Harmonic Analysis on Nilpotent Groups and Singular Integrals.
II. Singular Kernels Supported on Submanifolds*

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This paper is the second of a series begun in [18]. Here our principal objective will be to study singular convolution operators on general (simply connected) nilpotent Lie groups, and more specifically operators whose kernels are supported on possibly lower dimensional varieties, and which may also contain exponential polynomial oscillatory factors [17].

Let us describe some of our ideas in terms of a basic example of operators we consider. On the group $N$ let $T$ be the operator given by

$$(Tf)(x) = p.v. \int_N f(xy^{-1}) K(y) \, dy,$$

where $K$ is an appropriate singular kernel. In keeping with the classical theory of $\mathbb{R}^n$ one could expect to have boundedness on $L^p$ of the above operator whenever $K$ has a characteristic singularity described in terms of a critical degree of homogeneity.

Now, in the special case when $N$ happens to have automorphic dilations (i.e., $N$ is a homogeneous group), such a theory has been known for quite some time. One of our main results is that corresponding assertions are

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valid in the general case. This is connected with our previous paper, at least when \( N \) is a Heisenberg group and as far as \( L^2 \)-theory is concerned, because when we take the group Fourier transform (or the Fourier transform with respect to the center) we are led to oscillatory integrals related to the type studied in [18].

In the present paper, however, we do not reduce matters to such oscillatory integrals, but instead to convolution operators whose kernels are supported on lower-dimensional varieties, and in the end we can reverse the implications and generalize substantial parts of our earlier results about oscillatory integrals.

Our paper is organized as follows. In the first part we study in detail the singular integral operators on homogeneous nilpotent groups, defined by smooth kernels that are supported on lower-dimensional analytic submanifolds. We prove that such operators are bounded on every \( L^p \) for \( 1 < p < \infty \), at least when the manifold (which we always assume does not contain the origin) is connected.

The main reason that we are forced to restrict to analytic manifolds is that we make a strong use of iterated convolutions of the kernel with itself, so that the group that is generated by the manifold plays an important role. The analyticity condition is required by the fact that we need to know that any neighborhood of a point in the manifold generates the same group. It also implies that some negative powers of certain Jacobian determinants are integrable.

For the first reason we assume to begin with that the manifold is connected. This hypothesis can be eliminated by assuming that each component generates the same subgroup. As a matter of fact, the kernel may as well be a linear combination of kernels carried by manifolds of different dimensions. This last observation is used, together with the aid of an appropriate square-function operator, to prove \( L^p \)-boundedness for maximal operators along submanifolds.

Our results up to this point unify and extend part of the results in the literature [1, 8, 9, 16, 21, 22].

The second part of the paper deals with applications to operators of other kinds, such as

(a) singular integral operators related to dilations that are not group automorphisms,

(b) oscillatory singular integrals.

The fact that there should be results about singular integrals related to non-automorphic dilations is indicated, for instance, by the fact that the method of rotations implies that on the Heisenberg group \( H_n \) a kernel that is odd and homogeneous of degree \(-2n-1\) with respect to the isotropic
dilations \((z, t) \rightarrow (\delta z, \delta t)\) defines a bounded convolution operator on \(L^p\) for \(1 < p < \infty\). The same method shows that the maximal operator

\[
\mathcal{M} f(u) = \sup_{r > 0} r^{-2n-1} \int_{|z|^2 + |t|^2 < r^2} |f(u(z, t)^{-1})| \, dz \, dt
\]

is bounded on \(L^p\) for \(1 < p < \infty\).

A direct attack of these problems in general by classical methods is not possible because one cannot use the theory of spaces of homogeneous type and the related decomposition lemmas. Our proofs consist in realizing the non-homogeneous groups as quotients of homogeneous ones and applying a particular version of the more general method of transference [2]. This allows us to "push-forward" singular kernels from the larger group to its quotient together with their \(L^p\)-boundedness properties. If one places the kernel to be studied on a suitable homogeneous manifold in the larger group (so that \(L^p\)-boundedness follows by what was proved in the first part of the paper) and pushes it forward, the desired result follows.

The use of transference methods in connection with special cases of projections to quotient groups is not new [2, 4], but it seems not to have ever found applications in a general context. However, the idea of looking at non-homogeneous groups as quotients of homogeneous ones appears already implicitly in [19, 6] and more recently in [13], where it is exploited in different ways.

The above-mentioned application to oscillatory integrals also makes use of the transference method, and it is based on the model of the Schrödinger representations of the Heisenberg groups. We actually prove that, given any polynomial \(P(x, y)\) on any nilpotent group, it is possible to determine a representation of a larger group and a submanifold of it such that the representation transforms the singular integral operators with kernels supported on the manifold into oscillatory singular integrals on the original group containing the oscillatory factor \(e^{iP(x, y)}\). These considerations allow us to extend the results in [18] to the case of non-homogeneous groups and of oscillatory kernels supported on analytic manifolds.

\[
\text{PART I}
\]

1. Analytic Submanifolds Generating the Full Group

In this section we consider a general connected and simply connected Lie group \(G\). We denote by \(W\) a connected analytic submanifold of \(G\) that generates \(G\), in the sense that \(W\) is not contained in any proper closed
subgroup of $G$. A simple analytic continuation argument shows that if $W'$ is any open submanifold of $W$, then also $W''$ generates $G$.

**Proposition 1.1.** There is an integer $m$ such that $W^m$ contains an open subset of $G$.

**Proof.** We proceed by induction on the codimension of $W$ in $G$. Let $n = \dim G$, $k = \dim W$. If $k = n$ there is nothing to prove.

We cannot have $k = 0$, i.e., $W$ consisting of a single element. If it were so, $G$ would be Abelian, therefore isomorphic to $\mathbb{R}^n$, since it is connected and simply connected; but no single element can generate $\mathbb{R}^n$ as a group.

Let therefore $1 \leq k < n$, and define $\phi: W \times W \to G$ as $\phi(x, y) = xy$. $\phi$ is an analytic map whose rank is at least $k$ at each point. We prove, by contradiction, that $\phi$ has rank strictly larger than $k$ at some point.

Assume that the rank of $\phi$ equals $k$ at each point of $W \times W$. Then, by restricting $W$ if necessary, $W^2$ is a $k$-dimensional manifold. Let $x$ and $y$ be two elements of $W$. Since $xW$ and $Wy$ are $k$-dimensional manifolds contained in $W^2$, then, in a neighborhood of $xy$, we have

$$W^2 = xW = Wy,$$

i.e.,

$$x^{-1}W^2y^{-1} = W^{-1} = x^{-1}W$$

in a neighborhood of the identity.

For fixed $x_0 \in W$, let $S = x_0^{-1}W$, and let $\mathfrak{h}$ (as a subspace of the Lie algebra $\mathfrak{g}$ of $G$) be the tangent space to $S$ at the identity. Given $x \in W$, near the identity we have $Wx^{-1} = x_0^{-1}W = S$. Therefore

$$x^{-1}S = x^{-1}Wx^{-1} = x^{-1}W = x_0^{-1}W = S,$$

which shows that $\mathfrak{h}$ is $\text{Ad}(W)$-invariant. Since $W$ generates $G$, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. Let $H = \exp \mathfrak{h}$; then $S \subseteq H$. In fact, if $y = x_0^{-1}x \in S$, with $x \in W$, then $y^{-1}S = x^{-1}x_0S = x^{-1}W = S$ near the identity, and this implies that tangent vectors to $S$ at any point are left-translates of vectors in $\mathfrak{h}$. But then $W \subseteq x_0H$, so that a single element generates the full quotient group $G/H$. But this is impossible since $G/H$ is not trivial and is simply connected.

Let therefore $p > k$ be the maximum rank attained by $\phi$ on $W \times W$. If $p = n$ we are done. Otherwise, let $(x, y) \in W \times W$ be a point where $\phi$ has rank $p$. Let $U_x$ and $U_y$ be connected neighborhoods of $x$ and $y$, respectively, in $W$ such that $U_xU_y$ is a $p$-dimensional manifold. By the inductive hypothesis, the proof will be complete as soon as we show that $U_xU_y$ generates $G$.

Let $G'$ be the closed subgroup of $G$ generated by $U_xU_y$. Since $\phi$ maps
$U_x \times U_y$ into $G'$, it follows by analytic continuation that $W^2 \subset G'$. But $W^2$ generates $G$. To see this it is enough to show that $W^2$ generates a neighborhood of the identity. By using the exponential map, it is easily seen that any element close enough to the identity is a square, so it can be approximated by even products of elements in $W \cup W^{-1}$. At this point it is sufficient to observe that, if $y_1, y_2 \in W$, then $y_1 y_2^{-1} = y_1^2(y_2 y_1)^{-1} \in W^2(W^2)^{-1}$ to see that these products can be rearranged as products of elements in $W^2 \cup (W^2)^{-1}$. It follows that $G' = G$. Q.E.D.

**Corollary 1.2.** Let $m$ be as in Proposition 1.1. There is a neighborhood $U$ of the identity in $G$ such that if $x_0, x_1, \ldots, x_m \in U$, then $x_0 W x_1 W \cdots W x_m$ contains an open set.

**Proof.** For $\bar{x} = (x_0, x_1, \ldots, x_m) \in G^{m+1}$, let $\phi_{\bar{x}} : W \times W \cdots \times W \to G$ be defined as

$$\phi_{\bar{x}}(y_1, \ldots, y_m) = x_0 y_1 x_1 y_2 \cdots y_m x_m.$$  

Since $\phi_{(x_0, x_1, \ldots, x_m)}$ has rank $n = \dim G$ at some point, also $\phi_{\bar{x}}$ has rank $n$ at the same point for $x_0, x_1, \ldots, x_m$ close enough to the identity. Q.E.D.

**2. Smoothness of Measures Transported by Analytic Maps**

Let $\phi$ be an analytic function defined on the closure $\overline{B}$ of the unit ball in $\mathbb{R}^n$ with values in $\mathbb{R}^m$, $n \leq m$, and having rank $n$ at almost every point. Let also $\psi(x)$ be a $C^1$-function supported on the unit ball in $\mathbb{R}^m$, and let $d\mu(y) = \phi_*(\psi(x) \, dx)$ be the transported measure by $\phi$ of the measure $\psi(x) \, dx$ on $\mathbb{R}^m$, i.e., such that for $f \in C_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} f(y) \, d\mu(y) = \int_{\mathbb{R}^m} f(\phi(x)) \, \psi(x) \, dx.$$  

**Proposition 2.1.** The measure $d\mu$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^n$, and if $\rho(y)$ is its density, then $\rho$ satisfies an $L^1$-Hölder condition

$$\int_{\mathbb{R}^n} |\rho(y + t) - \rho(y)| \, dy \leq C \|t\|^\delta$$  

for some $\delta > 0$ depending only on $\phi$, and some $C > 0$ that depends also on the $C^1$-norm of $\psi$.

**Proof.** Let $J(x) = (\sum_k J_k(x)^2)^{1/2}$, where the $J_k$ are the minors of order $n$
of the Jacobian matrix $\phi'(x)$. Since $J^2$ is real-analytic, its zero set $Z$ has measure zero. Following an idea in [20], we consider a covering $\{B_j = B(x_j, r_j)\}$ of the complement of $Z$ in the unit ball in $\mathbb{R}^m$ such that

1. $r_j = cJ(x_j)$;
2. the balls $B^*_j = B(x_j, 2r_j)$ are disjoint from $Z$ and have the bounded overlapping property; i.e., there is an integer $N$ such that no point belongs to more than $N$ of the $B^*_j$.

For fixed $j$, let $J_k$ be a minor of maximum absolute value at $x_j$. Since the gradient $\nabla J_k$ is bounded on $\bar{B}$, if the constant $c$ in (i) is small enough, we have

$$|J_k(x)| > c'r_j$$

on all of $B^*_j$. Without loss of generality, we can assume that $J_k$ is determined by the first $n$ rows in $\phi'$, and we write $x \in \mathbb{R}^m$ as $x = (x', x'') \in \mathbb{R}^n \times \mathbb{R}^{m-n}$. If $B'_j = B(x'_j, r_j) \subset \mathbb{R}^n$ and $B''_j = B(x''_j, r_j) \subset \mathbb{R}^{m-n}$, then $B_j \subset B'_j \times B''_j \subset B^*_j$.

For each $x'' \in B''_j$, $\phi_{x''} = \phi_{|B''_j \setminus x''}$ has a Jacobian determinant that is larger than $c'r_j$. Moreover $\phi_{x''}$ is one-to-one, which follows from the inverse function theorem: if

$$G(x', y) = x' + \left[\phi_{x''}(x'_j)\right]^{-1} (y - \phi_{x''}(x')),$$

then

$$\frac{\partial G}{\partial x'} = I - \left[\phi_{x''}(x'_j)\right]^{-1} \phi_{x''}(x').$$

By Cramer's rule

$$\left\|\left[\phi_{x''}(x'_j)\right]^{-1}\right\| \leq \frac{K}{|J_k(x_j)|} = \frac{Kc}{r_j},$$

where $c$ is the constant in (i). By the boundedness of the second derivatives of $\phi$ on $\bar{B}$, it follows that

$$\left\|\phi_{x''}(x'_j) - \phi_{x''}(x')\right\| \leq Kr_j.$$

Therefore if the constant $c$ in (i) is small enough, the contraction condition on $G$ is satisfied on all of $B'_j$, so that $\phi_{x''}$ is invertible on $B'_j$.

Let now $\{\eta_j\}$ be a smooth partition of unity on $\bar{B} \setminus Z$ subordinated to the covering $\{B_j\}$ and such that $\|\nabla \eta_j\| \leq C r_j^{-1}$.
Let also $\psi_j = \psi_j y, \ d\mu_j(y) = \phi_j(\psi_j(x) dx), \ d\mu_{j,x'}(y) = (\phi_{j,x'})_*(\psi_j(x', x'') dx').$

Then

$$d\mu_{j,x'}(y) = |J_k(\phi_{j,x'}^{-1}(y), x'')|^{-1} \psi_j(\phi_{j,x'}^{-1}(y), x'') dy = \rho_{j,x'}(y) dy.$$ \hspace{1cm} (3)

The functions $\rho_{j,x'}$ satisfy the following two properties:

$$\int_{\mathbb{R}^n} |\rho_{j,x'}(y)| dy \leq C \int_{B_j} dx' = Cr^n; \hspace{1cm} (4)$$

$$\int_{\mathbb{R}^n} |\nabla \rho_{j,x'}(y)| dy \leq C r_j^{-2} \int_{B_j} dx' = Cr^{n-2}; \hspace{1cm} (5)$$

since

$$\nabla \rho_{j,x'}(\phi_{j,x'}(x')) \equiv [\phi_{j,x'}^{-1}(x')]^{-1} \nabla x'(J_k(x', x'')^{-1} \psi_j(x', x'')),$$

so that by (2) and Cramer's rule,

$$|\nabla \rho_{j,x'}(\phi_{j,x'}(x'))| \leq C r_j^{-2} |J_k(x', x'')|^{-1}.$$

It follows that the measure

$$\mu_j = \int_{B_j} \rho_{j,x'}(y) dx'$$

has a density

$$\rho_j(y) = \int_{B_j} \rho_{j,x'}(y) dx'$$

which satisfies the two conditions

$$\int_{\mathbb{R}^n} |\rho_j(y)| dy \leq Cr^n \hspace{1cm} (6)$$

$$\int_{\mathbb{R}^n} |\nabla \rho_j(y)| dy \leq Cr^{n-2}. \hspace{1cm} (7)$$

This implies that if $0 < \delta < 1$

$$\|\rho_j\|_{L^1} = \sup_{t \in \mathbb{R}^n} \|t\|^{-\delta} \int_{\mathbb{R}^n} |\rho(y + t) - \rho(y)| dy \leq Cr^{m-2\delta} \hspace{1cm} (8)$$

The sum $\sum_j r_j^{m-2\delta}$ is comparable with the integral $\int_{\mathbb{R}^n} J(x)^{-2\delta} dx$. This
integral is convergent for $\delta$ close to zero, as it follows from the fact that $I^2$ is real-analytic and the Weierstrass preparation theorem [12]. This implies that for $\delta > 0$ and small enough, $\rho(y) = \sum_i \rho_i(y)$ satisfies (1).

But $\psi(x) \int_x = \sum_i \psi_i(x) \int$, because $Z$ has measure zero, and this implies that

$$d\mu(y) = \sum_i d\mu_i(y) = \rho(y) \int.$$  

Q.E.D.

In order to apply Proposition 2.1 in the sequel, we have to consider the situation in which the range of the map $\phi$ is a Lie group and the smoothness of $\rho$ is referred to group translations. In this regard, the following property holds.

**Lemma 2.2.** Let $G$ be a unimodular Lie group, $\mathfrak{g}$ be its Lie algebra, and $\rho$ a function on $\exp_G B$, where $B$ is the unit ball in $\mathfrak{g}$ with respect to a given norm $\|\|$.

Then $\rho$ satisfies an $L^1$-Hölder condition

$$\int_G |\rho(x \exp G Y) - \rho(x)| \int_x \leq C \|Y\|^\delta$$  

if and only if the function $\tilde{\rho} = \rho \circ \exp_G$ on $\mathfrak{g}$ satisfies a condition

$$\int_{\mathfrak{g}} |\tilde{\rho}(X + Y) - \tilde{\rho}(X)| \int X \leq C' \|Y\|^\delta.$$  

Furthermore, right translation by $\exp_G Y$ in (9) can be replaced by left translation.

**Proof.** For $0 < t \leq 1$ let $B_t$ be the ball of radius $t$ in $\mathfrak{g}$ and let $\phi_t$ be a smooth function on $G$ supported on $\exp_G B_t$ such that $\phi_t \geq 0$, $\int G \phi_t(x) \int_x = 1$, and $\int G |(\phi_t \circ X)(x) | \int_x \leq Ct^{-1} \|X\|$ for every $X \in \mathfrak{g}$. Then $\{\phi_t\}$ is an approximate identity in $G$.

Assume that (9) holds. Then

$$\|\rho \circ \phi_t - \rho\|_{L^1(G)} = \int G \left( \int G (\rho(xy^{-1}) - \rho(x)) \phi_t(y) \int_y \right) \int_x \leq Ct^\delta.$$  

(11)

Since $\phi_t \circ X$ has mean value zero, we also have

$$\|\rho \circ \phi_t \circ X\|_{L^1(G)} = \int G \left( \int G (\rho(xy^{-1}) - \rho(x))(\phi_t \circ X)(y) \int_y \right) \int_x \leq Ct^\delta - 1 \|X\|.$$  

(12)
Let \( \hat{\rho}_t = (\rho \ast \varphi_t) \circ \exp_G \), and let \( y_1, ..., y_n \) be linear coordinates on \( g \). If \( \{X_1, ..., X_n\} \) is a basis of \( g \), then

\[
(\rho \ast \varphi_t \ast X_j) \circ \exp_G = \sum_i a_{ij}(y) \frac{\partial \hat{\rho}_t}{\partial y_i}
\]

with \( \det(a_{ij}(y)) \neq 0 \) for every \( y \). By applying Cramer's rule, we obtain that

\[
\|\nabla \hat{\rho}_t\|_{L^1(g)} \leq C t^{\delta - 1}
\]

for every \( t \leq 1 \). Therefore for every \( t \leq 1 \) and \( Y \in g \),

\[
\int_g |\hat{\rho}_t(X + Y) - \hat{\rho}_t(X)| \, dX \leq C t^\delta - 1 \| Y \|.
\]

(13)

Since

\[
\|\hat{\rho}_t - \hat{\rho}\|_{L^1(g)} \leq C \|\rho \ast \varphi_t - \rho\|_{L^1(G)} \leq C t^\delta.
\]

then

\[
\int_g |\hat{\rho}(X + Y) - \hat{\rho}(X)| \, dX \leq C (t^\delta - 1 \| Y \| + 2t^\delta).
\]

(14)

Taking \( t = \| Y \| \) when \( \| Y \| \leq 1 \), we obtain (10). For \( \| Y \| > 1 \), (10) follows from the trivial estimate

\[
\int_g |\hat{\rho}(X + Y) - \hat{\rho}(X)| \, dX \leq 2 \|\hat{\rho}\|_{L^1(g)}.
\]

By the same argument one proves that (10) implies (9), and the last statement is now trivial. Q.E.D.

**Corollary 2.3.** Let \( V_1, ..., V_k \) be connected analytic submanifolds of a unimodular Lie group \( G \), and assume that the product \( V_1 V_2 \cdots V_k \) contains an open set in \( G \). If for each \( j = 1 \cdots k \) we are given measures \( d\mu_j = \varphi_j \, d\sigma_j \), where \( d\sigma_j \) is the surface measure on \( V_j \) and \( \varphi_j \) is a smooth function with compact support on \( V_j \), then \( d\mu_1 \ast \cdots \ast d\mu_k \) is absolutely continuous with respect to Haar measure and its density \( \rho \) satisfies a right \( L^1 \)-Hölder condition as in (9), as well as a left \( L^1 \)-Hölder condition, for some exponent \( \delta > 0 \) that depends only on the manifolds.¹

¹ The fact that iterated convolutions of such singular measures can give rise to absolutely continuous measures has been known for various examples for quite some time, in particular in the work of F. John, J. E. Björck, O. C. McGehee, and G. S. Woodward, among others.
The proof follows easily from Proposition 2.1 and Lemma 2.2. In fact \( d\mu_1 \ast \cdots \ast d\mu_k \) is obtained by transporting the measure \( d\mu_1 d\mu_2 \cdots d\mu_k \) defined on \( V_1 \times V_2 \cdots \times V_k \) by means of \( \phi(x_1, x_2, \ldots, x_k) = x_1 x_2 \cdots x_k \).

3. Convolution Operators with Singular Kernels Supported on Manifolds

We assume now that the group \( G \) is nilpotent and has dilations \( \{D_\delta\}_\delta,\delta>0 \), i.e., automorphisms of \( G \) such that

(i) \( D_\delta D_\varepsilon = D_{\delta \varepsilon} \), for all \( \delta, \varepsilon > 0 \);

(ii) the differential of \( D_\delta \) diagonalizes on \( \mathfrak{g} \) with eigenvalues \( \delta^{x_1}, \ldots, \delta^{x_k} \), where \( \alpha_1, \ldots, \alpha_i \) are positive.

If \( \mathfrak{g}_{x_i} \) is the eigenspace of \( \mathfrak{g} \) relative to the exponent \( \alpha_i \) and \( d_j \) is its dimension, we denote by \( Q = \sum_{i=1}^k d_j \alpha_j \) the homogeneous dimension of \( G \).

We will use frequently the notation \( D_\delta x \) in place of \( D_\delta \cdot x \). We also denote by \( |x| \) a homogeneous gauge on \( G \), that we can take to be smooth away from the origin.

We will consider distributions \( K \) on \( G \) that satisfy the following condition.

Condition A. (i) The support of \( K \) is contained in \( V \cup \{0\} \), where \( V \) is a connected analytic homogeneous submanifold of \( G \) not containing the identity 0.

(ii) \( \langle K, f \rangle = p.v. \int_V f(x) K(x) \, d\sigma(x) \),

where \( d\sigma \) is surface measure on \( V \), \( K \) is a smooth function on \( V \) such that for \( 0 < a < b < \infty \)

\[ \int_{a < |x| < b} K(x) \, d\sigma(x) = 0 \]

and the measure \( K(x) \, d\sigma(x) \) on \( G \setminus \{0\} \) is homogeneous of degree \( -Q \).

It follows from (i) that if \( \Sigma = \{ x \in G: |x| = 1 \} \), then \( \text{supp} \, K \cap \Sigma \) is compact in \( V \cap \Sigma \). Furthermore, \( K \) is a homogeneous distribution on \( G \) of the critical degree \( -Q \).

It should be noted that in general the surface measure \( d\sigma(x) \) and the density \( K(x) \) do not separately have any property of homogeneity. A simple example of non-homogeneous surface measure is given by arc-length on the parabola \( y = x^2 \) in \( \mathbb{R}^2 \), where dilations are given by \( (x, y) \rightarrow (\delta x, \delta^2 y) \).
In addition to Condition A we will formulate a more general Condition B in Section 4.

Let now $V_0$ be the set of elements $x$ in $V$ such that $\frac{1}{2} < |x| < 2$, and let $\psi(x) = \hat{\varphi}(|x|)$ be a $C^\infty$-function supported on $V_0$ such that for every $x \neq 0 \sum_{j \in \mathbb{Z}} \psi(2^j x) = 1$. If we set

$$K_j(x) = K(x) \psi(2^j x),$$

then $K_j = K_j(x) \, d\sigma(x)$ is supported on $V_j = \{x \in V; 2^{-j-1} < |x| < 2^{-j+1}\}$. Furthermore

$$\int_G K_j(x) \, d\sigma(x) = 0$$

for every $j$ and $K = \sum_j K_j$. Together with $K_j$, we introduce the measures $K_j^*$ supported on $V_j^{-1}$ with densities $K_j^*(x) = K_j(x^{-1})$.

**Lemma 3.1.** Let $m$ be such that $V_0^m$ contains an open subset of $G$, according to Proposition 1.1. Then for every $x_0, x_1, \ldots, x_m \in G$ also $x_0 V_0 x_1 V_0 \cdots V_0 x_m$ contains an open subset of $G$.

**Proof.** Since $V$ is analytic, $x_0 V_0 x_1 \cdots V_0 x_m$ contains an open set if and only if $x_0 V x_1 \cdots V x_m$ does. So the set of $(m + 1)$-tuples $(x_0, \ldots, x_m)$ in $G^{m+1}$ for which this property holds is dilation-invariant. By Corollary 1.2 it contains a neighborhood of $(0, 0, \ldots, 0) \in G^{m+1}$, therefore it is all of $G^{m+1}$.

Q.E.D.

We denote by $f^{*m}$ the $m$th convolution power of $f$.

**Lemma 3.2.** Assume that $V$ generates $G$. Then there is an integer $m$ such that $(K_0 \ast K_0^*)^m$ has a density $\rho(x)$ on $G$ such that

$$\int_G |\rho(xy) - \rho(x)| \, dx \leq C |y|^\epsilon \tag{15}$$

and

$$\int_G |\rho(yx) - \rho(x)| \, dx \leq C |y|^\epsilon \tag{16}$$

for some positive constants $C$ and $\epsilon$.

**Proof.** If follows from Lemma 3.1 that $(V_0 V_0^{-1})^m$ contains an open sub-
set of $G$ for $m$ large. Let $\|Y\|$ be a norm on $g$. By Corollary 2.3, $(K_0 \ast K_0)^m$ has a density $\rho(x)$ on $G$ such that
\[
\int_G |\rho(x \exp_G Y) - \rho(x)| \, dx \leq C \|Y\|^\delta
\]
for some $C, \delta > 0$, and $\rho$ also satisfies a similar condition relative to left translations. As (15) and (16) are trivial for $|y| > 1$, it is enough to consider $|y| \leq 1$. But if $|y| \leq 1$, there is an exponent $\epsilon > 0$ such that if $y = \exp_G Y$, then $\|Y\|^\delta \leq C |y|^\epsilon$ [7], and this completes the proof. Q.E.D.

We will also need the following lemma.

**Lemma 3.3.** Let $f$ be a bounded function on $G$ supported on the unit ball, with mean value zero and satisfying the $L^1$-Hölder conditions (15) and (16). If $M$ is the singular kernel
\[
M(x) = \sum_{j \in \mathbb{Z}} 2^{jQ} f(2^j x)
\]
then the operator of (either left or right) convolution by $M$ is bounded on $L^p(G)$ for $1 < p < \infty$, with a norm that only depends on $p, \epsilon$, and the constant $C$ in (15) and (16).

The proof follows the same lines as that given in [8, Lemma 2.1] for the Heisenberg groups.

**Theorem 3.4.** Let $K$ be a distribution satisfying Condition A and suppose that $V$ generates $G$. Then the operator of (either left or right) convolution by $K$ is bounded on $L^p(G)$ for $1 < p < \infty$.

**Proof:** Let $\psi_j(x) = \psi(2^j x)$ be the partition of unity introduced before. If $c = \int_G \psi(x) \, dx$, define
\[
\eta_j(x) = c^{-1}(2^{(j+1)Q} \psi_{j+1}(x) - 2^j \psi_j(x)).
\]
Then for every $j_0 \in \mathbb{Z}$,
\[
\delta_0 = \sum_{j \geq j_0} \eta_j + c^{-1} 2^{j0Q} \psi_{j_0}
\]
on $G$. Away from the origin $K$ can be decomposed as
\[
K = \sum_{j \in \mathbb{Z}} K_j \ast \left( \sum_{l \geq j+10} \eta_l \right) \ast 1 \sum_{j \in \mathbb{Z}} c^{-1} 2^{(j+10)Q} K_j \ast \psi_{j+10}
\]
\[
= \sum_{k \geq 10} \left( \sum_{j \in \mathbb{Z}} K_j \ast \eta_{j+k} \right) + c^{-1} 2^{10Q} \sum_{j \in \mathbb{Z}} 2^{jQ} K_j \ast \psi_{j+10}.
\]
Then the distributions

\[ M_k = \sum_{j \in \mathbb{Z}} K_j * \eta_{j+k} \]

and

\[ N = c^{-1} 2^{10Q} \sum_{j \in \mathbb{Z}} 2^{jQ} K_j * \psi_{j+10} \]

are Calderón–Zygmund kernels that are smooth away from the origin. We will show that, if \( \|M_k\|_{p, p} \) denotes the convolution operator norm of \( M_k \) on \( L^p(G) \), the series \( \sum_{k \geq 10} \|M_k\|_{p, p} \) converges for every \( p, 1 < p < \infty \). Since this easily implies that the identity

\[ K = \sum_{k \geq 10} M_k + N \]

holds on all of \( G \), the proof will be complete.

We first estimate \( \|M_k\|_{p, p} \) by making use of Lemma 3.3. The functions \( K_0 * \eta_k \) are supported on a fixed ball for \( k \geq 10 \) and, if \( R_y \) denotes the operator of right translation by \( y \),

\[ \int_G |K_0 * \eta_k(xy) - K_0 * \eta_k(x)| \, dx = \|K_0 * (R_y \eta_k - \eta_k)\|_1 \]

\[ \leq C \|R_y \eta_k - \eta_k\|_1 \]

\[ = C \int_G |\eta_k(xy) - \eta_k(x)| \, dx \]

\[ = C \int_G |\eta_0(x(2^k y)) - \eta_0(x)| \, dx \]

\[ \leq C 2^{ek} |y|^\varepsilon \]

for every \( \varepsilon, 0 < \varepsilon \leq 1 \), since \( \eta_0 \) is smooth. It follows from Lemma 3.3 that for \( 0 < \varepsilon \leq 1 \)

\[ \|M_k\|_{p, p} \leq C_{\varepsilon, p} 2^{ek}. \quad (18) \]

We look now for a better estimate of the norms \( \|M_k\|_{2, 2} \), by using the non-commutative version of Cotlar’s lemma [15] combined with the iteration argument used in [1]. By the scale-invariance properties of \( M_k \), we have to estimate the norms

\[ \|\eta_{j+k}^* K_j^* * K_0 * \eta_k\|_{2, 2}, \quad \|K_0 * \eta_k * \eta_{j+k}^* * K_j^*\|_{2, 2} \quad (19) \]
for $k \geq 10$ and $j \geq 0$. Trivially, the norms in (19) are controlled by

$$\|\eta_{j+k}^* * K_j^* * K_0\|_{2,2}, \quad \|K_0 * \eta_k * \eta_{j+k}\|_{2,2},$$

(20)

respectively.

Now, if $A$ and $B$ are bounded linear operators on a Hilbert space, then

$$\|AB\| \leq \|A\|^{1/2} \|ABB^*\|^{1/2}.$$ Iterating $l$ times,

$$\|AB\| \leq \|A\|^{1-2^{-l}} \|A(BB^*)^{2l-1}\|^{2^{-l}}. \quad (21)$$

If we apply (21) to the first norm in (20), we obtain

$$\|\eta_{j+k}^* * K_j^* * K_0\|_{2,2} \leq \|\eta_{j+k}^* * K_j^*\|_{2,2}^{1-2^{-l}} \|\eta_{j+k}^* * K_j^* * (K_0 * K_0^*)^{2 l-1}\|_{2,2}^{2^{-l}}. \quad (22)$$

The norm $\|\eta_{j+k}^* * K_j^*\|_{2,2}$ is controlled by the product of the $L^1$-norm of $\eta_{j+k}$ and the measure norm of $K_j$, which are bounded independently of $j$ and $k$. By Lemma 3.2, we can take $l$ large enough so that $(K_0 * K_0^*)^{2 l-1}$ is a function $\rho(x)$ that has compact support and satisfies $L^1$ Hölder conditions (15) and (16). After convolution from the left with $K_j^*$, the right $L^1$ Hölder condition is preserved with constants that do not depend on $j$.

Using Lemma 2.2 and the fact that for $|y| < 1$, $y = \exp_G Y$,

$$C_1 \|Y\|^{q_1} \leq |y| \leq C_2 \|Y\|^{q_2}$$

(see [7]), we see that also some left $L^1$ Hölder condition is satisfied by $K_j^* * \rho,

$$\int_G |K_j^* * \rho(yx) - K_j^* * \rho(x)| \, dx \leq C |y|^{c^*}. \quad (23)$$

The constant $C > 0$ and the exponent $c^* > 0$ do not depend on $j$ because all the functions involved are supported on the same bounded set, by the assumption $j \geq 0$.

As $\eta_{j+k}^*$ has mean value zero and $\eta_{j+k}^*(x) = 2^{(j+k)^2} \eta_0^*(2^{j+k}x)$, it is easily seen that

$$\|\eta_{j+k}^* * K_j^* * \rho\|_1 \leq C 2^{-c'(j+k)}$$

and therefore

$$\|\eta_{j+k}^* * K_j^* * K_0 * \eta_k\|_{2,2} \leq C 2^{-\sigma(j+k)},$$
where $\sigma$ is some positive exponent. The second norm in (19) is estimated similarly. It follows that

$$\|M_k\|_{2,2} \leq C2^{-\alpha k}$$

(24)

which proves the theorem for $p = 2$.

Let now $p$ be a fixed exponent, $1 < p < 2$, and take $p_0$ such that $1 < p_0 < p$. We can then interpolate between (24) and the estimate (18) for $\|M_k\|_{p_0,p_0}$. If we choose the exponent $\varepsilon$ in (18) small enough, we obtain an exponential decay for $\|M_k\|_{p,p}$.

The result for $2 < p < \infty$ can now be proved by duality. Q.E.D.

Remarks. The hypotheses on $K$ in Theorem 3.4 (as well as in Theorem 4.1 in the next section) can be considerably relaxed.

Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be $C^1$-functions on $V$ supported on $V_0$, uniformly bounded, together with their first derivatives in the directions tangent to $V$. Let $K_j$ the measure supported on $V_j$ obtained by dilating $\phi_j\,d\sigma$ with a factor in front that preserves its total mass. Then, if each $\phi_j$ has mean value zero, the conclusions of Theorem 3.3 hold for $K = \sum_{j \in \mathbb{Z}} K_j$. A relevant example is provided by distributions with complex degree of homogeneity $-Q + i\gamma$, with $\gamma$ real.

The proof of Theorem 3.4 actually shows that if $\text{Re}\,z < \sigma$, then the kernel

$$K_z = \sum_{k=0}^{\infty} 2^{\alpha k} M_k$$

gives a bounded operator on $L^2(G)$, and for $\text{Re}\,z < 0$ but arbitrarily close to zero, $K_z$ gives a bounded operator on every $L^p(G)$, $1 < p < \infty$. In this sense the argument given here is analogous to the proof of Theorem 11(A) in [21].

4. Kernels Supported on Finite Unions of Manifolds and Maximal Operators

In Section 3 we have considered principal value distributions with smooth densities on analytic manifolds not containing the origin which are connected. For instance, Hilbert transforms along homogeneous curves do not fall under the scope of Theorem 3.4. However, they are known to be bounded on $L^p(G)$ for $1 < p < \infty$ [1, 21].

In order to include these operators, and many relevant others, in our discussion, we consider a more general class of kernels than those defined by Condition A.

Condition B. (i) The support of $K$ is contained in $V_1 \cup V_2 \cup \cdots \cup$
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$V_m \cup \{0\}$, where $V_1, V_2, \ldots, V_m$ are connected analytic homogeneous manifolds not containing the identity (they are not necessarily disjoint and do not necessarily have the same dimension);

(ii) $\langle K, f \rangle = \lim_{\varepsilon \to 0} \sum_{j=1}^{m} \int_{|x| > \varepsilon} f(x) K_j(x) d\sigma_j(x)$,

where $d\sigma_j$ is surface measure on $V_j$, $K_j$ is a smooth function on $V_j$ such that $(\text{supp } K_j) \cap \Sigma$ (where $\Sigma$ is the unit sphere in $G$) is compact in $V_j \cap \Sigma$; furthermore if $0 < a < b < \infty$

$$\sum_{j=1}^{m} \int_{a < |x| < b} K_j(x) d\sigma_j(x) = 0$$

and $K_j d\sigma_j$ are homogeneous measures on $G \setminus \{0\}$ of degree $-Q$.

The resulting distribution $K$ is then homogeneous of degree $-Q$.

**Theorem 4.1.** Assume that each $V_j$ generates the full group $G$ and that $K$ satisfies Condition B. Then the convolution operator determined by $K$ is bounded on $L^p(G)$ for $1 < p < \infty$.

**Proof.** Let $\psi$ be as in Section 3, and let $K_0(x) = K(x) \psi(x)$.

It is sufficient to show that for some integer $l$ the conclusion of Lemma 3.2 is valid for $K_0$.

If $K_0 = \sum_{j=1}^{m} K'_j$, then $(K_0 \ast K'_0) \ast \psi$ expands as a sum of $2l$ convolutions with factors taken from the $K'_0$ or from the $K'_j$.

For each $j$, let $l_j$ be such that $V_j^l$ contains an open set. If $l$ is large enough, every summand contains at least $l_j$ factors $K'_j$ for at least one $j$. By Lemma 3.1 and Corollary 2.3, the conclusion of Lemma 3.2 then holds for each summand separately. Q.E.D.

We now turn our attention to maximal operators along homogeneous analytic manifolds. These operators are defined in such a way that they are dilation-invariant. For simplicity we consider connected manifolds.

Let $\psi(x)$, $0 \leq \psi \leq 1$, be a function with compact support on $V \cup \{0\}$, and let $d\mu(x) = \psi(x) d\sigma(x)$, where $d\sigma$ is the surface measure on $V$. We define the measures $\mu_r$, for $r > 0$, by setting

$$\int_{G} f(x) d\mu_r(x) = \int_{G} f(r^{-1}x) d\mu(x)$$

and the maximal operator

$$(\mathcal{M}_\psi f)(x) = \sup_{r > 0} \int_{G} |f(xy^{-1})| d\mu_r(y).$$
Lemma 4.2. If $\psi$ is supported on $V_0$, then the operator $\mathcal{M}_\psi$ is bounded on $L^p(G)$ for $1 < p \leq \infty$, and its norm depends only on the support of $\psi$.

Proof. The statement is trivial for $p = \infty$, so we assume $1 < p < \infty$. We can find a smooth non-negative function $\psi^*$ on $V$, compactly supported, and depending only on $\text{supp } \psi$, with the property that, if $\frac{1}{2} \leq r < 1$, then

$$d\mu_r \leq \psi^* \ d\sigma = d\mu^*.$$ 

It follows that

$$(\mathcal{M}_\psi f)(x) \leq \sup_{j \in \mathbb{Z}} \int_{V} |f(xy^{-1})| \ d\mu^*_{2^{-j}}(y). \quad (27)$$

Let now $\varphi$ be a smooth non-negative function on $G$, supported on $\{x: \frac{1}{2} < |x| < 2\}$ such that

$$\int_{G} \varphi(x) \ dx = \int_{G} d\mu^*(x).$$

Then the maximal operator

$$(Mf)(x) = \sup_{j \in \mathbb{Z}} 2^{j} \int_{G} |f(xy^{-1})| \ \varphi(2^{j}y) \ dy$$

is bounded on $L^p(G)$ for $1 < p \leq \infty$ [7]. We define

$$dK_j(x) = d\mu^*(x) - 2^{j} \varphi(2^{j}x) \ dx$$

and consider the square-function operator

$$(Sf)(x) = \left( \sum_{j \in \mathbb{Z}} |f * K_j(x)|^2 \right)^{1/2}.$$ 

Since

$$(\mathcal{M}_\psi f)(x) \leq (Mf)(x) + (Sf)(x),$$

it is sufficient to show that $S$ is bounded on $L^p(G), 1 < p < \infty$.

For $j \in \mathbb{Z}$, let $\varepsilon_j = \pm 1$ and consider the singular kernel

$$\sum_{j \in \mathbb{Z}} \varepsilon_j K_j.$$ 

By Theorem 4.1 and one of the remarks at the end of Section 3,

$$\left\| f * \left( \sum_{j} \varepsilon_j K_j \right) \right\|_p \leq C_p \left\| f \right\|_p.$$
for $1 < p < \infty$, with $c_p$ independent of the choice of the $\varepsilon_j$. The conclusion follows from a standard argument involving Rademacher functions. Q.E.D.

Let now $M$ be a relatively compact open subset of $V_0$, and define $W = \bigcup_{\delta \leq 1} \delta M$. We want to extend the scope of Lemma 4.2 to include the case where $\psi$ is the characteristic function of $W$. In this case the support of $\psi$ contains the origin; however, its intersection with the unit sphere $\Sigma$ is compact in $V \cap \Sigma$.

**Theorem 4.3.** If $\psi$ is the characteristic function of $W$ and $M_\psi$ is defined by (26), then $M_\psi$ is bounded on $L^p(G)$ for $1 < p \leq \infty$.

**Proof.** Let $W_j = W \cap \{x: 2^{-j} < |x| \leq 2^{-j+1}\}$. Then $\chi_{W_j}$, the characteristic function of $W_j$, is the sum of the $\chi_{W_j}$, for $j \leq 0$. Therefore

$$M_{\chi_{W_j}} \leq \sum_{j \geq 0} M_{\chi_{W_j}} f.$$

The maximal function $M_{\chi_{W_j}}$ is the same as $M_{\psi_j}$, where $\psi_j$ is obtained by rescaling $\chi_{W_j} d\sigma$ by a factor $2^j$ according to (25). This means that if $f$ is a function on $V$,

$$\int_{V_j} f(x) \psi_j(x) d\sigma(x) = \int_{W_j} f(2^{-j} x) d\sigma(x)$$

$$= \int_{W_0} f(x) J_j(x) d\sigma(x),$$

where $J_j$ is the Jacobian determinant of the map from $V$ to itself that sends $x$ into $2^{-j} x$. A simple argument shows that there is an exponent $\gamma > 0$ such that if $x \in W_0$ and $j \geq 0$, then $J_j(x) \leq C 2^{-\gamma j}$.

By Lemma 4.2,

$$\|M_{xW_j} f\|_p = \|M_{f_j} f\|_p \leq C_\gamma 2^{\gamma} \|f\|_p$$

for $1 < p \leq \infty$, and therefore

$$\|M_{xW_j} f\|_p \leq C_\gamma \|f\|_p.$$  

Q.E.D.

**PART II**

5. **SINGULAR INTEGRALS ON GENERAL NILPOTENT GROUPS**

We shall now apply the results obtained in the first part of this paper to other settings, by means of the so-called "method of transference"
We first present a general transference Theorem for maximal operators, which is essentially contained in [2], but it is not stated there at the level of generality that we require.

We assume that $G$ is an amenable group (e.g., a connected nilpotent Lie group, as it will be in our applications) and that $\pi$ is an isometric representation of $G$ on $L^p(S)$, where $S$ is a $\sigma$-finite measure space and $1 \leq p \leq \infty$. If $\mu$ is a measure on $G$ with compact support, one defines the transferred operator $T_\mu$ on $L^p(S)$ according to the formula

$$
(T_\mu f)(s) = \int_G (\pi(x^{-1})f)(s) \, d\mu(x).
$$

(28)

**Proposition 5.1.** Suppose that for every $r > 0$ $\mu_r$ is a measure on $G$ with compact support. If the maximal operator

$$(\mu^* \varphi)(x) = \sup_{r > 0} |\varphi \ast \mu_r(x)|$$

(29)

is bounded on $L^p(G)$, then the transferred maximal operator

$$
(T^* f)(s) = \sup_{r > 0} |(T_{\mu_r} f)(s)|
$$

(30)

is bounded on $L^p(S)$ with a norm that depends only on the operator norm of $\mu^*$ on $L^p(G)$.

**Proof.** Given $M$ compact in $G$, consider the truncated maximal operators $\mu_M^*, T_M^*$, defined by taking the suprema in (29) and (30) only over those $r$ for which $\text{supp} \mu_r \subseteq M$.

Since for every $u \in G$ and $\varphi \in L^p(S) \| \varphi \|_p = \| \pi(u) \varphi \|_p$, if $U$ denotes a compact set in $G$ of positive measure, then

$$
\|T_M^* f\|_{L^p(S)}^p = \frac{1}{m(U)} \int_U \|\pi(u) T_M^* f\|_{L^p(S)}^p \, du
$$

$$
= \frac{1}{m(U)} \int_U \int_S \sup \left| \int_G (\pi(ux^{-1})f)(s) \, d\mu_\varphi(x) \right|^p \, ds \, du.
$$

Let $\varphi_s(x) = (\pi(x)f)(s)$, for fixed $s \in S$. Then

$$
\|T_M^* f\|_{L^p(S)}^p = \frac{1}{m(U)} \int_U \int_S \sup |\varphi_s \ast \mu_r(u)|^p \, ds \, du
$$

$$
= \frac{1}{m(U)} \int_U \int_S \sup |(\varphi_s \ast \mu_r) \ast \mu_r(u)|^p \, ds \, du.
$$
since supp \( \mu_e \subset M \). Therefore, if \( C \) is the operator norm of \( \mu^* \) on \( L^p(G) \),

\[
\|T_{\mu}^* f\|_{L^p(S)}^p = \frac{1}{m(U)} \int_S \int_G \mu_{M}^{\#}(\varphi_{s|_{UM^{-1}}})(u)^p \, du \, ds \\
\leq \frac{C^p}{m(U)} \int_S \int_{UM^{-1}} |\varphi_{s}(u)|^p \, du \, ds \\
= \frac{C^p}{m(U)} \int_{UM^{-1}} \int_S |(\pi(u)f)(s)|^p \, ds \, du \\
= C^p \frac{m(UM^{-1})}{m(U)} \|f\|_{L^p(S)}^p.
\]

Since \( G \) is amenable, it is possible to determine \( U \) such that \( m(UM^{-1})/m(U) \) is arbitrarily close to one. It follows that

\[
\|T_{\mu}^* f\|_{L^p(S)} \leq C \|f\|_{L^p(S)}
\]

for every compact subset \( M \) of \( G \). The conclusion then follows by the monotone convergence theorem.

Q.E.D.

Proposition 5.1 applies in particular to one single measure \( \mu \) with compact support in \( G \). In this case the statement is that the operator norm of \( T_{\mu} \) on \( L^p(S) \) is dominated by the operator norm of \( \mu \) as a right convolution operator on \( L^p(G) \).

Before treating the operators that are the main object of the second part of this paper, we show a simple application of the method of transference that allows to eliminate the hypothesis that the manifold \( V \) generates the full group in Theorem 3.4.

Assume that \( K \) satisfies Condition A. Then the principal value distribution

\[
\langle K, f \rangle = \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < \varepsilon^{-1}} K(x) f(x) \, d\sigma(x)
\]

is well defined also if \( V \) does not generate \( G \). Let \( H \) be the closed subgroup of \( G \) generated by \( V \). Then \( H \) is homogeneous, with the dilations induced by \( G \), and by Theorem 3.4 the operator of right convolution by \( K \) is bounded on \( L^p(H) \) for \( 1 < p < \infty \). The proof of Theorem 3.4 also shows that the truncated operators \( K_{\varepsilon} \) defined by (31) are uniformly bounded on \( L^p(H) \).

We now consider the representation \( \pi \) of \( H \) on \( L^p(G) \) given by restricting to \( H \) the right regular representation of \( G \) on \( L^p(G) \). Then \( \pi \) is isometric,
and if we apply Proposition 5.1 to each single measure $K_\varepsilon$, we see that the transferred operators $T_{K_\varepsilon}$ are uniformly bounded on $L^p(G)$. It is also easily seen that $T_{K_\varepsilon}$ is given by right convolution by $K_\varepsilon$ on $G$.

It follows by weak compactness that there is a sequence $\varepsilon_j$ tending to zero such that $T_{K_{\varepsilon_j}}$ weakly converges to a bounded linear operator $T$ on $L^p(G)$. $T$ obviously commutes with left translations, as the various $T_{K_\varepsilon}$ do, so $Tf = f \ast L$, where $L$ is some distribution on $G$.

But $L$ is the limit of the $K_\varepsilon$ in the sense of distributions. By (31) this implies that $L = K$.

The same argument can be applied to a distribution satisfying Condition B, so we can state the following generalization of Theorems 3.4 and 4.1.

**Theorem 5.2.** The conclusion of Theorem 3.4 still holds if $V$ does not generate the full group. Similarly, the conclusion of Theorem 4.1 is also valid when all the manifolds $V_j$ generate the same closed subgroup of $G$.

The weak compactness argument used to prove Theorem 5.2 also holds in the next applications and we will not mention it again.

We now turn to one of the main applications, which has to do with the representation $\pi$ of the group $G$ on $L^p$-functions on a quotient group $G/H$. This allows us to study singular integrals on nilpotent groups related to dilations that are not automorphisms.

Let $N$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Given a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{n}$, we consider dilations $D_\delta$, for $\delta > 0$, on $\mathfrak{n}$ defined by $D_\delta X_j = \delta^{\alpha_j} X_j$, where $\alpha_j > 0$ for $j = 1, \ldots, n$. We do not require that $D_\delta$ is an automorphism of $\mathfrak{n}$.

The given dilations on $\mathfrak{n}$ determine (non-automorphic) dilations on the group, once a set of canonical coordinates has been fixed. The dilations on the group depend in general on the chosen coordinates.

The coordinate systems we refer to are obtained by grouping the elements $X_1, \ldots, X_n$ in an arbitrary way, so that $\mathfrak{n}$ is decomposed into dilation-invariant subspaces $\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_k$. Then the generic element $y \in N$ is parametrized by $Y = Y_1 + \cdots + Y_k$, with $Y_1 \in \mathfrak{n}_1, \ldots, Y_k \in \mathfrak{n}_k$, by setting $y = \exp_{\mathfrak{n}_1} Y_1 \exp_{\mathfrak{n}_2} Y_2 \cdots \exp_{\mathfrak{n}_k} Y_k$.

The expression on the right-hand side will be denoted by $\exp Y$.

Given an integer $m$, the *step-$m$ free Lie algebra generated by* $\{X_1, \ldots, X_n\}$ *is the quotient of the free Lie algebra generated by these elements, modulo the ideal generated by all commutators of order* $m + 1$ (the notion of a free Lie algebra is in [14]; the notion of a free nilpotent Lie algebra was first formulated in [5]).

If $m$ is large enough, there is an ideal $\mathfrak{h}$ in the step-$m$ free Lie algebra $\mathfrak{g}$
generated by \( \{X_1, \ldots, X_n\} \) that is complementary to the subspace \( \hat{n} \) spanned by \( \{X_1, \ldots, X_n\} \) and such that \( g/h \cong n \). Therefore, if \( X' \) and \( X'' \) are in \( \hat{n} \), and we let \( [X', X'']_{\hat{n}} \) be the \( \hat{n} \)-component of \( [X', X''] \) relative to the decomposition \( g = h \oplus \hat{n} \), then \( \hat{n} \) inherits a Lie algebra structure that makes it isomorphic to \( n \).

From now on we will use the same notation for the elements of \( n \) and those of \( \hat{n} \).

Let \( G \) be the connected and simply connected group having \( g \) as its Lie algebra, and \( \sigma: G \to N \) be defined by lifting to the group level the quotient map of \( g \) onto \( g/h \).

\[
\exp G Z \exp G Y_1 \cdots \exp G Y_k = \exp Y, \tag{32}
\]

where \( Z \in h, Y_1 \in n_1, \ldots, Y_k \in n_k, Y = Y_1 + \cdots + Y_k \); \( \sigma \) is a surjective group homomorphism whose kernel is \( H = \exp G h \).

The reason for introducing the group \( G \) is that the dilations \( D_\delta \) on \( \hat{n} \) extend to automorphisms of \( g \), since every linear transformation on \( \hat{n} \) extends to an automorphism of the free nilpotent algebra.

Given a function \( f \) on \( N \) and \( x \in G \), we define

\[
(\pi(x)f)(y) = f(y\sigma(x)). \tag{33}
\]

For \( p < \infty \), \( \pi \) gives a representation of \( G \) on \( L^p(N) \) that is isometric. We want to realize explicitly what transferred operators from \( G \) to \( N \), according to (28), look like in this case.

For simplicity, we take an absolutely continuous measure \( \varphi(x) \, dx \) on \( G \), with \( \varphi \in L^1(G) \). Then

\[
(T_\varphi f)(y) = \int_G \varphi(x) f(y\sigma(x)^{-1}) \, dx
\]

\[
= \int_N f(yz^{-1}) \left( \int_H \varphi(\tilde{z}h) \, dh \right) \, dz,
\]

where \( \tilde{z} \) is any element in the coset \( \sigma^{-1}(z) \) and the Haar measures on \( H \) and \( N \cong G/H \) are suitably normalized. Therefore \( T_\varphi \) is a right convolution operator on \( N \) whose kernel is obtained by pushing forward the measure \( \varphi(x) \, dx \) by the map \( \sigma \).

The next theorem shows that transferring a singular integral operator on \( G \) with kernel supported on an appropriate submanifold gives a singular integral operator on \( N \) adapted to the non-automorphic dilations.

**Theorem 5.3.** Let \( K \) be a distribution on \( n \) that is smooth away from the
origin and homogeneous of the critical degree \(-Q = -\alpha_1 - \alpha_2 \cdots - \alpha_n\) with respect to the dilations \(D_\delta\). Then the convolution operator on \(N\)

\[
(Tf)(y) = \text{p.v.} \int_n f(y(\exp Y)^{-1}) K(Y) \, dY
\]

(34)

is bounded on \(L^p(N)\) for \(1 < p < \infty\).

**Proof.** Let \(V^\# = \exp_G n_1 \exp_G n_2 \cdots \exp_G n_k\). Then \(V^\#\) is a homogeneous analytic manifold in \(G\). As we can assume without loss of generality that \(\dim N > 1\), \(V^\# \setminus \{0\}\) is connected.

Let \(K^\#\) be the kernel supported on \(V^\#\) such that

\[
K^\#(\exp Y) \, d\sigma = K(Y) \, dY.
\]

We can apply Theorem 3.4 to conclude that the operator that maps \(f\) onto \(f \ast K^\#\) is bounded on \(L^p(G)\) for \(1 < p < \infty\). If we transfer this operator to \(L^p(N)\) by means of (33), we obtain

\[
(T_{K^\#} f)(y) = \text{p.v.} \int_{V^\#} K^\#(x)(\pi(x^{-1}) f)(y) \, d\sigma(x)
\]

\[
= \text{p.v.} \int_n K(Y) f(y(\exp Y)^{-1}) \, dY
\]

\[
= (Tf)(y).
\]

By Proposition 5.1, \(T\) is bounded on \(L^p(N)\) for \(1 < p < \infty\). Q.E.D.

The statement in Theorem 5.3 is not the most general one along these lines. It is easily seen that one can replace the kernel \(K\) supported on all of \(n\) with a homogeneous kernel on \(n\) that satisfies Condition B. As in the proof of Theorem 5.3, one reduces to transferring from \(G\) to \(N\) a smooth convolution kernel that is supported on the manifolds \(V_i^\# = \exp G V_i\).

A technical problem arises when the support of \(K\) has several analytic components \(V_1, \ldots, V_m\). In this case, in order to apply Theorem 4.1—or, more precisely, the second part of Theorem 5.2—one has to make sure that \(V_i^\#, \ldots, V_m^\#\) generate the same closed subgroup of the free nilpotent group \(G\). We indicate two situations in which this property is satisfied.

**Proposition 5.4.** Let \(\gamma(t)\) be a homogeneous curve (i.e., \(\gamma(\delta t) = D_\delta \gamma(t)\) for every \(t \in \mathbb{R}\) and \(\delta > 0\)) in \(n\) that is analytic at \(t = 0\). Let \(\gamma^\#\) be the curve \(\exp \gamma\) in \(G\); then the two half-curves \(\{\gamma^\#(t); t > 0\}\) and \(\{\gamma^\#(t); t < 0\}\) generate the same closed subgroup of \(G\).

**Proof.** Since \(\gamma^\#(t)\) is analytic at every \(t \in \mathbb{R}\), every arc of \(\gamma^\#\) generates the same closed subgroup. Q.E.D.
Suppose now we consider the first canonical coordinates on \( N \), i.e., given by the exponential map \( \exp_N : n \to N \). If \( V_1, \ldots, V_m \) are as above, then \( V^*_1 = \exp_G V_1, \ldots, V^*_m = \exp_G V_m \) in this case.

**Proposition 5.5.** \( V^*_i \) and \( V^*_j \) generate the same closed subgroup of \( G \) if and only if \( V_i \) and \( V_j \) span the same linear subspace of \( n \).

**Proof.** Let \( H_i \) and \( H_j \) be the closed subgroups generated by \( V^*_i \) and \( V^*_j \), respectively. Their Lie algebras \( h_i \) and \( h_j \) are those generated by \( V_i \) and \( V_j \), respectively. Moreover \( V_i \) and its linear span generate the same subalgebra, and similarly for \( V_j \).

The conclusion follows from the fact that in a free nilpotent algebra, two subspaces of the linear span of the set of generators generate the same subalgebra if and only if they are identical. Q.E.D.

Therefore, either if \( \gamma \) is a curve that is analytic through the origin, or if \( V_1, \ldots, V_m \) span the same linear subspace of \( n \) and first canonical coordinates are used, then any kernel satisfying Condition B gives bounded convolution operators on \( L^p(N) \) for \( 1 < p < \infty \).

We prove next a maximal theorem on \( N \) related to the non-automorphic dilations \( D_\sigma \).

On the Lie algebra \( n \) we fix a smooth homogeneous gauge \( || \ | \) and consider the corresponding balls centered at the origin in \( N \)

\[
B_r = \{ x \in N : x = \exp Y, |Y| < r \}.
\]

We define the maximal operator on \( N \)

\[
(\mathcal{M} f)(x) = \sup_{r > 0} \frac{1}{m(B_r)} \int_{B_r} |f(xy^{-1})| \, dy.
\] (35)

It is interesting to notice that in general the gauge does not satisfy a triangular inequality of the form \( |xy| \leq C(|x| + |y|) \) (where the gauge has been transported from \( n \) to \( N \) by the map \( \exp \)).

Therefore the theory of spaces of homogeneous type [3] cannot be applied. We will make use instead of the full strength of Proposition 5.1.

**Theorem 5.6.** The operator \( \mathcal{M} \) in (35) is bounded on \( L^p(N) \) for \( 1 < p \leq \infty \).

**Proof.** For \( p = \infty \) the statement is trivial, so we assume that \( 1 < p < \infty \).
Let \( f \in L^p(N) \), which, without loss of generality, we can assume to be non-negative. In this case

\[
(Mf)(x) = \sup_{r > 0} f \ast \left( \frac{1}{m(B_r)} \chi_{B_r} \right)(x).
\]

Let \( V^* = \exp \tilde{n} \) be the manifold in the free group \( G \) introduced in the proof of Theorem 5.3, and let \( \mu_r, r > 0, \) be the measure supported on \( V^* \) such that

\[
\int_G \varphi(x) \, d\mu_r(x) = \frac{1}{m(B_r)} \int_{|Y| < r} \varphi(\exp Y) \, dY.
\]

Then \( \mu_r \) is compactly supported and its total variation is bounded by one. Furthermore

\[
\int_G \varphi(x) \, d\mu_r(x) = \int_G \varphi(r^{-1}x) \, d\mu_1(x)
\]

so that by Theorem 4.3 the operator

\[
(\mu^* \varphi)(x) = \sup_{r > 0} \left| \int \varphi(xy^{-1}) \, d\mu_r(y) \right|
\]

is bounded on \( L^p(G) \). The conclusion follows from Proposition 5.1. Q.E.D.

### 6. Oscillatory Singular Integrals

Oscillatory singular integrals have already been treated in [18].

It is proven there that if \( K \) is a standard Calderón-Zygmund kernel on the nilpotent group \( N \), endowed with automorphic dilations, and \( P(x, y) \) is a real polynomial on \( N \times N \), then the operator

\[
(Tf)(x) = \text{p.v.} \int_N e^{iP(x, y)} K(y) f(xy^{-1}) \, dy
\]

is bounded on \( L^p(N) \) for \( 1 < p < \infty \), and its norm depends on \( K, p \) and only on the degree of the polynomial \( P \). We are using here a slightly different notation from [18]. The two notations match if \( P(x, y) \) in (36) is replaced by \( P(x, xy^{-1}) \). However, this difference has no relevance.

In this section we give an alternative proof of this fact, that makes use of our previous results and of the method of transference. This new proof does not allow one to treat also the more general kernels considered in [18]. On
the other hand it leads to generalizations in other directions, namely it allows one to consider kernels supported on analytic submanifolds and/or non-automorphic dilations on the group.

An extension of this technique, which allows one to cover also kernels with homogeneity above the critical degree, will be considered in a forthcoming paper.

In order to state our results in full generality, we assume that the Lie algebra \( n \) of \( N \) is endowed with (not necessarily automorphic) dilations \( D_\delta \) and that \( n \) is decomposed into \( k \) homogeneous subspaces \( n_1, \ldots, n_k \) which determine canonical coordinates on \( N \), by mapping \( Y = Y_1 + \cdots + Y_k \in n_1 \oplus \cdots \oplus n_k \) into \( \exp Y - \exp Y_1 \cdots \exp Y_k \in N \). Furthermore we assume that \( V \) is a connected analytic homogeneous submanifold of \( n \) not containing the origin and that \( K \) satisfies Condition A on \( V \).

**Theorem 6.1.** Let \( V \) and \( K \) be as above, and let \( P(x, y) \) be a real polynomial on \( N \times N \). Then the operator

\[
(Tf)(x) = p.v. \int_V e^{iP(x, \exp Y) K(Y)} f(x(\exp Y)^{-1}) \, d\sigma(Y) \tag{37}
\]

is bounded on \( L^p(N) \) for \( 1 < p < \infty \), with a bound that depends only on \( K, p \), and the degree of \( P \).

Before going into the proof, we observe that if we express \( x \) and \( y \) in any set of canonical coordinates, then \( P(x, y) \) reduces to a polynomial on \( n \times n \). The degree of this polynomial may depend on the coordinates; however, a change of coordinates will have the effect of increasing at most the degree by some fixed amount depending only on the degree in the original coordinates.

**Proof.** With a slight abuse of notation, we write, for \( X = X_1 + \cdots + X_k \) and \( Y = Y_1 + \cdots + Y_k \) in \( n \), \( P(X, Y) \) in place of \( P(\exp X, \exp Y) \).

Let

\[
P(X, Y) = \sum_{|\beta| \leq q} c_\beta X^\alpha Y^\beta. \tag{38}
\]

We can assume that each monomial in (38) has \( |\beta| \geq 1 \). In fact, if \( P \) contains a monomial \( c_\alpha X^\alpha \) depending only on \( X \), the corresponding exponential factor \( e^{i\alpha X^\alpha} \) can be taken outside the integral without changing the \( L^p \)-norm of \( Tf \).

We consider the free Lie algebra \( \mathfrak{g} \) over the following generators:

(i) the elements \( Z_1, \ldots, Z_n \) of a homogeneous basis for \( n \);

(ii) an element \( W_{x \beta} \) for each pair \( (\alpha, \beta) \) of multiindices with \( 0 \leq |\alpha| \leq q, 1 \leq |\beta| \leq q \).
We then define a representation \( \pi \) of \( \mathfrak{g} \) on \( \mathcal{S}(N) \) by setting
\[
(\pi(Z_j)f)(x) = (Z_j f)(x) \tag{39}
\]
\[
(\pi(W_{x\beta})f)(x) = -ic_{x\beta}X^x f(x), \tag{40}
\]
where \( c_{x\beta} \) is the coefficient in (38). The operator \( Z_j \) on the right-hand side in (39) refers to the left-invariant vector field on \( N \), and multiplication by \( X^x \) in (40) is consistent with the choice of coordinates \( x = \exp X \) on \( N \).

It is easily seen that \( \pi \) annihilates all commutators of the generators of high enough order. There is therefore an integer \( m \), depending only on \( q \), such that (39) and (40) define a representation, also denoted by \( \pi \), of the step-\( m \) free nilpotent algebra \( \mathfrak{g}_q \) over the generators given in (i) and (ii). The algebra \( \mathfrak{g}_q \) does not depend on the coefficients \( c_{x\beta} \).

We want to show that the representation \( \pi \) given by (39) and (40) can be exponentiated to the simply connected group \( G_q \) that has \( \mathfrak{g}_q \) as its Lie algebra. To do this, let \( \mathfrak{w} \) be the ideal in \( \mathfrak{g}_q \) generated by the \( W_{x\beta} \), let \( \mathfrak{z} \) be the subalgebra generated by the \( Z_j \), and let \( \mathfrak{w}, \mathfrak{z} \) be the corresponding subgroups of \( G_q \). Then \( G_q \) is the semi-direct product of \( \mathfrak{z} \) and the normal subgroup \( \mathfrak{w} \). From (39) we obtain, by exponentiating, the representation \( \tilde{\pi} \) of the free nilpotent group \( \mathfrak{z} \) as the left regular representation on its quotient group \( N \).

From (39) and (40) we obtain a representation \( \tilde{\pi} \) of \( \mathfrak{w} \) given by multiplication operators on \( L^2(N) \) by factors which are exponentials of purely imaginary polynomials. To see that these representations combine to give a representation of the semi-direct product \( G_q \), it suffices to prove only that if \( z \in \mathfrak{z} \) and \( w \in \mathfrak{w} \), then
\[
\tilde{\pi}(zwz^{-1}) = \tilde{\pi}(z) \tilde{\pi}(w) \tilde{\pi}(z^{-1}).
\]

Both sides represent multiplication operators, so it suffices to show that for \( W \in \mathfrak{w} \)
\[
\pi(\text{Ad}(z)W) = \tilde{\pi}(z) \pi(W) \tilde{\pi}(z^{-1}).
\]
for \( Y = Y_1 + \cdots + Y_k \in \mathfrak{n} \). We see that the representation \( \tilde{\pi} \) consists of isometries of every \( L^p(N), 1 \leq p < \infty \).

\[
\left( \tilde{\pi} \left( \exp_{G_q} \left( \sum_{x\beta} t_{x\beta} W_{x\beta} \right) \right) f \right)(x) = e^{-i\Sigma c_{x\beta} t_{x\beta} X^x} f(x) \tag{41}
\]
\[
(\tilde{\pi}(\exp_{G_q} Y_1 \cdots \exp_{G_q} Y_k) f)(x) = f(x \exp X) \tag{42}
\]
for \( Y = Y_1 + \cdots + Y_k \in \mathfrak{n} \), we see that the representation \( \tilde{\pi} \) consists of isometries of every \( L^p(N), 1 \leq p < \infty \).
If the dilations on \( n \) map \( Z_j \) into \( \delta^{t_j} Z_j \), we define dilations \( \mathcal{D}_\delta \) on \( \mathfrak{g}_q \) by setting
\[
\mathcal{D}_\delta Z_j = \delta^{t_j} Z_j, \quad \mathcal{D}_\delta W_{\alpha \beta} = \delta^{\gamma_j \beta} W_{\alpha \beta}.
\]
Since \( \mathfrak{g}_q \) is a free nilpotent algebra, \( \mathcal{D}_\delta \) extend to automorphisms of \( \mathfrak{g}_q \).

Let now
\[
V^* = \left\{ \exp_{\mathfrak{g}_q} Y_1 \cdots \exp_{\mathfrak{g}_q} Y_k \exp_{\mathfrak{g}_q} \left( \sum_{\beta} Y_{\beta} W_{\alpha \beta} \right); \ Y_1 + \cdots + Y_k \in \mathcal{V} \right\}.
\]

Then \( V^* \) is a connected analytic homogeneous manifold in \( \mathbf{G}_q \). As \( V \) parametrizes \( V^* \), we can define a kernel \( K^* \) on \( V^* \) by the requirement that \( K^* \ d\sigma^* = K \ d\sigma \), where \( d\sigma^* \) is the surface measure on \( V^* \). We obtain a homogeneous distribution supported on \( V^* \) to which Theorem 5.2 can be applied in order to obtain the boundedness of the corresponding right-convolution operator on \( L^p(\mathbf{G}_q) \) for \( 1 < p < \infty \).

We now transfer this result to \( L^p(\mathcal{N}) \) by means of the representation \( \hat{\pi} \). The resulting operator is
\[
(T_{K^*} f)(x) = p.v. \int_{V^*} K^*(y) (\hat{\pi}(y^{-1}) f)(x) \ d\sigma^*(y)
\]
\[
= p.v. \int_{V^*} K(Y) \left( \hat{\pi} \left( \exp_{\mathfrak{g}_q} \left( - \sum_{\beta} Y_{\beta} W_{\alpha \beta} \right) \exp_{\mathfrak{g}_q} (-Y_k) \right) \right)
\]
\[
\cdots \exp_{\mathfrak{g}_q} (-Y_1) f)(x) \ d\sigma(Y)
\]
\[
- p.v. \int_{V^*} K(Y) e^{i \gamma_{\alpha \beta} x_{\alpha \beta} Y_{\beta}} (\hat{\pi} \exp_{\mathfrak{g}_q} (-Y_k))
\]
\[
\cdots \exp_{\mathfrak{g}_q} (-Y_1) f)(x) \ d\sigma(Y)
\]
\[
= p.v. \int_{V^*} K(Y) e^{i p(x, Y) f(x(e^{ip} Y) -')} \ d\sigma(Y)
\]
\[
= (Tf)(x).
\]

It follows by transference that \( T \) is bounded on \( L^p(\mathcal{N}) \) for \( 1 < p < \infty \). Since the coefficients of \( P \) only intervene in the definition of \( \hat{\pi} \), which is always isometric, the norm of \( T \) does not depend on the choice of \( c_{x \beta} \), but only on \( q \).

Q.E.D.

In order to treat non-connected manifolds, it is necessary also in this case to make sure that the various components of \( V^* \) generate the same subgroup of \( \mathbf{G}_q \). In this regard the hypotheses of Propositions 5.4 and 5.5
are sufficient also when dealing with oscillatory singular integrals. However, since our results are deduced by transference from a larger group, these hypotheses have to be assumed even when $N$ is endowed with automorphic dilations.

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