Weak orderability of topological spaces
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\begin{abstract}
In 1951 Ernest Michael wrote a definitive seminal article on hyperspaces [E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951) 152–182] raising a general question that became known as Michael’s selection problem for hyperspaces. The present paper contains a detailed discussion on particular aspects of this problem, also some further open questions.
\end{abstract}

\section{1. The selection problem for hyperspaces}

The Vietoris hyperspace. For a $T_1$-space $X$, let $\mathcal{F}(X)$ be the set of all nonempty closed subsets of $X$. One of the best known topologies on $\mathcal{F}(X)$ is the Vietoris one $\tau_V$, and we usually refer to $(\mathcal{F}(X), \tau_V)$ as the Vietoris hyperspace of $X$. Recall that a base for $\tau_V$ is given by all collections of the form

$$\langle V \rangle = \left\{ S \in \mathcal{F}(X) : S \subseteq \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of $X$. If $\mathcal{V} = \{V_1, \ldots, V_k\}$ for some open subsets $V_1, \ldots, V_k \subset X$, we often write $\langle \mathcal{V} \rangle = \langle V_1, \ldots, V_k \rangle$ rather than $\langle \mathcal{V} \rangle = \langle \{V_1, \ldots, V_k\} \rangle$.

Let us make the explicit agreement that when $\mathcal{T}$ is a given topology on $X$ and the clarity seems to demand it, the prefix “$\mathcal{T}$-” will be broadly used to express concepts related to $\mathcal{T}$. For instance, $\mathcal{T}$-open sets for the members of $\mathcal{T}$; $\mathcal{T}$-closed sets; etc. In this regard, we will also write $\tau_V(\mathcal{T})$ for the Vietoris topology on the nonempty $\mathcal{T}$-closed subsets of $X$ generated by a topology $\mathcal{T}$ on $X$.

Michael’s selection problem. Suppose that $\Phi : Y \to \mathcal{F}(X)$ is a map, usually called a set-valued mapping, or a multimap, and, sometimes, a multification. Once $\mathcal{F}(X)$ has been topologized, $\Phi$ becomes a usual map between topological spaces, and it...
makes sense to talk about its continuity and other topological properties. In 1951 Ernest Michael [23] raised the following general question:

**Question 1.** ([23, Question 6.1]) Given \( \Phi : Y \to \mathcal{F}(X) \), when is it possible to find a continuous map \( f : Y \to X \) such that \( f(y) \in \Phi(y) \) for all \( y \in Y \) (i.e., a selection for \( \Phi \))?  

As he wrote in his paper [23], a sufficient condition that this be possible is that both the following hold: \( \Phi \) is continuous, and there exists a "selection" from \( \mathcal{F}(X) \) to \( X \). The problem is thus reduced to two simpler ones, the second of which is concerned only with the space \( X \) and has nothing to do with the space \( Y \) or the multifunction \( \Phi \). This second problem is now known as the Selection Problem for Hyperspaces.

**Selections and hyperspaces.** In the sequel, all spaces are assumed to be infinite and Hausdorff if not suggested otherwise, while any subset \( D \subset \mathcal{F}(X) \) will carry the relative Vietoris topology \( \tau_V \) as a subspace of the hyperspace \((\mathcal{F}(X), \tau_V)\). A map \( f : D \to X \) is a selection for \( D \) if \( f(S) \in S \) for every \( S \in D \). A selection \( f : D \to X \) is continuous if it is continuous with respect to the relative Vietoris topology \( \tau_V \) on \( D \). Sometimes, for reasons of convenience, we will also say that \( f \) is Vietoris continuous, or \( \tau_V \)-continuous, to emphasize that \( f \) is continuous with respect to the topology \( \tau_V \).

Related to selections, we will use the following special subsets of \( \mathcal{F}(X) \):

\[
\mathcal{F}_n(X) = \{ S \subset X : 1 \leq |S| \leq n \} \quad \text{and} \quad \{X\}^n = \{ S \subset X : |S| = n \}, \quad n \geq 1.
\]

We may identify \( X \) with the set \([X]^1 = \mathcal{F}_1(X)\) and, in fact, \( X \) is homeomorphic to the space \((\mathcal{F}_1(X), \tau_V)\). The latter means that the Vietoris topology is admissible in sense of [23].

**Orderable-like spaces.** A space \( X \) is orderable (or, linearly orderable) if the topology of \( X \) coincides with the open interval topology \( \mathcal{F}_\preceq \) on \( X \) generated by a linear ordering \( \preceq \) on \( X \). In this case, the order \( \preceq \) on \( X \) is called compatible for the topology of \( X \), or, merely, a compatible order for \( X \). Recall that all \( \preceq \)-open intervals

\[
(\preceq)(\preceq) \preceq \{ y \in X : y \prec x \} \quad \text{and} \quad (\preceq) \preceq \{ y \in X : x \preceq y \}, \quad x \in X,
\]

consider a subbase for \( \mathcal{F}_\preceq \). In the sequel, the term "orderable" will be explicitly reserved for orderable topological spaces.

Subspaces of orderable spaces are not necessarily orderable, they are called suborderable (or, generalized ordered). Here is another way how to define them. A subset \( B \) of a linearly ordered set \((X, \preceq)\) is \( \preceq \)-convex, or merely convex, if \( \{ x \in X : y \prec x \preceq z \} \subset B \) for every \( y, z \in B \), with \( y \preceq z \). Now, a space \((X, \mathcal{F})\) is suborderable if and only if there exists a linear order \( \preceq \) on \( X \) (called compatible for \( X \)) such that the corresponding open interval topology \( \mathcal{F}_\preceq \) is coarser than \( \mathcal{F} \) (i.e., \( \mathcal{F}_\preceq \subset \mathcal{F} \)), and \( \mathcal{F} \) has a base of \( \preceq \)-convex sets.

A space \((X, \mathcal{F})\) is weakly orderable (also called "Eilenberg orderable") if it admits a coarser open interval topology \( \mathcal{F}_\preceq \) for some linear ordering \( \preceq \) on \( X \) (called compatible for \( X \)). However, \( \mathcal{F}_\preceq \subset \mathcal{F} \) if and only if the identity map \( \text{id}_X : (X, \mathcal{F}) \to (X, \mathcal{F}_\preceq) \) is continuous. Hence, a space \( X \) is weakly orderable if and only if there exist an orderable space \( Y \) and a continuous injective map \( h : X \to Y \). Since a subbase for \( \mathcal{F}_\preceq \) is given by all \( \preceq \)-open intervals \((\preceq)(\preceq)\) and \((\preceq) \preceq \), \( x \in X \), the inclusion \( \mathcal{F}_\preceq \subset \mathcal{F} \) is also equivalent to the statement that \((\preceq)(\preceq)(\preceq)(\preceq)\) for every \( x \in X \).

**Van Mill and Wattel’s problem.** Every selection \( f : \mathcal{F}_2(X) \to X \) generates a natural order-like relation \( \preceq_f \) on \( X \) [23, Definition 7.1] defined for \( x, y \in X \) by \( x \preceq_f y \) if and only if \( f([x, y]) = x \). For convenience, we write \( x \preceq f y \) provided \( x \preceq_f y \) and \( x \neq y \).

The relation \( \preceq_f \) is very similar to a linear order on \( X \) in that it is both total and anti-symmetric, but, unfortunately, it may fail to be transitive. However, if \( f \) is also continuous, then all \( \preceq_f \)-open intervals \( \{ y \in X : y \prec_f x \} \) and \( \{ y \in X : x \prec_f y \} \), \( x \in X \), are open in \( X \). Thus, in 1981 Jan van Mill and Evert Wattel raised the following general conjecture.

**Question 2.** (van Mill and Wattel [24]) Let \( X \) be a space which has a continuous selection for \( \mathcal{F}_2(X) \). Then, is it true that \( X \) is weakly orderable?

In view of this possible relationship, a selection \( f \) for \( \mathcal{F}_2(X) \) is often called a weak selection for \( X \).

Recently, Michael Hrušák and Iván Martínez-Ruiz answered Question 2 in the negative by constructing a separable, first countable and locally compact space which admits a continuous weak selection but is not weakly orderable [21]. On the other hand, this question was resolved in the affirmative in a number of situations. Some of these results were previously discussed by the authors in [16], for other recent positive solutions the interested reader is refer to [10,11]. The purpose of this paper is to discuss different aspects of Question 2.
2. Weak selections and selection relations

Binary relations. Given a set $X$, a subset $E \subset X^2$ is usually called a binary relation, or merely a relation on $X$. For a relation $E$ on $X$, we write $xEy$ to express that $(x, y) \in E$. The inverse relation $E^{-1}$ of $E$ is the relation on $X$ defined by $xE^{-1}y$ if and only if $yEx$.

Any relation $E$ on $X$ has a natural extension to a relation on the subsets of $X$, denoted again by $E$, and defined for $B, C \subset X$ by $BEC$ if and only if $B \times C \subset E$. That is, $BEC$ if and only if $yEx$ for every $y \in B$ and $x \in C$. In this regard, for a point $x \in X$, we will write $xEC$ rather than $\{x\}EC$, etc., which may simplify some of our notation.

Selection relations. A relation $E$ on $X$ is called a selection relation [15] if $E$ is total and anti-symmetric. In these terms, a selection relation is a linear order if and only if it is transitive. The order-like relation $\approx_f$ associated to a weak selection $f$ for $X$ is a selection relation on $X$. The converse is also true. Namely, every selection relation $E \subset X^2$ defines a weak selection $f_E$ for $X$ by letting that $f_E((x, y)) = x$ if and only if $xEx$. Thus, the weak selections for $X$ are exactly the selection relations on $X$. In the sequel, we will often write $\approx_s$ for a selection relation on $X$. Also, for points $x, y \in X$, we will write $x \approx_s y$ to express that $x \approx_s y$ and $x \neq y$.

Relation intervals. To deal with selection relations, it is sometimes more convenient to work with “intervals” generated by them, just as in the case of linear orders. So, for a selection relation $\approx_s$ on $X$ and $x \in X$, we let

$$(\leftarrow, x)_{\approx_s} = \{y \in X: y \approx_s x\} \quad \text{and} \quad (x, \rightarrow)_{\approx_s} = \{y \in X: x \approx_s y\},$$

and will refer to these sets as $\approx_s$-closed intervals. In the same way, we consider the corresponding $\approx_s$-open intervals:

$$(\leftarrow, x)_{\approx_s} = \{y \in X: y \approx_s x\} \quad \text{and} \quad (x, \rightarrow)_{\approx_s} = \{y \in X: x \approx_s y\}.$$

Finally, for points $x, y \in X$, we consider also the following composite intervals:

$$(x, y)_{\approx_s} = (x, \rightarrow)_{\approx_s} \cap (\leftarrow, y)_{\approx_s},$$

$$(x, y)_{\approx_s} = [x, \rightarrow)_{\approx_s} \cap (\leftarrow, y]_{\approx_s}$$

and

$$(x, y)_{\approx_s} = [x, \rightarrow]_{\approx_s} \cap (\leftarrow, y)_{\approx_s}.$$
Proof. Let $x \in U$ if and only if there exists a nonempty finite subset $F \subset \mathcal{A}(a) \varnothing$ such that $x \in \bigcap F \subset U$. On the other hand, $x \in (a, b) \varnothing$ for some $a, b \in X$ because it is a $\varnothing \varnothing$-cut point. Then, $A = A_0 \cup \{a\}$ and $B = B_0 \cup \{b\}$ are as required, where

$$A_0 = \{y \in X: (y, \to) \varnothing \in \mathcal{F}\}$$ and
$$B_0 = \{z \in X: (-z, z) \varnothing \in \mathcal{F}\}.$$

Also, $A \cap B = \varnothing$ because $(A, B) \varnothing \neq \varnothing$ while $(z, z) \varnothing \varnothing = \varnothing$ for all $z \in X$. □

**Theorem 2.2.** ([15]) Let $\varnothing \varnothing$ be a selection relation on a set $X$. Then, the selection topology $\mathcal{F}(a)$ is regular.

Theorem 2.2 is an immediate consequence of the definition of the $\mathcal{F}(a)$ and the following lemma. In this lemma, $\varnothing \varnothing(a)$ denotes the $\varnothing \varnothing$-closure of a subset $B \subset X$.

**Lemma 2.3.** ([15]) Let $\varnothing \varnothing$ be a selection relation on a set $X$ and $x, y \in X$ be such that $x \varnothing \varnothing y$. Then, there is $U \in \mathcal{F}(a)$ such that

$$x \in U \quad \text{and} \quad \varnothing \varnothing(x \cap (-y)) \subset (-x, y) \varnothing.$$

In particular, $y \in V = X \setminus \varnothing \varnothing(U \cap (-y) \varnothing)$ and $\varnothing \varnothing(V \cap (x, y) \varnothing) \subset (x, y) \varnothing$.

**Proof.** In case $(x, y) \varnothing \varnothing \neq \varnothing$, set $U = (-z) \varnothing$ for some $z \in (x, y) \varnothing$. Then, $x \in U \in \mathcal{F}(a)$. Since $(-z) \varnothing$ and $(x, y) \varnothing$ are $\mathcal{F}(a)$-closed and $y \neq (-z) \varnothing$, we get that

$$\varnothing \varnothing(U \cap (-y) \varnothing) \subset (-z) \varnothing \cap (-y) \varnothing \subset (-y) \varnothing.$$

If $(x, y) \varnothing \varnothing = \varnothing$, then $y \varnothing \varnothing x$ for every $z \varnothing \varnothing y$. That is, $(x, y) \varnothing \varnothing \subset (-z) \varnothing$ and, therefore, $(x, y) \varnothing \varnothing = (x, x) \varnothing \cup (-y) \varnothing \in \mathcal{F}(a)$. Hence, $(-x) \varnothing$ is a $\mathcal{F}(a)$-clopen set which contains $x$ and doesn’t contain $y$, and we can now take $U = (x, y) \varnothing$. □

Closed linear orders. Following [4], for a space $X$ let $\Lambda : X^2 \to X^2$ be the map defined by

$$\Lambda(x, y) = (y, x), \quad (x, y) \in X^2.$$ (2.2)

By the definition of the Tychonoff product topology, $\Lambda$ is a homeomorphism.

We shall say that a binary relation $\varnothing$ on a space $X$ is closed (respectively, open) if $\varnothing \subset X^2$ is closed (respectively, open). The following simple observation relates open and closed selection relations.

**Proposition 2.4.** A selection relation $\varnothing \varnothing$ on a space $X$ is closed if and only if the relation $\varnothing$ is open. In particular, $\varnothing \varnothing$ is closed if and only if for every $x, y \in X$, with $x \varnothing \varnothing y$, there are open sets $U, V \subset X$ such that $x \in U, y \in V$ and $U \varnothing \varnothing V$.

**Proof.** We have that $\varnothing \varnothing = X^2 \setminus (\varnothing \varnothing)^{-1}$, where $(\varnothing \varnothing)^{-1}$ is the inverse relation. The proof now follows from that fact that the map $\Lambda : X^2 \to X^2$, defined as in (2.2), is a homeomorphism such that $\Lambda(\varnothing \varnothing) = (\varnothing \varnothing)^{-1}$. □

The following theorem is crucial to examine the difference between the existence of continuous weak selections and weak orderability, in particular to translate the latter property only in terms of closed relations. The equivalence of (b), (c) and (d) in this theorem is due to Eilenberg [4].

**Theorem 2.5.** ([4]) For a space $(X, \mathcal{F})$ and a linear order $\varnothing$ on $X$, the following are equivalent:

(a) The linear order $\varnothing$ is $\mathcal{F}$-closed.
(b) If $x, y \in X$ and $x \varnothing y$, then there are $\mathcal{F}$-open sets $U, V \subset X$ such that $x \in U, y \in V$ and $U \varnothing V$.
(c) If $x, y \in X$ and $x \varnothing y$, then there are $\mathcal{F}$-open sets $U, V \subset X$ such that $x \in U, y \in V, x \varnothing V$ and $U \varnothing y$.
(d) $\mathcal{F}(a) \subset \mathcal{F}$.

**Proof.** The implication (a) ⇒ (b) follows by Proposition 2.4, while (b) ⇒ (c) is obvious. Suppose that (c) holds, and take a point $x \in X$. Whenever $y \in (x, y) \varnothing$, by (c), there exists a $\mathcal{F}$-open set $V \subset X$ such that $y \in V$ and $x \varnothing z$ for every $z \in V$. Consequently, $V \subset (x, y) \varnothing$ which implies that $(x, y) \varnothing \in \mathcal{F}(a)$. In the same way, $(x, x) \varnothing \in \mathcal{F}(a)$. That is, $\mathcal{F}(a) \subset \mathcal{F}$ which is (d).

To show finally that (d) ⇒ (a), suppose that $\mathcal{F}(a) \subset \mathcal{F}$, and take points $x, y \in X$ such that $x \varnothing y$. In case $(x, y) \varnothing \neq \varnothing$, set $U = (x, y) \varnothing$ and $V = (y, x) \varnothing$ for some $z \in (x, y) \varnothing$. If $(x, y) \varnothing = \varnothing$, then set $U = (x, y) \varnothing$ and $V = (x, y) \varnothing$. Thus, in both cases, $x \in U \subset \mathcal{F}(a) \subset \mathcal{F}$, $y \in V \subset \mathcal{F}(a) \subset \mathcal{F}$ and $U \varnothing V$. According to Proposition 2.4, the linear order $\varnothing$ is $\mathcal{F}$-closed. □
Closed selection relations. Every selection \( g : [X]^2 \to X \) has a unique extension to a selection \( f \) for \( \mathcal{F}_2(X) \) defined by \( f([x]) = x, x \in X \). In fact, while the selection relation \( \preceq_f \) corresponds to a selection \( f \) for \( \mathcal{F}_2(X) \), the relation \( \prec_f \) corresponds simply to \( f^{-1}([X]^2) \). On the other hand, if \( f \) is a selection for \( \mathcal{F}_2(X) \), then \( (V) \cap \mathcal{F}_2(X) \subset f^{-1}(V) \) for every open \( V \subset X \). Hence, a selection \( f \) for \( \mathcal{F}_2(X) \) is continuous if and only if \( f^{-1}([X]^2) \) is continuous. Thus, the (continuous) weak selections for \( X \) are precisely the (continuous) selections for \( [X]^2 \). In view of that, we will often work with \( [X]^2 \) rather than \( \mathcal{F}_2(X) \), and will make no difference between the selections for \( [X]^2 \) and those for \( \mathcal{F}_2(X) \).

As it was already emphasized, the selection relations on a set \( X \) are the weak selections for \( X \) expressed in terms of order-like relations. The following theorem translates the continuity of weak selections in terms of closed relations, and incorporates a criterion for continuity of weak selections, see [13, Theorem 3.1] and [11, Proposition 2.1].

**Theorem 2.6.** For a weak selection \( f \) for a space \((X, \mathcal{T})\), the following are equivalent:

(a) The selection relation \( \preceq_f \) is \( \mathcal{T} \)-closed.
(b) \( f \) is \( \tau_{\mathcal{T}} \)-continuous.
(c) \( f \) is \( \tau_{\mathcal{T}} \)-continuous.

**Proof.** The implication (a) \( \Rightarrow \) (b) follows by Proposition 2.4. In order to show that (b) \( \Rightarrow \) (c), take distinct points \( x, y \in X \), say \( x \prec_f y \). Then, by (b), there are \( \mathcal{T} \)-open sets \( U, V \subset X \) such that \( x \in U, y \in V \) and \( U \prec_f V \). Hence, \( f((U \cap V, W)) \subset W \) for every \( \mathcal{T} \)-open set \( W \subset X \) containing \( x \). So, \( f \) is \( \tau_{\mathcal{T}} \)-continuous.

To show finally that (c) \( \Rightarrow \) (a), take points \( x, y \in X \) such that \( x \prec_f y \). Since \( x \neq y \), there are disjoint \( \mathcal{T} \)-open sets \( W_x, W_y \subset X \), with \( x \in W_x \) and \( y \in W_y \). Since \( f \) is \( \tau_{\mathcal{T}} \)-continuous and \( f(x) = x \in W_x \), there are also \( \mathcal{T} \)-open sets \( U, V \subset X \) such that \( x \in U \subset W_x \), \( y \in V \subset W_y \), and \( f((U, V)) \subset W_x \). Then, we have that \( f(s, t) = s \) (i.e., \( s \prec_f t \)) for every \( s \in U \) and \( t \in V \). That is, \( U \prec_f V \) and, by Proposition 2.4, the selection relation \( \preceq_f \) is \( \mathcal{T} \)-closed. \( \square \)

According to Theorems 2.5 and 2.6, we have the following immediate consequence, see [23, Lemma 7.5.1].

**Corollary 2.7.** ([23]) If \( X \) is a weakly orderable space with respect to a linear order \( \preceq \), then \( X \) has a continuous weak selection \( f \) such that \( \preceq_f = \preceq \).

By Theorem 2.6, we get also the following consequence.

**Corollary 2.8.** ([13]) If \( f \) is a continuous weak selection for a space \((X, \mathcal{T})\), then \( f \) is also continuous with respect to any topology on \( X \) which is finer than \( \mathcal{T} \).

3. Isbell–Mrówka spaces and weak selections

**Isbell–Mrówka spaces.** Let \( X \) be an infinite countable set, and let

\[
[X]^{<\omega} = \bigcup\{(X)^n: 1 \leq n < \omega\} \quad \text{and} \quad [X]^\omega = \{S \subset X: |S| = \omega\}.
\]

A family \( \mathcal{A} \subset [X]^\omega \) is called **almost disjoint** if \( A \cap B = \emptyset \) for every two distinct elements \( A, B \in \mathcal{A} \). An almost disjoint family \( \mathcal{A} \subset [X]^{<\omega} \) is called **maximal** (briefly, MAD) if it is maximal with respect to this property, i.e. if for every \( B \in [X]^{<\omega} \setminus \mathcal{A} \) there exists \( A \in \mathcal{A} \) such that \( A \cap B = \emptyset \). An **Isbell–Mrówka space** \( \Psi(\mathcal{A}) \) generated by an almost disjoint family \( \mathcal{A} \subset [X]^\omega \) is usually defined as follows: \( \Psi(\mathcal{A}) = \mathcal{A} \cup X \); each element of \( X \) is isolated in \( \Psi(\mathcal{A}) \); and each element \( A \in \mathcal{A} \) has a neighbourhood base of the form \( \{A \cup (A \setminus F)\} \), where \( F \) runs over the finite subsets of \( X \). According to this definition, \( \Psi(\mathcal{A}) \) is always a first countable locally compact space. Since \( \mathcal{A} \subset \Psi(\mathcal{A}) \) is a closed discrete subset, \( \Psi(\mathcal{A}) \) is not countably compact when \( \mathcal{A} \) is infinite. Finally, it is well known that \( \Psi(\mathcal{A}) \) is pseudocompact if and only if \( \mathcal{A} \) is MAD [27] (see, also, [28]).

Answering a question of T. Nogura (who asked if \( \Psi(\mathcal{A}) \) has a continuous weak selection for some (any) MAD family \( \mathcal{A} \subset [X]^\omega \)), [22], Hrušák, Szyptycki and Tomita proved that \( \Psi(\mathcal{A}) \) has no continuous weak selection for any MAD family \( \mathcal{A} \subset [X]^\omega \). This fact follows also from a more general result obtained by Artico, Marconi, Pelant, Rotter and Tkachenko [1].

In what follows, we are going to examine weak selections for almost disjoint families \( \mathcal{A} \subset [X]^\omega \) which give rise to continuous weak selections for \( \Psi(\mathcal{A}) \). In particular, we are going to examine the counterexample in [21] of van Mill and Wattel’s weak orderability problem (Question 2).

**Extension of continuous weak selections.** Suppose that \( \mathcal{A} \subset [X]^\omega \) is an almost disjoint family. Identifying \( X \) with the singletons of \( X \), i.e. \( X = [X]^1 \), we get that a subset \( U \subset \Psi(\mathcal{A}) = \mathcal{A} \cup [X]^1 \) is open if and only if for every \( \alpha \in U \) there exists a finite subset \( F \subset X \) such that \( \emptyset \neq \alpha \setminus F \subset U \).
Motivated by this, it makes sense to consider almost $\mathcal{P}$-families, where $\mathcal{P}$ is a (binary) relation on the nonempty subsets of a set $X$. Here is an example that will be important for continuity of weak selections for Isbell–Mrówka spaces. Let $\lessdot_{s}$ be a selection relation on a set $X$. A family $\mathcal{M}$ of subsets of $X$ is $\lessdot_{s}$-decisive [18] (see, also, [12]) if it consists of nonempty subsets of $X$ and $C \lessdot_{s} D$ or $D \lessdot_{s} C$ for every two distinct members $C, D \in \mathcal{M}$. Now, we shall say that a family $\mathcal{M}$ of nonempty subsets of $X$ is almost $\lessdot_{s}$-decisive if for every two distinct elements $P, Q \in \mathcal{M}$ there are finite subsets $F_P, F_Q \subset X$ such that $P \setminus F_P \not= \emptyset \not= Q \setminus F_Q$ and $P \setminus F_P \lessdot_{s} Q \setminus F_Q$ or $Q \setminus F_Q \lessdot_{s} P \setminus F_P$. The following property was stated by Michael Hrušák and Iván Martínez-Ruiz [21, Lemma 2.1] in slightly different terms.

**Proposition 3.1.** ([21]) Let $X$ be an infinite countable set, and $\mathcal{A} \subset [X]^\omega$ be an almost disjoint family. Then, a weak selection $\psi$ for $X$ can be extended to a continuous weak selection for $\psi(\mathcal{A})$ if and only if the family $\mathcal{A} \cup \{X\}$ is almost $\lessdot_{\psi}$-decisive.

**Proof.** Let $\psi$ be a continuous weak selection for $\psi(\mathcal{A})$, and let $\psi = \psi \upharpoonright [X]^2$. Take distinct elements $\alpha, \beta \in \psi(\mathcal{A})$, say $\alpha \lessdot_{\psi} \beta$. Since $\psi$ is continuous, according to Theorem 2.6, there are open subsets $U, V \subset \psi(\mathcal{A})$ such that $\alpha \in U$, $\beta \in V$ and $U \lessdot_{\psi} V$. According to the definition of the topology of $\psi(\mathcal{A})$, there now exist finite subsets $F_\alpha, F_\beta \subset X$ such that $\emptyset \not= \alpha \setminus F_\alpha \subset U$ and $\emptyset \not= \beta \setminus F_\beta \subset V$. Since $\psi = \psi \upharpoonright [X]^2$, we have that $\alpha \setminus F_\alpha \lessdot_{\psi} \beta \setminus F_\beta$. So, $\psi(\mathcal{A})$ is almost $\lessdot_{\psi}$-decisive.

To show the converse, suppose that $\psi(\mathcal{A})$ is almost $\lessdot_{\psi}$-decisive for some weak selection $\psi$ for $X$. Extend $\psi$ to a weak selection $\psi$ for $\psi(\mathcal{A})$ in the following manner. If $\alpha, \beta \in \psi(\mathcal{A})$ are distinct elements, then, by hypothesis, there are finite subsets $F_\alpha, F_\beta \subset X$ such that $\alpha \setminus F_\alpha \not= \emptyset \not= \beta \setminus F_\beta$ and $\alpha \setminus F_\alpha \lessdot_{\psi} \beta \setminus F_\beta$ or $\beta \setminus F_\beta \lessdot_{\psi} \alpha \setminus F_\alpha$. Now, set $\alpha \lessdot_{\psi} \beta$ if and only if $\alpha \setminus F_\alpha \lessdot_{\psi} \beta \setminus F_\beta$. The selection $\psi$ is well-defined, and, by Theorem 2.6, it is continuous. □

The Cantor tree and its branches. A partially ordered set $(T, \lessdot_{\triangle})$ is a tree if the set $\{s \in T: S \lessdot_{\triangle} t\}$ is well-ordered for every $t \in T$. A chain $\pi$ in a tree $(T, \lessdot_{\triangle})$ is a subset $\pi \subset T$ which is linearly ordered by $\lessdot_{\triangle}$. A maximal chain $\pi$ in $T$ is called a branch in $T$, and we denote by $\mathcal{B}(T)$ the set of all branches in $T$.

Here, we will be mainly interested in the following realization of the Cantor set as a branch set. Namely, let $S$ be a set which has at least 2 distinct elements, $S^\omega$ be the set of all maps $t: N \rightarrow S$, and let

$$S^{<\omega} = \bigcup(S^n: n < \omega).$$

Whenever $t \in S^{<\omega}$, let Dom$(t)$ be the domain of $t$. Consider the partial order $\subseteq$ on $S^{<\omega}$ defined for $s, t \in S^{<\omega}$ by $s \subseteq t$ if and only if

$$\text{Dom}(s) \subset \text{Dom}(t) \quad \text{and} \quad t \upharpoonright \text{Dom}(s) = s.$$

Then, $(S^{<\omega}, \subseteq)$ is a tree such that its branch set $\mathcal{B}(S^{<\omega})$ is identical with $S^\omega$. Namely, each branch $\beta \in \mathcal{B}(S^{<\omega})$ of $(S^{<\omega}, \subseteq)$ can be identified with the element $\beta^* \in S^\omega$ for which $\beta = [\beta^* \upharpoonright n: n < \omega]$. This correspondence is bijective, hence we will tacitly assume that $\mathcal{B}(S^{<\omega}) = S^\omega$. In particular, $\mathcal{B}(2^{<\omega}) = 2^\omega$ and, in the sequel, we will refer to the tree $(2^{<\omega}, \subseteq)$ as the Cantor tree.

**Selections for the Cantor tree.** The partial order $\subseteq$ on the Cantor tree $2^{<\omega}$ can be extended to a selection relation $\lessdot_{\sigma}$ on $2^{<\omega}$ in the following way. For $\subseteq$-incomparable elements $s, t \in 2^{<\omega}$, define

$$s \triangle t = \min\{k \in \text{Dom}(s) \cap \text{Dom}(t): s(k) \not= t(k)\}.$$ 

Then, let $s \lessdot_{\sigma} t$ if and only if $s(s \triangle t) < t(s \triangle t)$ or, in other words, if $s(s \triangle t) = 0$.

Next, for $s \in 2^{<\omega}$ and $\beta, \gamma \in 2^\omega$, with $s \not= \beta \not= \gamma$, let

$$s \triangle \beta = \min\{k \in \text{Dom}(s): s(k) \not= \beta(k)\} \quad \text{and} \quad \beta \triangle \gamma = \min\{k < \omega: \beta(k) \not= \gamma(k)\}.$$

**Proposition 3.2.** The branch set $2^\omega$ of the Cantor tree $(2^{<\omega}, \subseteq)$ is an almost disjoint family, while $\lessdot_{\sigma}$ is a selection relation on $2^{<\omega}$ such that, for every $\beta \in 2^\omega$ and $s \in 2^{<\omega} \setminus \beta$, we have

$$s \lessdot_{\sigma} \{\beta \upharpoonright k: k > s \triangle \beta\} \quad \text{if and only if} \quad s(s \triangle \beta) < \beta(s \triangle \beta).$$

In particular, $\lessdot_{\sigma}$ defines a weak selection $\sigma$ for $2^{<\omega}$ such that the family $2^\omega \cup 2^{<\omega}$ is almost $\lessdot_{\sigma}$-decisive.

**Proof.** The family $2^\omega$ is clearly almost disjoint. If $\beta \in 2^\omega$ and $s \in 2^{<\omega} \setminus \beta$, then $s \triangle (\beta \upharpoonright k) = s \triangle \beta$ for every $k > s \triangle \beta$. Consequently, $s \lessdot_{\sigma} \{\beta \upharpoonright k: k > s \triangle \beta\}$ if and only if $s(s \triangle \beta) < \beta(s \triangle \beta)$. □

According to Propositions 3.1 and 3.2, the weak selection $\sigma$ for $2^{<\omega}$ can be extended to a continuous weak selection for $\psi(2^\omega) = 2^\omega \cup 2^{<\omega}$. In our next considerations, we will be mostly interested in modifying $\sigma$ only on particular branches of the Cantor tree $(2^{<\omega}, \subseteq)$. 
Dense weak selections. Following the concept of a universal weak selection in [21], we shall say that a selection \( \varphi : [X]^{2} \to X \) is dense in \( X \) if for every two disjoint sets \( F, G \subseteq [X]^{<\omega} \) there is a point \( z \in X \), with \( F \prec_{\varphi} z \prec_{\varphi} G \). According to (2.1), a weak selection \( \varphi \) for \( X \) is dense in \( X \) if and only if \( (F, G) \prec_{\varphi} \emptyset \) for every two disjoint sets \( F, G \subseteq [X]^{<\omega} \). The following is a useful property of dense selections.

**Proposition 3.3.** Let \( X \) be an infinite set, \( \varphi \) be a dense weak selection for \( X \), and let \( \prec \) be a linear order on \( X \). Then, there are points \( x, y, z \in X \) such that \( z \prec_{\varphi} x \prec_{\varphi} y \), but \( x \prec \{y, z\} \).

**Proof.** Take points \( a, b \in X \) such that \( a \prec b \). Since \( \varphi \) is dense, there exists a point \( c \in X \) such that \( b \prec_{\varphi} c \prec_{\varphi} a \). Next, let \( x = \min_{<}(a, b, c) \). Since \( a \prec_{\varphi} b \prec_{\varphi} c \prec_{\varphi} a \), we may always assume that \( x = a \). Then, we take \( y = b \) and \( z = c \). These \( x, y \) and \( z \) are as required. \( \Box \)

For a selection \( \varphi : [X]^{2} \to X \), we let

\[
\mathcal{D}(\varphi) = \{ A \subseteq X : \varphi \mid [A]^{2} \text{ is dense in } A \}.
\]

The following fact about dense weak selections is due to Michael Hrušák and Iván Martín-Ruiz, see [21, Proposition 2.3].

**Proposition 3.4.** ([21]) For a dense selection \( \varphi : [X]^{2} \to X \), the following hold:

(a) if \( \mathcal{D} = \{ P_{0}, P_{1} \} \) is a partition of \( X \), then \( P_{i} \in \mathcal{D}(\varphi) \) for some \( i < 2 \),

(b) \( \{ x \in X : F \prec_{\varphi} x \prec_{\varphi} G \} \in \mathcal{D}(\varphi) \), whenever \( F, G \subseteq [X]^{<\omega} \) are disjoint sets.

**Proof.** We follow the proof of [21, Proposition 2.3]. Suppose that (a) fails, i.e. that there exists a partition \( \mathcal{P} = \{ P_{0}, P_{1} \} \) of \( X \) such that \( P_{0}, P_{1} \notin \mathcal{D}(\varphi) \). Since each \( \varphi \mid [P_{i}]^{2} \), \( i < 2 \), is not dense, there are disjoint sets \( F_{i}, G_{i} \subseteq [P_{i}]^{<\omega} \), \( i < 2 \), such that \( P_{0} \cap \bigcap_{F_{i}, G_{i} \subseteq [P_{i}]^{<\omega}} \emptyset = \emptyset = P_{1} \cap \bigcap_{F_{i}, G_{i} \subseteq [P_{i}]^{<\omega}} \emptyset \), see (2.1). On the other hand, \( P_{0} \) and \( P_{1} \) are disjoint, and therefore \( F_{0} \cup F_{1} \) and \( G_{0} \cup G_{1} \) are also disjoint. Hence, there is \( z \in X = P_{0} \cup P_{1} \) such that \( F_{0} \cup F_{1} \prec_{\varphi} z \prec_{\varphi} G_{0} \cup G_{1} \) because \( \varphi \) is dense. However, this is impossible because \( z \in P_{0} \) or \( z \in P_{1} \). A contradiction! Thus, (a) holds. To see (b), take disjoint sets \( F, G \subseteq [X]^{<\omega} \), if

\[
B = \{ x \in X : F \prec_{\varphi} x \prec_{\varphi} G \} \notin \mathcal{D}(\varphi),
\]

then, by (a), \( X \setminus B \in \mathcal{D}(\varphi) \) while, by the definition of the relation \( \prec_{\varphi} \), we have that \( F, G \subseteq [X] \setminus B \). Since \( \varphi \) is dense, there is \( x \in X \setminus B \), with \( F \prec_{\varphi} x \prec_{\varphi} G \). However, by the definition of \( B \), we also have that \( x \in B \). A contradiction! \( \Box \)

**Proposition 3.5.** Let \( \sigma \) be a weak selection for \( 2^{<\omega} \). Then, \( 2^{<\omega} \) has a weak selection \( \varphi \) such that

(a) \( 2^{\omega} \subseteq \mathcal{D}(\varphi) \),

(b) if \( \beta \in 2^{\omega} \) and \( s \in 2^{<\omega} \setminus \beta \), then \( \varphi((s, \beta \setminus k)) = \sigma((s, \beta \setminus k)) \) for \( k > s \triangle \beta \).

**Proof.** We will construct \( \varphi \) modifying the values of \( \sigma \) on certain branches of the tree \( (2^{<\omega}, \sqsubseteq) \). To this end, let \( \{(F_{n}, G_{n}) : n < \omega\} \) be the set of all ordered pairs of nonempty disjoint finite subsets of \( 2^{<\omega} \) such that, for every \( n < \omega \), there exists a branch \( \beta_{n} \in 2^{\omega} \), with \( F_{n} \cup G_{n} \subseteq \beta_{n} \). For convenience, for every \( n < \omega \), let

\[
\ell_{n} = \max\{k < \omega : \beta_{n} \mid k \subseteq F_{n} \cup G_{n}\}. \tag{3.1}
\]

Then, \( F_{n} \cup G_{n} \subseteq \beta_{n} \mid k \leq \ell_{n} \). Next, take a strictly increasing sequence \( \{d_{n} : n < \omega\} \subseteq \omega \) such that \( d_{n} > \ell_{n}, n < \omega \). Finally, define

\[
Z_{n} = \{ \gamma : d_{n} : \gamma \in 2^{\omega} \text{ and } \gamma \mid \ell_{n} = \beta_{n} \mid \ell_{n} \}, \quad n < \omega.
\]

Thus, we get a sequence \( \{Z_{n} : n < \omega\} \) of pairwise disjoint finite subsets \( Z_{n} \subseteq 2^{<\omega}, n < \omega \). For later use, let us observe that, for every \( \gamma \in 2^{\omega} \) and \( n < \omega \),

\[
\gamma \cap Z_{n} \neq \emptyset \quad \text{if and only if} \quad F_{n} \cup G_{n} \subseteq \gamma . \tag{3.2}
\]

Now, we may define for \( s, t \in 2^{<\omega} \) that

\[
\varphi((s, t)) = \begin{cases} 
  s & \text{if } s \in F_{n} \text{ and } t \in Z_{n} \text{ for some } n < \omega, \\
  t & \text{if } s \in G_{n} \text{ and } t \in Z_{n} \text{ for some } n < \omega, \\
  \sigma((s, t)) & \text{otherwise.}
\end{cases} \tag{3.3}
\]

This \( \varphi \) is as required. Indeed, if \( \gamma \in 2^{\omega} \) and \( F_{n} \cup G_{n} \subseteq \gamma \), then, by (3.2), \( \gamma \mid d_{n} \in Z_{n} \), so \( F_{n} \prec_{\varphi} \gamma \mid d_{n} \prec_{\varphi} G_{n} \). To show (b), take \( \beta \in 2^{\omega}, s \in 2^{<\omega} \setminus \beta, k > s \triangle \beta \) and \( n < \omega \) such that \( \beta \mid k \in F_{n} \cup G_{n} \). We claim that \( s \notin Z_{n} \). Indeed, if \( s \in Z_{n} \), then \( s = \gamma \mid d_{n} \)
for some $γ \in 2^ω$, hence $s \setminus εn = βn \setminus εn$. However, by (3.1), $k ≤ εn$ and $β \setminus κ = βn \setminus k$ because $β \setminus k ∈ Fn \cup Gn$. Consequently, $s \setminus k = β \setminus k$ and, therefore, $k ≤ s \setminus β$ but $k > s \setminus β$. A contradiction! Thus, $s \notin Zn$ and, by (3.3), $ψ(s, β \setminus k) = σ(s, β \setminus k)$. Finally, suppose that $β \setminus k ∈ Zn$ for some $n < ω$. Then, $k = d_n$ and, by (3.2), $Fn \cup Gn ⊆ β$. Hence, $s \notin Fn \cup Gn$ because $s \in 2^{ω \setminus β}$. So, by (3.3), we have again that $ψ(s, β \setminus k) = σ(s, β \setminus k)$. The proof is completed. □

**Killing linear orders on the Cantor tree.** The lemma below is a simplified version of [21, Lemma 2.6].

**Lemma 3.6.** ([21]) Let $ψ$ be a weak selection for $2^{<ω}$, $≤$ be a linear ordering on $2^{<ω}$, and let $β ∈ 2^{<ω}$ be such that $β ∈ 2(ψ)$. Then, there are sequences

$$\{x_n : n < ω\}, \{y_n : n < ω\} ⊆ β$$

such that, for every $n < ω$,

(a) $\{x_n : k ≤ n\} \prec ψ \{y_k : k ≤ n\}$,

(b) $x_n \prec ψ \{y_k : k > n\}$ and $\{y_k : k > n\} \prec ψ y_n$,

(c) $x_{2n} < y_{2n}$ and $y_{2n+1} < x_{2n+1}$.

**Proof.** Since $ψ \setminus [β]^2$ is dense in $β$, by Proposition 3.3, there are points $x_0, y_0 \in β$ such that $x_0 \prec ψ y_0$ and $x_0 < y_0$. We can proceed by induction. Namely, suppose that, for some $n > 0$, $x_n, y_n \in β$, $k < n$, have been already constructed so that $\{x_k : k < n\} \prec ψ \{y_k : k < n\}$. By Proposition 3.4, $ψ \setminus [A_n]^{-1}$ is dense in $A_n$, where $A_n = \{s ∈ β : \{x_k : k ≤ n\} \prec ψ s \prec ψ \{y_k : k < n\}\}$. Hence, by Proposition 3.3, there are points $x_n, y_n \in A_n$ as in (b) and (c), i.e. such that $x_n \prec ψ y_n$, while $x_n < y_n$ if $n$ is even and $y_n < x_n$ if $n$ is odd. Since $x_n, y_n \in A_n$, according to our assumption, we also get that $\{x_k : k ≤ n\} \prec ψ \{y_k : k < n\}$. The proof is completed. □

**Theorem 3.7.** ([21]) There exists an almost disjoint family $𝒮$ of subsets of $2^{<ω}$ such that the corresponding Isbell–Mrówka space $Ψ(𝒮) = 2^ω \cup 2^{<ω}$ has a continuous weak selection $ψ$ but is not weakly ordinal orderable.

**Proof.** Let $σ : [2^{<ω}]^2 → 2^{<ω}$ be as in Proposition 3.2, and let $ψ : [2^{<ω}]^2 → 2^{<ω}$ be as in Proposition 3.5 constructed with this particular $σ$. Then, whenever $β, γ ∈ 2^{<ω}$ are different branches and $β(β \setminus γ) < γ(β \setminus γ)$, we have that

$$[β \setminus n : n > β \setminus γ] \prec ψ [γ \setminus n : n > β \setminus γ].$$

(3.4)

Let $\{≤β : β ∈ 2^{<ω}\}$ be the set of all linear orders on $2^{<ω}$. By Lemma 3.6, for every $β ∈ 2^{<ω}$ there are sequences $\{x_{(β,n)} : n < ω\}, \{y_{(β,n)} : n < ω\} ⊆ β$ such that, for every $n < ω$,

$$\{x_{(β,k)} : k ≤ n\} \prec ψ \{y_{(β,k)} : k ≤ n\},$$

(3.5)

$$x_{(β,n)} \prec ψ x_{(β,k)} : k > n\} \prec ψ y_{(β,n)} \prec ψ y_{(β,n)},$$

(3.6)

$$x_{(β,2n)} < β y_{(β,2n)} \prec β x_{(β,2n+1)}.$$  

(3.7)

Set $β_0 = \{x_{(β,n)} : n < ω\}$ and $β_1 = \{y_{(β,n)} : n < ω\}, β ∈ 2^{<ω}$. Thus, we get an almost disjoint family $𝒮 = \{β_0, β_1 : β ∈ 2^{<ω}\}$ of subsets of $2^{<ω}$ because so is $2^{<ω}$. Let $Ψ(𝒮) = 2^{<ω} \cup 2^{<ω}$ be the Isbell–Mrówka space generated by $𝒮$. According to (3.4), (3.5) and (3.6), the family $𝒮 \cup 2^{<ω}$ is almost $≤ψ$-decisive. Hence, by Proposition 3.1, $ψ$ can be extended to a continuous weak selection $ψ$ for $Ψ(𝒮)$. Suppose that $Ψ(𝒮)$ is weakly ordinal orderable with respect to a linear ordering $≤$ on it. Then, $≤ [2^{<ω} × 2^{<ω}] = ≤β$ for some $β ∈ 2^{<ω}$. Consider the corresponding subsets $β_0, β_1 ⊆ β$ defined above, and say we have that $β_0 < β_1$. Thus, by Theorem 2.5, there are open sets $U_0, U_1 ⊂ Ψ(𝒮)$ such that $β_i \in U_i, i < 2$, and

$$U_0 < U_1.$$  

(3.8)

According to the definition of the topology of $Ψ(𝒮)$, there is $m < ω$ such that $\{x_{(β,k)} : k > m\} ⊆ U_0$ and $\{y_{(β,k)} : k > m\} ⊆ U_1$. In particular, we have that $x_{(β,2m+1)} ∈ U_0$ and $y_{(β,2m+1)} ∈ U_1$ and, therefore, by (3.8), we get that $x_{(β,2m+1)} < β y_{(β,2m+1)}$. However, by (3.7), we have that $y_{(β,2m+1)} < β x_{(β,2m+1)}$. A contradiction! □

**4. Van Mill and Wattel’s problem revised**

**More on the continuity of weak selections.** Let us explicitly mention that if $f$ is a continuous weak selection for a space $X$ and $x, y ∈ X$, with $x < f y$, then, by Theorem 2.6, there are open sets $U, V \subset X$ such that $x ∈ U, y ∈ V$, $x < f V$ and $U < f y$. However, in contrast to the case of weak orderability in Theorem 2.5, this condition doesn’t imply the continuity of weak selections.
Proposition 4.1. For a selection relation $\preceq_s$ on a space $(X, \mathcal{T})$, the following are equivalent:

(a) if $x, y \in X$ and $x \preceq_s y$, then there are $\mathcal{T}$-open sets $U, V \subset X$ such that $x \in U$, $y \in V$, $x \prec_s V$ and $U \prec_s y$,

(b) $\mathcal{T}_{\preceq_s} \subset \mathcal{T}$.

Proof. The implication (a) $\Rightarrow$ (b) is obvious, while (b) $\Rightarrow$ (a) follows from the fact that $(\iff, x) \preceq_s y (x, \rightarrow) \preceq_s$.

Now, on the one hand, by Theorem 2.6 and Proposition 4.1, we get the following consequence.

Corollary 4.2. ([23]) If $(X, \mathcal{T})$ is a space and $\preceq_s$ is a $\mathcal{T}$-closed relation on $X$, then $\mathcal{T}_{\preceq_s} \subset \mathcal{T}$. In particular, $(\iff, x) \preceq_s, (x, \rightarrow) \preceq_s \in \mathcal{T}$, for every $x \in X$.

Corollary 4.2 represents a very basic fact, and in the sequel we will freely rely on it without any explicit reference.

On the other hand, we have the following crucial example, see [13, Example 3.6].

Example 4.3. ([13]) There exist a space $(X, \mathcal{T})$ and a selection relation $\preceq_s$ on $X$ such that $\mathcal{T}_{\preceq_s} \subset \mathcal{T}$ but $\preceq_s$ is not $\mathcal{T}$-closed. In particular, $\preceq_s$ is also not $\mathcal{T}_{\preceq_s}$-closed.

Proof. For every $n < \omega$, let $x_n = 2^{-n} - 1$ and $y_n = 1 - 2^{-n}$. Then, $\{x_n : n < \omega\}$ is a strictly decreasing sequence convergent to $-1$, while $\{y_n : n < \omega\}$ is a strictly increasing sequence convergent to $1$. Set

$$X = \{-1, 1\} \cup \{x_n, y_n : n < \omega\},$$

and endow it with the open interval topology $\mathcal{T}_\preceq$ generated by the linear order $\preceq$ on $X$ inherit from the one of the real line $\mathbb{R}$. In fact, $\mathcal{T}_\preceq$ is the usual topology on $X$ as a subspace of the real line. Next, define a relation $\preceq_s$ on $X$ by letting $y_{n+1} \preceq_s x_n$ for every $n < \omega$, and in all other cases it to be equal to $\preceq$. Thus, $\preceq_s$ is total and anti-symmetric, hence it is a selection relation. Furthermore, let us observe that, for every $n < \omega$,

$$(\iff, x_n) \preceq_s = (\iff, x_n) \preceq \cup \{y_{n+1}\} \quad \text{and} \quad (y_n, \rightarrow) \preceq_s = (x_n, \rightarrow) \preceq \setminus \{y_{n+1}\}.$$

In the same way,

$$(\iff, y_{n+1}) \preceq_s = (\iff, y_{n+1}) \preceq \setminus \{x_n\} \quad \text{and} \quad (y_{n+1}, \rightarrow) \preceq_s = (y_{n+1}, \rightarrow) \preceq \cup \{x_n\}.$$

Since $x_0 = y_0$ and each $x_n, y_n, n < \omega$, is an isolated point of $(X, \mathcal{T}_\preceq)$, we get that $\mathcal{T}_{\preceq_s} \subset \mathcal{T}_\preceq$. However, the relation $\preceq_s$ is not closed with respect to $\mathcal{T}_\preceq$. Indeed, on the contrary of this, suppose that $\preceq_s$ is $\mathcal{T}_\preceq$-closed. Then, $y_{n+1} \preceq_s x_n$ for every $n < \omega$ and, therefore,

$$1 = \lim_{n \to \omega} y_{n+1} \preceq_s \lim_{n \to \omega} x_n = -1.$$

Hence $1 \preceq_s -1$ because $1 \neq -1$, but, by definition, $-1 \preceq_s 1$. A contradiction! □

Separately and properly continuous weak selections. Motivated by this, we shall say that a weak selection $f$ for a space $(X, \mathcal{T})$ is separately continuous if $\mathcal{T}_{\preceq_f} \subset \mathcal{T}$. Clearly, every continuous weak selection is separately continuous, but, according to Example 4.3, the converse fails.

Motivated by the same, we introduce also the following further property of continuity of weak selections.

Definition 4.4. We shall say that a weak selection $f$ for a space $(X, \mathcal{T})$ is properly continuous if

(i) $f$ is separately continuous (i.e., $\mathcal{T}_{\preceq_f} \subset \mathcal{T}$), and

(ii) the selection relation $\preceq_f$ is $\mathcal{T}_{\preceq_f}$-closed.

According to Theorem 2.6, a separately continuous weak selection $f$ for $X$ is properly continuous if and only if it is continuous with respect to the selection topology $\mathcal{T}_{\preceq_f}$, it generates (i.e., when $X$ is endowed with $\mathcal{T}_{\preceq_f}$, and $\mathcal{T}_2(X)$ with the Vietoris topology generated by $\mathcal{T}_{\preceq_f}$). In particular, by Corollary 2.8, every properly continuous selection is continuous. However, the weak selection in Example 4.3 is continuous with respect to the discrete topology on $X$, but is not properly continuous.

Concerning properly continuous selections, let us also explicitly remark that (i) in Definition 4.4 is important in order to define a continuity-like property of weak selections related to the topology of $X$. Namely, if $X$ is a space which is not weakly orderable, say $X = \mathbb{R}^2$, and $\preceq$ is a linear order on it, then $\preceq$ generates a weak selection $f$ for $X$ such that $\preceq_f = \preceq$ and $f$ is continuous with respect to the selection topology $\mathcal{T}_{\preceq_f}$. However, $f$ is not even separately continuous because $X$ is not weakly orderable, see Proposition 4.1.

If $\preceq$ is a closed linear order on a space $(X, \mathcal{T})$, then, by Theorem 2.5, $\preceq$ is $\mathcal{T}_{\preceq}$-closed and $\mathcal{T}_\preceq \subset \mathcal{T}$. Hence, we have the following immediate consequence.
Corollary 4.5. Every weakly orderable space has a properly continuous weak selection.

Continuity and dense weak selections.

Proposition 4.6. If \( f \) is a dense weak selection for an infinite set \( Z \), then each point of \( Z \) is \( \preceq_f \)-cut.

Proof. Recall that \( f \) is dense in \( Z \) if for every two disjoint sets \( F, G \subseteq [Z]^{<\omega} \) there exists a point \( z \in Z \), with \( F \prec_f z \prec_f G \), or, in other words, if \( (F, G)_{\preceq_f} \neq \emptyset \). Take a point \( z \in Z \) and another one \( x \in Z \setminus \{z\} \). Then, in particular, there are points \( s, t \in Z \) such that \( x \prec_f s \prec_f z \prec_f t \prec_f x \). The proof is completed. \( \square \)

Using this property, we now have the following simple observation about continuity of dense weak selections.

Proposition 4.7. If \( f \) is a dense weak selection for an infinite set \( Z \), then \( f \) is not continuous with respect to selection topology \( T_{\preceq_f} \) it generates. In particular, if \( Z \) is a space and \( f \) is a dense weak selection for \( Z \), then \( f \) is not properly continuous.

Proof. Suppose that \( f \) is continuous with respect to \( T_{\preceq_f} \), and take distinct points \( x, y \in Z \), say \( x \prec_f y \). By Proposition 4.6, both \( x \) and \( y \) are \( \preceq_f \)-cut points. Hence, by Proposition 2.1 and Theorem 2.6, there are nonempty finite disjoint subsets \( A_x, B_x \subseteq Z \) and nonempty finite disjoint subsets \( A_y, B_y \subseteq Z \) such that

\[
\begin{align*}
x \in (A_x, B_x)_{\preceq_f}, & \quad y \in (A_y, B_y)_{\preceq_f} \quad \text{and} \quad (A_x, B_x)_{\preceq_f} \prec_f (A_y, B_y)_{\preceq_f}.
\end{align*}
\]

(4.1)

According to Proposition 3.4, \( f \restriction (A_x, B_x)_{\preceq_f} \) is also dense and, in particular, the set \( (A_x, B_x)_{\preceq_f} \) is infinite. Hence, there exists a point \( s \in (A_x, B_x)_{\preceq_f} \setminus A_y \). Then, \( A_y \) and \( B_y \cup \{s\} \) are disjoint finite sets, so there exists a point \( t \in Z \) such that \( A_y \prec_f t \prec_f B_y \cup \{s\} \). Thus, \( t \prec_f s \). However, \( s \in (A_x, B_x)_{\preceq_f} \) and \( t \in (A_y, B_y)_{\preceq_f} \), and, by (4.1), \( s \prec_f t \). A contradiction! \( \square \)

Corollary 4.8. Let \( \mathcal{B} \) and \( \psi \) be as in Theorem 3.7. Then, \( \psi \) is not a properly continuous selection for \( \Psi(\mathcal{B}) = \mathcal{B} \cup 2^{<\omega} \).

Proof. Recall that \( \psi \restriction [2^{<\omega}]^2 = \varphi \), where \( \varphi \) was constructed as in Proposition 3.5. Take any branch \( \beta \in 2^{\omega} \). On the one hand, \( \beta \subset 2^{<\omega} \subset \Psi(\mathcal{B}) \). On the other hand, by (a) of Proposition 3.5, \( \psi \restriction [\beta]^2 = \varphi \restriction [\beta]^2 \) is dense in \( \beta \). Consequently, by Proposition 4.7, \( \psi \restriction [\beta]^2 \) is not properly continuous, and, in particular, \( \psi \) is not a properly continuous weak selection for \( \Psi(\mathcal{B}) \). \( \square \)

Weak orderability problem revised. Motivated by Corollary 4.8, we have the following revised version of Question 2.

Question 3. Let \( X \) be a space which has a properly continuous weak selection. Then, is it true that \( X \) is weakly orderable?

In fact, the first interesting case to test this question is related to the properties of Isbell–Mrówka spaces.

Question 4. Let \( X \) be a separable, first countable locally compact space which has a properly continuous weak selection. Then, is it true that \( X \) is weakly orderable?

Another issue in Question 2 of van Mill and Wattel is its possible dependence on separation axioms. The space in Theorem 3.7 is a special Isbell–Mrówka spaces which is not normal. Hence, we have the following further question.

Question 5. Let \( X \) be a normal space which has a (properly) continuous weak selection. Then, is it true that \( X \) is weakly orderable?

Normality and selection-dense sets. Let \( g \) be a weak selection for \( Y \). We shall say that a subset \( A \subseteq Y \) is \( g \)-dense if \( (F, G)_{\preceq_g} \cap A \neq \emptyset \) whenever \( F, G \subseteq [Y]^{<\omega} \) are disjoint sets. The following observation is an immediate consequence of the definition of the selection topology \( T_{\preceq_g} \) on \( Y \).

Proposition 4.9. Let \( g \) be a weak selection for a set \( Y \), and let \( A \subseteq Y \) be a \( g \)-dense subset. Then, \( A \) is \( T_{\preceq_g} \)-dense.

Involving selection-dense sets, we have the following construction which incorporates [8, Lemma 2.2] and the substantial part of the construction in [8, Example 2.3].

Lemma 4.10. Let \( Y \) be a second countable space without isolated points. Let \( A \) be a countable dense subset, and let \( X = Y \setminus A \). Then, any weak selection \( f \) for \( X \) can be extended to a weak selection \( g \) for \( Y \) such that \( A \) is \( T_{\preceq_g} \)-dense, while \( X \) is \( T_{\preceq_g} \)-closed.
Proof. Since $A$ is countable, $A = \{a_n: n < \omega\}$ for some one-to-one indexing with the elements of $\omega$. Take a countable close base $\mathcal{B}$ for $Y$, and let $(\mathcal{F}_n, \mathcal{G}_n): n < \omega$ be the set of all ordered pairs of elements $\mathcal{F}_n, \mathcal{G}_n \in [\mathcal{B}]^<\omega$ such that

$$\left(\bigcup \mathcal{F}_n\right) \cap \left(\bigcup \mathcal{G}_n\right) = \emptyset \neq Y \setminus \left(\bigcup (\mathcal{F}_n \cup \mathcal{G}_n)\right).$$

(4.2)

For convenience, set $F_n = \bigcup \mathcal{F}_n$, $G_n = \bigcup \mathcal{G}_n$ and $A_n = \{a_k: k \leq n\}$, $n < \omega$. By hypothesis, $Y$ has no isolated points. Hence, using (4.2), for every $n < \omega$ there are distinct points $b_n, c_n \in A$ such that $b_0, c_0 \notin A_0 \cup F_0 \cup G_0$ and

$$b_{n+1} \in A_n \setminus \{b_k, c_k: k \leq n\} \cup A_{n+1} \cup F_{n+1} \cup G_{n+1}, \quad n < \omega.$$  

Finally, take a weak selection $f$ for $X$, and extend it in an arbitrary way to a weak selection $h$ for $Y$. Then, define a weak selection $g$ for $Y$ by letting for distinct points $x, y \in Y$ that $g((x, y)) = x$ if, for some $n < \omega$,

$$\langle x, y \rangle \in (F_n \times \{c_n\}) \cup ((c_n) \times G_n) \cup (X \times \{b_n\}) \cup ((b_n) \times \{a_n\}),$$

and $g((x, y)) = h((x, y))$ otherwise. In fact, we are mostly interested in the following two properties of the selection $g$ that

$$F_n \triangleleft g c_n \triangleleft g G_n, \quad n < \omega,$$

(4.3)

and

$$X \triangleleft g b_n \triangleleft g a_n, \quad n < \omega.$$  

(4.4)

Note that if $F, G \subseteq X$ are nonempty finite disjoint sets, then there is $n < \omega$ such that $F \subseteq F_n$ and $G \subseteq G_n$ because $\mathcal{B}$ is a base for the topology of $Y$. Hence, by (4.3), $A$ is $g$-dense, and, by Proposition 4.9, it is also $\mathcal{T}_{\mathcal{E}_g}$-dense in $Y$. According to (4.4), $a_n \in (b_n, \rightarrow)_{\mathcal{E}_g}$ and $(b_n, \rightarrow)_{\mathcal{E}_g} \subseteq X \setminus \emptyset$ for every $n < \omega$. This implies that $X$ is $\mathcal{T}_{\mathcal{E}_g}$-closed. The proof is completed. 

Theorem 4.11. ([8]) Let $Y$ be a second countable space of cardinality continuum and without isolated points. Then, $Y$ has a weak selection $g$ such that $\mathcal{T}_{\mathcal{E}_g}$ is not normal.

Proof. Take a countable dense subset $A \subseteq Y$, and let $X = Y \setminus A$. Then, $X$ has a weak selection $f$ such that $\mathcal{T}_{\mathcal{E}_f}$ is discrete, see [8, Lemma 2.1]. According to Lemma 4.10, $f$ can be extended to a weak selection $g$ for $Y$ such that $A$ is $\mathcal{T}_{\mathcal{E}_g}$-dense, while $X$ is $\mathcal{T}_{\mathcal{E}_g}$-closed. Since $X$ is of cardinality of continuum, by [5, Corollary 2.1.10], $(Y, \mathcal{T}_{\mathcal{E}_g})$ is not normal. 

5. Weak orderability and countability

We shall say that a family $\mathcal{H}$ of subsets of a space $X$ is separating for the points of $X$ (also called $T_0$-separating) if for every two distinct points of $X$ there exists $H \in \mathcal{H}$ which contains the one point and doesn’t contain the other. The following simple observation is well known (see, for instance, [10, Remark 5.5]).

Proposition 5.1. Let $X$ be a space which has a countable family $\mathcal{H}$ of clopen subsets that is separating for the points of $X$. Then, $X$ is weakly orderable.

Proof. Every $H \in \mathcal{H}$ defines a continuous function $g_H: X \rightarrow 2 = \{0, 1\}$ such that $H = g_H^{-1}(0)$. Then, the diagonal map $g = \Delta(g_H: H \in \mathcal{H}): X \rightarrow 2^\omega$ is a continuous injective map because $\mathcal{H}$ is separating for the points of $X$. Since the Cantor set $2^\omega$ is orderable, $X$ must be weakly orderable. 

The idea of Proposition 5.1 was used in a number of situations related weak orderability of separable spaces, they are summarized below.

Weak orderability of countable spaces. The following result was obtained in [6, Theorem 3.1]. Here, we provide simple arguments based on the selection topology and Proposition 5.1.

Theorem 5.2. ([16]) A countable space $X$ is weakly orderable if and only if it has a continuous weak selection.

Proof. Let $f$ be a continuous weak selection for $X$, and let $\mathcal{T}_{\mathcal{E}_f}$ be the corresponding selection topology on $X$. Then, by Theorem 2.2, $(X, \mathcal{T}_{\mathcal{E}_f})$ is a regular space. Since $(X, \mathcal{T}_{\mathcal{E}_f})$ is also second-countable, it is normal and (strongly) zero-dimensional, see [5]. Hence, $(X, \mathcal{T}_{\mathcal{E}_f})$ has a countable clopen base, so $X$ has a countable clopen family which is separating for the points of $X$. According to Proposition 5.1, $X$ must be weakly orderable. 

Selection-jumps. A pair \((L, U)\) of subsets of a linearly ordered set \((X, \preceq)\) is called a cut, or a \(\sim\)-cut, if \(X = L \cup U, L \neq \emptyset \neq U\) and \(L \prec U\). In this case, \(L\) is called the lower section of the cut, and \(U\) — the upper section. A cut \((L, U)\) is called a jump if the lower section \(L\) has a maximal element and the upper section \(U\) has a minimal element. A cut \((L, U)\) is called a gap if the lower section \(L\) has no maximal element and upper section \(U\) has no minimal element.

Here, we consider jumps with respect to arbitrary selection relations. Namely, let \(\preceq_s\) be a selection relation on a set \(X\). We shall say that a pair of distinct points \(x, y \in X\) is a \(\preceq_s\)-jump of \(X\) if \(x, y \prec \preceq_s (x, y)\). If \(x, y \in [X]^2\) is a \(\preceq_s\)-jump of \(X\) such that \(x <_s y\), then \(z <_s x\) for every \(z <_s y\). In the same way, \(y <_s z\) for every \(z <_s X\), with \(x <_s z\). This implies the following simple observation.

**Proposition 5.3.** Let \(\preceq_s\) be a selection relation on a set \(X\) and \(x, y \in [X]^2\) be a \(\preceq_s\)-jump of \(X\), with \(x <_s y\). Then, \((-x) <_s y\) and \((y, x) <_s (x, y)\).

In our next considerations, we will use \(J(X, \preceq_s)\) to denote the set of all \(\preceq_s\)-jumps of \(X\). The following property was actually obtained in [2] (see, also, [10]).

**Lemma 5.4.** ([2,10]) Let \(X\) be a space, and let \(\preceq_s\) be a closed selection relation on \(X\). Then, \(J(X, \preceq_s)\) is a discrete subset of \([X]^2\).

**Proof.** We follow the proof of [10, Theorem 5.4]. Take a pair \(\beta = (x, y) \in [X]^2\). If \(\beta \in J(X, \preceq_s)\) and \(x <_s y\), set \(O_\beta = \{(-x) <_s y\} \cap \{x, y\} \preceq_s (x, y)\}. According to Proposition 5.3, \(O_\beta = \{(-x) <_s y\} \cap \{x, y\} \preceq_s (x, y)\}. Which means that \(\{(-x) <_s y\} \preceq_s (x, y)\}. Indeed, if \(s \in (-x) <_s y\), \(s \in \{x, y\} \preceq_s (x, y)\} and \(s \neq x \) or \(y \neq t\), then \(s \in (-x) <_s y\) and, therefore, \(s \neq x \) or \(y \neq t\). That is, \(\{(-x) <_s y\} \preceq_s (x, y)\} which completes the proof. \(\square\)

Let us explicitly mention the following immediate consequence of Lemma 5.4.

**Corollary 5.5.** Let \(X\) be a separable space such that \([X]^2\) is collectionwise Hausdorff, and let \(\preceq_s\) be a closed selection relation on \(X\). Then, the set \(J(X, \preceq_s)\) is countable. In particular, \(J(X, \preceq_s)\) is countable provided \(X\) is second countable.

Weaker oraribility of separable spaces.

**Proposition 5.6.** Let \(X\) be a space, let \(\preceq_s\) be a closed selection relation, and let \(p \in X\) be a \(G_\delta\)-point in the selection topology \(T_{\preceq_s}\) and an intersection of clopen subsets of \(X\). Then, \(p\) is a countable intersection of clopen sets.

**Proof.** It suffices to show that \(p\) is a countable intersection of clopen subsets of \([p, \rightarrow)_{\preceq_s}\. Indeed, the case with the interval \((-p)_{\preceq_s}\ is completely identical, while \(L \cup R\) is a clopen neighbourhood of \(p\) provided \(L\) is a clopen neighbourhood of \(p\) in \((-p)_{\preceq_s}\ and \(R\) is a clopen neighbourhood of \(p\) in \([p, \rightarrow)_{\preceq_s}\. In order to show that \(p\) is a countable intersection of clopen subsets of \([p, \rightarrow)_{\preceq_s}\, we may exclude the case when \(p\) is isolated in \((p, \rightarrow)_{\preceq_s}\. If \(p\) is nonisolated in \((p, \rightarrow)_{\preceq_s}\, then there exists a countable subset \(C \subset (p, \rightarrow)_{\preceq_s}\ such that \(p = \bigcap (p, y)_{\preceq_s} : y \in C\) because \(p\) is a \(G_\delta\)-point with respect to the selection topology \(T_{\preceq_s}\. However, \(p\) is also an intersection of clopen subsets of \(X\), hence, for every \(y \in C\) there exists a clopen subset \(V_y \subset X\ such that \(p \in V_y\ and \(y \notin V_y\). Then, \(U_y = (p, y)_{\preceq_s} \cap V_y = (p, y)_{\preceq_s} \cap \bigcap (p, y)_{\preceq_s}, y \in C\, are clopen subsets of \([p, \rightarrow)_{\preceq_s}\ and \([p, \rightarrow)_{\preceq_s} \cap \bigcap (p, y)_{\preceq_s} = \emptyset \). \(\square\)

**Theorem 5.7.** Let \(X\) be a separably totally disconnected space which has a closed selection relation \(\preceq_s\ such that the set \(J(X, \preceq_s)\) is countable. Then, \(X\) is weakly orderable.

**Proof.** Let \(D\) be a countable dense subset of \(X\). Then \(D\) is also dense in the selection topology \(T_{\preceq_s}\ which implies that each point of \(X\ is a \(G_\delta\)-point in the selection topology. Namely, let \(x \in X\ be such that \(x\) is nonisolated in \([x, \rightarrow)_{\preceq_s}\. Then, for every \(y \in [x, \rightarrow)_{\preceq_s}\ there is a point \(p \in D\ such that \(p \in (x, y)_{\preceq_s}\. Hence,

\[
(x) = \bigcap (p, x)_{\preceq_s} : p \in D and x <_s p \].

The case with the interval \((-x)_{\preceq_s}\ is completely identical, hence each point of \(X\ is a \(G_\delta\)-point in \(T_{\preceq_s}\. Thus, by Proposition 5.6, for each point \(p \in D\ there exists a countable family \(\mathcal{U}_p\ of clopen subsets of \(X\ such that \(\bigcap (p, x)_{\preceq_s} = \bigcap \mathcal{U}_p \). Since

\[\bigcap \mathcal{U}_p \setminus U = \bigcap \mathcal{U}_p \setminus U\ for every U \in \mathcal{U}_p\]

\(\mathcal{U}_p = \{(p, x)_{\preceq_s} \setminus U : U \in \mathcal{U}_p\}\) is also a countable family of clopen sets of \(X\, hence so is \(\mathcal{H}_0 = \bigcup \mathcal{H}_0 : p \in D\) because \(D\ is countable. Next, for every \(\beta = (s, t) \in J(X, \preceq_s)\), with \(s <_s t\), let \(L_\beta = (-s)_{\preceq_s} \setminus \mathcal{U}_p\ which, by Proposition 5.3, is clopen in \(X\.

Finally, set \(\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1\ where \(\mathcal{H}_1 = \{L_\beta : \beta \in J(X, \preceq_s)\}. By hypothesis, \(J(X, \preceq_s)\) is countable, hence \(\mathcal{H}\) is a countable
clopen family in $X$. Let us show that it is separating for the points of $X$. Take distinct points $x, y \in X$. If $(x, y)_{\leq s} = \emptyset = (y, x)_{\leq s}$ and $x \leq s y$, then $\{x, y\} = y$ for some $y \in \mathcal{Z}(X, \leq s)$, hence $x \in L_y \in \mathcal{C}$ and $y \notin L_y$. If $(x, y)_{\leq s} \neq \emptyset$ or $(y, x)_{\leq s} \neq \emptyset$, then there exist a point $p \in D \cap ((x, y)_{\leq s} \cup (y, x)_{\leq s})$ and $U \in \mathcal{U}_p$ such that $x, y \notin U$. In this case, $x \in (\leftarrow, p)_{\leq s} \setminus U \in \mathcal{C}$ if and only if $y \notin (\leftarrow, p)_{\leq s} \setminus U$. According to Proposition 5.1, $X$ is weakly orderable. □

By Corollary 5.5 and Theorem 5.7, we have the following immediate consequence which is a generalization of a result obtained by Camillo Costantini [2].

**Corollary 5.8.** Let $X$ be a separable totally disconnected space which has a continuous weak selections and let $[X]^2$ be collectionwise Hausdorff. Then, $X$ is weakly orderable.

Related to Corollary 5.8, let us explicitly mention that $[X]^2$ is collectionwise Hausdorff for every second countable space $X$. On the other hand, by [17, Theorem 2.1], every second countable totally disconnected space has a continuous weak selection. Hence, every second countable totally disconnected space is weakly orderable, see [10, Remark 5.6].

### 6. Weak orderability and connectedness

**Selection relations and components.** Given a space $X$ and $x \in X$, we will use $\mathcal{C}[x]$ to denote the component of $x$ in $X$, and $\mathcal{C}^*[x] = \mathcal{C}[x] \setminus \{x\}$; the corresponding quasi-component. Recall that

$$
\mathcal{C}[x] = \bigcup \{C \subseteq X : x \in C \text{ and } C \text{ is connected}\},
$$

$$
\mathcal{C}^*[x] = \bigcap \{C \subseteq X : x \in C \text{ and } C \text{ is clopen}\}.
$$

It is well known that $\mathcal{C}[x] \subset \mathcal{C}^*[x]$ but, in general, the converse is not necessarily true. The following result was obtained in [14, Theorem 4.1].

**Theorem 6.1.** ([14]) Let $X$ be a space which has a closed selection relation $\leq s$. Then, $\mathcal{C}^*[x] = \mathcal{C}[x]$ for every point $x \in X$.

The proof of Theorem 6.1 is based on the following lemma.

**Lemma 6.2.** ([14]) If $\leq s$ is a closed selection relation on a space $X$, $t \in X$ and $x, y \in \mathcal{C}^*[t]$, with $x \leq s y$, then $[x, y]_{\leq s} \subset \mathcal{C}^*[t]$ is connected.

**Proof.** Suppose that there exists $z \in [x, y]_{\leq s} \setminus \mathcal{C}^*[t]$. Since $z \notin \mathcal{C}^*[t]$, there exists a clopen subset $V \subseteq X$ such that $\mathcal{C}^*[t] \subseteq V$ and $z \notin V$. Then, $U = (\leftarrow, z)_{\leq s} \cap V$ is clopen in $X$ because $U = (\leftarrow, z)_{\leq s} \cap V$. However, this is impossible because $x \in U$ implies $\mathcal{C}^*[t] \subseteq U$, while $y \in X \setminus U$. Thus, $[x, y]_{\leq s} \subset \mathcal{C}^*[t]$. To show that $[x, y]_{\leq s}$ is connected, suppose on the contrary that there exists a clopen (in $[x, y]$) neighbourhood $W \subseteq [x, y]_{\leq s}$ of $y$ such that $[x, y]_{\leq s} \setminus W \neq \emptyset$. Take a point $z \in [x, y]_{\leq s} \setminus W$, and then set $T = W \cap [z, y]_{\leq s}$. Thus, we get a clopen (in $[z, y]$) neighbourhood $T$ of $y$, with $z \notin T$. Then, the set $G = T \cup [y, \rightarrow)_{\leq s}$ is clopen in $X$. Indeed, $G$ is closed in $X$ as a union of two closed sets. Since $T \subseteq (z, \rightarrow)_{\leq s}$, there exists an open subset $E \subseteq (z, \rightarrow)_{\leq s}$, with $E \cap [z, y]_{\leq s} = T$. Hence, $G = E \cup (y, \rightarrow)_{\leq s}$ is also open in $X$. However, this impossible because $z \in [x, y]_{\leq s} \setminus G$ and $[x, y]_{\leq s} \subset \mathcal{C}^*[t] \subseteq G$. A contradiction! □

**Proof of Theorem 6.1.** Let $x \in X$. According to Lemma 6.2,

$$
\mathcal{C}^*[x] = \bigcup \{[y, z]_{\leq s} : y, z \in \mathcal{C}^*[x], y \leq s z \text{ and } y \leq s x \leq s z\}.
$$

By the same lemma, the set $[y, z]_{\leq s}$ is connected for every $y, z \in \mathcal{C}^*[x]$, with $y \leq s z$. Hence, $\mathcal{C}^*[x]$ is connected as well. □

**Transitivity and clopen sets.** The clopen subsets of $X$ are an indication for the existence of nontransitive selection relations. The following theorem is based on [13, Proposition 2.2].

**Theorem 6.3.** If a space $X$ has a continuous weak selection, then it has a continuous weak selection $f$ such that $\leq f$ is not transitive if and only if

$$
\left|\{\mathcal{C}[x] : x \in X\}\right| \geq 3.
$$

**Proof.** Suppose that $f$ is a continuous weak selection for $X$ such that $\leq f$ is not transitive. Then, there are points $x, y, z \in X$ such that

$$
\ldots \prec f x \prec f y \prec f Z \prec f X \prec f \ldots.
$$
Take now $U_x = (\longleftrightarrow, y) \in (z, \rightarrow) \in X$, $U_y = (\leftrightarrow, z) \in (x, \rightarrow) \in X$, and, respectively, $U_z = (\longleftrightarrow, x) \in (y, \rightarrow) \in X$. Thus, $\{U_x, U_y, U_z\}$ is a pairwise disjoint family of open subsets of $X$ such that $x \in U_x$, $y \in U_y$ and $z \in U_z$. In fact, each of the sets $U_x$, $U_y$ and $U_z$ is also closed in $X$. Indeed,$$
abla_x = (\longleftrightarrow, y) \in (z, \rightarrow) \in X, \quad \nabla_y = (\leftrightarrow, z) \in (x, \rightarrow) \in X, \quad \nabla_z = (\longleftrightarrow, x) \in (y, \rightarrow) \in X.$$Similarly, for $U_y$ and $U_z$. Hence, $C'[x] \subset C'^{*}[x] \subset U_x$, $C'[y] \subset C'^{*}[y] \subset U_y$, and $C'[z] \subset C'^{*}[z] \subset U_z$.

To show the converse, suppose that $|\{\nabla[x]: x \in X\}| \geq 3$. According to Theorem 6.1, this implies that $X$ has points $x_1, x_2, x_3 \in X$ such that $C'[x_1] \cap \nabla^*[x_2] = \emptyset = \nabla^*[x_2] \cap \nabla^*[x_3] = \emptyset = \nabla^*[x_3] \cap \nabla^*[x_1]$.

So, $X$ has a partition $\Omega = \{Q_1, Q_2, Q_3\}$ consisting of nonempty clopen sets. Take a continuous weak selection $g$ for $X$, and define another weak selection $f$ for $X$ such that $f \cap \Omega(Q_i) = g \cap \Omega(Q_i)$, $i = 1, 2, 3$, and $Q_1 \prec_f Q_2 \prec_f Q_3 \prec_f Q_1$. Since $\Omega$ is an open partition of $X$ and $g$ is continuous, by Theorem 2.6, $f$ is continuous as well. However, $\preceq_f$ is not transitive because $Q_i \neq \emptyset, i = 1, 2, 3$.

If $X$ is connected, then $C'[x] = X$ for every $x \in X$. That is, we have the following immediate consequence of Theorems 2.5, 2.6 and 6.3.

**Corollary 6.4.** ([23]) If $X$ is a connected space and $\preceq_s$ is a closed selection relation on $X$, then $\preceq_s$ is transitive and, consequently, $X$ is weakly orderable with respect to $\preceq_s$. In particular, a connected space $X$ is weakly orderable if and only if it has a closed selection relation.

**Corollary 6.5.** (Eilenberg [4]) For an infinite space $X$, the following are equivalent:

(a) $X$ is disconnected.
(b) $P(X)$ contains a clopen set $D$ such that $\emptyset \neq D \neq P(X)$ and $\Lambda(D) = D$, where $\Lambda : X^2 \to X^2$ is defined as in (2.2).

**Proof.** (a) $\Rightarrow$ (b). Since $X$ is disconnected, it has an infinite clopen set $Y \subset X$ such that $Y \neq X$. Then, $D = P(Y)$ is as in (b).

(b) $\Rightarrow$ (a). We present the arguments in [4, (3.1)]. Suppose that $D \subset P(X)$ is as in (b) and, contrary to (a), that $X$ is connected. Let $C = P(X) \setminus D$ and, for every $x \in X$, set $C_x = \{y \in X : \langle x, y \rangle \in C\}$ and $D_x = \{y \in X : \langle x, y \rangle \in D\}$. Since $P(X)$ is open in $X^2$ and $(C, D)$ is a clopen partition of $P(X)$, we now have that $C_x$ and $D_x$ are disjoint open subsets of $X$ such that $X \setminus \{\emptyset\} = C_x \cup D_x$. Hence, both $C_x \cup \{x\}$ and $D_x \cup \{x\}$ are connected because $X$ is connected. First of all, let us show that

$$C_x = \emptyset \quad \text{or} \quad D_x = \emptyset \quad \text{for every} \quad x \in X. \quad (6.1)$$

Indeed, suppose that there exists a point $x \in X$ such that $C_x \neq \emptyset \neq D_x$, and take points $y \in C_x$ and $z \in D_x$. Then, $(D_x \cup \{x\}) \times \{y\} \subset P(X) = C \cup D$ because $x \neq y$ and $y \notin D_x$. However, $(x, y) \in C$ because $y \in C_x$, hence $(D_x \cup \{x\}) \times \{y\} \subset C$ because $C$ is clopen in $P(X)$ and $(D_x \cup \{x\}) \times \{y\} \subset C$ because $C$ is clopen in $P(X)$. Finally, according to (b), $\Lambda(D) = D$ and, therefore, $\langle z, y \rangle \in D$. That is, we now have $C \cap D \neq \emptyset$ which is impossible, so (6.1) holds.

To finish the proof, take a point $x \in X$. By (6.1), we have $C_x = \emptyset$ or $D_x = \emptyset$, say $C_x = \emptyset$. Then, $D_x = X \setminus \{x\}$ and $(x) \times D_x \subset D = \Lambda(D)$. Therefore, $\langle y, x \rangle \in D$ for every $y \in X \setminus \{x\}$. That is, $x \in D_y$ for every $y \in X \setminus \{x\}$, and, by (6.1), $C_y = \emptyset$ and $D_y = X \setminus \{y\}$ for every $y \in X \setminus \{x\}$. In particular, we now have that $D = P(X)$ while, by (b), $D \neq P(X)$. A contradiction, which implies that $X$ must be disconnected.

**Lemma 6.5** implies the following characterization of weak orderability of connected spaces.

**Corollary 6.6.** (Eilenberg [4]) Let $X$ be a connected space. Then,

(a) If $P(X)$ is not connected, then $P(X)$ has exactly two connected components $A$ and $B$ such that $\Lambda(A) = B$.

(b) If $f$ is a continuous weak selection for $X$, then

$$\prec_f = \{\langle x, y \rangle \in X^2 : x \prec_f y\} \quad \text{and} \quad (\prec_f)^{-1} = \{\langle x, y \rangle \in X^2 : y \prec_f x\}$$

are the connected components of $P(X)$. In particular, there exists exactly one other continuous weak selection $g$ for $X$ and the selection relation $\prec_g$ generated by $g$ is reverse to $\prec_f$.

(c) $X$ is weakly orderable if and only if $P(X)$ is not connected.
Proof. If \( P(X) \) is not connected, then it contains a clopen subset \( A \) such that \( \emptyset \neq A \neq P(X) \). In this case, \( A \) is connected and \( A \cap \Lambda(A) = \emptyset \). Indeed, if \( D = A \cap \Lambda(A) \neq \emptyset \), then \( D \neq P(X) \) and \( \Lambda(D) = D \). Hence, by Lemma 6.5, \( X \) must be disconnected which is not the case. Thus, \( A \cap \Lambda(A) = \emptyset \). If \( A \) is not connected, then it contains a clopen subset \( Q \) such that \( \emptyset \neq Q \neq A \).

In this case, \( D = Q \cup \Lambda(Q) \) is a clopen subset of \( P(X) \) such that \( \emptyset \neq D \neq P(X) \) and \( \Lambda(D) = D \). According to Lemma 6.5 once again, this is impossible because \( X \) is connected. This demonstrates (a).

If \( f \) is a continuous weak selection for \( X \), then, by Proposition 2.4 and Theorem 2.6, the relation \( \prec_f = \{(x, y) \in X^2 : x \prec_f y\} \) is open in \( X^2 \), hence so is the inverse one \( (\prec_f)^{-1} = \{(x, y) \in X^2 : y \prec_f x\} = \Lambda(\prec_f) \).

Consequently, \( P(X) \) is not connected and, by (a), \( \prec_f \) and \( (\prec_f)^{-1} \) are the connected components of \( P(X) \). This demonstrates (b).

If \( X \) is weakly orderable, by Corollary 2.7, it has a continuous weak selection, and, by (b), \( P(X) \) is not connected. If \( P(X) \) is not connected, then, by (a), it contains a clopen subset \( A \) such that \( P(X) \setminus A = \Lambda(A) \). Define a selection relation \( \preceq_A \) on \( X \) by \( x \preceq_A y \) if and only if \( (x, y) \in A \) or \( x = y \). Then, \( \preceq_A \) is a closed relation because \( \preceq_A = X^2 \setminus \Lambda(A) \) and, by Theorem 2.6 and Corollary 6.4, \( X \) is weakly orderable. This is (c).

By Corollaries 6.4 and 6.6 we have the following immediate consequence.

Corollary 6.7. ([4]) Every connected weakly orderable space has precisely two compatible orders which are inverse each other.

We also have the following consequence related to the hyperspace \( [X]^2 \).

Corollary 6.8. An infinite space \( X \) is connected if and only if the hyperspace \( [X]^2 \) is \( \tau_V \)-connected.

Proof. If \( X \) is not connected, then it contains an infinite clopen subset \( Y \) such that \( Y \neq X \). Then, \( |Y|^2 \) is \( \tau_V \)-clopen in \( [X]^2 \) and \( |Y|^2 \neq [X]^2 \), so \( [X]^2 \) is not \( \tau_V \)-connected. Suppose that \( X \) is connected, and consider the continuous map \( h : P(X) \to [X]^2 \) defined by \( h(x, y) = (x, y) \), \( (x, y) \in P(X) \). If \( P(X) \) is connected, then so is \( [X]^2 \). If \( P(X) \) is not connected, then, by Corollary 6.5, it has exactly two connected components \( A \) and \( B \) such that \( \Lambda(A) = B \). In particular, \( A \) is connected and \( h \upharpoonright A : A \to [X]^2 \) is onto. Hence, \( [X]^2 \) is \( \tau_V \)-connected.

Continuously orderable spaces. A linearly ordered set \((X, \preceq)\) is called densely ordered if no cut of \( X \) is a jump (i.e., if \((x, y), \neq \emptyset \) whenever \( x < y \)). If, moreover, no cut of \( X \) is a gap, then \((X, \preceq)\) is called continuously ordered. For instance, the set of the rational numbers \( \mathbb{Q} \) is densely ordered with respect to the usual order \( \leq \) on \( \mathbb{Q} \), but not continuously ordered. The set of the real numbers \( \mathbb{R} \) is continuously ordered with respect to the usual order \( \leq \) on \( \mathbb{R} \).

If \( X \) is weakly orderable with respect to a linear order \( \preceq \) and \((L, U)\) is a cut of \( X \), then we shall say that \((L, U)\) is a clopen cut of \( X \) (see, [11]) if both sets \( L \) and \( U \) are clopen (equivalently, open or closed) in \( X \). The following is a characterization of orderable spaces among the suborderable ones, see [11, Lemma 6.8].

Lemma 6.9. ([11]) A suborderable space \( X \) is orderable with respect to a compatible linear order \( \preceq \) on it if and only if each clopen \( \preceq \)-cut of \( X \) is either a gap or a jump.

Proof. Suppose that \( X \) is suborderable by a linear order \( \preceq \) such that each clopen \( \preceq \)-cut of \( X \) is either a gap or a jump. For convenience, let \( T \) be the topology on \( X \), and let \( T_{\preceq} \) be the open interval one. To show that \( T \subseteq T_{\preceq} \), take a point \( x \in X \) such that \( [x, \rightarrow)_{\preceq} \subseteq T \). Then, \([x, \rightarrow)_{\preceq} \) is clopen in \( X \) because \( T_{\preceq} \subseteq T \). If \( X = [x, \rightarrow)_{\preceq} \), then \([x, \rightarrow)_{\preceq} \subseteq T_{\preceq} \). If \( X \neq [x, \rightarrow)_{\preceq} \), then consider the clopen \( \preceq \)-cut \((D, E)\) of \( X \), where \( D = (\leftarrow, x)_{\preceq} \) and \( E = [x, \rightarrow)_{\preceq} \). Since \( x \) is the \( \preceq \)-minimal element of \( E \), the \( \preceq \)-cut \((D, E)\) must be a jump, consequently \( D \) has a \( \preceq \)-maximal element \( y \). In the same way, \((\leftarrow, x)_{\preceq} \subseteq T_{\preceq} \) is provided \((\leftarrow, x)_{\preceq} \subseteq T \). Hence, \( T_{\preceq} = T \).

Suppose now that \( X \) is an orderable space by a linear order \( \preceq \) and take a clopen \( \preceq \)-cut \((D, E)\) of \( X \). Further, suppose that \( E \) has a \( \preceq \)-minimal element \( x \in E \). Since \( E \) is open and \( x \in [x, \rightarrow)_{\preceq} \), there exists a \( y \in X \), with \( x \in (y, \rightarrow)_{\preceq} \subseteq E \). Hence, \((y, \rightarrow)_{\preceq} \subseteq [x, \rightarrow)_{\preceq} \) which implies that \( y \notin E \) and \((y, x)_{\preceq} = \emptyset \). That is, \( y \in D \) and it is the \( \preceq \)-maximal element of \( D \). Thus, \((D, E)\) is a jump. In the same way, \((D, E)\) is a jump if \( D \) has a \( \preceq \)-maximal element, which completes the proof.

The following is a simple description of continuously ordered spaces, see [4, (2.2)].

Corollary 6.10. If \( X \) is orderable with respect to a linear order \( \preceq \), then the following are equivalent:

(a) \( X \) is connected.
(b) \( \preceq \) is a connected subset of \( X^2 \).
(c) \(X\) does not contain a clopen \(\prec\)-cut.
(d) \((X, \prec)\) is continuously ordered.

**Proof.** The implication (a) \(\Rightarrow\) (b) follows by Corollary 6.6. To show that (b) \(\Rightarrow\) (c), suppose that \((L, U)\) a clopen \(\prec\)-cut of \(X\). Then, \(L \times U\) is a clopen subset of \(\prec\) such that \(\emptyset \neq L \times U \neq \prec\), hence \(\prec\) is not connected. The implication (c) \(\Rightarrow\) (d) follows from the fact that every jump or gap of \((X, \prec)\) must be a clopen \(\prec\)-cut of \(X\). To show finally that (d) \(\Rightarrow\) (a), suppose that \(X\) is not connected. Hence, it contains a clopen subset \(A \subset X\) and points \(x \in A\) and \(y \notin A\) such that \(x \prec y\). Set \(B = A \cap (\leftarrow, y)_\prec = A \cap (\leftarrow, y)\prec\) which is clopen as well, and then let

\[
L = \bigcup \{(\leftarrow, z)_\prec: z \in B\}.
\]

Since \(B\) is open, \(L\) is also open. Suppose that \(L\) is not closed, and take a point \(p \in \overline{L} \setminus L\) and a convex neighbourhood \(V\) of \(p\). Then, \(V \cap L \neq \emptyset\) while, by (6.2), \(z \prec p\) for every \(z \in L\) because \(p \notin L\). Since \(V\) is convex, this implies that \(V \cap B \neq \emptyset\). Hence, \(p \in B \subset L\) because \(B\) is closed, but this is impossible. Thus, \(L\) must be closed. Finally, let \(U = X \setminus L\) and observe that \(U \neq \emptyset\) because \(y \in U\). According to (6.2), we now have that \(L \prec U\), hence \((L, U)\) is a clopen \(\prec\)-cut of \(X\). Since \(X\) is orderable, by Lemma 6.9, \((L, U)\) must be either a gap or a jump. Hence, \((X, \prec)\) cannot be continuously orderable. The proof is completed. \(\square\)

7. Weak orderability and connected components

*Cut and noncut points.* A point \(p\) of a connected space \(Z\) is called a *cut point* if \(Z \setminus \{p\}\) is not connected, and it is called *noncut* if \(Z \setminus \{p\}\) is connected. If \(p\) is a cut point of \(Z\), then \(Z \setminus \{p\} = U \cup V\) for some nonempty disjoint open sets \(U, V \subset Z\) such that \(\overline{U} \cap \overline{V} = \{p\}\). In particular, both sets \(\overline{U}\) and \(\overline{V}\) must be connected.

**Proposition 7.1.** ([10]) Let \(\prec_s\) be a closed selection relation on space \(X\) and \(Z\) be a connected subset of \(X\). Then,

(a) \(Z \subset (\leftarrow, x)_{\prec_s}\) or \(Z \subset (x, \rightarrow)_{\prec_s}\) for every \(x \in X \setminus Z\).
(b) \(\emptyset \neq (y, z)_{\prec_s} \subset Z\) for every \(y, z \in Z\), with \(y \prec_s z\).

In particular, \([y, z]_{\prec_s}\) is a connected subset of \(X\) for every \(y, z \in Z\), with \(y \prec_s z\).

**Proof.** If \(x \in X \setminus Z\), then

\[
(\leftarrow, x)_{\prec_s} \cap Z = (\leftarrow, x)_{\prec_s} \cap Z \quad \text{and} \quad (x, \rightarrow)_{\prec_s} \cap Z = [x, \rightarrow)_{\prec_s} \cap Z.
\]

Hence, \((\leftarrow, x)_{\prec_s} \cap Z\) and \((x, \rightarrow)_{\prec_s} \cap Z\) are disjoint clopen subsets of \(Z\). Since \(Z\) is connected and \(Z \subset X \setminus \{x\} = (\leftarrow, x)_{\prec_s} \cup (x, \rightarrow)_{\prec_s}\), we get that \(Z \subset (\leftarrow, x)_{\prec_s}\) or \(Z \subset (x, \rightarrow)_{\prec_s}\). Thus, (a) holds. Let \(y, z \in Z\) be such that \(y \prec_s z\). According to (a), \([y, z]_{\prec_s} \subset X\) or \(X \prec_s [y, z]\) for every \(x \in X \setminus Z\). Hence, \((y, z)_{\prec_s} \subset Z\). If \((y, z)_{\prec_s} = \emptyset\), then \(U = (\leftarrow, y)_{\prec_s} \cup (\rightarrow, y)_{\prec_s}\) is a clopen subset of \(X\) such that \(y \in U\) and \(z \notin U\). A contradiction, which completes the verification of (b). The last part of this proposition now follows by Lemma 6.2 because \(Z \subset \prec_s^*[z]\) for every \(z \in Z\). \(\square\)

In the sequel, we will use \(ct(Z)\) to denote the set of all cut points of \(Z\), and \(net(Z)\) — that of all noncut points of \(Z\).

**Proposition 7.2.** ([10]) Let \(\prec_s\) be a closed selection relation on \(X, Z\) be a connected subset of \(X\), and let \(p \in Z\). Then,

(a) \(p \in net(Z)\) if and only if \(x \prec_s p\) for every \(x \in Z\) or \(p \prec_s x\) for every \(x \in Z\).
(b) \(p \in ct(Z)\) if and only if there are points \(s, t \in Z\), with \(s \prec_s p \prec_s t\).

In particular, \(|net(Z)| \leq 2\) and \(ct(Z)\) is open in \(X\).

**Proof.** First of all, let us observe that \((\leftarrow, p)_{\prec_s} \cap Z\) is a connected subset of \(X\). Indeed, whenever \((\leftarrow, p)_{\prec_s} \cap Z \neq \emptyset\), take a fixed point \(c \in (\leftarrow, p)_{\prec_s} \cap Z\). Then, the statement follows by Corollary 6.4 and Proposition 7.1 because

\[
(\leftarrow, p)_{\prec_s} \cap Z = \bigcup \{[x, y]_{\prec_s}: x, y \in Z \text{ and } x \prec_s c \prec_s y \prec_s p\}.
\]

In the same way, \((p, \rightarrow)_{\prec_s} \cap Z\) is a connected subset of \(X\). Finally, by Proposition 7.1, \(Z \setminus \{p\} \subset (\leftarrow, p)_{\prec_s}\) or \(Z \setminus \{p\} \subset (p, \rightarrow)_{\prec_s}\) provided \(Z \setminus \{p\}\) is connected. Consequently, \(Z \setminus \{p\}\) is connected if and only if \(Z \setminus \{p\} = (\leftarrow, p)_{\prec_s} \cap Z\) or \(Z \setminus \{p\} = (p, \rightarrow)_{\prec_s} \cap Z\), which is (a). Since (b) follows by (a), the proof is completed. \(\square\)
Purisch sets. Relying on a construction in [29], to every space \( X \) which has a closed selection relation one can associate a totally disconnected subset \( Z \subseteq X \) which incorporates a certain information about the components of \( X \).

**Definition 7.3.** ([10,29]) Let \( X \) be a space such that \( |\text{nct}(\mathcal{C}[x])| \leq 2 \) for every \( x \in X \). A subset \( Z \subseteq X \) is called a **Purisch set** if for every \( x \in X \) the following holds:

(a) \( \mathcal{C}[x] \subseteq Z \) provided \( \mathcal{C}[x] \) is a singleton.
(b) \( |\mathcal{C}[x] \cap Z| = 1 \) provided \( \text{nct}(\mathcal{C}[x]) = \emptyset \).
(c) \( |\mathcal{C}[x] \cap Z| = 2 \) and \( \text{nct}(\mathcal{C}[x]) \subseteq Z \) otherwise.

Below we present some basic properties of Purisch sets showing, in particular, that such sets are not as arbitrary as one might look at first.

**Proposition 7.4.** ([10]) Let \( X \) be a space which has a closed selection relation. Then, \( X \) has at least one Purisch subset, and any Purisch subset of \( X \) is totally disconnected.

**Proof.** According to Proposition 7.2 and Definition 7.3, \( X \) has at least one Purisch subset. Take such a subset \( Z \subseteq X \). On the one hand, the component of each point of \( Z \) is contained in the corresponding component of that point in \( X \). Consequently, by Definition 7.3, the components (in \( Z \)) of the points of \( Z \) must be singletons. On the other hand, \( Z \) has a closed selection relation being a subset of \( X \). Hence, by Theorem 6.1, \( Z \) must be totally disconnected.

**Proposition 7.5.** ([10]) Let \( X \) be a space which has a closed selection relation, and let \( Z \subseteq X \) be a Purisch subset. Then, \( Z \) is closed in \( X \).

**Proof.** If \( x \in X \setminus Z \), then, by Definition 7.3, \( x \in \text{ct}(\mathcal{C}[x]) \). Since \( |\mathcal{C}[x] \cap Z| \leq 2 \), by Proposition 7.2, \( U = \text{ct}(\mathcal{C}[x]) \setminus Z \) is an open subset of \( X \) which contains \( x \).

Suppose that \( Z \subseteq X \) is a Purisch set. Following [29], for every \( z \in Z \) we define a subset \( \text{nb}(z) \subseteq Z \) by setting \( \text{nb}(z) = \mathcal{C}[z] \cap Z \). The elements of \( \text{nb}(z) \) will be called neighbours. Clearly, \( y \in \text{nb}(z) \) if and only if \( \text{nb}(y) = \text{nb}(z) \). Of course, \( |\text{nb}(z)| \leq 2 \) for every \( z \in Z \). Let \( \mathcal{N}(Z) = \{ \text{nb}(z) \mid z \in Z \} \).

Recall that a pair of distinct point \( y, z \in Z \) is a \( \preceq_s \)-jump of \( Z \) for a selection relation \( \preceq_s \) on \( Z \), if \( (y, z) \preceq_s = \emptyset = (z, y) \preceq_s \). Also, that the set of all \( \preceq_s \)-jumps of \( Z \) was denoted by \( \mathcal{J}(X, \preceq_s) \).

**Proposition 7.6.** Let \( X \) be a space with a closed selection relation \( \preceq_s \), and let \( Z \subseteq X \) be a Purisch set of \( X \). Then,

\[
\{ v \in \mathcal{N}(Z) : |v| = 2 \} \subset \mathcal{J}(Z, \preceq_s).
\]

**Proof.** By Proposition 7.1, we have \( (x, y) \preceq_s, (y, x) \preceq_s \subseteq \mathcal{C}[z] \) for every \( z \in X \) and \( x, y \in \mathcal{C}[z] \), which completes the proof.

**Definition 7.7.** ([10]) A clopen subset \( W \) of a Purisch set \( Z \subseteq X \) is called **order-regular** if there exist an open set \( U \subseteq X \) and a point \( y \in W \) such that \( U \cap Z = W \) and \( U \subseteq U \cup \mathcal{C}[y] \).

We will use \( \mathcal{O}_r(Z) \) to denote the set of all order-regular subsets of \( Z \).

**Proposition 7.8.** Let \( X \) be a space which has a closed selection relation, \( Z \subseteq X \) is a Purisch set, \( W \in \mathcal{O}_r(Z) \) and let \( U \subseteq X \) be open such that \( U \cap Z = W \) and \( \overline{U} \subseteq U \cup \mathcal{C}[y] \) for some \( y \in W \). Then,

(a) For every \( x \in U \setminus \mathcal{C}[y] \) there is a clopen subset \( V \subseteq X \) such that \( x \in V \subseteq U \).
(b) If \( \text{nb}(y) \subseteq W \), then \( V = U \cup \mathcal{C}[y] \) is a clopen subset of \( X \).
(c) If \( z \in W \), then
   (i) \( \text{nb}(z) \subseteq W \) if and only if there is a clopen subset \( V \subseteq X \) such that \( z \in V \cap Z \subseteq W \).
   (ii) \( z = y \) provided \( \text{nb}(z) \setminus W \neq \emptyset \).

**Proof.** If \( x \in U \setminus \mathcal{C}[y] \), then \( \mathcal{C}[x] \neq \mathcal{C}[y] \) and, by Theorem 6.1, there exists a clopen subset \( O \subseteq X \) such that \( \mathcal{C}[x] \subseteq O \) and \( \mathcal{C}[y] \subseteq X \setminus O \). Then, \( V = O \cap U \) is also clopen because \( \mathcal{C}[y] \cap O = \emptyset \) and, therefore,

\[
\overline{V} \subseteq O \cap \overline{U} \subseteq O \cap (U \cup \mathcal{C}[y]) = O \cap U = V.
\]

Thus, (a) holds. To see (b), observe that \( V = U \cup \mathcal{C}[y] \) is always closed in \( X \) because \( \overline{V} \subseteq \overline{U} \cup \mathcal{C}[y] \subseteq U \cup \mathcal{C}[y] = V \). If \( \text{nb}(y) \subseteq W \), then, by Definition 7.3, \( \text{nct}(\mathcal{C}[y]) \subseteq \text{nb}(y) \subseteq U \cap Z \). Hence, by Proposition 7.2, \( V \) is also open because \( V = \)
Proof. \(\text{To see (c), suppose that } nb(z) \subseteq W \text{ for some } z \in W. \text{ If } z \notin nb(y), \text{ then } z \in U \setminus \mathcal{C}[y], \text{ and the statement follows from (a). If } nb(z) = nb(y), \text{ then, by (b), } V = U \cup \mathcal{C}[y] \text{ is a clopen subset of } X \text{ such that } V \cap Z = W. \text{ Suppose finally that } z \neq y \text{ and } nb(z) \cap W \neq \emptyset. \text{ Since } y, z \in W \text{ and } nb(z) \cap W \neq \emptyset, \text{ we now have that } z \in W \setminus \text{nb}(y) \text{ because } |nb(z)| = 2. \text{ Hence, by (a), there is a clopen set } V \subseteq X \text{ such that } z \in V \cap Z \subseteq W \text{ which, by the first item of (c), implies that } nb(z) \subseteq W. \text{ However, by assumption, } nb(z) \setminus W \neq \emptyset. \text{ A contradiction!} \ \square\)

Proposition 7.8 implies the following useful property of order-regular sets.

Corollary 7.9. Let \(X\) be a space which has a closed selection relation, \(Z \subseteq X\) be a Purisch set, and let \(V, W \in \mathcal{O}_r(X)\) and \(y \in Z\) be such that
\[
\text{nb}(y) \cap V \neq \emptyset \neq \text{nb}(y) \cap W \quad \text{and} \quad \text{nb}(y) \setminus V \neq \emptyset \neq \text{nb}(y) \setminus W.
\]
Then, \(V \cup W, V \cap W \in \mathcal{O}_r(X)\).

Proof. By hypothesis, \(\text{nb}(y) \cap V \neq \emptyset \neq \text{nb}(y) \cap W\). Hence, by Proposition 7.8, there is an open set \(O_V \subseteq X\) such that \(O_V \cap Z = V\) and \(\overline{O_V} \subseteq O_V \cup \mathcal{C}[y]\). In the same way, there is an open set \(O_W \subseteq X\) such that \(O_W \cap Z = W\) and \(\overline{O_W} \subseteq O_W \cup \mathcal{C}[y]\). Then, \((O_V \cup O_W) \cap Z = V \cup W\) and \(\overline{O_V} \cup \overline{O_W} \subseteq O_V \cup O_W \cup \mathcal{C}[y]\). Hence, \(V \cup W \in \mathcal{O}_r(X)\). Similarly, \(V \cap W \in \mathcal{O}_r(X)\) because \((O_V \cap O_W) \cap Z = V \cap W\) and \(\overline{O_V} \cap \overline{O_W} \subseteq \overline{O_V} \cap \overline{O_W} \subseteq (O_V \cap O_W) \cup \mathcal{C}[y]\). \ \square

Let us explicitly mention that the intersection of order-regular sets is not necessarily an order-regular set, but we always have the following property.

Proposition 7.10. Let \(X\) be a space which has a closed selection relation, \(Z \subseteq X\) be a Purisch set, and let \(V, W \in \mathcal{O}_r(X)\). Then, there exists a finite pairwise disjoint family \(\mathcal{W} \subseteq \mathcal{O}_r(X)\) such that \(V \cap W = \bigcup \mathcal{W}\).

Proof. Since \(V \in \mathcal{O}_r(X)\), there exist an open subset \(O_V \subseteq X\) and \(y \in V\) such that \(V = O_V \cap Z\) and \(\overline{O_V} \subseteq O_V \cup \mathcal{C}[y]\). In the same way, there exist an open subset \(O_W \subseteq X\) and \(z \in W\) such that \(W = O_W \cap Z\) and \(\overline{O_W} \subseteq O_W \cup \mathcal{C}[z]\). Then, \(O = O_V \cap O_W\), we get that \(\overline{O} \subseteq \overline{O_V} \cap \overline{O_W} \subseteq O \cup \mathcal{C}[y] \cup \mathcal{C}[z]\). If \(\mathcal{C}[y] = \mathcal{C}[z]\), by definition, \(V \cap W \in \mathcal{O}_r(X)\) because \(O \cap Z = V \cap W\). Suppose that \(\mathcal{C}[y] \neq \mathcal{C}[z]\). Then, by Theorem 6.1, there is a clopen set \(U \subseteq X\) such that \(\mathcal{C}[y] \subseteq U\) and \(\mathcal{C}[z] \subseteq X \setminus U\). In this case, \(\mathcal{W} = \{U \cap Z, U \cap W\}\) is a clopen disjoint cover of \(V \cap W, \text{ where } U \cap Z = O \cap U\) and \(U \cap W = O \cap \emptyset\). Also,
\[
\overline{U} \subseteq \overline{U} \cap U \subseteq (O \cup \mathcal{C}[y] \cup \mathcal{C}[z]) \cap U \subseteq (O \cap U) \cup \mathcal{C}[y] = U \cup \mathcal{C}[y]
\]
because \(\mathcal{C}[y] \subseteq X \setminus U\). Similarly, \(U \subseteq U \cup \mathcal{C}[z]\) because \(\mathcal{C}[y] \subseteq U\). Thus, by definition, \(\mathcal{W} \subseteq \mathcal{O}_r(X)\). \ \square

Proposition 7.11. Let \(X\) be a space which has a closed selection relation \(\preceq_S\), and let \(Z \subseteq X\) be a Purisch set. Then,
(a) \(V \cap Z \in \mathcal{O}_r(X)\) for every clopen subset \(V \subseteq X\).
(b) \((\leftarrow, \preceq_S) \cap Z, (\rightarrow, \preceq_S) \cap Z \in \mathcal{O}_r(X)\) for every \(x \in X \setminus Z\).

Proof. The statement of (a) follows by the definition. As for (b), let
\[
W = (\leftarrow, x) \preceq_S \cap Z.
\]
If \(\mathcal{C}[x] \cap W = \emptyset\), then \(U = (\leftarrow, x) \preceq_{S_x} \cap \mathcal{C}[x]\) is clopen in \(X\). Indeed, \(U\) is open because \(\mathcal{C}[x]\) is closed. According to Definition 7.3, \(x\) is a cut point of \(\mathcal{C}[x]\) because \(x \in \mathcal{C}[x] \setminus Z\). By the same reason, \(\mathcal{C}[x]\) has no \(\preceq_s\)-minimal element because \(\mathcal{C}[x] \cap W = \emptyset\). Hence, \(U = (\leftarrow, x) \preceq_{S_x} \cap \mathcal{C}[x]\) and, by Proposition 7.2, it must be closed. Thus, \(U\) is clopen and \(U \cap Z = W\), i.e. \(W \in \mathcal{O}_r(X)\). If \(\mathcal{C}[x] \cap W \neq \emptyset\), then \(W\) is clopen in \(Z\) because \((\leftarrow, x) \preceq_S \cap Z = (\leftarrow, x) \preceq_{S_x} \cap Z,\) and we have that \((\leftarrow, x) \preceq_{S_x} \cap (\leftarrow, x) \preceq_{S_x} \subseteq (\leftarrow, x) \preceq_{S_x} \cup \mathcal{C}[x]\), i.e. \(W \in \mathcal{O}_r(X)\). In the same way, \((\rightarrow, x) \preceq_{S_x} \cap Z \in \mathcal{O}_r(X)\). \ \square

Proposition 7.12. Let \(X\) be a space which has a closed selection relation, \(Z \subseteq X\) be a Purisch set, and let \(W \in \mathcal{O}_r(X)\). Then, \(Z \setminus W \in \mathcal{O}_r(X)\).

Proof. By Definition 7.7, there exist an open set \(U \subseteq X\) and \(y \in W\) such that \(U \cap Z = W\) and \(\overline{U} \subseteq U \cup \mathcal{C}[y]\). If \(nb(z) \subseteq W\) for every \(z \in W\), then, by Proposition 7.8, \(V = U \cup \mathcal{C}[y]\) is clopen in \(X\) and \(V \cap Z = W\). Hence, by Proposition 7.11, \(Z \setminus W = (X \setminus W) \cap Z \in \mathcal{O}_r(X)\). If \(nb(z) \setminus W \neq \emptyset\) for some \(z \in W\), then Proposition 7.8 implies that \(z = y\). Let \(p \in nb(y) \setminus W\), and let \(\preceq_{S_y}\) be a closed selection relation on \(X\). Assume, for instance, that \(p \prec_S y\). Then, by Proposition 7.1, there is \(x \in X\) such
that $p \prec_s x \prec_s y$. By Proposition 7.11, $(\leftarrow, x) \leq_s Z \in \mathcal{O}_r(Z)$. Since $(\leftarrow, x) \leq_s (\leftarrow, x) \leq_s \cup [\leq, x] \leq_s \cup [\leq, y]$, by Corollary 7.9, we now have that $(\leftarrow, x) \leq_s W = (\leftarrow, x) \leq_s Z \cap W \in \mathcal{O}_r(Z)$. However, $nb(z) \subset (\leftarrow, x) \leq_s Z \cap W$ for every $z \in (\leftarrow, x) \leq_s Z \cap W$. Hence, by Proposition 7.8, there exists a clopen subset $G \subset X$ such that $G \cap Z = (\leftarrow, x) \leq_s Z \cap W$. Finally, set $V = G \cup (U \cap (\leftarrow, y) \leq_s ) \cup ((\leftarrow, y) \leq_s \cap \mathcal{O}[y])$. Then, $V$ is an open subset of $X$ such that $V \cap Z = W$ and $V \subset U \cup \{x\}$. Indeed, $V$ is open in $X$ because $V = G \cup (U \cap (\leftarrow, y) \leq_s ) \cup ((\leftarrow, y) \leq_s \cap \mathcal{O}[y]))$. On the other hand,

$$U \cap (x, \rightarrow) \leq_s \subset (U \cup \mathcal{C}[y]) \cap (x, \rightarrow) \leq_s = (U \cap [x, \rightarrow] \leq_s ) \cup (\mathcal{C}[y] \cap [x, \rightarrow] \leq_s ).$$

Hence, $V \subset U \cup (U \cap (\leftarrow, y) \leq_s ) \cup (\mathcal{C}[y] \cap [x, \rightarrow] \leq_s ) = V \cup [x]$. Then, $T = X \setminus V$ is open in $X$ such that $T \cap Z = Z \setminus W$ and $T \subset X \setminus V \subset T \cup [x] \subset T \cup \mathcal{O}[y]$. Since $\mathcal{O}[p] = \mathcal{O}[y]$ and $p \in Z \setminus W$, by Definition 7.7, $Z \setminus W \in \mathcal{O}_r(Z)$. □

Weak orderability and Purisch spaces. Let $X$ be a space which has a closed selection relation, and let $Z \subset X$ be a Purisch set. According to Propositions 7.11, 7.10 and 7.12, the family $\mathcal{O}_r(Z)$ is a base for a zero-dimensional topology $\mathcal{T}_r(Z)$ on $Z$. In the sequel, the topological space $(Z, \mathcal{T}_r(Z))$ will be called a Purisch space associated to $X$.

The following theorem summarizes the idea of [29] (see, also, [10, Theorem 4.1]).

**Theorem 7.13.** Let $X$ be a space which has a closed selection relation, and let $Z \subset X$ be a Purisch set. Then, the following are equivalent:

(a) $X$ is weakly orderable.

(b) The Purisch space $(Z, \mathcal{T}_r(Z))$ is weakly orderable so that each $v \in \mathcal{N}(Z)$ is a convex subset of $Z$.

**Proof.** If $\prec_s$ is a closed linear order on $X$, then, by Proposition 7.6, each $v \in \mathcal{N}(Z)$ is $\prec_s$-convex. Take a point $y \in Z$ and $z \in T = (y, \rightarrow) \leq_s Z \cap T$. If $y$ and $z$ are neighbours, then $y, z \in \mathcal{C}[y]$ and $y \prec_s z$. Hence, by Proposition 7.1, there exists a point $x \in [y, z] \subset X \setminus Z$. Then, by Proposition 7.11, $(x, \rightarrow) \leq_s Z \in \mathcal{O}_r(Z)$ and

$$z \in (x, \rightarrow) \leq_s Z \subset (y, \rightarrow) \leq_s Z = T.$$

If $y$ and $z$ are not neighbours, then $\mathcal{C}[y] \neq \mathcal{C}[z]$. Hence, by Theorem 6.1, there exists a clopen subset $O \subset X$ such that $\mathcal{C}[z] \subset O$ and $\mathcal{C}[y] \subset X \setminus O$. In this case, $V = O \cap [y, \rightarrow) \leq_s = O \cap (y, \rightarrow) \leq_s$ is clopen in $X$ and, by Proposition 7.11, $V \cap Z \in \mathcal{O}_r(Z)$, while $z \in V \cap Z \subset T$. Thus, $(Z, \mathcal{T}_r(Z))$ is also weakly orderable with respect to $\prec_s$, which completes the verification of (a) $\Rightarrow$ (b).

To see that (b) $\Rightarrow$ (a), let $Z$ be weakly orderable with respect to a linear order $\leq_s$ such that each $v \in \mathcal{N}(Z)$ is $\leq_s$-convex. Also, let $\leq_s$ be a closed selection relation on $X$, and for every $v \in \mathcal{N}(Z)$ let $C_v$ be the corresponding connected component in $X$ of some (every) point of $v$. Since $|v| \leq 2$ for every $v \in \mathcal{N}(Z)$, we now have that $\leq_s |v|^2 = \leq_s |v|^2$ or $\leq_s |v|^2 = (\leq_s |v|^2)$ for every $v \in \mathcal{N}(Z)$. On the other hand, by Corollary 6.4, each $C_v$, $v \in \mathcal{N}(Z)$, is weakly orderable with respect to $\leq_s$. Hence, each $C_v$, $v \in \mathcal{N}(Z)$, is weakly orderable with respect to a closed selection relation $\leq_s$ on $X$ such that $\leq_s |v|^2 = \leq_s |v|^2$. Finally, consider the lexicographical order $\prec$ on $X$ generated by $\leq_s$ and $\leq_s |\cdot|^2$. By Proposition 7.1, the $\leq_s$-order on $X$ is $\leq_s$-orderable on $X$.

- **(WO1).** $y$ is a cut point of $\mathcal{C}[y]$. In this case, there are points $s, t \in \mathcal{C}[y]$ such that $x \prec_s y \prec_s t$. Indeed, if $x \neq \mathcal{C}[y]$, then, by the definition of $\leq_s$, we have that $x \prec_s z$ for every $z \in \mathcal{C}[y]$, hence Proposition 7.2 implies the statement. If $x \in \mathcal{C}[y]$, then $x \prec_s y$, and the existence of these $s, t \in \mathcal{C}[y]$ now follows by Propositions 7.1 and 7.2. Finally, $(s, t) \leq_s \subset \mathcal{C}[y]$ by Propositions 7.1: $(s, t) \leq_s$ is open in $X$; and $y \in (s, t) \leq_s \subset (x, \rightarrow) \leq_s$ by the definition of $\leq_s$.

- **(WO2).** $y$ is a noncut point of $\mathcal{C}[y]$ and $x \neq \mathcal{C}[y]$. By Definition 7.3, $y$ has a neighbour $z \in Z$, i.e. there exists a point $z \in v$, with $z \neq y$. By Proposition 7.2, this implies that $z \prec y$ because $x \prec y$. Then, there exists an order-regular set $W \in \mathcal{O}_r(Z)$ such that $y \notin W \subset (z, \rightarrow) \leq_s$. Hence, by Definition 7.7 and Proposition 7.8, there exists an open subset $U \subset X$ such that $U \cap Z = W$ and $U \subset U \cup \{y\}$. Finally, let us show that $V = U \cap (x, \rightarrow) \leq_s \subset (x, \rightarrow) \leq_s$. Take a point $t \in V$. If $t \in \mathcal{C}[y]$, then $x \prec t$ because $t \in (x, \rightarrow) \leq_s$. If $t \notin \mathcal{C}[y]$, by Proposition 7.1, $\mathcal{C}[t] \cap (x, \rightarrow) \leq_s \subset$, while, by Proposition 7.8, $\mathcal{C}[t] \subset U$. That is, $\emptyset \neq \mathcal{C}[t] \cap Z \subset W$. Take a point $s \in \mathcal{C}[t] \cap W$. Then, $z \prec s$ because $s \in W \subset (z, \rightarrow) \leq_s$ and, by the definition of $\leq_s$, we finally get that $x \prec t$ because $x \in \mathcal{C}[y] = \mathcal{C}[z]$ and $t \in \mathcal{C}[s]$.

- **(WO3).** $y$ is a noncut point of $\mathcal{C}[y]$ and $x \neq \mathcal{C}[y]$. Take a $z \in \mathcal{C}[x] \cap Z$. Then, $y \prec z$ because $x \prec y$. Since $y$ and $z$ are not neighbours, by Corollary 7.9, there exists an order-regular set $W \in \mathcal{O}_r(Z)$ such that $nb(y) \subset W \subset (z, \rightarrow) \leq_s$. Hence, by Propo-
sition 7.8, there exists a clopen subset $V \subset X$ such that $y \in V \cap Z \subset W$. In order to show that $V \subset (x, \to)_\lessdot$, take a point $t \in V$. Then,

$$\emptyset \neq \mathcal{C}[t] \cap Z \subset V \cap Z \subset W$$

and, therefore, there exists $s \in W$ such that $t \in \mathcal{C}[s]$. Hence, by the definition of $\lessdot$, we have that $x \lessdot t$ because $z \lessdot s$ while $x \in \mathcal{C}[z]$ and $t \in \mathcal{C}[s]$.

This completes the verification that $(x, \to)_\lessdot$ is open in $X$. Since the verification that $(\leftarrow, x)_\lessdot$ is open in $X$ is completely analogous, the proof is completed. \(\square\)

**Weak orderability of separable spaces II.** The following theorem improves the construction in Proposition 5.1, see [10, Theorem 4.1].

**Theorem 7.14.** Let $X$ be a space which has a closed selection relation, and let $Z \subset X$ be a Purisch set which has a countable family $\mathcal{H} \subset \mathcal{O}(Z)$ that is separating for the points of $Z$. Then, the Purisch space $(Z, \mathcal{O}(Z))$ is weakly orderable with respect to a linear order $\lessdot$ such that any member of $\mathcal{N}(Z)$ is $\lessdot$-convex.

Suppose that $X$ and $Z \subset X$ are as in Theorem 7.14. A linear order $\lessdot$ on a pairwise disjoint cover $\mathcal{W} \subset \mathcal{O}(Z)$ of $Z$ is $\mathcal{W}$-ordering [29] if two distinct members of $\mathcal{W}$ which contain neighbours have no other member of $\mathcal{W}$ between them with respect to $\lessdot$. Let $\mathcal{W} \subset \mathcal{O}(Z)$ be a pairwise disjoint cover of $Z$, $\leq_{\mathcal{W}}$ be a $\mathcal{W}$-ordering on $\mathcal{W}$, and let $\mathcal{V} \subset \mathcal{O}(Z)$ be another pairwise disjoint cover of $Z$ which is a refinement of $\mathcal{W}$. Whenever $W \in \mathcal{W}$, set $\mathcal{Y}(W) = \{V \in \mathcal{V} : V \subset W\}$. A $\mathcal{Y}$-ordering $\leq_\mathcal{Y}$ on $\mathcal{V}$ will be called $\leq_{\mathcal{W}}$-compatible if $V_1 \leq_\mathcal{Y} V_2$ provided $V_1 \in \mathcal{Y}(W)$, $i = 1, 2$, for some $W_1, W_2 \in \mathcal{W}$ such that $W_1 \leq_\mathcal{W} W_2$.

**Proof of Theorem 7.14.** Let $X, Z \subset X$ and $\mathcal{H} \subset \mathcal{O}(Z)$ be as in that theorem. For every $H \in \mathcal{H}$, let $\mathcal{W}_H = [H, Z \setminus H]$. According to Proposition 7.12, this defines a countable sequence $\{\mathcal{W}_H \subset \mathcal{O}(Z) : H \in \mathcal{H}\}$ of pairwise disjoint open covers of $(Z, \mathcal{O}(Z))$ which is separating the points of $Z$ in sense of [10] (i.e., for every two distinct points $y, z \in Z$ there is $H \in \mathcal{H}$ and distinct members $U, V \in \mathcal{W}_H$ such that $y \in U$ and $z \in V$). Then, according to Theorem 7.10, we get a sequence $\{\mathcal{W}_n : n < \omega\}$ of (finite) pairwise disjoint open covers of $Z$ such that

$$\{\mathcal{W}_n : n < \omega\} \text{ is separating the points of } Z, \quad (7.1)$$

$$\text{each } \mathcal{W}_{n+1} \text{ is a refinement of } \mathcal{W}_n, \quad n < \omega. \quad (7.2)$$

By Proposition 7.8, $|\{z \in W : \text{nb}(z) \setminus W \neq \emptyset\}| \leq 1$ for every $W \in \mathcal{O}(Z)$. Hence, for every $n < \omega$ there exists a $\mathcal{W}_n$-ordering $\leq_n$ on $\mathcal{W}_n$ such that

$$\text{the } \mathcal{W}_{n+1} \text{-ordering } \leq_{n+1} \text{ is } \leq_n \text{-compatible, } \quad n < \omega. \quad (7.3)$$

Finally, define a relation $\leq$ on $Z$ by letting for $y, z \in Z$ that $y \leq z$ if $y = z$ or there exist an $n < \omega$ and members $V, W \in \mathcal{W}_n$ such that $y \in V, z \in W$ and $V \lessdot_n W$. According to (7.1), (7.2) and (7.3), $\lessdot$ is a well-defined linear order on $Z$. Take $y \in Z$ and $z \in (y, \to)_\lessdot$. Then, $y \lessdot z$ and, by definition, there is an $n < \omega$ and members $U, V \in \mathcal{W}_n$ such that $z \in U, y \in V$ and $V \lessdot_n U$. In this case, $z \in U \subset (y, \to)_\lessdot$, hence $(y, \to)_\lessdot$ is open in $(Z, \mathcal{O}(Z))$. In the same way, $(\leftarrow, y)_\lessdot$ is also open. That is, $(Z, \mathcal{O}(Z))$ is weakly orderable. Finally, take $v \in \mathcal{N}(Z)$. If $v$ is a singleton, then it is $\lessdot$-convex in an obvious manner. Suppose that $y, z \in v$ and $y < z$. If $x \in Z \setminus v$, then, by (7.1), (7.2) and (7.3), there exist an $n < \omega$ and distinct members $U, V, W \in \mathcal{W}_n$ such that $x \in U, y \in V$ and $z \in W$. Since $\leq_n$ is a $\mathcal{W}_n$-ordering, we now have that $U \lessdot_n V$ or $W \lessdot_n U$ because $V \lessdot_n W$ and there is no other member of $\mathcal{W}_n$ between $V$ and $W$. Consequently, $x \notin [y, z]_\lessdot$, i.e. $v$ is convex. The proof is completed. \(\square\)

For a space $Y$, we will use $\ell(Y)$ to denote the Lindelöf number of $Y$.

**Corollary 7.15.** ([10]) Let $X$ be a space which has a closed selection relation and $\ell([X]^2) \leq \omega$. Then, $X$ is weakly orderable. In particular, a second countable space is weakly orderable if and only if it has a closed selection relation.

**Proof.** Take a Purisch set $Z \subset X$. According to Theorems 7.13 and 7.14, it suffices to show that there exists a countable family $\mathcal{H} \subset \mathcal{O}(Z)$ which is separating for the points of $Z$. To this end, for every $W \in \mathcal{O}(Z)$, let $\alpha_W = (W, Z \setminus W)$. Thus, by Propositions 7.11 and 7.12, we get an open cover $\{\alpha_W : W \in \mathcal{O}(Z)\}$ of $[Z]^2$. By Proposition 7.5, $Z$ is a closed subset of $X$, hence $\ell([Z]^2) \leq \omega$. Therefore, there exists a countable subset $\mathcal{H} \subset \mathcal{O}(Z)$ such that $[Z]^2 \subset \bigcup\{\alpha_H : H \in \mathcal{H}\}$. The family $\mathcal{H} \subset \mathcal{O}(Z)$ is as required. \(\square\)

The following question was posed in [10, Question 2].
**Question 6.** ([10]) Let $X$ be a space which has a closed selection relation and $\ell(X^2) \leq \omega$. Then, is $X$ weakly orderable?

The answer to Question 6 is “Yes” if $\Delta(X) = \{(x, x): x \in X\}$ is a $G_\delta$-set in $X^2$. In this case, $\ell(X^2 \setminus \Delta(X)) \leq \omega$ which implies that $\ell(|X|^2) \leq \omega$ because the map $h : P(X) = X^2 \setminus \Delta(X) \to |X|^2$, defined by $h(x, y) = (x, y)$, $(x, y) \in X^2 \setminus \Delta(X)$, is a continuous surjection.

**Corollary 7.16.** ([10]) Let $X$ be a separable space which has a closed selection relation $\preceq_s$ and countably many $\preceq_s$-jumps. Then, $X$ is weakly orderable. In particular, every separable space $X$ which has a closed selection relation and $[X]^2$ is collectionwise Hausdorff is weakly orderable.

**Proof.** The second part of this statement follows by Corollary 5.5. To show the first part, take a Purisch set $Z \subset X$. Just like in the previous proof, it suffices to show that there exists a countable family $\mathcal{H} \subset \mathcal{O}(Z)$ that is separating for the points of $Z$. To this end, take a countable dense subset $D \subset X$, and a pair of distinct points $y = \{s, t\} \subset [D]^2$, with $s \prec_s t$. If $\mathcal{C}[s] = \mathcal{C}[t]$, by Proposition 7.1, there exists a point $x \in \mathcal{C}[s] \setminus Z$, with $x \in (s, t) \preceq_s$. In this case, by Proposition 7.11, $H_y = (\leftarrow, x)_{\preceq_s} \cap Z \in \mathcal{O}(Z)$ and, by Proposition 7.1,

$$\left(\leftarrow, s\right)_{\preceq_s} \cap Z \subset H_y \subset \left(\leftarrow, t\right)_{\preceq_s} \cap Z.$$  

If $\mathcal{C}[s] \neq \mathcal{C}[t]$, by Theorem 6.1, there exists a clopen set $V \subset X$ such that $\mathcal{C}[s] \subset V$ and $V \cap \mathcal{C}[t] = \emptyset$. Set $W = (\leftarrow, t)_{\preceq_s} \cap V$ which is clopen in $X$ because $W = (\leftarrow, t)_{\preceq_s} \cap V$. Finally, by Proposition 7.11, $H_y = (\leftarrow, x)_{\preceq_s} \cup W \cap Z \in \mathcal{O}(Z)$ and, by construction,

$$\left(\leftarrow, s\right)_{\preceq_s} \cap Z \subset H_y \subset \left(\leftarrow, t\right)_{\preceq_s} \cap Z.$$  

Thus, we get a family $\{H_y : y \in [D]^2\} \subset \mathcal{O}(Z)$ such that if $y = \{s, t\} \in [D]^2$ and $s \prec_s t$, then

$$\left(\leftarrow, s\right)_{\preceq_s} \cap Z \subset H_y \subset \left(\leftarrow, t\right)_{\preceq_s} \cap Z. \quad (7.4)$$  

Let $D_0$ be the set of all isolated points of $D$ (hence, of $X$ as well), and for every singleton $\sigma \in [D_0]^1$, $\sigma = \{s\}$, let

$$H_\sigma = (\leftarrow, s)_{\preceq_s} \cap Z. \quad (7.5)$$

Since $\sigma = \{s\}$ is a clopen set in $X$, by Proposition 7.11, $H_\sigma \in \mathcal{O}(Z)$.

Finally, for every $\preceq_s$-jump $y = \{x, y\} \in \mathcal{Z}(X, \preceq_s)$, with $x \prec y$, set

$$H_y = (\leftarrow, x)_{\preceq_s} \cap Z. \quad (7.6)$$

According to Propositions 5.3 and 7.11, $H_y \in \mathcal{O}(Z)$. Since $D$ and $\mathcal{Z}(X, \preceq_s)$ are countable, we get a countable family

$$\mathcal{H} = \{H_y : y \in [D]^2 \cup [D_0]^1 \cup \mathcal{Z}(X, \preceq_s)\} \subset \mathcal{O}(Z).$$

Let us show that it is separating for the points of $Z$, so take distinct points $x, y \in Z$. If one of the intervals $(x, y)_{\preceq_s}$ or $(y, x)_{\preceq_s}$ is nonempty, say $(x, y)_{\preceq_s}$, then $(x, y)_{\preceq_s} \cap D \neq \emptyset$. If $|D| \geq 2$, then there is a pair $y = \{s, t\} \in [D]^2$ such that $s_{\prec_s} t$ and $s \in (y)_{\preceq_s}$. By construction, $H_y \in \mathcal{H}$ while, by (7.4), $x \in H_y$ and $y \notin H_y$. In case $|D| = 1$, $(x, y)_{\preceq_s}$ contains an isolated point $s \in D_0$. Then, $H_\sigma = \mathcal{H}$, where $\sigma = \{s\}$, and, by (7.5), $x \in H_\sigma$ while $y \notin H_\sigma$. Suppose finally that $(x, y)_{\preceq_s} = \emptyset = (y, x)_{\preceq_s}$ and, for convenience, that $x \prec y$. Then, $y = \{x, y\} \in \mathcal{Z}(X, \preceq_s)$ and, by (7.6), $x \in H_y$ while $y \notin H_y$. Since $H_y \in \mathcal{H}$, the proof is completed.

The following question was motivated by Corollary 7.16, see [10, Question 3].

**Question 7.** ([10]) Let $X$ be a separable collectionwise Hausdorff space which has a closed selection relation. Then, is $X$ weakly orderable?

The next question is motivated by Theorem 5.7 and Corollary 7.16.

**Question 8.** Let $X$ be a separable space which has a closed selection relation $\preceq_s$ such that $\mathcal{Z}(X, \preceq_s)$ is countable. Then, is $X$ weakly orderable with respect to linear ordering $\preceq$ such that any member $y \in \mathcal{Z}(X, \preceq_s)$ is $\preceq$-convex?

8. **Orderability and connectedness**

*Orderability and local connectedness.* A map $h : Y \to X$ between linearly ordered sets $(Y, \preceq)$ and $(X, \preceq)$ is called order preserving (respectively, reversing) if $h(y) \preceq h(z)$ (respectively, $h(z) \preceq h(y)$) for every $y, z \in Y$ with $y \preceq z$. The following result is due to Eilenberg.
Lemma 8.1. ([4]) Let \( (Y, \preceq) \) be a weakly orderable space, let \( (X, \preceq) \) be a connected weakly orderable space, and let \( C \subseteq X \) be a nonempty open connected subset. If \( h : Y \to X \) is bijective and order preserving (or, reversing), then \( h^{-1}(C) \) is open in \( Y \).

Proof. We follow the arguments in [4, (5.1)]. According to Proposition 7.1, \( x \prec C \) or \( C \prec x \) for every \( x \in X \setminus C \). Hence, the sets
\[
L = \{ x \in X : x \prec C \} \quad \text{and} \quad R = \{ x \in X : C \prec x \}
\]
compose a partition of \( X \setminus C \) such that both \( L \cup C \) and \( C \cup R \) are open in \( X \). Indeed, \( C \neq \emptyset \) which implies that
\[
L \cup C = \bigcup \{ (\leftarrow, x) \prec : x \in C \} = \bigcup \{ (\leftarrow, x) \prec \cup C : x \in C \}
\]
because \( L \subseteq (\leftarrow, x) \prec \) and \( R \subseteq (x, \rightarrow) \prec \), \( x \in C \). By hypothesis, \( C \) is open, hence \( L \cup C \) is open as well. In the same way, \( C \cup R \) is also open. That is, both \( L \) and \( R \) are closed. In fact, \( L \) must have a maximal element if it is nonempty. Indeed, suppose that \( L \neq \emptyset \) but it has no maximal element. Then, \( L \) will be open in \( X \) because \( L = \bigcup \{ (\leftarrow, x) \prec : x \in L \} \). Consequently, \( \{ L, C \cup R \} \) will be a partition of \( X \) consisting of nonempty clopen subsets which is impossible because \( X \) is connected. Exactly in the same way, \( R \) must have a minimal element if it is nonempty. Then \( h^{-1}(C) = Y \) provided \( L = \emptyset = R \); \( h^{-1}(C) = (h^{-1}(p), \rightarrow) \prec \) provided \( p \) is the maximal element of \( L \) and \( R = \emptyset \); \( h^{-1}(C) = (\leftarrow, h^{-1}(q)) \prec \) provided \( L \neq \emptyset \) and \( q \) is the minimal element of \( R \); and \( h^{-1}(C) = (h^{-1}(p), h^{-1}(q)) \prec \) provided \( p \) is the maximal element of \( L \) and \( q \) is the minimal element of \( R \). Thus, \( h^{-1}(C) \) is open. \( \square \)

Corollary 8.2. ([4]) A connected weakly orderable space \( (X, \prec) \) is orderable if and only if it is locally connected.

Proof. Suppose that \( (X, \prec) \) is orderable, and take points \( x, y \in X \) such that \( x \prec y \). Since \( X \) is connected, by Proposition 7.1, there exists \( p \in (x, y) \prec \). Then, by Lemma 6.2, \( (x, y) \prec \) is connected because
\[
(x, y) \prec = \bigcup \{ [s, t] \prec : s, t \in X \text{ and } x \prec s \prec p \prec t \prec y \}.
\]
In the same way, both \( (\leftarrow, x) \prec \) and \( (x, \rightarrow) \prec \) are also connected. Then, \( X \) is locally connected because it has a base of connected sets. To show the converse, suppose that \( X \) is locally connected. Since \( (X, \prec) \) is weakly orderable, the identity map \( h = \text{id}_X : X \to (X, \mathcal{T}_\prec) \) is a continuous order preserving bijection. Since \( X \) has a base of connected sets (being locally connected), by Lemma 8.1, \( h \) is an open map. Hence, \( h \) is a homeomorphism and \( (X, \prec) \) is orderable. \( \square \)

The following is an immediate consequence of Corollary 6.10.

Corollary 8.3. ([4]) A weakly orderable connected space \( X \) is continuously ordered if and only if it is locally connected.

Orderability and local compactness. An element \( u \) of an ordered set \( (X, \preceq) \) is called the least upper bound of a subset \( A \subseteq X \), if \( x \preceq u \) for every \( x \in A \) and \( u \preceq v \) for every \( v \in X \) satisfying \( x \preceq v \) for every \( x \in A \). The greatest lower bound of a subset \( A \subseteq X \) is defined analogously. The least upper bound of \( X \), if it exists, is the maximal element of \( X \), while the least upper bound of \( \emptyset \), if it exists, is the minimal element of \( X \).

If \( (X, \preceq) \) is a weakly orderable space, \( A \subseteq X \) and \( u \in X \) is an upper bound of \( A \), then \( A \subseteq (\leftarrow, u) \preceq \). Hence, \( A \subseteq (\leftarrow, u) \preceq \) and, therefore, \( u \) is also the least upper bound of \( A \).

The following observation was actually established in [11, Proposition 6.1].

Proposition 8.4. An orderable space \( (X, \preceq) \) is compact if and only if every clopen subset \( F \subseteq X \) has a least upper bound.

Proof. If \( X \) is compact, then every subset \( F \subseteq X \) has a least upper bound, Haar and König [20] (see, also, Engelking [5]). To show the converse, suppose that every clopen subset \( F \subseteq X \) has the least upper bound property. Then, \( X \) has a minimal element and a maximal one. Hence, by the mention result of Haar and König, it suffices to show that every nonempty closed subset \( F \subseteq X \) has a least upper bound. Suppose if possible that this fails for some nonempty closed subset \( F \subseteq X \). Consider the set \( E = \bigcup \{ (\leftarrow, x) \prec : x \in F \} \) which is open in \( X \) and \( F \subseteq E \) because \( F \) has no least upper bound. In fact, if \( u \in F \), then \( z \preceq u \) for every \( z \in E \) if and only if \( x \preceq u \) for every \( x \in F \). Consequently, \( E \) also has no least upper bound and, by hypothesis, it cannot be closed. That is, there exists a point \( u \in \mathbb{E} \setminus E \). Take a \( \preceq \)-convex neighbourhood \( U \) of \( u \). Then, \( U \cap (\leftarrow, x) \preceq \neq \emptyset \) for some \( x \in F \), which implies that \( x \in U \cap F \) because \( U \) is \( \preceq \)-convex and \( z \preceq u \) for every \( z \in E \). This finally implies that \( u \in F \) because \( F \) is closed. However, \( x \preceq u \) for every \( x \in F \). A contradiction! \( \square \)

Corollary 8.5. Every connected and locally connected weakly orderable space \( (X, \preceq) \) is locally compact. In particular, every locally connected weakly orderable space is locally compact.

Proof. By Corollary 8.2, \( (X, \preceq) \) is orderable. It now suffices to show that \( [x, y] \prec \) is compact for every \( x, y \in X \), with \( x \preceq y \). So, take points \( x, y \in X \) such that \( x \preceq y \). According to Lemma 6.2, \( [x, y] \prec \) is a connected subset of \( X \) and, clearly, it is
orderable because so is \((X, \preccurlyeq)\). Since \(X\) is the minimal element of \([x, y]_\preccurlyeq\) and \(y\) is the maximal one, by Proposition 8.4, \([x, y]_\preccurlyeq\) is compact. □

The converse of Corollary 8.5 was established in [1, Proposition 1.18].

**Proposition 8.6.** ([1]) Every connected and locally compact weakly orderable space \((X, \preccurlyeq)\) is orderable.

**Proof.** We present the proof in [1, Proposition 1.18]. Take an open subset \(U \subseteq X\) such that \(U\) is compact, a point \(x \in U\) and suppose that \((x, \rightarrow)_\preccurlyeq \neq \emptyset\). Then, there exists a point \(y \in (x, \rightarrow)_\preccurlyeq\) such that \([x, y]_\preccurlyeq \subseteq \overline{U}\). To show this, suppose on the contrary that it fails, and consider the space \(Z = [x, y]_\preccurlyeq\) which, by Lemma 6.2, is connected because \(Z = \bigcup \{[x, y]_\preccurlyeq: y \in Z\}\). Set \(K = Z \cap \overline{U}\) and \(V = Z \cap U\). Then, by assumption, \([x, y]_\preccurlyeq \setminus K \neq \emptyset\) for every \(y \in (x, \rightarrow)_\preccurlyeq\). Hence, \(x\) is the greatest lower bound for the set \(Z \setminus K\). Keeping in mind this, for every \(z \in Z \setminus K\), let \(F_z = [x, z]_\preccurlyeq \cap (K \setminus V)\), and let us observe that \(F_z \neq \emptyset\). Indeed, the set \(T_z = [x, z]_\preccurlyeq \cap K\) is closed in \(Z\) because \(z \notin K\) and we have \(T_z = [x, z]_\preccurlyeq \cap K\). If \(F_z = \emptyset\), then \(T_z\) will be also open in \(Z\) because \(T_z = [x, z]_\preccurlyeq \cap V\). However, this is impossible because \(Z\) is connected, while \(x \in T_z\) and \(z \notin T_z\). Thus, each \(F_z\), \(z \in Z \setminus K\), is a nonempty subset of \(K\). Also, the family \(\{F_z: z \in Z \setminus K\}\) has the finite intersection property. Since \(K\) is compact and \(x\) is the greatest lower bound for the set \(Z \setminus V\), we finally get that

\[
\emptyset \neq \bigcap \{F_z: z \in Z \setminus K\} \subseteq \bigcap \{[x, z]_\preccurlyeq: z \in Z \setminus K\} = \{x\}.
\]

This is however impossible because \(x \in V\), hence \(x \notin F_z\) for every \(z \in Z \setminus K\). Thus, \([x, y]_\preccurlyeq \subseteq \overline{U}\) for some \(y \in (x, \rightarrow)_\preccurlyeq\). Exactly in the same way, we get that if \((\leftarrow, x)_\preccurlyeq \neq \emptyset\), then \(([y, x]_\preccurlyeq \subseteq \overline{U}\) for some \(y \in (\leftarrow, x)_\preccurlyeq\). Because \(X\) is weakly orderable, this implies that it is also orderable. □

The following theorem summarizes the statements of Corollaries 8.2, 8.3 and 8.5, and Proposition 8.6.

**Theorem 8.7.** For a connected weakly orderable space \(X\), the following are equivalent:

(a) \(X\) is orderable.
(b) \(X\) is continuously ordered.
(c) \(X\) is locally connected.
(d) \(X\) is locally compact.

Semi-orderability and local connectedness. Recall that a family \(\mathcal{P}\) of subsets of a set \(X\) is called a partition of \(X\) if it is a pairwise disjoint cover of \(X\). If \(X\) is a topological space, we say that \(\mathcal{P}\) is a clopen partition of \(X\) if it consists of clopen (equivalently, open) subsets of \(X\). If \(\mathcal{P}\) is a clopen partition of \(X\), then \(X\) is, in fact, the topological sum \(\bigcup \mathcal{P}\) of the elements of the partition.

Let us mention that orderability of topological spaces is not invariant with respect to topological sums. Here is a very simple example. Take, for instance, \(X = \{0\} \cup (1, 2) \subset \mathbb{R}\). Then, \(X\) is the sum of two orderable spaces, but it is itself not orderable. Motivated by this, a topological space \(X\) was called semi-orderable in [11] if it has a clopen partition into two orderable spaces, or, equivalently, if it is the topological sum of two orderable spaces. Every orderable space is semi-orderable, while every semi-orderable space is suborderable. However, no one of these implications is invertible. As mentioned above, there are semi-orderable spaces which are not orderable. On the other hand, the Sorgenfrey line and the Michael line are examples of suborderable spaces which are not semi-orderable, see [11, Example 4.12].

The following theorem was proved in [17, Theorem 4.2].

**Theorem 8.8.** ([11]) A space \(X\) is semi-orderable if and only if it has a clopen partition consisting of orderable spaces.

According to Corollary 8.5, every locally connected space which has a continuous weak selection must be locally compact. Such a space must be also semi-orderable.

**Corollary 8.9.** A locally connected space is weakly orderable if and only if it is semi-orderable.

**Proof.** Every semi-orderable space is suborderable, hence weakly orderable as well. Suppose that \(X\) is a locally connected weakly orderable space. Then, the family \(\mathcal{C}\{x\}: x \in X\) of the connected components of \(X\) is a clopen partition of \(X\). By Corollary 8.2, each \(\mathcal{C}\{x\}, x \in X\), is orderable because it is connected and locally connected. Thus, \(X\) has a clopen partition by orderable sets and, by Theorem 8.8, it is semi-orderable. □

9. Orderability and compactness-like properties

Orderability and compactness. Since every weakly orderable compact space is orderable, we have the following consequence of Corollary 6.4.
Corollary 9.1. ([4,23]) A compact connected space is orderable if and only if it has a continuous weak selection.

Corollary 9.1 was generalized for arbitrary compact spaces by van Mill and Wattel in 1981 [24].

Theorem 9.2. ([24]) A compact space is orderable if and only if it has a continuous weak selection.

For a completely regular space $X$, let $\beta X$ be the Čech–Stone compactification of $X$. The following further result was obtained by van Mill and Wattel in 1984 [25].

Theorem 9.3. ([25]) For a completely regular space $X$, the following are equivalent:

(a) $X$ is suborderable.
(b) $X$ has a continuous weak selection $g$ such that for every $p \in \beta X \setminus X$, $g$ can be extended to a continuous weak selection for $X \cup \{p\}$.

Orderability and compactifications. In what follows, $X$ will be at least a completely regular space. The following result was obtained by Venkataraman, Rajagopalan and Soundararajan [30].

Proposition 9.4. ([30]) If $\beta X$ is orderable, then $X$ is normal and pseudocompact. Hence, $X$ is also countably compact.

On the other hand, we have the following result which is due to Eric van Douwen [3].

Proposition 9.5. ([3]) A countably compact space $X$ with a continuous weak selection is sequentially compact. Hence, $X^2$ is also sequentially compact.

According to Propositions 9.4 and 9.5, we have the following consequence.

Corollary 9.6. If $\beta X$ has a continuous weak selection, then $X^2$ is pseudocompact.

Finally, let us recall the Glicksberg’s theorem [9].

Theorem 9.7. ([9]) If $X^2$ is pseudocompact, then $\beta(X \times X) = \beta X \times \beta X$.

Combining all these results, we get the following consequence.

Corollary 9.8. ([1,26]) For a completely regular space $X$, the following are equivalent:

(a) $\beta X$ has a continuous weak selection.
(b) $\beta X$ is orderable.
(c) $X$ is suborderable and pseudocompact.
(d) $X$ is countably compact and has a continuous weak selection.
(e) $X$ is sequentially compact and has a continuous weak selection.
(f) $X^2$ is pseudocompact and $X$ has a continuous weak selection.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) follow by Theorem 9.2, Propositions 9.4 and 9.5, and Corollary 9.6. To see that (f) $\Rightarrow$ (a), suppose that $f$ is a continuous weak selection for $X$. Then, $g(x, y) = f([x, y]), x, y \in X$, defines a continuous symmetric map $g : X \times X \to X$ such that $g(x, y) \in [x, y], x, y \in X$. Hence, it can be extended to a continuous $\beta g : \beta(X \times X) \to \beta X$. However, $X \times X$ is pseudocompact and, by Theorem 9.7, $\beta(X \times X) = \beta X \times \beta X$. Thus, $\beta g : \beta X \times \beta X \to \beta X$ is also a symmetric map such that $\beta g(x, y) \in [x, y], x, y \in \beta X$. We may now define $h([x, y]) = \beta g(x, y), x, y \in \beta X$, which is a continuous weak selection for $\beta X$. 

The solution of the orderability problem of the Čech–Stone compactification was finally accomplished by García-Ferreira and Sanchis in 2004 [7].

Theorem 9.9. ([7]) If $X$ is pseudocompact and has a continuous weak selection, then $X \times Y$ is pseudocompact for every pseudocompact space $Y$. In particular, in this case, $X^2$ is also pseudocompact.
Semi-orderability and orderability. Related to suborderability of pseudocompact spaces, the following result was obtained by Artico, Marconi, Pelant, Rotter and Tkachenko [1, Corollary 1.20].

**Proposition 9.10.** ([1]) If \( X \) is a connected pseudocompact space which has a continuous weak selection, then \( X \) is orderable.

On the other hand, there are even sequentially compact spaces which are semi-orderable but not orderable.

**Example 9.11.** There exists a sequentially compact space which is the topological sum of two connected orderable spaces but is itself not orderable.

**Proof.** Let \( \mathbb{L}_+ \) be the closed long ray, i.e. \( \mathbb{L}_+ \) is the set \( \omega_1 \times \{0, 1\} \) endowed with the open interval topology generated by the lexicographical order on \( \omega_1 \times \{0, 1\} \). Take another disjoint copy of the closed long ray \( \mathbb{L}_- \), call it \( \mathbb{L}_- \), and endow it with the reverse lexicographical order on \( \omega_1 \times \{0, 1\} \). Next, let \( \mathbb{L} \) be the orderable space obtained from the topological sum \( \mathbb{L}_- \cup \mathbb{L}_+ \) by identifying the points \( (0, 0) \) from each of these closed long rays. The resulting space is usually refer to as the long line. It is well known that the long line \( \mathbb{L} \) is a connected sequentially compact space, but is not compact. In fact, \( \mathbb{L} \) has neither a minimal element nor a maximal one. Now, we can take \( X \) to be the topological sum \( \mathbb{L}_- \cup \mathbb{L}_+ \), where \( I = \{0, 1\} \). Then, \( X \) is a semi-orderable sequentially compact space which, by Lemma 6.9, is not orderable. Namely, if \( \preceq \) is a compatible linear order on \( X \) and \( (L, U) \) is a clopen \( \preceq \)-cut of \( X \), then \( (L, U) = (\mathbb{L}_-, I) \) or \( (L, U) = (I, \mathbb{L}_+) \). Hence, in either case \( (L, U) \) is neither a gap nor a jump. \( \square \)

In this regard, we have the following natural question.

**Question 9.** Does there exist a pseudocompact space which has a continuous weak selection but is not semi-orderable?

Related to semi-orderability and compactness-like properties, let us also mention the following recent result obtained in [11, Theorem 5.1].

**Theorem 9.12.** ([11]) A locally-compact paracompact space is semi-orderable if and only if it has a continuous weak selection.

To understand properly the difference between orderable and semi-orderable spaces, let us recall that an orderable space \( X \) is anti-compact orderable [11] if for every compatible order \( \preceq \) on \( X \), no clopen subset of \( X \) has a maximal element, see Proposition 8.4. If \( X \) is orderable with respect to a linear ordering \( \preceq \), then it is also orderable with respect to the reverse one \( (\preceq)^{-1} = \succeq \). Hence, any clopen subset of an anti-compact orderable space has no minimal and maximal elements. For instance, the real line \( \mathbb{R} \), also the long line \( \mathbb{L} \) in Example 9.11, are anti-compact orderable spaces.

The following result was obtained in [11, Theorem 6.3].

**Theorem 9.13.** ([11]) Let \( X \) be a semi-orderable space which is not orderable. Then, \( X \) is the topological sum of a nonempty compact orderable space and a nonempty anti-compact orderable one.

In case of locally compact spaces, there is a natural topological description of the possible compact and anti-compact “components” of semi-orderable spaces [11, Proposition 6.11].

**Proposition 9.14.** ([11]) Let \( X \) be a locally compact space which is the topological sum of a compact orderable space \( K \) and an anti-compact orderable space \( L \). Then,

\[
K = \{ x \in X : \mathrm{nect}(\mathcal{C}[x]) \neq \emptyset \} \quad \text{and} \quad L = \{ x \in X : \mathrm{nect}(\mathcal{C}[x]) = \emptyset \}.
\]

If \( X \) is a totally disconnected space, then \( \mathrm{nect}(\mathcal{C}[x]) \neq \emptyset \) for every \( x \in X \). Hence, according to Theorems 9.12 and 9.13 and Proposition 9.14, we have the following immediate consequence.

**Corollary 9.15.** ([11]) Every semi-orderable locally compact totally disconnected space \( X \) is orderable. In particular, a locally compact totally disconnected paracompact space \( X \) is orderable if and only if it has a continuous weak selection.

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