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Approximations to -, di- and tri-logarithms

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Abstract

We propose hypergeometric constructions of simultaneous approximations to polylogarithms. These approximations suit for computing the values of polylogarithms and satisfy 4-term Apéry-like (polynomial) recursions.

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The series for the logarithm function

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad |z| < 1,$$

is so simple and nice that mathematicians immediately generalize it by introducing the polylogarithms

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \qquad |z| < 1, \quad s = 1, 2, \ldots, \tag{1}$$

and considering then multiple, q-basic, p-adic and any other possible generalizations, just to have a serious mathematical research (i.e., to have some fun). It is so that nobody could now overview the whole amount of the results around all these generalizations of the logarithm, since the literature on the subject increases to infinity as a geometric progression (almost hypergeometrically).

It is not surprising that the transcendence number theory also dreams of getting new and new results for the values of the polylogarithms, especially after Lindemann's proof of the transcendence of $\log x$ for any algebraic x different from 0 and 1. The main problems (or, if you like, intrigues) are therefore extensions of the result to the values of (1), where Lindemann's argument based on proving the transcendence for the inverse, exponential, function does not work in an

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obvious manner any more. And we now have so far irrationality and linear independence results for the polylogarithm values at non-zero rational points close to zero, thanks to contributions of Maier [10], Galochkin [6], Nikishin [12], Chudnovsky [5], Hata [7,8], Rhin and Viola [15]. The very last piece of news is the irrationality of $\mathrm{Li}_2(1/q)$ for q integer, $q \leqslant -5$ or $q \geqslant 6$, obtained by the powerful arithmetic method in [15], which improves the range of [8] by adding $q = 6$ (the work [15] also includes quantitative improvements of the irrationality in other cases, but we do not touch this subject in this short note). Another direction of arithmetic investigations are the values of (1) at $z = 1$ (or $z = -1$), so-called *zeta values*. This goes back to Euler's time, who has contributed by the formula

$$\mathrm{Li}_{2k}(1) = \zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!} \quad \text{for } k = 1, 2, \ldots, \tag{2}$$

where $B_{2k} \in \mathbb{Q}$ are the Bernoulli numbers, thus Lindemann's proof of the transcendence of π results in the transcendence of the numbers (2). Apéry [1] has shown that $\zeta(3)$ is irrational, and since that time, thanks to Ball and Rivoal [3], we dispose of only partial irrationality information for other values of $\mathrm{Li}_{2k+1}(1) = \zeta(2k + 1)$ if $k = 2, 3, 4, \ldots$.

All known achievements in this subject are closely related to hypergeometric series and also multiple and complex integrals originated from the series. This is a general concept of the hypergeometric method developed for arithmetic study of the values of the polylogarithms; we refer the reader to a brief exposition of this concept in [11].

Here, we would like to present some new ingredients of the hypergeometric method. We cannot achieve some new number-theoretic results by these means, and for the moment this note may be viewed as a methodological contribution. Nevertheless, approximations to the values of the polylogarithms that we derive here are quite reasonable from the computational point of view, and, in this sense, we continue our previous work on deducing curious Apéry-like recurrences.

We hope that the reader is somehow familiar with our work on the hypergeometric method in arithmetic study of zeta values (at least with the preprint [16]).

1. Simultaneous approximations to the logarithm and dilogarithm

For each $n = 0, 1, \ldots$, consider the rational function

$$R_n(t) = \frac{((t-1)(t-2)\cdots(t-n))^2}{n! \cdot t(t+1)\cdots(t+n)}.$$

Since degree of the numerator is greater than degree of the denominator, we will have a polynomial part while decomposing into partial fractions. The arithmetic properties of this decomposition are given in the following statement; D_n denotes the least common multiple of the numbers $1, 2, \ldots, n$.

Lemma 1. *We have*

$$R_n(t) = \frac{((t-1)(t-2)\cdots(t-n))^2}{n! \cdot t(t+1)\cdots(t+n)} = \sum_{k=0}^{n} \frac{A_k}{t+k} + B(t),$$

where numbers A_k are all integers and $D_n \cdot B(t)$ is an integer-valued polynomial of degree $n - 1$.

Proof. Write this decomposition as follows:

$$R_n(t) = \sum_{k=0}^{n} \frac{A_k}{t+k} + \sum_{j=0}^{n-1} B_j \frac{t(t+1)\cdots(t+j-1)}{j!}$$

(the empty product for $j = 0$ is 1). The coefficients A_k are easily determined by the standard procedure:

$$A_k = R_n(t)(t+k)|_{t=-k} = (-1)^k \binom{n}{k} \binom{n+k}{k}^2, \quad k = 0, 1, \ldots, n, \tag{3}$$

while the remaining group of unknown coefficients requires some work. Denote

$$F_l(t) = (t+l) \cdot \sum_{k=0}^{n} \frac{A_k}{t+k}, \quad l = 0, 1, \ldots, n.$$

Then

$$F_l(t) = \sum_{k=0}^{n} A_k \left(1 - \frac{k-l}{t+k}\right)$$

and

$$\frac{\mathrm{d}F_l(t)}{\mathrm{d}t} = \sum_{\substack{k=0\\k\neq l}}^{n} A_k \frac{k-l}{(t+k)^2}.$$

Therefore,

$$\left.\frac{\mathrm{d}F_l(t)}{\mathrm{d}t}\right|_{t=-l} = \sum_{\substack{k=0\\k\neq l}}^{n} \frac{A_k}{k-l}, \quad l = 0, 1, \ldots, n. \tag{4}$$

What will happen if we do the same with the polynomial tail? Define

$$G_l(t) = (t+l) \cdot \sum_{j=0}^{n-1} B_j \frac{t(t+1)\cdots(t+j-1)}{j!}, \quad l = 0, 1, \ldots, n-1.$$

Then

$$\left.\frac{\mathrm{d}G_l(t)}{\mathrm{d}t}\right|_{t=-l} = \frac{\mathrm{d}}{\mathrm{d}t}\left.\left(\sum_{j=0}^{l} B_j \frac{t(t+1)\cdots(t+j-1)}{j!} \cdot (t+l)\right)\right|_{t=-l}$$

and, since for a polynomial $P(t)$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}(P(t)(t+l))\right|_{t=-l} = P(-l),$$

we finally obtain

$$\left.\frac{\mathrm{d}G_l(t)}{\mathrm{d}t}\right|_{t=-l} = \sum_{j=0}^{l} (-1)^j \binom{l}{j} B_j, \quad l = 0, 1, \ldots, n-1,$$

and hence

$$B_l = \left.\frac{\mathrm{d}G_l(t)}{\mathrm{d}t}\right|_{t=-l} - (-1)^l \sum_{j=0}^{l-1} (-1)^j \binom{l}{j} B_j, \quad l = 0, 1, \ldots, n-1. \tag{5}$$

Furthermore, we have

$$F_l(t) + G_l(t) = R_n(t)(t+l) = \left(\frac{(t-1)\cdots(t-n)}{n!}\right)^2 \cdot \frac{n!(t+l)}{t(t+1)\cdots(t+n)},$$
$$l = 0, 1, \ldots, n-1,$$

hence

$$D_n \cdot \left.\frac{\mathrm{d}}{\mathrm{d}t}(F_l(t) + G_l(t))\right|_{t=-l} \in \mathbb{Z}, \quad l = 0, 1, \ldots, n-1,$$

where D_n denotes the least common multiple of the numbers $1, 2, \ldots, n$. These inclusions and inclusions

$$D_n \cdot \left.\frac{\mathrm{d}F_l(t)}{\mathrm{d}t}\right|_{t=-l} \in \mathbb{Z}, \quad l = 0, 1, \ldots, n-1,$$

which follow from formulae (3) and (4), together with the induction on l on the basis of (5), show that

$$D_n \cdot B_l \in \mathbb{Z}, \quad l = 0, 1, \ldots, n-1,$$

and the proof follows. □

Since for an integer-valued polynomial $P(t)$ of degree at most n its derivative multiplied by D_n is again an integer-valued polynomial, we also have

Lemma 2. *The following decomposition is valid*:

$$-\frac{\mathrm{d}R_n(t)}{\mathrm{d}t} = \sum_{k=0}^{n} \frac{A_k}{(t+k)^2} + \widetilde{B}(t),$$

where numbers A_k are all integers and $D_n^2 \cdot \widetilde{B}(t)$ is an integer-valued polynomial of degree $n-2$.

Let z be a rational number with $0 < |z| < 1$. We are now interested in the following two hypergeometric-type series:

$$r_n = r_n(z) = \sum_{v=1}^{\infty} z^v R_n(t)|_{t=v}, \quad \widetilde{r}_n = \widetilde{r}_n(z) = -\sum_{v=1}^{\infty} z^v \left.\frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\right|_{t=v}.$$

Lemma 3. *We have*

$$\begin{gathered} r_n(z) = a_n \mathrm{Li}_1(z) - b_n, \quad \widetilde{r}_n(z) = a_n \mathrm{Li}_2(z) - \widetilde{b}_n, \\ z_1^n a_n \in \mathbb{Z}, \quad (z_1 z_2)^n D_n b_n \in \mathbb{Z}, \quad (z_1 z_2)^n D_n^2 \widetilde{b}_n \in \mathbb{Z}, \end{gathered} \tag{6}$$

where z_1 and z_2 are the denominators of the numbers $1/z$ and $z/(1-z)$, respectively.

Proof. Let us write the polynomials $B(t)$ and $\widetilde{B}(t)$ in the form

$$B(t) = \sum_{j=0}^{n-1} B_j \frac{(t-1)(t-2)\cdots(t-j)}{j!}, \quad \widetilde{B}(t) = \sum_{j=0}^{n-2} \widetilde{B}_j \frac{(t-1)(t-2)\cdots(t-j)}{j!},$$

where

$$D_n \cdot B_j \in \mathbb{Z}, \quad D_n^2 \cdot \widetilde{B}_j \in \mathbb{Z}, \quad j = 0, \ldots, n-1 \tag{7}$$

(this is guaranteed by the theorem of choosing a basis in the $\mathbb{Z}$-space of integer-valued polynomials). Then

$$\begin{aligned} r_n &= \sum_{v=1}^{\infty} z^v \left(\sum_{k=0}^{n} \frac{A_k}{v+k} + \sum_{j=0}^{n-1} B_j \frac{(v-1)(v-2)\cdots(v-j)}{j!} \right) \\ &= \sum_{k=0}^{n} A_k z^{-k} \sum_{v=1}^{\infty} \frac{z^{v+k}}{v+k} + \sum_{j=0}^{n-1} B_j z^{j+1} \sum_{v=1}^{\infty} \frac{(v-1)(v-2)\cdots(v-j)}{j!} z^{v-j-1} \\ &= \sum_{k=0}^{n} A_k z^{-k} \left(\sum_{l=1}^{\infty} - \sum_{l=1}^{k} \right) \frac{z^l}{l} - \sum_{j=0}^{n-1} B_j z^{j+1} \cdot \frac{1}{(z-1)^{j+1}} \\ &= \sum_{k=0}^{n} A_k z^{-k} \cdot \mathrm{Li}_1(z) - \sum_{k=0}^{n} A_k \sum_{l=1}^{k} \frac{z^{-(k-l)}}{l} - \sum_{j=0}^{n-1} B_j \left(\frac{z}{z-1} \right)^{j+1}. \end{aligned}$$

In the same vein,

$$\widetilde{r}_n = \sum_{k=0}^{n} A_k z^{-k} \cdot \mathrm{Li}_2(z) - \sum_{k=0}^{n} A_k \sum_{l=1}^{k} \frac{z^{-(k-l)}}{l^2} - \sum_{j=0}^{n-2} \widetilde{B}_j \left(\frac{z}{z-1}\right)^{j+1}.$$

Using (7) and integrality of A_k for all $k = 0, 1, \ldots, n$, we arrive at the desired claim. □

Remark. As follows from (3) and the above proof, we have the following explicit formula:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^2 \left(-\frac{1}{z}\right)^k.$$

As we see from Lemma 3, the sequences $r_n(z)$ and $\widetilde{r}_n(z)$, $n = 0, 1, \ldots$, realize simultaneous rational approximations to $\mathrm{Li}_1(z)$ and $\mathrm{Li}_2(z)$, although approximation 'tails' are not simply polynomials in $1/z$, but sums of two polynomials from $\mathbb{Q}[1/z]$ and $\mathbb{Q}[z/(z-1)]$. Although we worked in the region $|z| < 1$, the result of Lemma 3 remains valid in the closed disc $|z| \leqslant 1$ except the point $z = 1$ by analytic continuation.

Running the Gosper–Zeilberger algorithm of creative telescoping [13, Chapter 6] with the input $R_n(t)z^t$, one can find the difference operator of order 3, which annihilates linear forms (6) and their coefficients $a_n, b_n, \widetilde{b}_n$. In order not to frighten the reader, we indicate only the characteristic polynomial of this operator

$$z(z-1)\lambda^3 - (3z^2 - 20z + 16)\lambda^2 + z(3z+8)\lambda - z^2$$

(containing all details on the asymptotic behaviour of the approximants, due to Poincaré's theorem), and the partial case $z = -1$ of the corresponding recurrence.

Theorem 1. *The coefficients a_n, b_n and $\widetilde{b}_n$ of the simultaneous approximations*

$$\begin{aligned} r_n &= r_n(-1) = a_n \mathrm{Li}_1(-1) - b_n = -a_n \log 2 - b_n, \\ \widetilde{r}_n &= \widetilde{r}_n(-1) = a_n \mathrm{Li}_2(-1) - \widetilde{b}_n = -a_n \frac{\pi^2}{12} - \widetilde{b}_n, \end{aligned} \qquad n = 0, 1, \ldots,$$

as well as the approximations themselves satisfy the recurrence relation

$$\begin{aligned} &2(59n-24)(n+1)^2 a_{n+1} - (2301n^3 + 1365n^2 - 376n - 240)a_n \\ &\quad - (295n^3 - 120n^2 - 60n + 35)a_{n-1} - (59n+35)(n-1)^2 a_{n-2} = 0, \quad n = 2, 3, \ldots, \end{aligned}$$

of order 3, *and the necessary initial data are as follows*:

$$\begin{aligned} &a_0 = 1, \quad a_1 = 5, \quad a_2 = 55, \\ &b_0 = 0, \quad b_1 = -\tfrac{7}{2}, \quad b_2 = -\tfrac{305}{8}, \quad \widetilde{b}_0 = 0, \quad \widetilde{b}_1 = -4, \quad \widetilde{b}_2 = -\tfrac{181}{4}. \end{aligned}$$

In addition,

$$\begin{aligned} &\lim_{n\to\infty} |r_n|^{1/n} = \lim_{n\to\infty} |\widetilde{r}_n|^{1/n} = |\lambda_{1,2}| = 0.15960248\ldots, \\ &\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |b_n|^{1/n} = \lim_{n\to\infty} |\widetilde{b}_n|^{1/n} = \lambda_3 = 19.62866250\ldots, \end{aligned}$$

where $\lambda_{1,2} = -0.06433125\ldots \pm \mathrm{i}0.14606314\ldots$ and λ_3 are zeros of the characteristic polynomial $2\lambda^3 - 39\lambda^2 - 5\lambda - 1$.

One can run the algorithm of creative telescoping with the input $R_n(t)$ (that is, $z=1$, a non-sense!) to obtain a much simpler recurrence

$$(n+1)^2a_{n+1}+(11n^2+11n+3)a_n-n^2a_{n-1}=0, \quad n=1,2,\ldots,$$

of order 2, which may be recognized in view of Apéry's proof of the irrationality of $\zeta(2)$ (see [1,14]). In fact, we have the identity

$$a_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}^2=(-1)^n\sum_{k=0}^{n}\binom{n+k}{k}\binom{n}{k}^2$$

thanks to Thomae's transformation of ${}_3F_2(1)$-hypergeometric series (see [2, Section 3.2]), and the latter sum gives (up to the sign factor) the denominators of Apéry's approximations to $\zeta(2)$. In order to give the necessary sense to the substitution $z=1$, we should introduce the following complex Barnes integral:

$$\frac{1}{2\pi\mathrm{i}}\int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty}R_n(t)\Big(\frac{\pi}{\sin\pi t}\Big)^2z^t\,\mathrm{d}t=\widetilde{r}_n(z)-r_n(z)\log z, \tag{8}$$

where C is an arbitrary constant in the interval $0<C<n+1$. The integral converges in the whole disc $|z|\leqslant 1$, hence we should have the limit in the right-hand side of (8), i.e., (in notations of the proof of Lemma 3) the limit

$$\lim_{z\to 1}\left(\sum_{j=0}^{n-2}\widetilde{B}_j\Big(\frac{z}{z-1}\Big)^{j+1}-\log z\cdot\sum_{j=0}^{n-1}B_j\Big(\frac{z}{z-1}\Big)^{j+1}\right)$$

exists and is equal to a certain rational constant depending on n. On the other hand, the complex integral in (8) admits a real double-integral representation thanks to [11, Theorem 2]. Taking $m=3$, $r=2$ and $a_1=a_2=a_3=n+1$, $b_2=b_3=2n+2$ in this Nesterenko's theorem, we obtain

$$\widetilde{r}_n(z)-r_n(z)\log z=\iint_{[0,1]^2}\frac{x^n(1-x)^ny^n(1-y)^n}{(1-x+zxy)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y.$$

Substituting $z=1$ and multiplying by $(-1)^n$ reduces the latter integral to Beukers' famous double integral [4] for Apéry's approximations to $\zeta(2)$.

2. Simultaneous approximations to $\zeta(2)$ and $\zeta(3)$

Our first natural generalization of the construction in the previous section is based on the rational function

$$R_n(t)=\frac{((t-1)(t-2)\cdots(t-n))^3}{n!^2\cdot t(t+1)\cdots(t+n)}.$$

Then 'reasonable' approximations to the first three polylogarithms are given by the series

$$r_n(z)=\sum_{\nu=1}^{\infty}z^\nu R_n(t)|_{t=\nu}=a_n\mathrm{Li}_1(z)-b_n,$$

$$\widetilde{r}_n(z)=-\sum_{\nu=1}^{\infty}z^\nu\frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\bigg|_{t=\nu}=a_n\mathrm{Li}_2(z)-\widetilde{b}_n,$$

$$\widetilde{\widetilde{r}}_n(z)=\frac{1}{2}\sum_{\nu=1}^{\infty}z^\nu\frac{\mathrm{d}^2R_n(t)}{\mathrm{d}t^2}\bigg|_{t=\nu}=a_n\mathrm{Li}_3(z)-\widetilde{\widetilde{b}}_n,$$

where

$$z_1^n a_n \in \mathbb{Z}, \quad (z_1 z_2)^n D_n b_n \in \mathbb{Z}, \quad (z_1 z_2)^n D_n D_{2n} \widetilde{b}_n \in \mathbb{Z}, \quad (z_1 z_2)^n D_n D_{2n}^2 \widetilde{\widetilde{b}}_n \in \mathbb{Z} \tag{9}$$

and z_1 and z_2 are the denominators of the numbers $1/z$ and $z/(1-z)$, respectively. The reason of having the multiples D_{2n} in (9) is the higher degree $2n-1$ of the polynomial in the decomposition of $R_n(t)$ into partial fractions, and it is required to derivate it for getting the representation of $\widetilde{r}_n$ and $\widetilde{\widetilde{r}}_n$. The explicit formula for the coefficient a_n is as follows:

$$a_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^3 \left(-\frac{1}{z}\right)^k.$$

This time we are interested in the particular 'non-sense' case $z=1$ of the construction. Without tiring reader's eyes by writing two complex integrals converging in the disc $|z| \leqslant 1$ and guaranteeing the existence of limits for corresponding series expansions, we just present the final result for the approximation sequences

$$\widetilde{r}_n = \widetilde{r}_n(1) = a_n \zeta(2) - \widetilde{b}_n, \quad \widetilde{\widetilde{r}}_n = \widetilde{\widetilde{r}}_n(1) = a_n \zeta(3) - \widetilde{\widetilde{b}}_n, \quad n = 0, 1, \ldots.$$

Theorem 2. *The above sequences $\widetilde{r}_n, \widetilde{\widetilde{r}}_n$ as well as the coefficients a_n, $\widetilde{b}_n$ and $\widetilde{\widetilde{b}}_n$ satisfy the recurrence relation*

$$\begin{aligned}
&2(946n^2 - 731n + 153)(2n+1)(n+1)^3 a_{n+1} \\
&\quad - 2(104060n^6 + 127710n^5 + 12788n^4 - 34525n^3 - 8482n^2 + 3298n + 1071) a_n \\
&\quad + 2(3784n^5 - 1032n^4 - 1925n^3 + 853n^2 + 328n - 184) n a_{n-1} \\
&\quad - (946n^2 + 1161n + 368) n (n-1)^3 a_{n-2} = 0, \quad n = 2, 3, \ldots,
\end{aligned}$$

of order 3, *and the necessary initial data is as follows*:

$$\begin{aligned}
&a_0 = 1, \quad a_1 = 7, \quad a_2 = 163, \\
&\widetilde{b}_0 = 0, \quad \widetilde{b}_1 = \tfrac{23}{2}, \quad \widetilde{b}_2 = \tfrac{2145}{8}, \quad \widetilde{\widetilde{b}}_0 = 0, \quad \widetilde{\widetilde{b}}_1 = \tfrac{17}{2}, \quad \widetilde{\widetilde{b}}_2 = \tfrac{3135}{16}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\limsup_{n\to\infty} |\widetilde{r}_n|^{1/n} = \limsup_{n\to\infty} |\widetilde{\widetilde{r}}_n|^{1/n} = |\lambda_{1,2}| = 0.067442248\ldots, \\
&\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |\widetilde{b}_n|^{1/n} = \lim_{n\to\infty} |\widetilde{\widetilde{b}}_n|^{1/n} = \lambda_3 = 54.96369509\ldots,
\end{aligned}$$

where $\lambda_{1,2} = 0.018152450\ldots \pm \mathrm{i}0.064953409\ldots$ *and* λ_3 *are zeros of the characteristic polynomial* $4\lambda^3 - 220\lambda^2 + 8\lambda - 1$.

Based on the recurrence, we have observed experimentally and we are able to show that the correct form of the inclusions (9) in this special case $z=1$ is

$$a_n = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \binom{n+2k}{n} \in \mathbb{Z},$$

$$D_n D_{2n} \widetilde{b}_n \in \mathbb{Z}, \quad D_n^3 \widetilde{\widetilde{b}}_n \in \mathbb{Z}.$$

Remark. Normalizing the approximations of Theorem 3 by multiplying them by the factor $\binom{2n}{n}$, we arrive at the recurrence previously obtained in [17, Theorem 4] by means of a certain implicit construction. Our new explicit consideration gives an answer to arithmetic observations posed in [17].

3. Well-poised approximations

The arithmetic study of zeta values was strongly influenced by *well-poised* hypergeometric series. They are a 'heart' of the proof in [3] and of many other similar results, and we cannot avoid considering a well-poised generalization of the construction in Section 1.

Take

$$\begin{aligned} R_n(t) &= (-1)^{n+1}\left(t+\frac{n}{2}\right)\frac{((t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n))^2}{n!\cdot(t(t+1)\cdots(t+n))^3} \\ &= (-1)^n R_n(-t-n), \end{aligned}$$

which has now a quite complicated partial-fraction decomposition:

$$R_n(t) = \sum_{k=0}^{n}\left(\frac{A_k}{(t+k)^3}+\frac{A'_k}{(t+k)^2}+\frac{A''_k}{t+k}\right)+B(t), \tag{10}$$

where (repeating arguments of the proof of Lemma 1) $2A_k, 2D_nA'_k, 2D_n^2A''_k$ are integers for all $k=0,1,\ldots,n$, while $2D_n^3B(t)$ is an integer-valued polynomial. Gathering this arithmetic knowledge and proceeding as in the proof of Lemma 3, we deduce that the series

$$r_n(z) = \sum_{\nu=1}^{\infty} z^\nu R_n(t)|_{t=\nu}, \quad \tilde{r}_n(z) = -\sum_{\nu=1}^{\infty} z^\nu \left.\frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\right|_{t=\nu}$$

are certain linear forms involving certain polylogarithms (up to $\mathrm{Li}_4(z)$). Not so exciting, but we would like to deal with the construction at the only one point, $z=-1$. The well-poised thread (take $-t-n$ in place of t in (10)) results in equalities $(-1)^kA_k = -(-1)^{n-k}A_{n-k}$ and $(-1)^kA''_k = -(-1)^{n-k}A''_{n-k}$, $k=0,1,\ldots,n$, and they are the circumstance, which makes $r_n(-1)$ and $\tilde{r}_n(-1)$ linear forms in $\mathrm{Li}_2(-1) = -\pi^2/12$, 1 and $2\mathrm{Li}_3(-1) = -3\zeta(3)/2$, 1, respectively, with the same leading coefficient. We write this final production as follows:

$$r_n = r_n(-1) = a_n\frac{\pi^2}{12} - b_n, \quad \tilde{r}_n = \tilde{r}_n(-1) = a_n\frac{3\zeta(2)}{2} - \tilde{b}_n, \quad n=0,1,\ldots,$$

where

$$2D_na_n \in \mathbb{Z}, \quad 2^nD_n^3b_n \in \mathbb{Z}, \quad 2^nD_n^4b_n \in \mathbb{Z}, \quad n=0,1,\ldots, \tag{11}$$

the nth powers of 2 appear since the two is the denominator of $z/(z-1)$ when $z=-1$. Applying the algorithm of creative telescoping we arrive at the following result.

Theorem 3. *The sequences $r_n, \tilde{r}_n$ and the coefficients a_n, b_n and $\tilde{b}_n$ satisfy the recurrence relation*

$$\begin{aligned} &(1457n^2-1363n+348)(n+1)^4a_{n+1} \\ &\quad -(148614n^6+158202n^5-9295n^4-61894n^3-11111n^2+8932n+2784)a_n \\ &\quad +(97619n^6-91321n^5-9443n^4+35343n^3-5440n^2-5678n+1768)a_{n-1} \\ &\quad -3(1457n^2+1551n+442)(3n-2)(3n-4)(n-1)^2a_{n-2} = 0, \quad n=2,3,\ldots, \end{aligned}$$

of order 3, *and the initial values are as follows*:

$$a_0=1, \quad a_1=8, \quad a_2=264,$$

$$b_0=0, \quad b_1=\tfrac{13}{2}, \quad b_2=\tfrac{1737}{8}, \quad \tilde{b}_0=0, \quad \tilde{b}_1=\tfrac{29}{2}, \quad \tilde{b}_2=\tfrac{7617}{16}.$$

In addition,

$$\limsup_{n\to\infty}|r_n|^{1/n}=\limsup_{n\to\infty}|\widetilde{r}_n|^{1/n}=|\lambda_{1,2}|=0.51616460\ldots,$$
$$\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}|b_n|^{1/n}=\lim_{n\to\infty}|\widetilde{b}_n|^{1/n}=\lambda_3=101.34149804\ldots,$$

where $\lambda_{1,2}=0.32925097\ldots\pm \mathrm{i}0.39751691\ldots$ *and* λ_3 *are zeros of the characteristic polynomial* $\lambda^3-102\lambda^2+67\lambda-27$.

On the basis of the recurrence relation we find much better inclusions than (11), namely

$$\widetilde{\Phi}_n^{-1}a_n\in\mathbb{Z},\quad 2\widetilde{\Phi}_n^{-1}D_n^2b_n\in\mathbb{Z},\quad 2\widetilde{\Phi}_n^{-1}D_n^3b_n\in\mathbb{Z},\quad n=0,1,\ldots, \tag{12}$$

where $\widetilde{\Phi}_n$ is the following product over primes:

$$\widetilde{\Phi}_n=\prod_{\substack{p\leqslant n\\ 2/3\leqslant\{n/p\}<1}}p,$$

$\{\cdot\}$ denotes the fractional part of a number. Inclusions (12) are quite expected in view of 'denominator conjectures' around linear forms in zeta values (see [9, Section 17.1] about the difficulties in proving the correct arithmetic in similar cases). But why do we get the cancellation of 2^n? This might be also caused by the well-poised origin of the series used by us. At least the integrality of a_n is an immediate consequence of the following explicit formulae:

$$a_n=(-1)^n\sum_{j=0}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n}{2}-t\right)\binom{n}{t}^3\binom{n+t}{n}^2\binom{2n-t}{n}^2\Bigg|_{t=j}$$
$$=\sum\sum_{0\leqslant i\leqslant j\leqslant n}(-1)^{n+j}\binom{n}{i}^2\binom{n}{j}\binom{2n-i}{n}\binom{n+j}{n}\binom{n+j-i}{n} \tag{13}$$
$$=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}^2\binom{n}{j}^2\binom{n+i}{n}\binom{i+j}{i}, \tag{14}$$

where equality (13) follows from [9, Proposition 5] and equality (14) is communicated to us by G. Almkvist.

4. Final remarks

As already promised, no new irrationality and linear independence results were presented. We just have tried to give some sense to certain hypergeometric-type series that are expressed in terms of polylogarithms and are divergent when one formally plugs z with $|z|=1$. Transforming a non-terminating single hypergeometric series into a multiple one (some kind of 'identités non-terminées gigantesques', cf. [9, Section 17.5]) often meets convergence troubles for the latter series, i.e., it is just a formal transformation, which we could never use in a rigorous proof. As an option to proceed in such troubling cases, we see dealing with transformations for complex Barnes (multiple) integrals and further decompositions of the integrals into sums involving (multiple) zeta values. This does not look an easy program, but we do not believe that deducing new results for zeta values and polylogarithms might be simple.

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