n-1}}=f_{n-1}$. If $f_\infty:(Z,\tau_{\mathcal{Q}})\to I$ is equal to f_n on $F\cup Z_n$, $n=0,1,\ldots,$ then f_∞ is continuous on every $Q\in\mathcal{Q}$ and so is continuous. Thus $(Z,\tau_{\mathcal{Q}})$ is normal (and Hausdorff). Hence $(Z,\tau_{\mathcal{Q}})$ is perfectly normal.

If we have $(***)$ for any $Q\in\mathcal{Q}$ then $\dim Z_0\leqslant r$ and $\dim F_{Qni}\leqslant r$ for any $Q\in\mathcal{Q}_n$, $n,i\in\mathbb{N}$, Since the family $\{F_{Qni}:\ Q\in\mathcal{Q}_n\}$ is discrete in $F_{ni}\subset(Z,\tau_{\mathcal{Q}})$ we have that $\dim F_{ni}\leqslant r$. By the sum theorem, $\dim(Z,\tau_{\mathcal{Q}})\leqslant r$. $\square$

The following is evident.

Lemma 1.3. *Let a map $g:(Y,\tau')\to(Z,\tau)$ and families $\mathcal{Q}'$ and $\mathcal{Q}$ of subsets of Y and Z, respectively, be such that for any $F'\in\mathcal{Q}'$, we have $gF'\subset F$ for some $F=F(F')\in\mathcal{Q}$. If $g_{\mathcal{Q}'\mathcal{Q}}:(Y,\tau_{\mathcal{Q}'})\to(Z,\tau_{\mathcal{Q}})$ is such that for the identical maps $\mathrm{id}_{\mathcal{Q}'}:(Y,\tau_{\mathcal{Q}'})\to(Y,\tau')$ and $\mathrm{id}_{\mathcal{Q}}:(Z,\tau_{\mathcal{Q}})\to(Z,\tau)$, we have $\mathrm{id}_{\mathcal{Q}}\circ g_{\mathcal{Q}'\mathcal{Q}}=g\circ\mathrm{id}_{\mathcal{Q}'}$ then $g_{\mathcal{Q}'\mathcal{Q}}$ is continuous.*

2. σ-products

If Z is a subset of the Cartesian product of sets Z_α, $\alpha \in \mathcal{A}$, and $z \in Z$ then z_α denotes the αth coordinate of z.

We shall consider pointed spaces, i.e., spaces with a fixed point. For a space Z with a fixed point, if this point is not denoted specially it will be denoted by 0_Z (note that always $0_{[0,1]} = 0$) and the set $Z \setminus \{0_Z\}$ will be denoted by $\operatorname{co} 0_Z$. For a space Z with a fixed point and a map $f : X \to Z$, the set $f^{-1} \operatorname{co} 0_Z$ will be denoted by $\operatorname{coz} f$. For a system of maps $[f] = \{f_\alpha\colon \alpha \in \mathcal{A}\}$ of a space X to pointed spaces Z_α, $\alpha \in \mathcal{A}$, $\operatorname{coz}[f]$ will denote the family $\{\operatorname{coz} f_\alpha\colon \alpha \in \mathcal{A}\}$.

Let we have a system $[Z]$ of pointed spaces Z_α, $\alpha \in \mathcal{A}$. The subspace $(\sigma_{[Z]} \equiv \sigma Z) = \sigma\{Z_\alpha\colon \alpha \in \mathcal{A}\}$ of the Tychonoff product Π of all Z_α consisting of all points $z \in \Pi$ such that $|\{\alpha \in \mathcal{A}\colon z_\alpha \neq 0_{Z_\alpha}\}| < \omega$ is called the σ-product of the system $[Z]$. The point z of σZ such that $z_\alpha = 0_{Z_\alpha}$ for any α will be denoted by $0_{\sigma_{[Z]}} \equiv 0_{\sigma Z}$ or $0_{[Z]}$.

If for a system of maps $[f] = \{f_\alpha\colon \alpha \in \mathcal{A}\}$ of a space X to pointed spaces Z_α, $\alpha \in \mathcal{A}$, the family $\operatorname{coz}[f]$ is point-finite (in particular, locally finite) then, evidently, $\Delta[f]X \subset \sigma_{[Z]}$ for the diagonal product $\Delta[f]$ of all f_α. In such situations, we shall suppose that $\Delta[f]$ is the map to $\sigma_{[Z]}$.

Let we have a system $[Z]$ of pointed spaces Z_α, $\alpha \in \mathcal{A}$. For $\sigma_{[Z]}$ and $a \in \mathcal{A}^*$, let $(Q_{\sigma_{[Z]}a} \equiv Q_{[Z]a}) = \{z \in \sigma_{[Z]}\colon z_\alpha = 0_{Z_\alpha}$ for any $\alpha \in \mathcal{A} \setminus a\}$ (thus $Q_{[Z]\emptyset} = \{0_{\sigma_{[Z]}}\}$).

Then $(\mathcal{Q}_{\sigma_{[Z]}\chi} \equiv \mathcal{Q}_{[Z]\chi}) = \{Q_{[Z]a}\colon a \in \mathcal{A}^*\}$ and $(\mathcal{Q}_{[Z]\chi})_n = \{\mathcal{Q}_{[Z]a} \in \mathcal{Q}_{[Z]\chi}\colon |a| \leqslant n\}$, $n = 0, 1, \ldots,$ will be called, respectively, the *canonical family of subsets of the σ-product* $[Z]$ and the *nth part, of this family*. (Note that $(\mathcal{Q}_{[Z]\chi})_0 = \{Q_{[Z]\emptyset}\}$.)

If $Z \subset \sigma_{[Z]}$ then $\mathcal{Q}_{Z\chi} = \{Z_a\colon a \in \mathcal{A}^*\}$, where $Z_\emptyset = Q_{[Z]\emptyset}$ and $Z_a = Z \cap Q_{[Z]a}$ for $Q_{[Z]a} \in \mathcal{Q}_{[Z]\chi}$, $|a| > 0$, and $(\mathcal{Q}_{Z\chi})_n = \{Z_a\colon a \in \mathcal{A}^*,\ |a| \leqslant n\}$, $n = 0, 1, \ldots,$ will be called, respectively, the *canonical family of subsets of* $Z \subset \sigma_{[Z]}$ and the *nth part of this family*.

Corollary 2.1. *Let (Z, τ) be a subspace of the σ-product $\sigma_{[Z]}$ of spaces Z_α with fixed points $0_\alpha = 0_{Z_\alpha}$, $\alpha \in \mathcal{A}$; $\mathcal{Q}_{Z\chi}$ be the canonical family of subsets of $(Z, \tau) \subset \sigma_{[Z]}$; $(\mathcal{Q}_{Z\chi})_n$ be its nth part. Let also all finite products of spaces Z_α be perfectly normal.*

Then, for $\mathcal{Q} = \mathcal{Q}_{Z\chi}$, $\mathcal{Q}_n = (\mathcal{Q}_{Z\chi})_n$ and $Z_n = \bigcup \mathcal{Q}_n$, $n = 0, 1, \ldots,$ the space $(Z, \tau_\mathcal{Q})$ is perfectly normal and if $\dim(Q \setminus Z_{n-1}) \leqslant r$ for any $Q \in \mathcal{Q}_n$, $n = 1, 2, \ldots,$ then $\dim(Z, \tau_\mathcal{Q}) \leqslant r$, $r = 0, 1, \ldots.$

Since $\dim Z_\emptyset \leqslant \dim Q_\emptyset = 0 \leqslant r$, the formulated corollary follows from Proposition 1.2.

3. Formulation of the main theorem

From this place of the paper **P** is a class of perfectly normal spaces such that

(1) **P** is *hereditary*, i.e., if $X \in \mathbf{P}$ and $A \subset X$ then $A \in \mathbf{P}$;
(2) **P** is *finitely productive*, i.e., finite topological products of elements of **P** are again elements of **P**;
(3) the *weak factorization theorem for maps to elements of* **P** holds, i.e., for a map f of a space X to a space $Z \in \mathbf{P}$ there exist a space $Y \in \mathbf{P}$ and maps $g : X \to Y$, $h : Y \to Z$ such that, $f = h \circ g$ and $\dim Y < \dim X$; and
(4) for any $X \in \mathbf{P}$ and any open in X set U, there exist a pointed space $R \in \mathbf{P}$ and a map $g : X \to R$ such that $U = g^{-1} \operatorname{co} 0_R$ and the corestriction of $g|_U$ to $\operatorname{co} 0_R$ is a homeomorphism.

Since the subset theorem is true for the dimension dim in the class of perfectly normal spaces, we can suppose that

> *g in point* (3) *is an onto map*.

Let we have a system $[Z]$ of pointed spaces Z_α and a system $[f]$ of maps f_α of a space X to Z_α, $\alpha \in \mathcal{A}$.

Suppose that $\operatorname{coz}[f]$ is locally finite,

Let $(Z \equiv Z_{[f]}) = \Delta[f]X$. Since $Z \subset \sigma_{[Z]}$, we have the canonical family $(\mathcal{Q} \equiv \mathcal{Q}_{[f]}) = \mathcal{Q}_{Z\chi}$ of subsets of Z, its nth parts $(\mathcal{Q}_n \equiv (\mathcal{Q}_{[f]})_n) = (\mathcal{Q}_{Z\chi})_n$, the space $Z_{[f]\mathcal{Q}} = (Z_{[f]}, (\tau_{\mathcal{Q}[f]} \equiv \tau_{\mathcal{Q}_{[f]}}))$ and the identical map $\mathrm{id}_{\mathcal{Q}[f]}$ of $(Z_{[f]}, \tau_{\mathcal{Q}[f]})$ onto the subspace Z of $\sigma_{[Z]}$.

Lemma 1.1 implies the following assertion.

Proposition 3.1. *If $f_{[f]}$ is the corestriction of $\Delta[f]$ to $Z_{[f]}$ then the mapping $f_{[f]\mathcal{Q}} : X \to Z_{[f]\mathcal{Q}}$ such that*

$$f_{[f]} = \mathrm{id}_{\mathcal{Q}[f]} \circ f_{[f]\mathcal{Q}}$$

is continuous.

Definition 3.1. For a space X, we shall say that systems $[Z_{\mathcal{A}(i)}]$ of pointed spaces $Z_\alpha \in \mathbf{P}$, $\alpha \in \mathcal{A}(i)$, and systems of maps $[f_{\mathcal{A}(i)}] = \{(f_\alpha : X \to Z_\alpha)\colon \alpha \in \mathcal{A}(i)\}$, $i \in \mathbb{N}$, are **P**-selecting (or they **P**-select) a subset G of X if all systems $\operatorname{coz}[f_{\mathcal{A}(i)}]$ are locally finite and, for $\mathcal{Q}_i = \mathcal{Q}_{[f_{\mathcal{A}(i)}]}$, $[f_i] = [f_{\mathcal{A}(i)}]$ and the diagonal product $f : X \to (Z_G = \prod\{Z_{[f_i]\mathcal{Q}_i}\colon i \in \mathbb{N}\})$ of all $f_{[f_i]\mathcal{Q}_i}$, we have that $G = f^{-1}H$ for some open set H in Z_G.

Definition 3.2. For a subspace X of a space Y, we shall say that systems $[Z_{\mathcal{A}(i)}]$ of pointed spaces $Z_\alpha \in \mathbf{P}$, $\alpha \in \mathcal{A}(i)$, and systems of maps $[f_{\mathcal{A}(i)}] = \{(f_\alpha : X \to Z_\alpha)\colon \alpha \in \mathcal{A}(i)\}$, $i \in \mathbb{N}$, are piecewise **P**-selecting (or they **P**-select piecewise) a subset G of X in Y if these systems **P**-select the subset G of X and, for any $\alpha \in \mathcal{A} = \bigcup\{\mathcal{A}(i)\colon i \in \mathbb{N}\}$, there exists a continuous extension f'_α of $f_\alpha|_{\operatorname{coz} f_\alpha}$ over Y.

Note that for $W_\alpha = \operatorname{coz} f_\alpha$ and $W'_\alpha = \operatorname{coz} f'_\alpha$, W_α is closed–open in $W'_\alpha \cap X$.

Definition 3.3. A subspace X of a space Y will be called **P**d-right (in Y) if, for any fo subset G of X, there exist piecewise **P**-selecting G in Y systems $[Z(i)]_G$ of pointed spaces and systems $[f(i)]_G$ of maps of X to elements of $[Z(i)]_G$, $i \in \mathbb{N}$.

Theorem 3.2 *(The main theorem). If a subspace X of a space Y is* **P***d-right then*

$$\dim X \leqslant \dim Y.$$

The proof of the theorem is a complicated variant of the proofs of Theorem 17 from [3] and Theorem 1 from [5]. It will be given below.

Let us indicate one possible variant of the class **P**.

Recall that μ-spaces are (topologically) subspaces of the countable products of F_σ-metrizable paracompacta. All μ-spaces are perfectly normal and paracompact. Let $\mathbf{P}_\mu$ be the class of all μ-spaces.

Evidently, the class $\mathbf{P}_\mu$ of all μ-spaces is hereditary and finitely (even countably) productive. The factorization (and so the weak factorization) theorem for maps to elements of $\mathbf{P}_\mu$ is proved in [2]. Pass to property 4 of **P**.

First, let X be an F_σ-metrizable paracompactum, F be its closed subset and $U = X \setminus F$. Let X/F be the disjoint union of U and a one-point set $\{0_X\}$ and $q_F : X \to X/F$ be equal to id_U on U and $q_F(F) = \{0_X\}$. Take on X/F the topology so that q_F will become quotient. It is not difficult to prove that the corestriction of $q_F|_U$ to $U \subset X/F$ is a homeomorphism and that X/F is an F_σ-metrizable paracompactum.

Now let $X \in \mathbf{P}_\mu$, F be its closed subset and $U = X \setminus F$. Then we can suppose that X is a subspace of the Tychonoff product Π' of F_σ-metrizable paracompacta $X(i)$, $i \in \mathbb{N}$. Let pr_i be the projection of Π' to $X(i)$. Take a map $f : X \to I = [0,1]$ such that $F = f^{-1}0$. Then for the diagonal product g_i of $\mathrm{pr}_i|_X$ and f, $F = (g_i)^{-1}(X(i) \times \{0\})$, the set $X(i) \times \{0\}$ is closed in $X(i) \times I$ and $X(i) \times I$ is an F_σ-metrizable paracompactum. Hence, without loss of generality, we can suppose that $\mathrm{pr}_i X = X(i)$ and there exists a closed set $F(i)$ in $X(i)$ such that $F = \mathrm{pr}_i^{-1} F(i)$ and so for $U(i) = X(i) \setminus F(i)$, $U = \mathrm{pr}_i^{-1} U(i)$. Take spaces $X(i)/F(i)$ and maps $q_{F(i)}$. Then the diagonal product $\Delta : X \to (\Pi = \prod\{X(i)/F(i)\colon i \in \mathbb{N}\})$ of all $q_{F(i)} \circ \mathrm{pr}_i|_X$ is such that $\Delta(F) = \{0_R = (0_{X(i)})_{i\in\mathbb{N}}\}$ and $\Delta|_U$ is a topological embedding. Let $R = \Delta(X)$, $V = R \setminus \{0_R\}$ and g be the corestriction of $\Delta|_X$ to R. Then R as a subspace of Π is a μ-space, $U = g^{-1}V = g^{-1}\operatorname{co} 0_R$ and the corestriction of $g|_U$ to $\operatorname{co} 0_R$ is a homeomorphism.

Corollary 3.3. *If a subspace X of a space Y is $(\mathbf{P}_\mu)d$-right then* $\dim X \leqslant \dim Y$.

Let us prove that the **P**d-rightness is a generalization of the countable accessibility of X in Y.

In the definition of the countable accessibility cited in the beginning of the paper, the openness of G in the topology τ means that there exists an open set H' in the Tychonoff product $\Pi = \prod\{I_{is} = [0,1]\colon i \in \mathbb{N},\ s \in S\}$ such that $G = f^{-1}H'$, where f is the diagonal product $\Delta[f]$ of the system $[f]$ of all f_{is}. We can consider Π as the Tychonoff product of the Tychonoff products $\Pi_i = \prod\{I_{is}\colon s \in S\}$, $i \in \mathbb{N}$. Then f is the diagonal product of the diagonal products $f_i = \Delta[f_i]$ of the systems of maps $[f_i] = \{f_{is}\colon s \in S\}$, $i \in \mathbb{N}$. Let $Z_i = Z_{[f_i]} = f_i X$ and id_i be the identical embedding of Z_i in Π_i. Then (see Proposition 3.1 and Definition 3.1), for $\mathcal{Q}_i = \mathcal{Q}_{[f_i]}$, f_i is the composition of maps $(g_i = f_{[f_i]\mathcal{Q}_i}) : X \to Z_{[f_i]\mathcal{Q}_i}$ and $(h_i = \mathrm{id}_i \circ \mathrm{id}_{\mathcal{Q}_i[f_i]}) : Z_{[f_i]\mathcal{Q}_i} \to \Pi_i$. If $h : (Z_G = \prod\{Z_{[f_i]\mathcal{Q}_i}\colon i \in \mathbb{N}\}) \to \Pi$ is the product of maps h_i, $i \in \mathbb{N}$, and g is the diagonal product of g_i, $i \in \mathbb{N}$, then $f = h \circ g$, $H = h^{-1}H'$ is open in Z_G and $g^{-1}H = G$. It follows from this that, for example, the $\mathbf{P}_\mu$-rightness is a generalization of the countable accessibility.

It was proved in [5] that for a space Y with $\dim Y = 0$ and its subspace X, $\dim X \leqslant \dim Y$ iff X is d-right in Y. Hence in this case i.e., for $\dim Y = 0$, the countable accessibility of X in Y and the **P**d-rightness of X in Y are equivalent to the d-rightness of X in Y.

Problem 3.4. For what X and Y (and various **P**) are the d-rightness, the countable accessibility and the **P**d-rightness or some of these properties of X in Y equivalent?

Problem 3.5. When, for Tychonoff X and Y (and various **P**), are some of the following properties:

$X \times Y$ is piecewise rectangular (see [4,5]),
$X \times Y$ is countably accessible in $\beta X \times \beta Y$,
$X \times Y$ is **P**d-right in $\beta X \times \beta Y$

equivalent?

Note that *as variants of* **P** *may be taken the class* $\mathbf{P}_\rho$ *of all metrizable spaces and the class* $\mathbf{P}_{\rho\omega}$ *of all separable metrizable spaces.*

4. Inverse superspectra, their graphs, maps of spaces to superspectra and to their graphs

A system $[R] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{B}\}$, where

$\mathcal{B}$ is a directed set,
R_a is a space, 0_a is its fixed point, $U_a = \operatorname{co} 0_a = R_a \setminus \{0_a\}$, $a \in \mathcal{B}$,
maps $p_{ba} : U_b \to U_a$ are defined for $b, a \in \mathcal{B}$, $a \leqslant b$,

will be called an *inverse superspectrum* (*with the finite precedence* if $L(a) = \{a' \in \mathcal{A}\colon\ a' \leqslant a\}$ is finite for each $a \in \mathcal{B}$) if

$$p_{aa} = \mathrm{id}_{U_a}, \qquad p_{ba} \circ p_{cb} = p_{ca} \quad \text{if } a \leqslant b \leqslant c. \tag{1}$$

Remark 4.1. For an inverse superspectrum $[R] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{B}\}$ the system $\{U_a, p_{ba}; \mathcal{B}\}$ is an inverse spectrum ($\equiv$ an inverse system) of spaces.

If we have an inverse superspectrum $[R] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{B}\}$ then the σ-product $\sigma_{[R]} = \sigma\{(R_a, 0_a)\colon\ a \in \mathcal{B}\}$ is defined. Let (see Section 2), for $A \in \mathcal{B}^*$, $Q_{RA} = Q_{\sigma_{[R]}A}$ (in particular, $Q_{R\emptyset} = (\{0_{[R]} = 0_{\sigma_{[R]}}\})$), π_{RA} be the projection of $\sigma_{[R]}$ to the face Q_{RA} (in particular, $\pi_{R\emptyset}(\sigma_{[R]}) = \{0_{[R]}\}$); π_{RBA} be the projection of Q_{RB} to Q_{RA} for $A \subset B \in \mathcal{B}^*$. Let $\mathcal{Q}_{R\chi} = \mathcal{Q}_{\sigma_{[R]}\chi}$. Thus $\mathcal{Q}_{R\chi}$ is the canonical family of subsets of the σ-product $\sigma_{[R]}$.

Suppose now that $[R]$ is a superspectrum with the finite precedence.

Take $\langle \Gamma_{Ra} \rangle = \{t \in Q_{RL(a)}\colon\ t_b = p_{ab} t_a \text{ for any } b \leqslant a\}$, $a \in \mathcal{B}$.

The subspaces $\Gamma_{[R]} = \{0_{[R]}\} \cup (\bigcup\{\langle \Gamma_{Ra} \rangle\colon\ a \in \mathcal{B}\})$ and $(\Gamma_{[R]})_n = \{0_{[R]}\} \cup (\bigcup\{\langle \Gamma_{Ra} \rangle\colon\ a \in \mathcal{B},\ |L(a)| \leqslant n\})$ of $\sigma_{[R]}$ will be called the *graph of* $[R]$ and the *n-graph of* $[R]$, respectively, $n = 1, 2, \ldots$; $(\Gamma_{[R]})_0 = \{0_{[R]}\}$ will be called the *0-graph of* $[R]$.

Note that for any $t \in \langle \Gamma_{Ra} \rangle$, we have that $t_b \in U_b$ (and so $t_b \neq 0_b$) if $b \leqslant a$ (and $t_b = 0_b$ if $b \notin L(a)$); $\langle \Gamma_{Ra} \rangle \cap \langle \Gamma_{Rb} \rangle = \emptyset$ if $a \neq b$. Since $p_{aa} = \mathrm{id}_{U_a}$, the diagonal product of all p_{ab}, $b \leqslant a$, is a homeomorphism of U_a onto $\langle \Gamma_{Ra} \rangle$ and the restriction to $\langle \Gamma_{Ra} \rangle$ of the projection of the product $Q_{RL(a)}$ onto its factor R_a is a homeomorphism of $\langle \Gamma_{Ra} \rangle$ onto U_a. (Note that for $|L(a)| > 1$, $\langle \Gamma_{Ra} \rangle$ coincides with the graph of the diagonal product of all p_{ab}, $b \leqslant a$, $b \neq a$.) Hence

$$\dim \langle \Gamma_{Ra} \rangle = \dim U_a. \tag{2}$$

If, additionally, R_a is perfectly normal then

$$\dim \langle \Gamma_{Ra} \rangle = \dim U_a \leqslant \dim R_a. \tag{3}$$

Put $\Gamma_{Ra} = \Gamma_{[R]} \cap Q_{RL(a)}$, $a \in \mathcal{B}$. Evidently, $\Gamma_{Ra} = \{0_{[R]}\} \cup (\bigcup\{\langle \Gamma_{Rb} \rangle\colon\ b \leqslant a\})$; all Γ_{Ra} are closed in $\Gamma_{[R]}$; and for $b \geqslant a$ and $\pi_{Rba} = \pi_{RL(b)L(a)}$, the following holds

$$\pi_{Rba} \Gamma_{Rb} = \Gamma_{Ra}; \tag{4}$$

$\Gamma_{[R]} = \bigcup\{\Gamma_{Ra}\colon\ a \in \mathcal{B}\}$; $(\Gamma_{[R]})_n = \bigcup\{\Gamma_{Ra}\colon\ a \in \mathcal{B},\ |L(a)| \leqslant n\}$, $n = 0, 1, \ldots$, and $(\Gamma_{[R]})_n$ is closed in $\Gamma_{[R]}$; and for $|L(a)| = n > 0$, the sets $\langle \Gamma_{Ra} \rangle = \Gamma_{Ra} \setminus (\Gamma_{[R]})_{n-1}$ are open in $(\Gamma_{[R]})_n$.

Note that Γ_{Ra} are perfectly normal (even $\Gamma_{Ra} \subset Q_{RL(a)} \in \mathbf{P}$) if $R_a \in \mathbf{P}$ for all $a \in \mathcal{B}$.

Corollary 4.1. *Let we have an inverse superspectrum $[R] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{B}\}$ with the finite precedence and with $R_a \in \mathbf{P}$, $a \in \mathcal{B}$. If $\dim R_a \leqslant r$ for any $a \in \mathcal{B}$ then $\dim \Gamma_{Ra} \leqslant r$ for any $a \in \mathcal{B}$.*

If $S \subset \Gamma_{[R]}$ and $\mathcal{Q}S = \{(S_a = S \cap \Gamma_{Ra})\colon\ a \in \mathcal{B}^*\}$, then $(S, \tau_{\mathcal{Q}S})$ is a perfectly normal space with $\dim(S, \tau_{\mathcal{Q}S}) \leqslant r$.

Proof. Follows from Proposition 1.2 and (2). □

Proposition 4.2. *Let we have a set $\mathcal{A}$, an inverse superspectrum $[R] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{A}^*_\emptyset\}$ and a system $[h]$ of maps h_α of spaces $R_{\{\alpha\}}$ to spaces Z_α with fixed points $0'_\alpha$ and open sets $V_\alpha = \operatorname{co} 0'_\alpha$ such that $(h_\alpha)^{-1} V_\alpha = U_\alpha$, $\alpha \in \mathcal{A}$. Then for $[Z] = \{Z_\alpha\colon\ \alpha \in \mathcal{A}\}$, there exists a map $(h = ([R], [h])) : \Gamma_{[R]} \to \sigma_{[Z]}$ such that*

$$\mathrm{pr}_{\sigma_{[Z]}\alpha} \circ h = h_\alpha \circ \pi_{R\{\alpha\}}|_{\Gamma_{[R]}}, \quad \alpha \in \mathcal{A}, \tag{5}$$

where $\mathrm{pr}_{\sigma_{[Z]}\alpha}$ is the projection of $\sigma_{[Z]}$ to its factor Z_α and

$$h(0_{[R]}) = 0_{[Z]}, \qquad h(\Gamma_{Ra}) \subset Q_{[Z]a}, \quad a \in \mathcal{A}^*_\emptyset. \tag{6}$$

Let, additionally, $S \subset \Gamma_{[R]}$, $\mathcal{Q}S = \{(S_a = S \cap \Gamma_{Ra})\colon\ a \in \mathcal{A}^*\}$, $Z = hS$, h_S be the corestriction of $h|_S$ to Z and $\mathcal{Q} = \mathcal{Q}_{Z\chi}$. Then

$$\mathrm{pr}_{\sigma_{[Z]}\alpha} \circ h_S = h_\alpha \circ \pi_{R\{\alpha\}}|_S, \quad \alpha \in \mathcal{A}, \tag{7}$$

and there exists a map $h_{(\mathcal{Q}S)\mathcal{Q}} : (S, \tau_{\mathcal{Q}S}) \to (Z, \tau_{\mathcal{Q}})$ such that

$$h_S \circ \mathrm{id}_{\mathcal{Q}S} = \mathrm{id}_{\mathcal{Q}} \circ h_{(\mathcal{Q}S)\mathcal{Q}}. \tag{8}$$

Proof. For any $t \in \Gamma_{[R]}$, put $h(t) = \{h_\alpha(\pi_{R\{\alpha\}}(t))\}_{\alpha\in\mathcal{A}}$. It is easy to verify that this h and the correspondent h_S are the desired maps.

The rest follows from Lemma 1.3. □

For a space X, a system $[g]$ of maps $g_a : X \to R_a$, $a \in \mathcal{B}$, will be called a *map of X to the superspectrum* $[R]$ if, for the sets $W_a = g_a^{-1}U_a$, we have the following

$$W_b \subset W_a \quad \text{and} \quad p_{ba} \circ g_b|_{W_b} = g_a|_{W_b} \quad \text{if } a \leqslant b. \tag{9}$$

Recall that

if the family $\nu = \operatorname{coz}[g] = \{W_a = g_a^{-1}U_a\colon a \in \mathcal{B}\}$ *is point-finite then the diagonal product* $\Delta[g] = \Delta\{g_a\colon a \in \mathcal{B}\} : X \to \prod\{R_a\colon a \in \mathcal{B}\}$ *is a map to* $\sigma_{[R]}$.

In this case the corestriction of $\Delta[g]$ to $\Delta[g](X)$ will be denoted by $g = g([g])$.

Proposition 4.3. *Let we have a space X, a system $[Z]$ of spaces Z_α with fixed points $0'_\alpha$ and open sets $V_\alpha = Z_\alpha \setminus \{0'_\alpha\}$, a system $[f_\mathcal{A}]$ of maps $f_\alpha : X \to Z_\alpha$, $\alpha \in \mathcal{A}$, an inverse superspectrum $[R_\mathcal{A}] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{A}^*_\emptyset\}$, a system $[h_\mathcal{A}]$ of maps $h_\alpha : R_{\{\alpha\}} \to Z_\alpha$, $\alpha \in \mathcal{A}$, and a map $[g_\mathcal{A}] = \{(g_a : X \to R_a)\colon a \in \mathcal{A}^*_\emptyset\}$ of X to $[R_\mathcal{A}]$ such that*

$$f_\alpha = h_\alpha \circ g_{\{\alpha\}}, \quad \alpha \in \mathcal{A};$$

$$U_{\{\alpha\}} = (h_\alpha)^{-1}V_\alpha, \quad \alpha \in \mathcal{A}; \quad \textit{and}$$

$$\textit{the family } \nu = \{W_\alpha = (f_\alpha)^{-1}V_\alpha\colon \alpha \in \mathcal{A}\} \quad \textit{is locally finite}.$$

Let $\mathcal{C} \subset \mathcal{A}$. Then we have the inverse superspectrum $[R_\mathcal{C}] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{C}^*_\emptyset\}$, the system $[Z_\mathcal{C}]$ of spaces Z_α, $\alpha \in \mathcal{C}$, the systems $[f_\mathcal{C}]$ of maps $f_\alpha : X \to Z_\alpha$ and $[h_\mathcal{C}]$ of maps $h_\alpha : R_{\{\alpha\}} \to Z_\alpha$, $\alpha \in \mathcal{C}$, and the map $[g_\mathcal{C}] = \{(g_a : X \to R_a)\colon a \in \mathcal{C}^*_\emptyset\}$ of X to $[R_\mathcal{C}]$;

$$\bigl(S_\mathcal{C} = \Delta[g_\mathcal{C}](X)\bigr) \subset \Gamma_{[R_\mathcal{C}]}; \tag{10}$$

for $g_\mathcal{C} = g_\mathcal{C}([g_\mathcal{C}])$ and any $\alpha \in \mathcal{C}$,

$$\pi_{(R_\mathcal{C})\{\alpha\}}|_{S_\mathcal{C}} \circ g_\mathcal{C} = g_{\{\alpha\}}; \tag{11}$$

for $\mathcal{Q}S_\mathcal{C} = \{(S_{\mathcal{C}a} = S_\mathcal{C} \cap \Gamma_{Ra})\colon a \in \mathcal{C}^*\}$, there exists a map $g_{\mathcal{Q}S_\mathcal{C}} : X \to (S_\mathcal{C}, \tau_{\mathcal{Q}S_\mathcal{C}})$ such that

$$g_\mathcal{C} = \mathrm{id}_{\mathcal{Q}S_\mathcal{C}} \circ g_{\mathcal{Q}S_\mathcal{C}}; \tag{12}$$

there exists a map $h_\mathcal{C}$ of $S_\mathcal{C}$ onto a subspace $Z_\mathcal{C}$ of the σ-product $\sigma_{[Z_\mathcal{C}]} = \sigma\{(Z_\alpha, 0'_\alpha)\colon \alpha \in \mathcal{C}\}$ such that

$$\mathrm{pr}_{(\sigma_{Z_\mathcal{C}})\alpha} \circ h_\mathcal{C} = h_\alpha \circ \pi_{R_\mathcal{C}\{\alpha\}}|_{S_\mathcal{C}}, \quad \alpha \in \mathcal{C}, \tag{13}$$

and, for $\mathcal{Q}_\mathcal{C} = (\mathcal{Q}_\mathcal{C})_{Z_\mathcal{C}\chi}$, there exists a map $h_{(\mathcal{Q}S_\mathcal{C})\mathcal{Q}_\mathcal{C}} : (S_\mathcal{C}, \tau_{\mathcal{Q}S_\mathcal{C}}) \to (Z_\mathcal{C}, \tau_{\mathcal{Q}_\mathcal{C}})$ such that

$$h_\mathcal{C} \circ \mathrm{id}_{\mathcal{Q}S_\mathcal{C}} = \mathrm{id}_{\mathcal{Q}_\mathcal{C}} \circ h_{(\mathcal{Q}S_\mathcal{C})\mathcal{Q}_\mathcal{C}}; \tag{14}$$

for $(\Delta[f_\mathcal{C}]X = X_{[f_\mathcal{C}]}) = Z_\mathcal{C}$ and the corestriction $f_\mathcal{C}$ of $\Delta[f_\mathcal{C}]$ to $Z_\mathcal{C}$,

$$f_\mathcal{C} = h_\mathcal{C} \circ g_\mathcal{C}; \tag{15}$$

there exists a map $f_{\mathcal{Q}_\mathcal{C}} : X \to (Z_\mathcal{C}, \tau_{\mathcal{Q}_\mathcal{C}})$ such that

$$f_\mathcal{C} = \mathrm{id}_{\mathcal{Q}_\mathcal{C}} \circ f_{\mathcal{Q}_\mathcal{C}}; \tag{16}$$

$$f_{\mathcal{Q}_\mathcal{C}} = h_{(\mathcal{Q}S_\mathcal{C})\mathcal{Q}_\mathcal{C}} \circ g_{\mathcal{Q}S_\mathcal{C}}. \tag{17}$$

For the projection $\pi_{R\mathcal{A}\mathcal{C}}$ of $\sigma_{[R_\mathcal{A}]}$ onto $\sigma_{[R_\mathcal{C}]}$,

$$\pi_{R\mathcal{A}\mathcal{C}} S_\mathcal{A} = S_\mathcal{C}; \tag{18}$$

for the corestriction $\psi_{S\mathcal{A}\mathcal{C}}$ of $\pi_{R\mathcal{A}\mathcal{C}}|_{S_\mathcal{A}}$ to $S_\mathcal{C}$,

$$g_\mathcal{C} = \psi_{S\mathcal{A}\mathcal{C}} \circ g_\mathcal{A}; \tag{19}$$

there exists a map $\psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}} : (S_\mathcal{A}, \tau_{\mathcal{Q}S_\mathcal{A}}) \to (S_\mathcal{C}, \tau_{\mathcal{Q}S_\mathcal{C}})$, such that

$$\psi_{S\mathcal{A}\mathcal{C}} \circ \mathrm{id}_{\mathcal{Q}S_\mathcal{A}} = \mathrm{id}_{\mathcal{Q}S_\mathcal{C}} \circ \psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}} \tag{20}$$

and

$$g_{\mathcal{Q}S_\mathcal{C}} = \psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}} \circ g_{\mathcal{Q}S_\mathcal{A}}, \tag{21}$$

$$f_{\mathcal{Q}_\mathcal{C}} = h_{(\mathcal{Q}S_\mathcal{C})\mathcal{Q}_\mathcal{C}} \circ \psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}} \circ g_{\mathcal{Q}S_\mathcal{A}}. \tag{22}$$

Proof. Take $x \in X$. Put $a(x\mathcal{C}) = \{\alpha \in \mathcal{C}\colon\ x \in W_\alpha\}$. Then $\Delta[g_{\mathcal{C}}](x) = 0_{\sigma[R_{\mathcal{C}}]}$ if $a(x\mathcal{C}) = \emptyset$. Let $a(x\mathcal{C}) \neq \emptyset$. Then $\Delta[g_{\mathcal{C}}](x) \in Q_{RL(a(x\mathcal{C}))}$, $(\Delta[g_{\mathcal{C}}](x))_{a(x\mathcal{C})} = \pi_{(R_{\mathcal{C}})a(x\mathcal{C})}(\Delta[g_{\mathcal{C}}](x)) = g_{a(x\mathcal{C})}x \in U_{a(x\mathcal{C})}$ and, for any $b \in L(a(x\mathcal{C}))$, we have $p_{a(x\mathcal{C})b}((\Delta[g_{\mathcal{C}}](x))_{a(x\mathcal{C})}) = p_{a(x\mathcal{C})b}(g_{a(x\mathcal{C})}x) = g_b x = (\Delta[g_{\mathcal{C}}](x))_b$. Hence $\Delta[g_{\mathcal{C}}](x) \in \Gamma_{[R_{\mathcal{C}}]a(x\mathcal{C})}$. Thus $(S_{\mathcal{C}} = \Delta[g_{\mathcal{C}}](X)) \subset \Gamma_{[R_{\mathcal{C}}]}$, we can consider the corestriction $g_{\mathcal{C}}$ of $\Delta[g_{\mathcal{C}}]$ to $S_{\mathcal{C}}$ and, evidently, $\pi_{(R_{\mathcal{C}})\{\alpha\}}|_{S_{\mathcal{C}}} \circ g_{\mathcal{C}} = g_{\{\alpha\}}$.

For the family $\mathcal{Q}S_{\mathcal{C}}$ indicated above, the existence of the required map $g_{\mathcal{Q}S_{\mathcal{C}}}$ follows from the local finiteness of ν and Lemma 1.1.

The existence of the required maps $h_{\mathcal{C}}$ and $h_{(\mathcal{Q}S_{\mathcal{C}})\mathcal{Q}_{\mathcal{C}}}$ follows from Proposition 4.2.

Take $x \in X$. Then $\Delta[f_{\mathcal{C}}]x = \{f_\alpha(x)\}_{\alpha\in\mathcal{C}} = \{h_\alpha \circ g_{\{\alpha\}}(x)\}_{\alpha\in\mathcal{C}} = \{h_\alpha \circ \pi_{(R_{\mathcal{C}})\{\alpha\}} \circ g_{\mathcal{C}}(x)\}_{\alpha\in\mathcal{C}} = \{\mathrm{pr}_{(\sigma_{[Z_{\mathcal{C}}]})\alpha} \circ h_{\mathcal{C}} \circ g_{\mathcal{C}}(x)\}_{\alpha\in\mathcal{C}} = h_{\mathcal{C}} \circ g_{\mathcal{C}}(x)$. Hence $f_{\mathcal{C}} = h_{\mathcal{C}} \circ g_{\mathcal{C}}$ and $\Delta[f_{\mathcal{C}}]X = Z_{\mathcal{C}}$.

It follows from Lemma 1.1 that there exists a map $f_{\mathcal{Q}_{\mathcal{C}}}: X \to (Z_{\mathcal{C}}, \tau_{\mathcal{Q}_{\mathcal{C}}})$ such that $f_{\mathcal{C}} = \mathrm{id}_{\mathcal{Q}_{\mathcal{C}}} \circ f_{\mathcal{Q}_{\mathcal{C}}}$. Hence $\mathrm{id}_{\mathcal{Q}_{\mathcal{C}}} \circ f_{\mathcal{Q}_{\mathcal{C}}} = f_{\mathcal{C}} = h_{\mathcal{C}} \circ g_{\mathcal{C}} = h_{\mathcal{C}} \circ \mathrm{id}_{\mathcal{Q}S_{\mathcal{C}}} \circ g_{\mathcal{Q}S_{\mathcal{C}}} = \mathrm{id}_{\mathcal{Q}_{\mathcal{C}}} \circ h_{(\mathcal{Q}S_{\mathcal{C}})\mathcal{Q}_{\mathcal{C}}} \circ g_{\mathcal{Q}S_{\mathcal{C}}}$ and so $f_{\mathcal{Q}_{\mathcal{C}}} = h_{(\mathcal{Q}S_{\mathcal{C}})\mathcal{Q}_{\mathcal{C}}} \circ g_{\mathcal{Q}S_{\mathcal{C}}}$.

Relations (18) and (19) are evident. Also it is evident that there exists a not necessary continuous mapping $\psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}}: (S_{\mathcal{A}}, \tau_{\mathcal{Q}S_{\mathcal{A}}}) \to (S_{\mathcal{C}}, \tau_{\mathcal{Q}S_{\mathcal{C}}})$, such that (20) is true. If $0_{[R_{\mathcal{A}}]} \in S_{\mathcal{A}}$ then $\psi_{\mathcal{Q}S\mathcal{A}\mathcal{B}} 0_{[R_{\mathcal{A}}]} = 0_{[R_{\mathcal{C}}]}$. Let $g_{\mathcal{A}}x \in \langle \Gamma_{R_{\mathcal{A}}a}\rangle$ and $b = a \cap \mathcal{C}$. If $b = \emptyset$ then $\psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}} \circ g_{\mathcal{A}}x = g_{\mathcal{C}}x = 0_{[R_{\mathcal{C}}]}$. If $b \neq \emptyset$ then $\psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}} \circ g_{\mathcal{A}}x = g_{\mathcal{C}}x \in \langle \Gamma_{R_{\mathcal{C}}b}\rangle \cap S_{\mathcal{C}b} \subset S_{\mathcal{C}b}$. By Lemma 1.3, $\psi_{QS\mathcal{A}\mathcal{C}}$ is continuous.

At last, (21) is a simple consequence of (19) and $f_{\mathcal{Q}_{\mathcal{C}}} = h_{(\mathcal{Q}S_{\mathcal{C}})\mathcal{Q}_{\mathcal{C}}} \circ g_{\mathcal{Q}S_{\mathcal{C}}} = h_{(\mathcal{Q}S_{\mathcal{C}})\mathcal{Q}_{\mathcal{C}}} \circ \psi_{\mathcal{Q}S\mathcal{A}\mathcal{C}} \circ g_{\mathcal{Q}S_{\mathcal{A}}}$. □

Corollary 4.4. *Let we have a space X, a system $[Z_{\mathcal{A}}]$ of spaces Z_α with fixed points $0'_\alpha$ and open sets $V_\alpha = Z_\alpha \setminus \{0'_\alpha\}$, $\alpha \in \mathcal{A}$, a system $[f_{\mathcal{A}}]$ of maps $f_\alpha: X \to Z_\alpha$, $\alpha \in \mathcal{A}$, an inverse superspectrum $[R_{\mathcal{A}}] = \{(R_a, 0'_a, U_a), p_{ba}; \mathcal{A}^*_\emptyset\}$, a system $[h_{\mathcal{A}}]$ of maps $h_\alpha: R_{\{\alpha\}} \to Z_\alpha$, $\alpha \in \mathcal{A}$, and a map $[g_{\mathcal{A}}] = \{(g_a: X \to R_a)\colon\ a \in \mathcal{A}^*_\emptyset\}$ of X to $[R_{\mathcal{A}}]$ such that*:

$$f_\alpha = h_\alpha \circ g_{\{\alpha\}}, \quad \alpha \in \mathcal{A};$$

$$U_{\{\alpha\}} = (h_\alpha)^{-1} V_\alpha,\ \alpha \in \mathcal{A}; \quad \textit{and}$$

$$\textit{the family } \nu = \{W_\alpha = (f_\alpha)^{-1} V_\alpha\colon\ \alpha \in \mathcal{A}\} \quad \textit{is the union of locally finite families } \nu_i = \{W_\alpha\colon\ \alpha \in \mathcal{A}(i)\}, \quad i \in \mathbb{N}.$$

Then, for any $N \in \mathbb{N}^*_\emptyset$ and $\mathcal{B}(N) = \bigcup\{\mathcal{A}(i)\colon\ i \in N\}$, the family $\nu_N = \bigcup\{\nu_i\colon\ i \in N\}$ is locally finite; we have the inverse superspectrum $[R_{\mathcal{B}(N)}] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{B}(N)^*_\emptyset\}$, the system $[Z_{\mathcal{B}(N)}]$ of spaces Z_α, $\alpha \in \mathcal{B}(N)$, the systems $[f_{\mathcal{B}(N)}]$ of maps $f_\alpha: X \to Z_\alpha$ and $[h_{\mathcal{B}(N)}]$ of maps $h_\alpha: R_{\{\alpha\}} \to Z_\alpha$, $\alpha \in \mathcal{B}(N)$; the map $[g_{\mathcal{B}(N)}] = \{(g_a: X \to R_a)\colon\ a \in \mathcal{B}(N)^*_\emptyset\}$ of X to $[R_{\mathcal{B}(N)}]$;

$$\big(S_{\mathcal{B}(N)} = \Delta[g_{\mathcal{B}(N)}](X)\big) \subset \Gamma_{[R_{\mathcal{B}(N)}]}; \tag{10$'$}$$

for $g_{\mathcal{B}(N)} = g_{\mathcal{B}(N)}([g_{\mathcal{B}(N)}])$ and any $\alpha \in \mathcal{B}(N)$,

$$\pi_{(R_{\mathcal{B}(N)})\{\alpha\}}|_{S_{\mathcal{B}(N)}} \circ g_{\mathcal{B}(N)} = g_{\{\alpha\}}; \tag{11$'$}$$

for $\mathcal{Q}S_{\mathcal{B}(N)} = \{(S_{\mathcal{B}(N)a} = S_{\mathcal{B}(N)} \cap \Gamma_{Ra})\colon\ a \in \mathcal{B}(N)^*\}$, there exists a map $g_{\mathcal{Q}S_{\mathcal{B}(N)}}: X \to (S_{\mathcal{B}(N)}, \tau_{\mathcal{Q}S_{\mathcal{B}(N)}})$ such that

$$g_{\mathcal{B}(N)} = \mathrm{id}_{\mathcal{Q}S_{\mathcal{B}(N)}} \circ g_{\mathcal{Q}S_{\mathcal{B}(N)}}; \tag{12$'$}$$

there exists a map $h_{\mathcal{B}(N)}$ of $S_{\mathcal{B}(N)}$ onto a subspace $Z_{\mathcal{B}(N)}$ of the σ-product $\sigma_{[Z_{\mathcal{B}(N)}]} = \sigma\{(Z_\alpha, 0'_\alpha)\colon\ \alpha \in \mathcal{B}(N)\}$ such that

$$\mathrm{pr}_{(\sigma_{[Z_{\mathcal{B}(N)}]})\alpha} \circ h_{\mathcal{B}(N)} = h_\alpha \circ \pi_{R_{\mathcal{B}(N)}\{\alpha\}}|_{S_{\mathcal{B}(N)}}, \quad \alpha \in \mathcal{B}(N), \tag{13$'$}$$

and, for $\mathcal{Q}_{\mathcal{B}(N)} = (\mathcal{Q}_{\mathcal{B}(N)})_{Z_{\mathcal{B}(N)}\chi}$, there exists a map $h_{(\mathcal{Q}S_{\mathcal{B}(N)})\mathcal{Q}_{\mathcal{B}(N)}}: (S_{\mathcal{B}(N)}, \tau_{\mathcal{Q}S_{\mathcal{B}(N)}}) \to (Z_{\mathcal{B}(N)}, \tau_{\mathcal{Q}_{\mathcal{B}(N)}})$ such that

$$h_{\mathcal{B}(N)} \circ \mathrm{id}_{\mathcal{Q}S_{\mathcal{B}(N)}} = \mathrm{id}_{\mathcal{Q}_{\mathcal{B}(N)}} \circ h_{(\mathcal{Q}S_{\mathcal{B}(N)})\mathcal{Q}_{\mathcal{B}(N)}}; \tag{14$'$}$$

for $(\Delta[f_{\mathcal{B}(N)}]X = Z_{[f_{\mathcal{B}(N)}]}) = Z_{\mathcal{B}(N)}$ and the corestriction $f_{\mathcal{B}(N)}$ of $\Delta[f_{\mathcal{B}(N)}]$ to $Z_{\mathcal{B}(N)}$,

$$f_{\mathcal{B}(N)} = h_{\mathcal{B}(N)} \circ g_{\mathcal{B}(N)}; \tag{15$'$}$$

there exists a map $f_{\mathcal{Q}_{\mathcal{B}(N)}}: X \to (Z_{\mathcal{B}(N)}, \tau_{\mathcal{Q}_{\mathcal{B}(N)}})$ such that

$$f_{\mathcal{B}(N)} = \mathrm{id}_{\mathcal{Q}_{\mathcal{B}(N)}} \circ f_{\mathcal{Q}_{\mathcal{B}(N)}}; \tag{16$'$}$$

$$f_{\mathcal{Q}_{\mathcal{B}(N)}} = h_{(\mathcal{Q}S_{\mathcal{B}(N)})\mathcal{Q}_{\mathcal{B}(N)}} \circ g_{\mathcal{Q}S_{\mathcal{B}(N)}}. \tag{17$'$}$$

For $N \subset M \in \mathbb{N}^*_\emptyset$ and the projection π_{MN} of $\sigma_{[R_{\mathcal{B}(M)}]}$ onto $\sigma_{[R_{\mathcal{B}(N)}]}$,

$$\pi_{MN} S_{\mathcal{B}(M)} = S_{\mathcal{B}(N)}; \tag{18$'$}$$

for the corestrictions ψ_{MN} of $\pi_{MN}|_{S_{\mathcal{B}(M)}}$ to $S_{\mathcal{B}(N)}$,

$$g_{\mathcal{B}(N)} = \psi_{MN} \circ g_{\mathcal{B}(M)}; \tag{19$'$}$$

there exists a map $\psi_{\mathcal{Q}MN}: (S_M = (S_{\mathcal{B}(N)}, \tau_{\mathcal{Q}S_{\mathcal{B}(M)}})) \to (S_N = (S_{\mathcal{B}(N)}, \tau_{\mathcal{Q}S_{\mathcal{B}(N)}}))$ such that

$$\psi_{MN} \circ \mathrm{id}_{\mathcal{Q}S_{\mathcal{B}(M)}} = \mathrm{id}_{\mathcal{Q}S_{\mathcal{B}(N)}} \circ \psi_{\mathcal{Q}MN}, \tag{20$'$}$$

and

$$g_{\mathcal{Q}S_{\mathcal{B}(N)}} = \psi_{\mathcal{Q}MN} \circ g_{\mathcal{Q}S_{\mathcal{B}(M)}}, \tag{21'}$$

$$f_{\mathcal{Q}\mathcal{B}(N)} = h_{(\mathcal{Q}S_{\mathcal{B}(N)})\mathcal{Q}_{\mathcal{B}(N)}} \circ \psi_{\mathcal{Q}MN} \circ g_{\mathcal{Q}S_{\mathcal{B}(M)}}. \tag{22'}$$

For $N \subset M \subset L \in \mathbb{N}^*_{\emptyset}$,

$$\psi_{\mathcal{Q}MN} \circ \psi_{\mathcal{Q}LM} = \psi_{\mathcal{Q}LN}. \tag{23}$$

For the limit S *of the countable inverse spectrum* $Sp = \{S_N, \psi_{\mathcal{Q}MN}; N \in \mathbb{N}^*_{\emptyset}\}$, *its projections* $\Psi_N : S \to S_N$, *the limit* $g : X \to S$ *of maps* $g_{\mathcal{Q}S_{\mathcal{B}(N)}}$ *(i.e.,* $g_{\mathcal{Q}S_{\mathcal{B}(N)}} = \Psi_N \circ g$, $N \in \mathbb{N}^*_{\emptyset}$*), and* $(h_i = h_{(\mathcal{Q}S_{\mathcal{B}(\{i\})})\mathcal{Q}_{\mathcal{B}(\{i\})}} \circ \Psi_{\mathcal{B}(\{i\})} = h_{(\mathcal{Q}S_{\mathcal{A}(i)})\mathcal{Q}_{\mathcal{A}(i)}} \circ \Psi_{\mathcal{A}(i)}) : S \to ((Z_{\mathcal{B}(\{i\})} = Z_{[f_{\mathcal{B}(\{i\})}]}, \tau_{\mathcal{Q}_{\mathcal{B}(\{i\})}}) = (Z_{[f_{\mathcal{A}(i)}]}, \tau_{\mathcal{Q}_{\mathcal{A}(i)}}) = Z_{[f_{\mathcal{A}(i)}]\mathcal{Q}_{\mathcal{A}(i)}})$,

$$h_i \circ g = f_{[f_{\mathcal{A}(i)}]\mathcal{Q}_{[f_{\mathcal{A}(i)}]}}, \quad i \in \mathbb{N}. \tag{24}$$

If $R_a \in \mathbf{P}$ *and* $\dim R_a \leqslant r$, $a \in \mathcal{A}^*_{\emptyset}$, *then* S_i *and* S *are perfectly normal spaces with* $\dim S_i \leqslant r$ *and* $\dim S \leqslant r$, $i \in \mathbb{N}$.

Proof. Relations (10′)–(22′) follow from the previous proposition and (23) follows from the equality $\pi_{MN} \circ \pi_{LM} = \pi_{LN}$.

Note that $f_{\mathcal{Q}_{\mathcal{B}(\{i\})}}$ coincides with $f_{[f_{\mathcal{A}(i)}]\mathcal{Q}_{[f_{\mathcal{A}(i)}]}}$ and $h_i \circ g = h_{(\mathcal{Q}S_{\mathcal{B}(\{i\})})\mathcal{Q}_{\mathcal{B}(\{i\})}} \circ \Psi_{\mathcal{B}(\{i\})} \circ g = h_{(\mathcal{Q}S_{\mathcal{B}(\{i\})})\mathcal{Q}_{\mathcal{B}(\{i\})}} \circ g_{(\mathcal{Q}S_{\mathcal{B}(\{i\})})} = f_{\mathcal{Q}_{\mathcal{B}(\{i\})}}$. This implies (24).

It follows from Corollary 4.1 that all spaces S_N are perfectly normal and $\dim S_N \leqslant r$, $N \in \mathbb{N}^*_{\emptyset}$. Since the spectrum Sp has a cofinal part that is an inverse sequence, we have that S, by Charalambous's theorem on covering dimension of the limit of an inverse sequence of perfectly normal spaces, is a perfectly normal space with $\dim S \leqslant r$. □

Proposition 4.5. *Let we have a space* Y; *its subspace* X; *spaces* Z_α *with fixed points* 0_α *and open sets* $V_\alpha = \operatorname{co} 0_\alpha$ *and a system* $[f'_{\mathcal{A}}]$ *of maps* $f'_\alpha : Y \to Z_\alpha$, $\alpha \in \mathcal{A}$; *an inverse superspectrum* $[R_{\mathcal{A}}] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{A}^*_{\emptyset}\}$; *a system* $[h_{\mathcal{A}}]$ *of maps* $h_\alpha : R_{\{\alpha\}} \to Z_\alpha$, $\alpha \in \mathcal{A}$; *a map* $[g'_{\mathcal{A}}] = \{(g'_a : Y \to R_a)\colon a \in \mathcal{A}^*_{\emptyset}\}$ *of* Y *to* $[R_{\mathcal{A}}]$; *open–closed subsets* W_α *of* $X \cap (W'_\alpha = (f'_\alpha)^{-1} V_\alpha)$, $\alpha \in \mathcal{A}$, *such that*

$$f'_\alpha = h_\alpha \circ g'_{\{\alpha\}}, \quad \alpha \in \mathcal{A};$$

$$U_{\{\alpha\}} = (h_\alpha)^{-1} V_\alpha \quad \big(\text{and so } W'_\alpha = (g'_{\{\alpha\}})^{-1} U_{\{\alpha\}}\big), \quad \alpha \in \mathcal{A}; \quad \text{and}$$

$$\Big(W'_a = \bigcap\{W'_\alpha\colon \alpha \in a\}\Big) = (g'_a)^{-1} U_a, \quad a \in \mathcal{A}^*_{\emptyset}.$$

Let $f_\alpha : X \to Z_\alpha$ *be equal to* f'_α *on* W_α *and to* 0_α *on* $X \setminus W_\alpha$; $g_a : X \to R_a$ *be equal to* g'_a *on* $W_a = \bigcap\{W_\alpha\colon \alpha \in a\}$ *and to* 0_a *on* $X \setminus W_a$.

Then

$$W_\alpha = (f_\alpha)^{-1} V_\alpha, \quad \alpha \in \mathcal{A};$$

$$f_\alpha = h_\alpha \circ g_{\{\alpha\}}, \quad \alpha \in \mathcal{A};$$

$$W_\alpha = g^{-1}_{\{\alpha\}} U_{\{\alpha\}}, \quad \alpha \in \mathcal{A};$$

$$W_a = (g_a)^{-1} U_a \quad \text{and} \quad W_a \quad \text{is open–closed in } W'_a, \quad a \in \mathcal{A}^*_{\emptyset};$$

$$\text{the system } [g_{\mathcal{A}}] = \big\{(g_a : X \to R_a)\colon a \in \mathcal{A}^*_{\emptyset}\big\} \quad \text{is a map of } X \text{ to } [R_{\mathcal{A}}].$$

Proof. The proof is simple. □

5. Factorization of systems of maps by means of superspectra

We shall start with some preliminary considerations.

First we shall obtain the following ("pointed") version of the weak factorization theorem.

Proposition 5.1. *For any map* f *of a space* X *with* $\dim X = r$ *to a pointed space* $Z \in \mathbf{P}$, *there exist a pointed space* $Y \in \mathbf{P}$ *and maps* $g : X \to Y$, $h : Y \to Z$ *such that* $f = h \circ g$, $\dim Y \leqslant r$, $g(\operatorname{coz} f) = \operatorname{co} 0_Y$ *and* $h^{-1} 0_Z = \{0_Y\}$, $h^{-1} \operatorname{co} 0_Z = \operatorname{co} 0_Y$.

Lemma 5.2. *Let* f *be a map of a space* X *with* $\dim X = r$ *to a pointed perfectly normal space* Z. *Then there exist a pointed space* Y *and maps* $g : X \to Y$, $h : Y \to Z$ *such that* $f = h \circ g$, $\dim Y \leqslant r$, $g(\operatorname{coz} f) = \operatorname{co} 0_Y$ *and* $h^{-1} 0_Z = \{0_Y\}$, $h^{-1} \operatorname{co} 0_Z = \operatorname{co} 0_Y$.

Proof. Let $F = f^{-1} 0_Z$, $W = f^{-1} \operatorname{co} 0_Z$ and Y be the disjoint union of W and a one-point set $\{0_Y\}$. Take mappings $g : X \to Y$ and $h : Y \to Z$ such that $g|_W = \mathrm{id}_W$ and $gF \subset \{0_Y\}$, $h|_W = f|_W$ and $h 0_Y = 0_Z$. Evidently, $f = h \circ g$, $g(\operatorname{coz} f) = \operatorname{co} 0_Y$ and $h^{-1} 0_Z = \{0_Y\}$, $h^{-1} \operatorname{co} 0_Z = \operatorname{co} 0_Y$. Take the topology τ on Y with the subbase consisting of all open subsets of W as a subspace of X and all sets $h^{-1}O$, where O is open in Z. Then mappings $g : X \to (Y,\tau)$ and $h : (Y,\tau) \to Z$ are continuous and the corestriction of $g|_W$ to $W \subset Y$ is a homeomorphism.

Let us prove that $\dim Y \leqslant r$.

Take a finite fo cover $\nu = \{O_1,\dots,O_k\}$ of Y. Without loss of generality we can suppose that there exists a neighborhood G of 0_Z such that $(V = h^{-1}G) \subset O_k$ and $V \cap O_i = \emptyset$, $i < k$. There exists a finite fo refinement μ of $g^{-1}\nu = \{g^{-1}O_i\colon i = 1,\dots,k\}$ of order $\leqslant r$. Again without loss of generality we can suppose that $\mu = \{U_1,\dots,U_k\}$ and $U_i \subset g^{-1}O_i$, $i = 1,\dots,k$. Evidently, $g^{-1}V \subset U_k$. Since G is fo in Z, we have that V is fo in Y. Hence we can take a zero-set F in Y such that $0_Y \in \operatorname{int} F$, $F \subset V$. Then the set $(U'_k = U_k \setminus g^{-1}F) \equiv gU'_k$ is fo in Y and so $(U''_k = U'_k \cup V) = gU_k \cup \{0_y\}$ is fo in Y too. Evidently, all $gU_i \equiv U_i$, $i < k$, are also fo in Y and $\{U_i\colon i < k\} \cup \{U''_k\}$ is a refinement of ν of order $\leqslant r$. □

Proof of Proposition 5.1. Let we have a map f of a space X to a pointed space $Z \in \mathbf{P}$. By the previous lemma, there exist a pointed space Y', a map $g' : X \to Y'$ and a map $h' : Y' \to Z$ such that $f = h' \circ g'$, $\dim Y' \leqslant r$, $g' \operatorname{coz} f = \operatorname{co} 0_{Y'}$ and $(h')^{-1}0_Z = \{0_{Y'}\}$, $(h')^{-1}\operatorname{co} 0_Z = \operatorname{co} 0_{Y'}$. By property 3 of $\mathbf{P}$, there exist a space $Y \in \mathbf{P}$ and maps $g'' : Y' \to Y$ and $h : Y \to Z$ such that $h' = h \circ g''$ and $\dim Y \leqslant \dim Y' \leqslant r$. Since $\mathbf{P}$ is hereditary, we can suppose that g'' is an onto map. Evidently, for $g = g'' \circ g'$, we have that $f = h \circ g$, $g(\operatorname{coz} f) = \operatorname{co} 0_Y$ and $h^{-1}0_Z$ consists of one point. Let it be 0_Y. Then $h^{-1}0_Z = \{0_Y\}$ and $h^{-1}\operatorname{co} 0_Z = \operatorname{co} 0_Y$. □

Lemma 5.3. *Let we have a space Y of dimension $\dim Y = r$; a finite set a with $|a| > 1$; a space $R_b \in \mathbf{P}$ with a fixed point 0_b and $U_b = \operatorname{co} 0_b$ and a map $g'_b : Y \to R_b$ with $g'_b(W'_b = \operatorname{coz} g'_b) = U_b$ and $W'_b = (g'_b)^{-1}U_b$ for any $b \subset a$, $\emptyset \neq b \neq a$; maps $p_{bc} : U_b \to U_c$ for $c \subset b$ such that*

$$p_{bb} = \mathrm{id}_{U_b}, \qquad p_{bd} = p_{cd} \circ p_{bc} \quad \text{for } d \subset c \subset b;$$
$$W'_b = \bigcap\bigl\{W'_c\colon c \subset b,\ |c| = 1\bigr\} \quad \text{and} \quad g'_c|_{W'_b} = p_{bc} \circ g'_b|_{W'_b}.$$

Then there exist $R_a \in \mathbf{P}$ of dimension $\dim R_a \leqslant r$ with a fixed point 0_a and $U_a = \operatorname{co} 0_a$; a map $g'_a : Y \to R_a$ with $g'_a(W'_a = \operatorname{coz} g'_a) = U_a$ and $W'_a = (g'_a)^{-1}U_a$; maps $p_{ab} : U_a \to U_b$ for $b \subset a$ such that

$$p_{aa} = \mathrm{id}_{U_a} \quad \text{and} \quad p_{ac} = p_{bc} \circ p_{ab} \quad \text{for } c \subset b \subset a;$$
$$W'_a = \bigcap\bigl\{W'_b\colon b \subset a,\ |b| = 1\bigr\} \quad \text{and} \quad g'_b|_{W'_a} = p_{ab} \circ g'_a|_{W'_a}.$$

Proof. Let $\prod_a$ be the product of all R_b for $b \subset a$; pr_b be the projection of $\prod_a$ to R_b; Δ_a be the diagonal product of all g'_b. Then $\prod_a \in \mathbf{P}$ and $g'_b = \mathrm{pr}_b \circ \Delta_a$. Let $V'_a = \bigcap\{(\mathrm{pr}_b)^{-1}U_b\colon\ b \subset a,\ \ |b| = 1\}$ and $W'_a = (\Delta_a)^{-1}V'_a$. Then $W'_a = \bigcap\{(g'_b)^{-1}U_b\colon b \subset a,\ |b| = 1\} = \bigcap\{W'_b\colon b \subset a,\ |b| = 1\}$.

By property 3 of $\mathbf{P}$, there exist a space $R'_a \in \mathbf{P}$ of dimension $\dim R'_a \leqslant r$, a map $g''_a : Y \to R'_a$ and a map $h''_a : R'_a \to \Pi_a$ such that $\Delta_a = h''_a \circ g''_a$. Since $\mathbf{P}$ is hereditary and the subset theorem is true for perfectly normal spaces, we can suppose that g''_a is an onto map. Let $U'_a = (h''_a)^{-1}V'_a$. Evidently, $U'_a = g''_a W'_a$ and $W'_a = (g''_a)^{-1}U'_a$.

By property 4 of $\mathbf{P}$, there exist a pointed space $R_a \in \mathbf{P}$ with a fixed point 0_a and $U_a = \operatorname{co} 0_a$ and a map $\psi : R'_a \to R_a$ such that $U'_a = \psi^{-1}U_a$ and the corestriction χ of $\psi|_{U'_a}$ to U_a is a homeomorphism. Since $\dim U_a \leqslant \dim R'_a \leqslant r$, we have that $\dim R_a \leqslant r$. Evidently, for $g'_a = \psi \circ g''_a$, $\operatorname{coz} g'_a = (g'_a)^{-1}U_a = W'_a$ and $g'_a W'_a = U_a$. It is easy to see that g'_a, $p_{aa} = \mathrm{id}_{U_a}$, $p_{ab} = \mathrm{pr}_b \circ h''_a \circ \chi^{-1}$ and W'_a have the required properties. For example, for $c \subset b \subset a$, we have (because $W'_a \subset W'_b \subset W'_c$)

$$\begin{aligned} p_{ac} \circ g'_a|_{W'_a} &= \mathrm{pr}_c \circ h''_a \circ g''_a|_{W'_a} = \mathrm{pr}_c \circ \Delta_a|_{W'_a} = g'_c|_{W'_a} = p_{bc} \circ g'_b|_{W'_a} \\ &= p_{bc} \circ \mathrm{pr}_b \circ \Delta_a|_{W'_a} = p_{bc} \circ \mathrm{pr}_b \circ h''_a \circ g''_a|_{W'_a} = p_{bc} \circ p_{ab} \circ g'_a|_{W'_a}. \end{aligned}$$

Since $g'_a W'_a = U_a$, we have that $p_{ac} = p_{bc} \circ p_{ab}$. □

Proposition 5.4. *Let we have spaces $Z_\alpha \in \mathbf{P}$ with fixed points 0_α and $V_\alpha = \operatorname{co} 0_\alpha$, $\alpha \in \mathcal{A}$; a space Y with $\dim Y = r$; and a system of maps $[f'] = \{f'_\alpha : Y \to Z_\alpha\colon \alpha \in \mathcal{A}\}$.*

Then there exist an inverse superspectrum $[R] = \{(R_a, 0_a, U_a), p_{ba}; \mathcal{A}^*_\emptyset\}$ a map $[g'] = \{(g'_a : Y \to R_a)\colon a \in \mathcal{A}^*_\emptyset\}$ of Y to $[R]$ and a system $[h]$ of maps $h_\alpha : R_{\{\alpha\}} \to Z_\alpha$, $\alpha \in \mathcal{A}$, such that $R_a \in \mathbf{P}$, $\dim R_a \leqslant r$, $a \in \mathcal{A}^*_\emptyset$, $(U_{\{\alpha\}} = \operatorname{co} 0_{\{\alpha\}}) = (h_\alpha)^{-1}V_\alpha$ and $f'_\alpha = h_\alpha \circ g'_{\{\alpha\}}$, $\alpha \in \mathcal{A}$. If $W'_\alpha = \operatorname{coz}(f'_\alpha) = (f'_\alpha)^{-1}V_\alpha$ and $W'_a = \bigcap\{W'_\alpha\colon \alpha \in a\}$ then $W'_a = \operatorname{coz}(g'_a) = (g'_a)^{-1}U_a$ and $g'_a W'_a = U_a$, $a \in \mathcal{A}^*_\emptyset$.

Proof. By Proposition 5.1, there exist $R_{\{\alpha\}} \in \mathbf{P}$ of dimension $\dim R_{\{\alpha\}} \leqslant r$ with fixed points $0_{\{\alpha\}}$ and $U_{\{\alpha\}} = \operatorname{co} 0_{\{\alpha\}}$; maps $g'_{\{\alpha\}} : Y \to R_{\{\alpha\}}$ and $h_\alpha : R_{\{\alpha\}} \to Z_\alpha$ such that $f'_\alpha = h_\alpha \circ g'_{\{\alpha\}}$, $g'_{\{\alpha\}} W'_\alpha = U_{\{\alpha\}}$, $h_\alpha^{-1}0_\alpha = \{0_{\{\alpha\}}\}$, $U_{\{\alpha\}} = h_\alpha^{-1}V_\alpha$. Then, for $W'_{\{\alpha\}} = W'_\alpha$, $g'_{\{\alpha\}} W'_{\{\alpha\}} = U_{\{\alpha\}}$ and $W'_{\{\alpha\}} = (g'_{\{\alpha\}})^{-1}U_{\{\alpha\}}$.

The required inverse superspectrum $[R]$ is constructed using the previous lemma (induction on $|a|$, $a \in \mathcal{A}^*_\emptyset$). □

6. Proof of the main theorem

Proposition 6.1. *Let X be a subspace of a space Y and let, for any $j \in \mathbb{N}$, systems $[Zji]$ of spaces $Z_\alpha \in \mathbf{P}$ with fixed points 0_α and $V_\alpha = \operatorname{co} 0_\alpha$, $\alpha \in \mathcal{A}ji$, and systems of maps $[fji] = \{(f_\alpha : X \to Z_\alpha)\colon \alpha \in \mathcal{A}ji\}$, $i \in \mathbb{N}$, are piecewise $\mathbf{P}$-selecting a subset G_j of X.*

Then there exists a perfectly normal space S, a map $g : X \to S$ and open subsets U_j of S such that $\dim S \leqslant r$ and $G_j = g^{-1}U_j$, $j \in \mathbb{N}$.

Proof. Let $\mathcal{A} = \bigcup\{\mathcal{A}ji\colon\ i, j \in \mathbb{N}\}$ and $W_\alpha = f_\alpha^{-1} V_\alpha$, $\alpha \in \mathcal{A}$.

By Definitions 3.1 and 3.2, for $\mathcal{Q}ji = \mathcal{Q}_{[fji]}$ and the diagonal product $f_j : X \to (Z_{G_j} = \prod\{Z_{[fji]\mathcal{Q}ji}\colon\ i \in \mathbb{N}\})$ of all $f_{ji} = f_{[fji]\mathcal{Q}ji}$, we have that $G_j = f_j^{-1} H_j$ for some open set H_j in Z_{G_j}; there exists a system $[f'_{\mathcal{A}}] = \{(f'_\alpha : Y \to Z_\alpha)\colon\ \alpha \in \mathcal{A}\}$ of continuous extensions f'_α of $f_\alpha|_{\operatorname{coz} f_\alpha}$, and for any α, W_α is open–closed in $(W'_\alpha = (f'_\alpha)^{-1} V_\alpha) \cap X$.

By Propositions 5.4 and 4.5, there exist an inverse superspectrum $[R_{\mathcal{A}}] = \{(R_a, 0_{Ra}, U_a), p_{ba}; \mathcal{A}^*_\emptyset\}$; a map $[g_{\mathcal{A}}] = \{(g_a : Y \to R_a)\colon\ a \in \mathcal{A}^*_\emptyset\}$ of Y to $[R_{\mathcal{A}}]$ and a system $[h_{\mathcal{A}}]$ of maps $h_\alpha : R_{\{\alpha\}} \to Z_\alpha$, $\alpha \in \mathcal{A}$, such that

$$\dim R_a \leqslant r, \quad a \in \mathcal{A}^*_\emptyset;$$

$$W_\alpha = (f_\alpha)^{-1} V_\alpha, \quad \alpha \in \mathcal{A};$$

$$f_\alpha = h_\alpha \circ g_{\{\alpha\}}, \quad \alpha \in \mathcal{A}; \quad \text{and}$$

$$U_{\{\alpha\}} = (h_\alpha)^{-1} V_\alpha.$$

Since all families $\nu_{ij} = \{W_\alpha\colon\ \alpha \in \mathcal{A}_{ij}\}$ are locally finite, by Corollary 4.4, there exist a perfectly normal space S with $\dim S \leqslant r$ and maps $g : X \to S$ and $h_{ji} : S \to Z_{[fji]\mathcal{Q}ji}$ such that $f_{ji} = h_{ji} \circ g$, $j, i \in \mathbb{N}$. Let h_j be the diagonal product $\Delta\{h_{ji}\colon\ i \in \mathbb{N}\}$. Then $f_j = h_j \circ g$ and, for $U_j = (h_j)^{-1} H_j$, we have the relation $g^{-1} U_j = G_j$, $j \in \mathbb{N}$. $\square$

Proof of the main theorem. Let $\dim Y = r$. Take a finite fo cover $\varepsilon = \{G_j\colon\ j = 1, \ldots, k\}$ of X. By the previous proposition, there exist a perfectly normal space S, a map $g : X \to S$ and open subsets U_j of S such that $\dim S \leqslant r$ and $G_j = g^{-1} U_j$, $j = 1, \ldots, k$. Since S is perfectly normal, we can suppose that $S = gX$. Then $\eta = \{U_j\colon\ j = 1, \ldots, k\}$ is a cover of S and so there exists a finite refinement ζ of ε of order $\leqslant r$. Then $\delta = g^{-1}\zeta$ is a finite fo refinement of ε of order $\leqslant r$. $\square$

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