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Symmetric duality for a class of nondifferentiable multi-objective fractional variational problems ☆

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Abstract

We introduce a symmetric dual pair for a class of nondifferentiable multi-objective fractional variational problems. Weak, strong, converse and self duality relations are established under certain invexity assumptions. The paper includes extensions of previous symmetric duality results for multi-objective fractional variational problems obtained by Kim, Lee and Schaible [D.S. Kim, W.J. Lee, S. Schaible, Symmetric duality for invex multiobjective fractional variational problems, J. Math. Anal. Appl. 289 (2004) 505–521] and symmetric duality results for the static case obtained by Yang, Wang and Deng [X.M. Yang, S.Y. Wang, X.T. Deng, Symmetric duality for a class of multiobjective fractional programming problems, J. Math. Anal. Appl. 274 (2002) 279–295] to the dynamic case.

Keywords: Nonlinear programming; Symmetric duality; Variational problems; Generalized convexity

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1. Introduction

The classical dual in linear programming is symmetric in the sense that the dual of the dual is the original linear program. Such symmetry is usually not found in duality concept for nonlinear programming [11], not even in quadratic programming which is symmetric [3,5]. Extending these results to general convex programming, Dantzig et al. [4] formulated a symmetric dual pair and established weak and strong duality relations for the problem. Mond and Weir [19] established symmetric duality results under generalized convexity for a new type of dual pair. Weir and Mond [27] introduced two distinct symmetric duals for multi-objective programs. Under additional assumptions the multi-objective programs are shown to be self-dual.

Mond and Hanson [16] first extended the symmetric duality results of [15] to variational problems by introducing continuous analogues of the earlier concepts. Smart and Mond [25] extended symmetric duality results to variational problems by employing a continuous version of invexity introduced by Hanson [6], see also [7–9,14,20,21].

Mond and Schechter [18] studied nondifferentiable symmetric duality in which the objective function contains a support function. Following Mond and Schechter [18], Yang et al. [29] presented a pair of symmetric dual nonlinear fractional programming problems and established duality theorems under pseudo-convexity/pseudo-concavity assumptions on the kernel function. Later, Yang et al. [28] formulated a pair of nondifferentiable multi-objective fractional programming problems and proved duality theorems under a generalized invexity assumption. The results obtained by Yang et al. [28,29] include, as a special case duality results for multi-objective programs given by Weir and Mond [27] and for single objective fractional programs discussed by Chandra et al. [1], Mond et al. [17] as well as Mond and Schechter [18].

It is well known due to the works of Schaible [22–24] that duality results for convex programming do not apply to fractional programs in general. Duality concepts for such programs had to be defined separately [12,13,22–24,26,28,29]. Most duals in fractional programming are not symmetric [12,13,22–24,26,28,29]. For the multi-objective case of a static nonlinear fractional program symmetric duality was introduced by Weir [26] as an extension of an earlier work of Weir and Mond [27]. Weir [26] also established weak and strong duality results under convexity assumptions.

Very recently, Kim et al. [10] introduced a symmetric dual for multi-objective fractional variational problems which is different from the one proposed by Chen [2]. Kim et al. [10] established weak, strong, converse and self-duality theorems under invexity assumptions.

In this paper, we focus on symmetric duality for a class of nondifferentiable fractional variational problems. We introduced a symmetric dual pair for a class of nondifferentiable multi-objective fractional variational problems. We establish weak, strong, converse and self duality theorems under certain invexity assumptions. The results obtained in this paper extend the very recent results established by Kim et al. [10] to the nondifferentiable case and also extend an earlier work of Yang et al. [28] to the dynamic case. Moreover, these results also include, as special cases, the symmetric duality results of Yang et al. [29], Mond and Schechter [18], Weir and Mond [27] and others.

2. Notations and definitions

For $x, y \in R^n$, by $x \leqq y$ we mean $x_i \leqq y_i$ for all i, $x \leqslant y$ means $x_i \leqq y_i$ for all i and $x_j < y_j$ for at least one j, $1 \leqq j \leqq n$. By $x < y$ we mean $x_i < y_i$ for all i and by $x \nleqslant y$ we mean the negation of $x \leqslant y$.

Let $I=[a,b]$ be a real interval, $f: I \times R^n \times R^n \times R^m \times R^m \to R^k$ and $g: I \times R^n \times R^n \times R^m \times R^m \to R^k$. Consider the vector-valued function $f(t,x,\dot{x},y,\dot{y})$, where $t \in I$, x and y are functions of t with $x(t) \in R^n$ and $y(t) \in R^m$ and $\dot{x}$ and $\dot{y}$ denote the derivatives of x and y, respectively, with respect to t.

Assume that f has continuous fourth-order partial derivatives with respect to $x, \dot{x}, y$ and $\dot{y}$. Let f_x and $f_{\dot{x}}$ denote the $k \times n$ matrices of first-order partial derivatives with respect to x and $\dot{x}$; i.e.,

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n},\right) \quad \text{and} \quad f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n},\right), \quad i = 1,2,\ldots,k.$$

Similarly, f_y and $f_{\dot{y}}$ denote the $k \times m$ matrices of first-order partial derivatives with respect to y and $\dot{y}$.

For a multi-objective fractional variational problem:

$$\begin{aligned}
\text{(FVP)} \quad \text{Minimize} \quad & \frac{\int_a^b f(t,x(t),\dot{x}(t))\,dt}{\int_a^b g(t,x(t),\dot{x}(t))\,dt} \\
& = \left(\frac{\int_a^b f_1(t,x(t),\dot{x}(t))dt}{\int_a^b g_1(t,x(t),\dot{x}(t))\,dt}, \ldots, \frac{\int_a^b f_k(t,x(t),\dot{x}(t))\,dt}{\int_a^b g_k(t,x(t),\dot{x}(t))\,dt}\right) \\
\text{subject to} \quad & x(a)=\alpha, \quad x(b)=\beta, \\
& h\big(t,x(t),\dot{x}(t)\big) \leqq 0, \\
\text{where } & h: I \times R^n \times R^n \to R^l.
\end{aligned}$$

Assume that $g_i(t,x,\dot{x}) > 0$ and $f_i(t,x,\dot{x}) \geqq 0$ for all $i=1,2,\ldots,k$. Let X denote the set of all feasible solutions of (FVP).

Definition 2.1. A point $x^* \in X$ is said to be an *efficient* (Pareto optimal) solution of (FVP) if for all $x \in X$,

$$\frac{\int_a^b f(t,x,\dot{x})\,dt}{\int_a^b g(t,x,\dot{x})\,dt} \nleq \frac{\int_a^b f(t,x^*,\dot{x}^*)\,dt}{\int_a^b g(t,x^*,\dot{x}^*)\,dt}.$$

Definition 2.2. A point $x^* \in X$ is said to be a *properly efficient* solution of (FVP) if it is efficient for (FVP) and if there exists a scalar $M>0$ such that, for all $i \in \{1,2,\ldots,k\}$,

$$\frac{\int_a^b f_i(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_i(t,x^*,\dot{x}^*)\,dt} - \frac{\int_a^b f_i(t,x,\dot{x})\,dt}{\int_a^b g_i(t,x,\dot{x})\,dt} \leqq M\left(\frac{\int_a^b f_j(t,x,\dot{x})\,dt}{\int_a^b g_j(t,x,\dot{x})\,dt} - \frac{\int_a^b f_j(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_j(t,x^*,\dot{x}^*)\,dt}\right)$$

for some j, such that

$$\frac{\int_a^b f_j(t,x,\dot{x})\,dt}{\int_a^b g_j(t,x,\dot{x})\,dt} > \frac{\int_a^b f_j(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_j(t,x^*,\dot{x}^*)\,dt}$$

whenever $x \in X$, and

$$\frac{\int_a^b f_i(t,x,\dot{x})\,dt}{\int_a^b g_i(t,x,\dot{x})\,dt} < \frac{\int_a^b f_i(t,x^*,\dot{x}^*)\,dt}{\int_a^b g_i(t,x^*,\dot{x}^*)\,dt}.$$

Definition 2.3. A point $x^* \in X$ is said to be a *weakly efficient* solution of (FVP) if there exists no other feasible point x for which

$$\frac{\int_a^b f(t, x^*, \dot{x}^*)\, dt}{\int_a^b g(t, x^*, \dot{x}^*)\, dt} > \frac{\int_a^b f(t, x, \dot{x})\, dt}{\int_a^b g(t, x, \dot{x})\, dt}.$$

Now we recall the invexity for continuous case as follows:

Definition 2.4. The vector of functionals $\int_a^b f = (\int_a^b f_1, \ldots, \int_a^b f_k)$ is said to be *invex* in x and $\dot{x}$ if for each $y : [a, b] \to R^m$, with $\dot{y}$ piecewise smooth, there exists a function $\eta : [a, b] \times R^n \times R^n \times R^n \times R^n \to R^n$ such that $\forall i = 1, 2, \ldots, k$,

$$\begin{aligned}&\int_a^b \bigl\{ f_i(t, x, \dot{x}, y, \dot{y}) - f_i(t, u, \dot{u}, y, \dot{y}) \bigr\}\, dt \\ &\quad \geqq \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[f_i(t, u, \dot{u}, y, \dot{y}) - \frac{d}{dt} f_{i\dot{x}}(t, u, \dot{u}, y, \dot{y}) \right] dt\end{aligned}$$

for all $x : [a, b] \to R^n$, $u : [a, b] \to R^n$, where $(\dot{x}(t), \dot{u}(t))$ is piecewise smooth on $[a, b]$.

Definition 2.5. The vector of functionals $-\int_a^b f$ is said to be *invex* in y and $\dot{y}$ if for each $x : [a, b] \to R^n$, with $\dot{x}$ piecewise smooth, there exists a function $\xi : [a, b] \times R^m \times R^m \times R^m \times R^m \to R^m$ such that $\forall i = 1, 2, \ldots, k$,

$$\begin{aligned}&-\int_a^b \bigl\{ f_i(t, x, \dot{x}, v, \dot{v}) - f_i(t, x, \dot{x}, y, \dot{y}) \bigr\}\, dt \\ &\quad \geqq -\int_a^b \xi(t, v, \dot{v}, y, \dot{y})^T \left[f_{iy}(t, x, \dot{x}, y, \dot{y}) - \frac{d}{dt} f_{i\dot{y}}(t, x, \dot{x}, y, \dot{y}) \right] dt\end{aligned}$$

for all $v : [a, b] \to R^m$, $y : [a, b] \to R^m$, where $(\dot{v}(t), \dot{y}(t))$ is piecewise smooth on $[a, b]$.

Definition 2.6. The vector of functionals $\int_a^b f = (\int_a^b f_1, \ldots, \int_a^b f_k)$ is said to be *pseudo invex* in x and $\dot{x}$ if for each $y : [a, b] \to R^m$, with $\dot{y}$ piecewise smooth, there exists a function $\eta : [a, b] \times R^n \times R^n \times R^n \times R^n \to R^n$ such that $\forall i = 1, 2, \ldots, k$,

$$\begin{aligned}&\int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[f_i(t, u, \dot{u}, y, \dot{y}) - \frac{d}{dt} f_{i\dot{x}}(t, u, \dot{u}, y, \dot{y}) \right] dt \geqq 0 \\ &\quad \Rightarrow \quad \int_a^b \bigl\{ f_i(t, x, \dot{x}, y, \dot{y}) - f_i(t, u, \dot{u}, y, \dot{y}) \bigr\}\, dt \geqq 0,\end{aligned}$$

for all $x : [a, b] \to R^n$, $u : [a, b] \to R^n$, where $(\dot{x}(t), \dot{u}(t))$ is piecewise smooth on $[a, b]$.

Definition 2.7. The vector of functionals $-\int_a^b f$ is said to be *pseudo invex* in y and $\dot{y}$ if for each $x:[a,b]\to R^n$, with $\dot{x}$ piecewise smooth, there exists a function $\xi:[a,b]\times R^m\times R^m\times R^m\times R^m\to R^m$ such that $\forall i=1,2,\dots,k$,

$$-\int_a^b \xi(t,v,\dot{v},y,\dot{y})^T\left[f_{iy}(t,x,\dot{x},y,\dot{y})-\frac{d}{dt}f_{i\dot{y}}(t,x,\dot{x},y,\dot{y})\right]dt \geqq 0$$

$$\Rightarrow\quad -\int_a^b \left\{f_i(t,x,\dot{x},v,\dot{v})-f_i(t,x,\dot{x},y,\dot{y})\right\}dt \geqq 0,$$

for all $v:[a,b]\to R^m$, $y:[a,b]\to R^m$, where $(\dot{v}(t),\dot{y}(t))$ is piecewise smooth on $[a,b]$.

In the sequel, we will write $\eta(x,u)$ for $\eta(t,x,\dot{x},u,\dot{u})$ and $\xi(v,y)$ for $\xi(t,v,\dot{v},y,\dot{y})$.

We consider the problem of finding functions $x:[a,b]\to R^n$ and $y:[a,b]\to R^m$, where $(\dot{x}(t),\dot{y}(t))$ is piecewise smooth on $[a,b]$, to solve the following pair o symmetric dual multiobjective nondifferentiable fractional variational problems introduced as follows:

$$\text{(MNFP)}\quad \text{Min}\quad \frac{\int_a^b\{f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))+s(x(t)|C)-y(t)^Tz\}\,dt}{\int_a^b\{g(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-s(x(t)|E)+y(t)^Tr\}\,dt}$$

$$=\left(\frac{\int_a^b\{f_1(t,x(t),\dot{x}(t),y(t),\dot{y}(t))+s(x(t)|C_1)-y(t)^Tz_1\}\,dt}{\int_a^b\{g_1(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-s(x(t)|E_1)+y(t)^Tr_1\}\,dt},\dots\right.$$
$$\left.\frac{\int_a^b\{f_k(t,x(t),\dot{x}(t),y(t),\dot{y}(t))+s(x(t)|C_k)-y(t)^Tz_k\}\,dt}{\int_a^b\{g_k(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-s(x(t)|E_k)+y(t)^Tr_k\}\,dt}\right)$$

subject to
$$x(a)=0=x(b),\quad y(a)=0=y(b),$$
$$\dot{x}(a)=0=\dot{x}(b),\quad \dot{y}(a)=0=\dot{y}(b),$$
$$\sum_{i=1}^k\tau_i\big\{[f_{iy}-Df_{i\dot{y}}-z_i]G_i(x,y)$$
$$-[g_{iy}-Dg_{i\dot{y}}+r_i]F_i(x,y)\big\}\leqq 0,$$
$$\int_a^b y(t)^T\sum_{i=1}^k\tau_i\big\{[f_{iy}-Df_{i\dot{y}}-z_i]G_i(x,y)$$
$$-[g_{iy}-Dg_{i\dot{y}}+r_i]F_i(x,y)\big\}\,dt\geqq 0,$$
$$\tau>0,\quad \tau^Te=1,\quad t\in I,$$
$$z_i\in D_i,\quad r_{\rm i}\in H_i,\quad i=1,2,\dots,k.$$

$$\text{(MNFD)}\quad \text{Max}\quad \frac{\int_a^b\{f(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-s(v(t)|D)+u(t)^Tw\}\,dt}{\int_a^b\{g(t,u(t),\dot{u}(t),v(t),\dot{v}(t))+s(v(t)|H)-u(t)^Ts\}\,dt}$$

$$=\left(\frac{\int_a^b\{f_1(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-s(v(t)|D_1)+u(t)^Tw_1\}\,dt}{\int_a^b\{g_1(t,u(t),\dot{u}(t),v(t),\dot{v}(t))+s(v(t)|H_1)-u(t)^Ts_1\}\,dt},\dots\right.$$

$$\left.\frac{\int_a^b\{f_k(t,u(t),\dot u(t),v(t),\dot v(t))-s(v(t)|D_k)+u(t)^T w_k\}\,dt}{\int_a^b\{g_k(t,u(t),\dot u(t),v(t),\dot v(t))+s(v(t)|H_k)-u(t)^T s_k\}\,dt}\right)$$

subject to
$$u(a)=0=u(b),\quad v(a)=0=v(b),$$
$$\dot u(a)=0=\dot u(b),\quad \dot v(a)=0=\dot v(b),$$
$$\sum_{i=1}^{k}\tau_i\big\{[f_{iu}-Df_{i\dot u}+w_i]G_i^*(u,v)$$
$$-[g_{iu}-Dg_{i\dot u}-s_i]F_i^*(u,v)\big\}\geqq 0,$$
$$\int_a^b u(t)^T\sum_{i=1}^{k}\tau_i\big\{[f_{iu}-Df_{i\dot u}+w_i]G_i^*(u,v)$$
$$-[g_{iu}-Dg_{i\dot u}-s_i]F_i^*(u,v)\big\}\leqq 0,$$
$$\tau>0,\quad \tau^T e=1,\quad t\in I,$$
$$w_i\in C_i,\quad s_i\in E_i,\quad i=1,2,\dots,k,$$

where $f_i: I\times R^n\times R^n\times R^m\times R^m\to R_+$ and $g_i: I\times R^n\times R^n\times R^m\times R^m\to R_+\backslash\{0\}$ are continuously differentiable functions and

$$F_i(x,y)=\int_a^b\big\{f_i(t,x,\dot x,y,\dot y)+s\big(x(t)|C_i\big)-y(t)^T z_i\big\}\,dt;$$

$$G_i(x,y)=\int_a^b\big\{g_i(t,x,\dot x,y,\dot y)-s\big(x(t)|E_i\big)+y(t)^T r_i\big\}\,dt;$$

$$F_i^*(u,v)=\int_a^b\big\{f_i(t,u,\dot u,v,\dot v)-s\big(v(t)|D_i\big)+u(t)^T w_i\big\}\,dt;$$

and

$$G_i^*(u,v)=\int_a^b\big\{g_i(t,u,\dot u,v,\dot v)+s\big(v(t)|H_i\big)-u(t)^T s_i\big\}\,dt.$$

In the above problems (MNFP) and (MNFD), the numerators are nonnegative and denominators are positive; the differential operator D is given by

$$y=Dx\quad\Leftrightarrow\quad x(t)=\alpha+\int_a^t y(s)\,ds,$$

and $x(a)=\alpha$, $x(b)=\beta$ are given boundary values; thus $D=d/dt$ except at discontinuities. Let $f_x=f_x(t,x(t),\dot x(t),y(t),\dot y(t))$, $f_{\dot x}=f_{\dot x}(t,x(t),\dot x(t),y(t),\dot y(t))$, etc.

All the above statements for F_i, G_i, F_i^* and G_i^* will be assumed to hold for subsequent results. It is to be noted that

$$Df_{i\dot y}=f_{i\dot y t}+f_{i\dot y y}\dot y+f_{i\dot y\dot y}\ddot y+f_{i\dot y x}\dot x+f_{i\dot y\dot x}\ddot x$$

and consequently

$$\frac{\partial}{\partial y} Df_{i\dot y} = Df_{i\dot y y}, \quad \frac{\partial}{\partial \dot y} Df_{i\dot y} = Df_{i\dot y\dot y} + f_{i\dot y y}, \quad \frac{\partial}{\partial \ddot y} Df_{i\dot y} = f_{i\dot y\dot y},$$
$$\frac{\partial}{\partial x} Df_{i\dot y} = Df_{i\dot y x}, \quad \frac{\partial}{\partial \dot x} Df_{i\dot y} = Df_{i\dot y\dot x} + f_{i\dot y x}, \quad \frac{\partial}{\partial \ddot x} Df_{i\dot y} = f_{i\dot y\dot x}.$$

In order to simplify the notations we introduce

$$p_i = \frac{F_i(x,y)}{G_i(x,y)} = \frac{\int_a^b \{f_i(t,x,\dot x,y,\dot y) + s(x(t)|C_i) - y(t)^T z_i\}\,dt}{\int_a^b \{g_i(t,x,\dot x,y,\dot y) - s(x(t)|E_i) + y(t)^T r_i\}\,dt}$$

and

$$q_i = \frac{F_i^*(u,v)}{G_i^*(u,v)} = \frac{\int_a^b \{f_i(t,u,\dot u,v,\dot v) - s(v(t)|D_i) + u(t)^T w_i\}\,dt}{\int_a^b \{g_i(t,u,\dot u,v,\dot v) + s(v(t)|H_i) - u(t)^T s_i\}\,dt}$$

and express problems (MNFP) and (MNFD) equivalently as follows:

(EMSP) Min $p = (p_1,\ldots,p_k)^T$

subject to
$$x(a) = 0 = x(b), \quad y(a) = 0 = y(b), \tag{1}$$
$$\dot x(a) = 0 = \dot x(b), \quad \dot y(a) = 0 = \dot y(b), \tag{2}$$
$$\int_a^b \bigl\{f_i(t,x,\dot x,y,\dot y) + s(x|C_i) - y^T z_i\bigr\}\,dt - p_i \int_a^b \bigl\{g_i(t,x,\dot x,y,\dot y) - s(x|E_i) + y^T r_i\bigr\}\,dt = 0; \tag{3}$$
$$\sum_{i=1}^k \tau_i \bigl\{[f_{iy} - Df_{i\dot y} - z_i] - p_i[g_{iy} - Dg_{i\dot y} + r_i]\bigr\} \leqq 0, \quad t \in I, \tag{4}$$
$$\int_a^b y(t)^T \sum_{i=1}^k \tau_i \bigl\{[f_{iy} - Df_{i\dot y} - z_i] - p_i[g_{iy} - Dg_{i\dot y} + r_i]\bigr\} \geqq 0, \quad t \in I, \tag{5}$$
$$\tau > 0, \quad \tau^T e = 1, \quad t \in I, \tag{6}$$
$$z_i \in D_i, \quad r_{\mathrm{i}} \in H_i, \quad i = 1,2,\ldots,k. \tag{7}$$

(EMSD) Max $q = (q_1,\ldots,q_k)^T$

subject to
$$u(a) = 0 = u(b), \quad v(a) = 0 = v(b), \tag{8}$$
$$\dot u(a) = 0 = \dot u(b), \quad \dot v(a) = 0 = \dot v(b), \tag{9}$$
$$\int_a^b \bigl\{f_i(t,u,\dot u,v,\dot v) - s(v|D_i) + u^T w_i\bigr\}\,dt$$

$$-q_i \int_a^b \{g_i(t,u,\dot u,v,\dot v)+s(v|H_i)-u^T s_i\}\,dt=0; \tag{10}$$

$$\sum_{i=1}^k \tau_i\{[f_{iu}-Df_{i\dot u}+w_i]-q_i[g_{iu}-Dg_{i\dot u}-s_i]\} \geqq 0, \quad t\in I, \tag{11}$$

$$\int_a^b u(t)^T \sum_{i=1}^k \tau_i\{[f_{iu}-Df_{i\dot u}+w_i]-q_i[g_{iu}-Dg_{i\dot u}-s_i]\} \leqq 0, \quad t\in I, \tag{12}$$

$$\tau>0, \quad \tau^T e=1, \quad t\in I, \tag{13}$$

$$w_i\in C_i, \quad s_{\mathrm{i}}\in E_i, \quad i=1,2,\ldots,k. \tag{14}$$

In the above problems (EMSP) and (EMSD), it is to be noted that p and q are also nonnegative.

3. Duality theorems

In this section, we state duality theorems for problems (EMSP) and (EMSD) which lead to corresponding relations between (MNFP) and (MNFD). We establish weak, strong, converse and self duality relations between (EMSP) and (EMSD).

Theorem 3.1 *(Weak duality). Let $(x(t),y(t),p,\tau,z_1,z_2,\ldots,z_k,r_1,r_2,\ldots,r_k)$ be feasible for* (EMSP) *and let $(u(t),v(t),q,\tau,w_1,w_2,\ldots,w_k,s_1,s_2,\ldots,s_k)$ be feasible for* (EMSD). *Assume that $\int_a^b(f_i+\cdot^T w_i)\,dt$ and $-\int_a^b(g_i-\cdot^T s_i)\,dt$ are invex in x and $\dot x$ with respect to $\eta(x,u)$, and $-\int_a^b(f_i-\cdot^T z_i)\,dt$ and $\int_a^b(g_i+\cdot^T r_i)\,dt$ are invex in y and $\dot y$, with respect to $\xi(v,y)$, and $\eta(x,u)+u(t)\geqq 0$ and $\xi(v,y)+y(t)\geqq 0$, $\forall t\in I$, except possibly at corners of $(\dot x(t),\dot y(t))$ or $(\dot u(t),\dot v(t))$. Then one has $p\nleq q$.*

Proof. Since $\int_a^b(f_i+\cdot^T w_i)\,dt$ and $-\int_a^b(g_i-\cdot^T s_i)\,dt$ are invex in x and $\dot x$ with respect to $\eta(x,u)$, we have

$$\begin{aligned}
&\int_a^b \big[\{f_i(t,x(t),\dot x(t),v(t),\dot v(t))+x^T w_i\}\\
&\qquad - q_i\{g_i(t,x(t),\dot x(t),v(t),\dot v(t))-x^T s_i\}\big]\,dt\\
&\qquad - \int_a^b \big[\{f_i(t,u(t),\dot u(t),v(t),\dot v(t))+u^T w_i\}\\
&\qquad - q_i\{g_i(t,u(t),\dot u(t),v(t),\dot v(t))-u^T s_i\}\big]\,dt\\
&\quad \geqq \int_a^b \eta(x,u)^T\big[\{f_{ix}(t,u(t),\dot u(t),v(t),\dot v(t))+w_i\}\\
&\qquad - q_i\{g_{ix}(t,u(t),\dot u(t),v(t),\dot v(t))-s_i\}
\end{aligned}$$

$$- D\big\{f_{i\dot{x}}\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)+w_i\big\} - q_i\big\{g_{i\dot{x}}\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)-s_i\big\}\big]\,dt$$
$$= \int_a^b \eta(x,u)^T\big[\big\{\big(f_{ix}\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)+w_i\big)$$
$$- D\big(f_{i\dot{x}}\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)+w_i\big)\big\}$$
$$- q_i\big\{\big(g_{ix}\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)-s_i\big) - D\big(g_{i\dot{x}}\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)-s_i\big)\big\}\big]\,dt.$$

From (6), (11) and (12) with $\eta(x,u)+u(t) \geqq 0$, we obtain

$$\sum_{i=1}^k \tau_i \int_a^b \big[\big\{f_i\big(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\big)+x^T w_i\big\}$$
$$- q_i\big\{g_i\big(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\big)-x^T s_i\big\}\big]\,dt$$
$$\geqq \sum_{i=1}^k \tau_i \int_a^b \big[\big\{f_i\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)+u^T w_i\big\}$$
$$- q_i\big\{g_i\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)-u^T s_i\big\}\big]\,dt. \tag{15}$$

Since $x^T s_i \leqslant s(x|E_i)$, $s_i \in E_i$, and $x^T w_i \leqslant s(x|C_i)$, $w_i \in C_i$, (15) can be written as

$$\sum_{i=1}^k \tau_i \int_a^b \big[\big\{f_i\big(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\big)+s(x|C_i)\big\}$$
$$- q_i\big\{g_i\big(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\big)-s(x|E_i)\big\}\big]\,dt$$
$$\geqq \sum_{i=1}^k \tau_i \int_a^b \big[\big\{f_i\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)+u^T w_i\big\}$$
$$- q_i\big\{g_i\big(t,u(t),\dot{u}(t),v(t),\dot{v}(t)\big)-u^T s_i\big\}\big]\,dt. \tag{16}$$

By the invexity of $-\int_a^b (f_i - \cdot^T z_i)\,dt$ and $\int_a^b (g_i + \cdot^T r_i)\,dt$ are invex in y and $\dot{y}$, with respect to $\xi(v,y)$, for fixed x, we have

$$\int_a^b \big[\big\{f_i\big(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\big)-v^T z_i\big\} - p_i\big\{g_i\big(t,x(t),\dot{x}(t),v(t),\dot{v}(t)\big)+v^T r_i\big\}\big]\,dt$$
$$- \int_a^b \big[\big\{f_i\big(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\big)-y^T z_i\big\}$$
$$- p_i\big\{g_i\big(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\big)+y^T r_i\big\}\big]\,dt$$
$$\leqq \int_a^b \xi(v,y)^T\big[\big\{\big(f_{iy}\big(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\big)-z_i\big)$$
$$- D\big(f_{i\dot{y}}\big(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\big)-z_i\big)\big\}$$
$$- p_i\big\{\big(g_{iy}\big(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\big)+r_i\big) - D\big(g_{i\dot{y}}\big(t,x(t),\dot{x}(t),y(t),\dot{y}(t)\big)+r_i\big)\big\}\big]\,dt.$$

From (4), (5) and (13) along with $\xi(v, y) + y(t) \geqq 0$, $\forall t \in I$, we obtain

$$\sum_{i=1}^{k} \tau_i \int_a^b \Big[\big\{ f_i\big(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\big) - v^T z_i\big\} - p_i \big\{ g_i\big(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\big) + v^T r_i\big\}\Big]\,dt$$
$$\leqq \sum_{i=1}^{k} \tau_i \int_a^b \Big[\big\{ f_i\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) - y^T z_i\big\} - p_i \big\{ g_i\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) + y^T r_i\big\}\Big]\,dt. \tag{17}$$

Since $v^T r_i \leqslant s(v|H_i)$, $r_i \in H_i$, and $v^T z_i \leqslant s(v|D_i)$, $z_i \in D_i$, (17) can be written as

$$\sum_{i=1}^{k} \tau_i \int_a^b \Big[\big\{ f_i\big(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\big) - s(v|D_i)\big\} - p_i \big\{ g_i\big(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\big) + s(v|H_i)\big\}\Big]\,dt$$
$$\leqq \sum_{i=1}^{k} \tau_i \int_a^b \Big[\big\{ f_i\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) - y^T z_i\big\} - p_i \big\{ g_i\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) + y^T r_i\big\}\Big]\,dt. \tag{18}$$

From (16) and (18), we get

$$\sum_{i=1}^{k} \tau_i \int_a^b (p_i - q_i) g_i(t, x, \dot{x}, v, \dot{v})\,dt$$
$$\geqq \sum_{i=1}^{k} \tau_i \Bigg[\int_a^b \big\{ f_i(t, u, \dot{u}, v, \dot{v}) - s(v|D_i) + u^T w_i \big\}\,dt - q_i \int_a^b \big\{ g_i(t, u, \dot{u}, v, \dot{v}) + s(v|H_i) - u^T s_i \big\}\,dt \Bigg]$$
$$- \sum_{i=1}^{k} \tau_i \Bigg[\int_a^b \big\{ f_i(t, x, \dot{x}, y, \dot{y}) + s(x|C_i) - y^T z_i \big\}\,dt - p_i \int_a^b \big\{ g_i(t, x, \dot{x}, y, \dot{y}) - s(x|E_i) + y^T r_i \big\}\,dt \Bigg]. \tag{19}$$

From (3) and (10), (19) yields

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t, x, \dot{x}, v, \dot{v})\,dt \geqq 0. \tag{20}$$

If for some i, $p_i < q_i$ and $\forall j \neq i$, $p_i \leqq q_i$, then $\int_a^b g_i(t, x, \dot{x}, v, \dot{v})\,dt > 0$, $i = 1, 2, \ldots, k$, implies that

$$\sum_{i=1}^{k} \tau_i (p_i - q_i) \int_a^b g_i(t, x, \dot{x}, v, \dot{v})\,dt < 0,$$

which contradicts (20). Hence $p \nleq q$. □

Theorem 3.2 *(Weak duality). Let* $(x(t), y(t), p, \tau, z_1, z_2, \ldots, z_k, r_1, r_2, \ldots, r_k)$ *be feasible for* (EMSP) *and let* $(u(t), v(t), q, \tau, w_1, w_2, \ldots, w_k, s_1, s_2, \ldots, s_k)$ *be feasible for* (EMSD). *Assume that* $\sum_{i=1}^{k} \tau_i \int_a^b \{(f_i + \cdot^T w_i) - q_i(g_i - \cdot^T s_i)\}\,dt$ *is pseudo-invex in* x *and* $\dot{x}$ *with respect to* $\eta(x, u)$, *and* $-\sum_{i=1}^{k} \tau_i \int_a^b \{(f_i - \cdot^T z_i) - p_i(g_i + \cdot^T r_i)\}\,dt$ *is pseudo-invex in* y *and* $\dot{y}$, *with respect to* $\xi(v, y)$, *with* $\eta(x, u) + u(t) \geqq 0$ *and* $\xi(v, y) + y(t) \geqq 0$, $\forall t \in I$, *except possibly at corners of* $(\dot{x}(t), \dot{y}(t))$ *or* $(\dot{u}(t), \dot{v}(t))$. *Then one has* $p \nleq q$.

Proof. Using the condition $\eta(x, u) + u(t) \geqq 0$, $\forall t \in I$, and duality constraint (12), we get

$$\int_a^b \eta(x, u)^T \sum_{i=1}^{k} \tau_i \bigl\{[f_{iu} - Df_{i\dot{u}} + w_i] - q_i[g_{iu} - Dg_{i\dot{u}} - s_i]\bigr\}\,dt$$
$$= -\int_a^b u(t)^T \sum_{i=1}^{k} \tau_i \bigl\{[f_{iu} - Df_{i\dot{u}} + w_i] - q_i[g_{iu} - Dg_{i\dot{u}} - s_i]\bigr\}\,dt \geqq 0.$$

Since $\sum_{i=1}^{k} \tau_i \int_a^b \{(f_i + \cdot^T w_i) - q_i(g_i - \cdot^T s_i)\}\,dt$ is pseudo-invex with respect to $\eta(x, u)$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{k} \tau_i \int_a^b \bigl[\bigl\{f_i\bigl(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\bigr) + x^T w_i\bigr\} \\
&\qquad - q_i\bigl\{g_i\bigl(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\bigr) - x^T s_i\bigr\}\bigr]\,dt \\
&\quad \geqq \sum_{i=1}^{k} \tau_i \int_a^b \bigl[\bigl\{f_i\bigl(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\bigr) + u^T w_i\bigr\} \\
&\qquad - q_i\bigl\{g_i\bigl(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\bigr) - u^T s_i\bigr\}\bigr]\,dt.
\end{aligned} \tag{21}$$

Since $x^T s_i \leqslant s(x|E_i)$, $s_i \in E_i$, and $x^T w_i \leqslant s(x|C_i)$, $w_i \in C_i$, (21) can be written as

$$\begin{aligned}
&\sum_{i=1}^{k} \tau_i \int_a^b \bigl[\bigl\{f_i\bigl(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\bigr) + s(x|C_i)\bigr\} \\
&\qquad - q_i\bigl\{g_i\bigl(t, x(t), \dot{x}(t), v(t), \dot{v}(t)\bigr) - s(x|E_i)\bigr\}\bigr]\,dt \\
&\quad \geqq \sum_{i=1}^{k} \tau_i \int_a^b \bigl[\bigl\{f_i\bigl(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\bigr) + u^T w_i\bigr\} \\
&\qquad - q_i\bigl\{g_i\bigl(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\bigr) - u^T s_i\bigr\}\bigr]\,dt.
\end{aligned} \tag{22}$$

By $\xi(v, y) + y(t) \geqq 0$, $\forall t \in I$, and primal constraint (5), we get

$$\int_a^b \xi(x,u)^T \sum_{i=1}^k \tau_i \{[f_{iy} - Df_{i\dot{y}} - z_i] - p_i[g_{iy} - Dg_{i\dot{y}} + r_i]\}\,dt$$
$$= -\int_a^b y(t)^T \sum_{i=1}^k \tau_i \{[f_{iy} - Df_{i\dot{y}} - z_i] - p_i[g_{iy} - Dg_{i\dot{y}} + r_i]\}\,dt \leqq 0.$$

By pseudo-invexity of $-\sum_{i=1}^k \tau_i \int_a^b \{(f_i - \cdot^T z_i) - p_i(g_i + \cdot^T r_i)\}\,dt$ with respect to $\xi(v, y)$, we get

$$\sum_{i=1}^k \tau_i \int_a^b \big[\{f_i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) - v^T z_i\}$$
$$- p_i\{g_i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) + v^T r_i\}\big]\,dt$$
$$\leqq \sum_{i=1}^k \tau_i \int_a^b \big[\{f_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - y^T z_i\}$$
$$- p_i\{g_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) + y^T r_i\}\big]\,dt. \tag{23}$$

Since $v^T r_i \leqslant s(v|H_i)$, $r_i \in H_i$, and $v^T z_i \leqslant s(v|D_i)$, $z_i \in D_i$, (23) can be written as

$$\sum_{i=1}^k \tau_i \int_a^b \big[\{f_i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) - s(v|D_i)\}$$
$$- p_i\{g_i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) + s(v|H_i)\}\big]\,dt$$
$$\leqq \sum_{i=1}^k \tau_i \int_a^b \big[\{f_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - y^T z_i\}$$
$$- p_i\{g_i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) + y^T r_i\}\big]\,dt. \tag{24}$$

From (22) and (24), we get

$$\sum_{i=1}^k \tau_i \int_a^b (p_i - q_i) g_i(t, x, \dot{x}, v, \dot{v})\,dt$$
$$\geqq \sum_{i=1}^k \tau_i \Bigg[\int_a^b \{f_i(t, u, \dot{u}, v, \dot{v}) - s(v|D_i) + u^T w_i\}\,dt$$
$$- q_i \int_a^b \{g_i(t, u, \dot{u}, v, \dot{v}) + s(v|H_i) - u^T s_i\}\,dt\Bigg]$$
$$- \sum_{i=1}^k \tau_i \Bigg[\int_a^b \{f_i(t, x, \dot{x}, y, \dot{y}) + s(x|C_i) - y^T z_i\}\,dt$$

$$- p_i \int_a^b \left\{ g_i(t, x, \dot{x}, y, \dot{y}) - s(x|E_i) + y^T r_i \right\} dt \Bigg]. \tag{25}$$

From (3) and (10), (25) yields

$$\sum_{i=1}^k \tau_i (p_i - q_i) \int_a^b g_i(t, x, \dot{x}, v, \dot{v})\, dt \geqq 0. \tag{26}$$

If for some i, $p_i < q_i$ and $\forall j \neq i$, $p_i \leqq q_i$, then $\int_a^b g_i(t, x, \dot{x}, v, \dot{v})\, dt > 0$, $i = 1, 2, \ldots, k$, implies that

$$\sum_{i=1}^k \tau_i (p_i - q_i) \int_a^b g_i(t, x, \dot{x}, v, \dot{v})\, dt < 0,$$

which contradicts (26). Hence $p \nleq q$. □

The following strong duality Theorem 3.3 and converse duality Theorem 3.4 can be established on the lines of the proofs of Theorems 3.3 and 3.4 given by Kim et al. [10] in the light of the discussions given above in this section.

Theorem 3.3 *(Strong duality). Let* $(\bar{x}(t), \bar{y}(t), \bar{p}, \bar{\tau}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k, \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_k)$ *be a properly efficient solution for* (EMSP) *and fix* $\tau = \bar{\tau}$ *in* (EMSD)*, and define*

$$\bar{p}_i = \frac{\int_a^b \{f_i(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}) + s(\bar{x}(t)|C_i) - \bar{y}(t)^T \bar{z}_i\}\, dt}{\int_a^b \{g_i(t, \bar{x}, \dot{\bar{x}}, \bar{y}, \dot{\bar{y}}) - s(\bar{x}(t)|E_i) + \bar{y}(t)^T \bar{r}_i\}\, dt}, \quad i = 1, 2, \ldots, k.$$

Suppose that all the conditions in Theorem 3.1 or 3.2 are fulfilled. Furthermore, assume that

(I)

$$\begin{aligned} \sum_{i=1}^k \bar{\tau}_i \int_a^b \psi(t)^T \Big[& \big\{ \big[(f_{iyy} - z_i) - \bar{p}_i (g_{iyy} + r_i) \big] - D\big[(f_{i\dot{y}y} - z_i) - \bar{p}_i (g_{i\dot{y}y} + r_i) \big] \big\} \\ & - D\big\{ \big[(f_{iy\dot{y}} - z_i - D f_{i\dot{y}\dot{y}} - f_{i\dot{y}y}) - \bar{p}_i (g_{iy\dot{y}} + r_i - D g_{i\dot{y}\dot{y}} - g_{i\dot{y}y}) \big] \big\} \\ & + D^2 \big\{ -\big[(f_{i\dot{y}\dot{y}} - z_i) - \bar{p}_i (g_{i\dot{y}\dot{y}} + r_i) \big] \big\} \Big] \psi(t)^T\, dt = 0 \end{aligned}$$

implies that $\psi(t) = 0$, $\forall t \in I$, and

(II) $$\left[\int_a^b \left\{ (f_{1y} - z_1) - \bar{p}_1 (g_{1y} + r_1) \right\} dt, \ldots, \int_a^b \left\{ (f_{ky} - z_k) - \bar{p}_k (g_{ky} + r_k) \right\} dt \right]$$

is linearly independent.

Then there exist $\bar{w}_i \in R^n$, $\bar{s}_i \in R^m$, $i = 1, 2, \ldots, k$, such that $(\bar{x}(t), \bar{y}(t), \bar{p}, \bar{\tau}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k)$ is a properly efficient solution of (EMSD).

Theorem 3.4 *(Converse duality). Let* $(\bar{x}(t), \bar{y}(t), \overline{p}, \bar{\tau}, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k, \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_k)$ *be a properly efficient solution for* (EMSD) *and fix* $\tau = \bar{\tau}$ *in* (EMSP)*, and define* $\overline{p}_i$ *as in Theorem* 3.3. *Suppose that all the conditions in Theorem* 3.1 *or* 3.2 *are fulfilled. Furthermore, assume that* (I) *and* (II) *of Theorem* 3.3 *are satisfied. Then there exist* $\overline{w}_i \in R^n, \bar{s}_i \in R^m$*,* $i = 1, 2, \ldots, k$*, such that* $(\bar{x}(t), \bar{y}(t), \overline{p}, \bar{\tau}, \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_k, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_k)$ *is a properly efficient solution of* (EMSP).

Following Kim et al. [10], we also present self-duality for (MNFP) and (MNFD) instead of for (EMSP) and (EMSD). Assume that $x(t)$ and $y(t)$ have the same dimension. The function $f(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ is said to be skew-symmetric if

$$f\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = -f\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big)$$

for all $x(t)$ and $y(t)$ in the domain of f and the function $g(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ will be called symmetric if

$$g\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) = g\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big).$$

In order to establish the self-duality some conditions are required. We assume that $C = D$, $E = H$ and

$$\begin{aligned} &g\big(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\big) + s(v|E) - u(t)^T s \\ &\quad = g\big(t, v(t), \dot{v}(t), u(t), \dot{u}(t)\big) - s(u|E) + v(t)^T s. \end{aligned}$$

Theorem 3.5 *(Self-duality). If* $f(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ *is skew-symmetric and* $g(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ *is symmetric along with the assumptions* $C = D$*,* $E = H$ *and*

$$\begin{aligned} &g\big(t, u(t), \dot{u}(t), v(t), \dot{v}(t)\big) + s(v|E) - u(t)^T s \\ &\quad = g\big(t, v(t), \dot{v}(t), u(t), \dot{u}(t)\big) - s(u|E) + v(t)^T s, \end{aligned}$$

then (MNFP) *and* (MNFD) *are self-dual. If* (MNFP) *and* (MNFD) *are dual problems, then with* $(x^0(t), y^0(t), p^0, \tau^0, w^0, s^0)$ *also* $(y^0(t), x^0(t), p^0, \tau^0, w^0, s^0)$ *are a joint optimal solution and the common optimal value is zero.*

Proof. As in Kim et al. [10], we have

$$\begin{aligned} f_x\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= -f_y\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big), \\ f_y\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= -f_x\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big), \\ f_{\dot{x}}\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= -f_{\dot{y}}\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big), \\ f_{\dot{y}}\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= -f_{\dot{x}}\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big) \end{aligned}$$

and with g symmetric, we have

$$\begin{aligned} g_x\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= g_y\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big), \\ g_y\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= g_x\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big), \\ g_{\dot{x}}\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= g_{\dot{y}}\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big), \\ g_{\dot{y}}\big(t, x(t), \dot{x}(t), y(t), \dot{y}(t)\big) &= g_{\dot{x}}\big(t, y(t), \dot{y}(t), x(t), \dot{x}(t)\big). \end{aligned}$$

Expressing the dual problem (MNFD) as a minimization problem and making use of above relations and conditions given in the theorem, we have

$$\text{Min}\quad \frac{\int_a^b \{f(t,v(t),\dot v(t),u(t),\dot u(t)) + s(u(t)|C) - v(t)^T w\}\,dt}{\int_a^b \{g(t,v(t),\dot v(t),u(t),\dot u(t)) - s(u(t)|E) + v(t)^T s\}\,dt}$$

$$\begin{aligned}
\text{subject to}\quad & u(a)=0=u(b),\quad v(a)=0=v(b),\\
& \dot u(a)=0=\dot u(b),\quad \dot v(a)=0=\dot v(b),\\
& \sum_{i=1}^k \tau_i\big\{[f_{iv}-Df_{i\dot v}-w_i]G_i(v,u)-[g_{iv}-Dg_{i\dot v}+s_i]F_i(v,u)\big\} \leqq 0,\\
& \int_a^b v(t)^T \sum_{i=1}^k \tau_i\big\{[f_{iv}-Df_{i\dot v}-w_i]G_i(v,u)-[g_{iv}-Dg_{i\dot v}+s_i]F_i(v,u)\big\} \geqq 0,\\
& \tau>0,\quad \tau^T e=1,\quad t\in I,\\
& w_i\in C_i,\quad s_{\mathrm{i}}\in E_i,\quad i=1,2,\dots,k,
\end{aligned}$$

which is just the primal problem (MNFP). Thus if $(x^0(t), y^0(t), p^0, \tau^0, w^0, s^0)$ is an optimal solution for (MNFD), then $(y^0(t), x^0(t), p^0, \tau^0, w^0, s^0)$ is an optimal solution for (MNFD).

Since f is skew-symmetric, g is symmetric, $C=D$, $E=H$ and

$$\begin{aligned}
& g\big(t,u(t),\dot u(t),v(t),\dot v(t)\big)+s(v|E)-u(t)^T s\\
&\quad = g\big(t,v(t),\dot v(t),u(t),\dot u(t)\big)-s(u|E)+v(t)^T s,
\end{aligned}$$

we have

$$\begin{aligned}
& \frac{\int_a^b \{f(t,y(t),\dot y^0(t),x^0(t),\dot x^0(t)) + s(x^0(t)|C) - y^0(t)^T w^0\}\,dt}{\int_a^b \{g(t,y^0(t),\dot y^0(t),x^0(t),\dot x^0(t)) - s(x^0(t)|E) + y^0(t)^T s^0\}\,dt}\\
&\quad = -\frac{\int_a^b \{f(t,x^0(t),\dot x^0(t),y(t),\dot y^0(t)) - s(y^0(t)|C) + x^0(t)^T w^0\}\,dt}{\int_a^b \{g(t,x^0(t),\dot x^0(t),y^0(t),\dot y^0(t)) + s(y^0(t)|E) - x^0(t)^T s^0\}\,dt}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{\int_a^b \{f(t,x^0(t),\dot x^0(t),y(t),\dot y^0(t)) - s(y^0(t)|C) + x^0(t)^T w^0\}\,dt}{\int_a^b \{g(t,x^0(t),\dot x^0(t),y^0(t),\dot y^0(t)) + s(y^0(t)|E) - x^0(t)^T s^0\}\,dt}\\
&\quad = \frac{\int_a^b \{f(t,y(t),\dot y^0(t),x^0(t),\dot x^0(t)) + s(x^0(t)|C) - y^0(t)^T w^0\}\,dt}{\int_a^b \{g(t,y^0(t),\dot y^0(t),x^0(t),\dot x^0(t)) - s(x^0(t)|E) + y^0(t)^T s^0\}\,dt}\\
&\quad = -\frac{\int_a^b \{f(t,x^0(t),\dot x^0(t),y(t),\dot y^0(t)) - s(y^0(t)|C) + x^0(t)^T w^0\}\,dt}{\int_a^b \{g(t,x^0(t),\dot x^0(t),y^0(t),\dot y^0(t)) + s(y^0(t)|E) - x^0(t)^T s^0\}\,dt},
\end{aligned}$$

and so

$$\begin{aligned}
& \frac{\int_a^b \{f(t,x^0(t),\dot x^0(t),y(t),\dot y^0(t)) - s(y^0(t)|C) + x^0(t)^T w^0\}\,dt}{\int_a^b \{g(t,x^0(t),\dot x^0(t),y^0(t),\dot y^0(t)) + s(y^0(t)|E) - x^0(t)^T s^0\}\,dt}\\
&\quad = \frac{\int_a^b \{f(t,y(t),\dot y^0(t),x^0(t),\dot x^0(t)) + s(x^0(t)|C) - y^0(t)^T w^0\}\,dt}{\int_a^b \{g(t,y^0(t),\dot y^0(t),x^0(t),\dot x^0(t)) - s(x^0(t)|E) + y^0(t)^T s^0\}\,dt} = 0. \qquad \square
\end{aligned}$$

4. The static case

If the time dependence of programs (MNFP) and (MNFD) is removed and the functions involved are considered to have domain $R^n \times R^m$, we obtain the symmetric dual fractional pair given by

$$
\text{(SNMFP)}\quad \text{Minimize}\quad \left(\frac{f_1(x,y)+s(x|C_1)-y^T z_1}{g_1(x,y)-s(x|E_1)+y^T r_1},\ldots,\frac{f_k(x,y)+s(x|C_k)-y^T z_k}{g_k(x,y)-s(x|E_k)+y^T r_k}\right)
$$

$$
\text{subject to}\quad \sum_{i=1}^{k}\tau_i\Bigg[\nabla_y f_i(x,y)-z_i - \frac{f_i(x,y)+s(x|C_i)-y^T z_i}{g_i(x,y)-s(x|E_i)+y^T r_i}\big(\nabla_y g_i(x,y)+r_i\big)\Bigg]\leqslant 0,
$$

$$
y^T\sum_{i=1}^{k}\tau_i\Bigg[\nabla_y f_i(x,y)-z_i - \frac{f_i(x,y)+s(x|C_i)-y^T z_i}{g_i(x,y)-s(x|E_i)+y^T r_i}\big(\nabla_y g_i(x,y)+r_i\big)\Bigg]\geqslant 0,
$$

$$
z_i\in D_i,\quad r_i\in F_i,\quad 1\leqslant i\leqslant k,\quad \tau>0,\quad \tau^T e=1,\quad x\geqslant 0.
$$

$$
\text{(SNMFD)}\quad \text{Maximize}\quad \left(\frac{f_1(u,v)-s(v|D_1)+u^T w_1}{g_1(u,v)+s(v|F_1)-u^T t_1},\ldots,\frac{f_k(u,v)-s(v|D_k)+u^T w_k}{g_k(u,v)+s(v|F_k)-u^T t_k}\right)
$$

$$
\text{subject to}\quad \sum_{i=1}^{k}\tau_i\Bigg[\nabla_u f_i(u,v)+w_i - \frac{f_i(u,v)-s(v|D_i)+u^T w_i}{g_i(u,v)+s(v|F_i)-u^T t_i}\big(\nabla_u g_i(u,v)-t_i\big)\Bigg]\geqslant 0,
$$

$$
u^T\sum_{i=1}^{k}\tau_i\Bigg[\nabla_u f_i(u,v)+w_i - \frac{f_i(u,v)-s(v|D_i)+u^T w_i}{g_i(u,v)+s(v|F_i)-u^T t_i}\big(\nabla_u g_i(u,v)-t_i\big)\Bigg]\leqslant 0,
$$

$$
w_i\in C_i,\quad t_i\in E_i,\quad 1\leqslant i\leqslant k,\quad \tau>0,\quad \tau^T e=1,\quad v\geqslant 0.
$$

The pair of problems (SNMFP) and (SNMFD) obtained above is exactly the pair of problems (FP) and (FD) considered by Yang et al. [28].

If we set $k=1$, and our problems are time independent, we get the following pair of problems:

$$
\text{(SNFP)}\quad \text{Minimize}\quad \left(\frac{f(x,y)+s(x|C)-y^T z}{g(x,y)-s(x|E)+y^T r}\right)
$$

$$
\text{subject to}\quad \Bigg[\nabla_y f(x,y)-z-\frac{f(x,y)+s(x|C)-y^T z}{g(x,y)-s(x|E)+y^T r}\big(\nabla_y g(x,y)+r\big)\Bigg]\leqslant 0,
$$

$$y^T\bigg[\nabla_y f(x,y) - z - \frac{f(x,y)+s(x|C)-y^Tz}{g(x,y)-s(x|E)+y^Tr}\big(\nabla_y g(x,y)+r\big)\bigg] \geqslant 0,$$

$$z\in D, \quad r\in F, \quad x\geqslant 0.$$

$$\text{(SNFD)} \quad \text{Maximize} \quad \left(\frac{f(u,v)-s(v|D)+u^Tw}{g(u,v)+s(v|F)-u^Tt}\right)$$

$$\text{subject to} \quad \bigg[\nabla_u f(u,v)+w-\frac{f(u,v)-s(v|D)+u^Tw}{g(u,v)+s(v|F)-u^Tt}\big(\nabla_u g(u,v)-t\big)\bigg] \geqslant 0,$$

$$u^T\bigg[\nabla_u f(u,v)+w - \frac{f(u,v)-s(v|D)+u^Tw}{g(u,v)+s(v|F)-u^Tt}\big(\nabla_u g(u,v)-t\big)\bigg] \leqslant 0,$$

$$w\in C, \quad t\in E, \quad v\geqslant 0.$$

The pair of problems (SNFP) and (SNFD) is exactly the pair of problems (FP) and (FD) considered by Yang et al. [29].

If we remove the nondifferentiable terms of our problems, we get the problems discussed in Section 4 of Kim et al. [10].

5. Conclusion

In this paper, we have extended an earlier work of Kim et al. [10] to nondifferentiable case. Our results also extend an earlier work of Yang et al. [28] to continuous-time case. Many other results on symmetric duality in literature are particular cases of the results obtained in the present paper.

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