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Partiality, Cartesian Closedness, and Toposes

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INTRODUCTION

We present a basic, equational framework for categories of partial morphisms and systematically relate those structures to the classically known total ones. Partial functions have become a subject of interest, among others, in the context of proof theory (Scott's (1979) logic of partial elements), recursion theory (Di Paola and Heller, 1984; Rosolini, 1986), and semantics of programming languages (Plotkin, 1985). Recently Eugenio Moggi (1985, and also his forthcoming thesis) presented a detailed account of partiality in the λ-calculus, including difficult syntactic aspects (Church–Rosser). We refer to those works for an informal account of the importance of formalisms of partial functions in the respective application fields.

What is presented here is the categorical counterpart of Moggi's λ_p-calculus, although we do not present a formal equivalence proof (we refer to Pino, 1987). Thus as in previous work of Curien (1987, 1986) we adopt as much as possible the categories-as-syntax point of view. This paradigm, applied to the usual λ-calculus and cartesian closed categories, has already led to a new method of compiling and executing functional programs (Categorical abstract machine (Curien *et al.*, 1987).

The work should have applications in the area of type systems for programming languages (Obtułowicz (1982) investigated in his thesis categories of partial maps as models of intuitionistic logic). Also, due to their equational nature, the structures presented here might be a field of experimentation for automated categorical reasoning.

In our axiomatic presentation we start from an ordered-enriched

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category structure, where the order models the "less defined than" intuition. We insist on the partial counterpart of the terminal object, which plays the role of a domain classifier. We investigate the connection between the well-known total structures (product, exponential) and their partial counterparts (partial product, partial exponential). However more than the cartesian closed structure is needed to build partial exponentials out of a total structure. We show that important notions developed by other authors, such as liftings and complete objects, domains and ranges, can be developed inside our formalism. Much of the material in the paper was originally presented in Obtułowicz (1982) and circulated by the second author in various meetings as early as the year 1981. The only merit of the presentation here lies in important simplifications both in definitions and proofs, more work on equational presentations, and the quoted synthesis effort. The notion of *partial topos*, a category with partial products and partial exponentials where all partial monos have partial inverses, was not present in (Obtolowicz, 1982) (it appears, independently, in Carboni, 1985). Many of our constructions are also (independently) pointed out in (Rosolini, 1986; Carboni, 1985). The result that partial toposes can be described equationally (8.7) is specific to our work.

The plan is as follows: the first section introduces the loosest concept of partiality: a (pre)order structure on arrows of a category together with distinguished sets of maximal arrows. Section 2 introduces the partial cartesian (pCC) structure, which allows the development of the notions of domains, restrictions, and ranges in Section 3. Partial exponentials are introduced in Section 4. Total structures and partial structures are related in Sections 5 (total from partial; a very simple partial inversibility axiom suffices to recover a topos) and 6 (partial from total): the constructions are "inverse" enough to exploit properties of toposes to get more properties of partial toposes using known folklore of toposes, and reciprocally to gain more insight on toposes by proving that a cartesian closed category (CCC) is a topos iff its associated partial category is partially cartesian closed.

Our main interest in partiality arises from the fact that partial toposes enjoy *equational* descriptions (like CCCs, but unlike toposes, for which extended notions of equationality over graphs are needed, as was shown in (Burroni, 1981), which are discussed in Section 8. The pCC structure enjoys a number of quite different equational presentations, and, in particular, partially cartesian categories are closely related to Rosolini's *p*-categories, Hoehnke's *dht*-symmetric categories (Hoehnke, 1977; Schreckenberger, 1984), and Carboni's (1985) bicategories of partial maps. Some of these comparisons are also carried out in a recent survey of Robinson and Rosolini (1986), so we refer to this paper for some (parts of) proofs.

1. Partial Morphisms and Liftings

We define categories of partial morphisms, and prove basic properties of those very loose structures. Roughly we show, slightly generalizing constructions of Giuseppe Longo and Eugenio Moggi, how one can "add a bottom" to a domain as a trick to handle a partial function as a total one.

Our notation for arrows is as follows:

$$f:a\to b,\qquad \mathrm{Id}_a:a\to a,\qquad (f:a\to b);(g:b\to c):\quad a\to c$$

$$a\lhd b\qquad \text{via } ((r:a\to b),(s:b\to a))\qquad \text{when}\quad r;s=\mathrm{Id}.$$

Definition 1.1. A *category of partial morphisms*, or pC, is a category $\mathbb{C}$ endowed with the following structure:

— $\mathbb{C}$ has an order-enriched category structure, i.e., every homset $a\to b$ is equipped with a partial order $\leqslant$, called *restriction*; moreover, a distinguished subset of maximal arrows, called *total*, is given; identity arrows are total; composition is monotonic and preserves total arrows.

Of course in our examples we intend that the subset of total arrows be proper. Take the category PSET of sets and partial functions (it will enjoy all the properties discussed in the paper), then clearly $\lambda x.1/x:\mathbb{R}\to\mathbb{R}$ is not total.

Actually the results of this section hold in an even more general setting, replacing "partial order" by "preorder" (and keeping "maximal" w.r.t. the preorder, not the induced order, hence f maximal means $f\leqslant g\Rightarrow f=g$, not only $g\leqslant f$).

An important subcase of pC's weakened in this way is provided by pointed categories, as introduced in Di Paola and Heller (1984).

Definition 1.2. A *pointed* category is a category $\mathbb{C}$ having arrows $O_{a,b}:a\to b$ for all objects a, b, s.t. for all $f:a'\to a,\ g:b\to b'$,

$$f;O_{a,b}=O_{a',b},\qquad O_{a,b};g=O_{a,b'}.$$

The same category PSET can also be viewed as a pointed category. Take as O the everywhere undefined functions.

Fact. *A pointed category has (essentially, see above) a pC structure, where zeros are minimum.*

Proof. Take as total arrows those f s.t. for all h,

$$h;f=O\Rightarrow h=O.$$

Define $f \leqslant g$ iff for all h s.t. $h; f$ is total,

$$h; f = h; g.$$

$\leqslant$ is clearly a preorder. Suppose that f is total and $f \leqslant g$. Then $\mathrm{Id}; f = \mathrm{Id}; g$, since $\mathrm{Id}; f$ is total. Suppose that f, g are total; then

$$h; f; g = O \Rightarrow h; f = O \Rightarrow h = O.$$

For the monotonicity we first notice that if $f; g$ is total, then f is total. Indeed,

$$h; f = O \Rightarrow h; f; g = O \Rightarrow h = O.$$

Suppose $f \leqslant f'$, $g \leqslant g'$, and $h; f; g$ total. Then $h; f; g = h; f; g'$, and, $h; f$ being total, $h; f = h; f'$. Finally we show $O \leqslant f$ for all f. Suppose that $h; O = O$ is total. Then $\mathrm{Id}; O = O$ implies $\mathrm{Id} = O$, whence we deduce, for any $k : a \to b$, $k = k; \mathrm{Id} = k; O = O$, so that we are necessarily in the degenerate case of one point homsets $a \to b$ for all a, b, where obviously $h; O = h; f$ holds. ▌

Remark. The pC structure on PSET induced by its pointed category structure coincides with the one introduced above directly. But, as one of the referees kindly pointed out, this may not be the case for subcategories of PSET closed under the pointed category structure: take the category of topological spaces and open (i.e., mapping opens to opens) partial functions. Then total arrows are only maps defined on a dense subset, while the partial order is more than the restriction order (if $f \leqslant g$ then g must be undefined at some points where f is also undefined, for instance, at points in

$$\overline{\mathrm{dom}(f)} - \mathrm{dom}(f)).$$

Notation. In a pC total arrows give rise to a subcategory $\mathbb{C}_T$ of $\mathbb{C}$, having all the objects of $\mathbb{C}$ and s.t. the arrows in $a \to_T b$ are the total arrows in $a \to b$.

Of course not much can be said with so few axioms. Nevertheless they seem to have the adequate level of generality for basic properties relating partial and total arrows to be stated. The following notions (1.3, 1.4) of liftings and complete objects are adapted from Longo and Moggi (1984) and Asperti and Longo (1985). The notion of lifting will be central in relating partial structures to total ones (see in particular the proof of 5.5).

Definition 1.3. A pC has *liftings* when

— for any object a there is an object $a^{\uparrow}$, called *lifting* of a, such that for each object b there is a bijection $\tau_{b,a}$ (written simply τ_b (or even τ) when the reference to a is clear) from $b \to a$ to $b \to_T a^\dagger$ s.t. for all $f: c \to b$, $g: b \to a$, one has

$$f; \tau_b(g) \leqslant \tau_c(f; g).$$

Category theorists refer to that kind of weak naturality as *lax naturality*. More specifically in an order-enriched category, given two functors F, G, a lax natural transformation between F, G is a family of arrows $\delta_a: Fa \to Ga$ s.t. for any $f: a \to b$,

$$Ff; \delta_b \leqslant \delta_a; Gf.$$

Thus the definition says that the family τ_b^{-1} is lax natural from $\lambda b.b \to_T a^\dagger$ to $\lambda b.b \to a$.

Remark. Notice that when f is total, then the equality holds in the definition. Thus lifting is right adjoint to the forgetful functor, i.e., inclusion $\subseteq: \mathbb{C}_T \to \mathbb{C}$. The functorialization of † is, as usual, for $f: a \to b$,

$$f^\dagger = \tau_{a^\dagger, b}(ex_a; f).$$

Warning. Notice that the adjunction property is not a definition of lifting, since the adjunction deals only with the naturality arising as a special case of lax naturality (cf. definition).

However, in at least two special cases the definition reduces to the adjunction property. This is true when the pC structure is induced by a pointed category structure (as in Longo and Moggi, 1984; Asperti and Longo, 1985). This is also true when coreflexives split. We defer this second case to Section 5 and deal now with the first.

FACT. *If $\mathbb{C}$ is a pointed category, then lax naturality is a consequence of naturality ranging over total arrows $f: c \to b$.*

Proof. Take any $f: c \to b$ and suppose that $h; f; \tau_b(g)$ is total. Then $h; f$, h are total, hence

$$(h; f); \tau_b(g) = \tau(h; f; g), \qquad h; \tau_b(f; g) = \tau(h; f; g). \quad \blacksquare$$

Here are some basic properties of a pC with liftings.

FACT. *Lifting is unique up to isomorphism.*

Proof. By lax naturality. $\blacksquare$

Remark. All the other structures introduced in the paper can be easily checked to be also unique up to isomorphism. This will not be mentioned anymore.

FACT. *$a \lhd a^\dagger$ via* $(\mathrm{in}_a, \mathrm{ex}_a)$, *where*

$$\mathrm{in}_a = \tau_a(\mathrm{Id}), \qquad \mathrm{ex}_a = (\tau_{a^\dagger})^{-1}(\mathrm{Id})$$

(*notice that* in_a *is total*) *and where* $\lhd$ *is a retraction in* $\mathbb{C}$ (*if the context is mainly total, then* $\lhd$ *should be stressed into* $\lhd_P$).

Proof. By naturality. ∎

Remark. Actually we have more than a retraction, we have an adjunction

$$\mathrm{in}_a \dashv \mathrm{ex}_a$$

(i.e., in_a; $\mathrm{ex}_a = \mathrm{Id}$ ($\Leftrightarrow \mathrm{in}_a; \mathrm{ex}_a \geqslant \mathrm{Id}$) and $\mathrm{ex}_a; \mathrm{in}_a \leqslant \mathrm{Id}$), by lax naturality (not mere naturality!).

What follows makes only sense in the framework of retractions.

FACT. *τ_b^{-1} extends to $b \to a^\dagger$.*

Proof. Set $\tau'_b(g) = g$; ex_a for $g: b \to a^\dagger$; it is easily checked that the restriction of τ'_b to total arrows is τ_b^{-1}. ∎

We are now interested in a stronger condition.

DEFINITION 1.4. If $a \lhd_T a^\dagger$, then we say that a is *complete* (again if the context is mainly total one may write $\lhd$ instead of $\lhd_T$).

Notice that the difference here is that we want the projection to be total. Indeed one may keep the same injection.

FACT. *If $a \lhd_T a^\dagger$ via (i, j), then also $a \lhd_T a^\dagger$ via* $(\mathrm{in}_a, \mathrm{out}_a)$ *for some* out_a, *where* in_a *is as above.*

Proof. Take $\mathrm{out}_a = \tau_{a^\dagger}(\mathrm{ex}; i; \mathrm{ex}); j$. ∎

The complete objects have the following characterization.

PROPOSITION 1.5. *Complete objects are exactly those objects a satisfying*

$$\forall b, \forall f: b \to a, \qquad \exists \mathbf{f}: b \to_T a \cdot f \leqslant \mathbf{f}.$$

Proof. Take $\mathbf{f} = \tau_b(f)$; out_a. By definition

$$f; \mathrm{in}_a = f; \tau_a(\mathrm{Id}) \leqslant \tau_b(f; \mathrm{Id}) = \tau_b(f).$$

Hence

$$f = f; \mathrm{in}_a; \mathrm{out}_a \leqslant \tau_b(f); \mathrm{out}_a = \mathbf{f}.$$

Reciprocally, take the pair (in_a, $\mathbf{ex}_a$). Then

$$\text{in}_a ; \mathbf{ex}_a \geqslant \text{in}_a ; \text{ex}_a = \text{Id}.$$

Hence $\text{in}_a ; \mathbf{ex}_a = \text{Id}$, Id being maximal. ∎

FACT. *Liftings are complete objects: one has* $a^{\uparrow} \lhd_T a^{\uparrow\uparrow}$ *through* ($\text{in}_{a^\uparrow}$, $\tau_{a^{\uparrow\uparrow}}(\text{ex}_{a^\uparrow} ; \text{ex}_a)$).

Proof. Remark $\text{ex}_{a^\uparrow} = \text{ex}_{a^\uparrow} ; \tau_{a^\uparrow}(\text{ex}_a) \leqslant \tau_{a^{\uparrow\uparrow}}(\text{ex}_{a^\uparrow} ; \text{ex}_a)$. ∎

FACT. *Total retracts of complete objects are complete.*

Proof. Use the above characterization: if $b \lhd_T a$ through (i, j), for any $f: c \to b$ $\mathbf{f} ; \mathbf{i} ; j$ is a total extension of f, since

$$f = f ; i ; j \leqslant \mathbf{f} ; \mathbf{i} ; j. \quad ∎$$

2. PARTIAL CARTESIAN STRUCTURE

We introduce a partial counterpart of cartesian structure and list a number of easy consequences of the definition. The basic ideas of this section as well as the following one can be illustrated in PSET:

— the terminal object $\{\cdot\}$ of SET becomes a domain classifier: a partial function $f: a \to \{\cdot\}$ is just a subset a' of a.

— There is a maximum, total! $= \lambda x . \{\cdot\} : a \to \{\cdot\}$.

— If $f: a \to b$ is a partial function, then $f ; !$ is just the domain of definition of f.

— when $f: a \to b$, $g: a \to c$ are partial functions, the partial function $\langle f, g \rangle : a \to b \times c$ is defined exactly when f, g are; moreover $\langle f, g \rangle$; Fst (where Fst is the first projection, a total function) is not f in general, but the restriction of f to the domain of g. This provides an alternative approach to the description of the domain of definition of a partial function.

— If $f: a \to b$ is a partial function, then $\langle \text{Id}, f \rangle$; Fst is the restriction of the identity to the domain of definition of f.

DEFINITION 2.1. A pC is called *partially cartesian*, or *pCC*, when it has the following additional structure:

— a distinguished object t, called *domain classifier*, and for every object a an arrow $!_a : a \to t$ s.t.

(t1) $!_a$ is maximum in $a \to t$ for every a

(t2) the total arrows are exactly those arrows $f: a \to b$ s.t. $f; !_b = !_a$

(t3) if $f, f', g: a \to b$ are s.t. $f, f' \leqslant g$ and $f; !_b \leqslant f'; !_b$, then $f \leqslant f'$

(t4) if $h, h': a \to b$ are s.t. $h \leqslant h'$, then for any $g: b \to t$ $h; g = (h'; g) \cap (h; !_b)$;

— for all objects a, b, a *partial product* given by an object $a \times b$, *projection* arrows $\mathrm{Fst}_{a,b}: a \times b \to a$, $\mathrm{Snd}_{a,b}: a \times b \to b$, and a *pairing* operation, associating $\langle f, g \rangle: a \to b \times c$ with $f: a \to b$, $g: b \to c$ s.t.

(p1) $\mathrm{Fst}_{a,b}$, $\mathrm{Snd}_{a,b}$ are total

(p2) pairing is monotonic in both arguments

(p3) for any $h: a \to b \times c$, $\langle h; \mathrm{Fst}_{b,c}, h; \mathrm{Snd}_{b,c} \rangle = h$

(p4) for all $f: a \to b$, $g: a \to c$, $\langle f, g \rangle; \mathrm{Fst}_{b,c} \leqslant f$, $\langle f, g \rangle; \mathrm{Snd}_{b,c} \leqslant g$

(p5) naturality: for all $f: a \to b$, $g: a \to c$, $h: d \to c$, $h; \langle f, g \rangle = \langle h; f, h; g \rangle$

(p6) for all $f: a \to b$, $g: a \to c$, $\langle f, g \rangle; !_{b \times c} = (f; !_b) \cap (g; !_c)$.

We have to check that the axiom (t2) is sound. Suppose $f; !_b = !_a$, and $f \leqslant f'$. Then $f': !_b \geqslant !_a$, hence $f': !_b = !_a$; $f = f'$ follows by (t3). This definition of total arrows has the desired stability properties: $\mathrm{Id}_a; !_a = !_a$; suppose $f: a \to b$, $g: b \to c$ are s.t. $f; !_b = !_a$, $g; !_c = !_b$; then $f; g; !_c = f; !_b = !_a$.

Remark. The division between axioms (ti) and (pi) is rather artificial. It will be clear from Sections 3 and 8 that (t2), (t3), (t4) can be rephrased in the (pi) style. In Section 8, a purely equational description will be given.

Remark. Hidden in (t1) is the lax naturality of !,

$$f; ! \leqslant !$$

which plays an important role in other axiomatizations of the same concept (especially by Aurelio Carboni (1985), see Section 8).

Notation. A useful abbreviation is $f \times g = \langle \mathrm{Fst}; f, \mathrm{Snd}; g \rangle$; this is justified because it turns $\times$ into a functor.

Now we show some properties of a pCC. First, the definition of totality ensures

FACT. *If $f; g$ is total, then f is total.*

Proof. One has $g; ! \leqslant !$ by (t1); hence by monotonicity $! = f; g; ! \leqslant b; !$, and $f; ! = !$ by maximality. ∎

The partial product structure provides a characterization of the restriction ordering:

FACT. $f \leqslant g$ *iff* $\langle f, g \rangle$; Snd $= f$.

Proof. Suppose $\langle f, g \rangle$; Snd $= f$; by (p4) $f \leqslant g$. Suppose $f \leqslant g$; by (p4) $\langle f, g \rangle$; Snd $\leqslant g$, and by monotonicity $f; ! \leqslant g; !$. By (p1), (p6),

$$\langle f, g \rangle; \mathrm{Snd}; ! = \langle f, g \rangle; ! = (f; !) \cap (g; !) = f; !.$$

Hence $\langle f, g \rangle$; Snd $= f$ by (t3). ▮

Here are some identities:

FACT. — $!_t = \mathrm{Id}_t$, $!_{t \times a} = \mathrm{Fst}_{t,a}$, $!_{a \times t} = \mathrm{Snd}_{a,t}$
— $\langle f, f \rangle$; Snd $= f$, $\langle f, g \rangle$; Fst $= \langle g, f \rangle$; Snd
— $\langle f, g \rangle = \langle g, f \rangle$; $\langle$Snd, Fst$\rangle$.

Proof. When there is a maximum, any maximal is maximum; for the next identities, use $\langle f, f \rangle$; Snd $\leqslant f$, $\langle f, g \rangle$; Fst, $\langle g, f \rangle$; Snd $\leqslant f$; the final one is proved by (p5), the preceding ones, and (p3). ▮

As a consequence of the first identity in the last fact we get

FACT. *For each* a, $a \to t$ *is an* inf-*semi-lattice.*

Proof. By specializing (p6). ▮

Notice also

FACT. — *if* $f \leqslant g$, *then* $\langle f, g \rangle$; Fst $= f$
— $f \leqslant g$ *iff* $\langle f, g \rangle = \langle f, f \rangle$.

Proof. $\langle f, g \rangle$; Fst $\leqslant f = \langle f, f \rangle$; Fst $\leqslant \langle f, g \rangle$; Fst. For the second statement use (p3). ▮

The following is our first statement on comparing partial and total structures. There will be others (Sections 5 and 7).

PROPOSITION 2.2. *The pCC structure in* $\mathbb{C}$ *yields a cartesian structure in* $\mathbb{C}_T$.

Proof. For every object b, $!_b$ is the unique total arrow in $b \to t$, since any total arrow, being maximal, must be the maximum; if f, g are total, clearly

$$\langle f, g \rangle; \mathrm{Fst} = f, \qquad \langle f, g \rangle; \mathrm{Snd} = g \qquad \text{by (p4), (p6), (t3).}$$ ▮

In Section 5 we shall show some kind of inverse: how to recover a pCC out of a cartesian category, and how far the constructions are "inverse."

We end the section by quoting that partial products of two given objects (and the domain classifier object) are easily shown to be all isomorphic, so that the property of being a pCC is indeed a property of the order-enriched category. This can be done either by direct reasoning on the axioms, or by indirect arguments using the cartesian category $\mathbb{C}_T^{\wedge}$ built out of $\mathbb{C}$ (see Section 5), as in (Carboni, 1985).

3. Domains, Restrictions, and Ranges

We discuss the categorical description of the notion of domain of a partial function, of range of a (partial) function, and partial notions of epi, mono and iso w.r.t. ranges and domains. We also discuss the notion of internal equality, which is somehow the partial counterpart of pullbacks (see Section 5). For each notion we derive elementary consequences of the definition which will be needed later.

As already discussed at the beginning of Section 2, in a pCC there are two natural candidates for expressing the idea of the domain of $f: a \to b$:

— $f: !$

— $\langle \mathrm{Id}, f \rangle; \mathrm{Fst}$ ($\leqslant \mathrm{Id}$).

The two points of view happen to be isomorphic. For every $\varphi: a \to t$, $\alpha: a \to a \leqslant \mathrm{Id}$ let

$$\varphi^0 = \langle \mathrm{Id}_a, \varphi \rangle; \mathrm{Fst}, \qquad \alpha^0 = \alpha;!.$$

FACT. *$\lambda\varphi.\varphi^0$ and $\lambda\alpha.\alpha^0$ are inverse order isomorphisms between $a \to t$ and $\{\alpha: a \to a \mid \alpha \leqslant \mathrm{Id}\}$.*

This isomorphism connects exactly the two definitions of domains.

DEFINITION 3.1. For $h: a \to b$ we write

$$\mathrm{Dom}^O(h) = h; ! \qquad \text{and} \qquad \mathrm{Dom}(h) = \langle \mathrm{Id}, h \rangle; \mathrm{Fst}.$$

FACT. $\mathrm{Dom}(h) = \mathrm{Dom}^O(h)^0$.

This allows us to speak freely about *domains*, and we shall write safely $\mathrm{Dom}(h) \leqslant \mathrm{Dom}(k)$ when it holds, or (see below) $h \restriction (\mathrm{Dom}(k) \cap \mathrm{Dom}(l))$. We shall use whichever definition of domain will be convenient in each case.

Together with each of these formalizations of domains there comes a notion of restriction.

DEFINITION 3.2. For $h: a \to b$, $\varphi: a \to t$, $\alpha: a \to a \leqslant \mathrm{Id}$, we write

$$h \restriction^O \varphi = \langle h, \varphi \rangle; \mathrm{Fst}, \qquad h \restriction^D \alpha = \alpha; h.$$

To bring these definitions into agreement, we define more generally a notion of *restriction* w.r.t. an arrow $g: a \to c$ (i.e., we assume only that g has the same source as h); we write

$$h \upharpoonright g = \langle h, g \rangle; \text{Fst}.$$

FACT. — $h \upharpoonright g \leqslant h$
— *$\upharpoonright$ is an extension of both $\upharpoonright^O$ and $\upharpoonright^D$.*

Proof. The only nontrivial check is $\upharpoonright^D$: since $h \upharpoonright^D \alpha$, $h \upharpoonright \alpha \leqslant h$ we only need to check

$$\alpha; h; ! = h; ! \cap \alpha; !$$

which is an instance of (t4), using $\alpha \leqslant \text{Id}$. ▮

The following identities are easily checked:

FACT. — $h \upharpoonright \varphi = h \upharpoonright \varphi^0$
— $h \upharpoonright g = h \upharpoonright \text{Dom}^O(g) = h \upharpoonright \text{Dom}(g)$.

The order on arrows $\alpha \leqslant \text{Id}$ has a characterization in terms of composition:

FACT. *If $\alpha, \beta: a \to a \leqslant \text{Id}$, then*

$$\alpha; \beta = \beta; \alpha = \alpha \cap \beta.$$

As an immediate consequence $\alpha; \alpha = \alpha$, $(\alpha \leqslant \beta \Leftrightarrow \alpha; \beta = \alpha)$ hold.

Proof. By symmetry we need only show $\alpha; \beta = \alpha \cap \beta$. Using $\alpha, \beta \leqslant \text{Id}$ we get $\alpha; \beta \leqslant \alpha, \beta$. We prove the full property first for $\alpha = \beta$, i.e., $\alpha; \alpha = \alpha$. We only need to check $\alpha; \alpha; ! = \alpha; !$, which is done by (t3) ($f \leqslant \text{Id}$). Coming back to the general case, suppose $h \leqslant \alpha, \beta$; then $h = h; h \leqslant \alpha; \beta$. ▮

The key properties of restrictions are, for all f, g, h of appropriate types:

FACT. — $h; (f \upharpoonright \varphi) = (h; f) \upharpoonright (h; \varphi)$
— $(f \upharpoonright g); h = (f; h) \upharpoonright g$.

Proof. The first is obvious by (p5); for the second we can replace g by $\text{Dom}(g)$, $\upharpoonright$ by $\upharpoonright^D$. ▮

As consequences of the second equality we get

FACT. — *$f \upharpoonright g = f$ if g is total and, more technically,*
— $\langle f, g \rangle; \langle \text{Fst}; \varphi, \text{Snd}; \psi \rangle; ! = (f; \varphi) \cap (g; \psi)$.

Proof. For the first take $h = !$ and replace g by $\mathrm{Dom}^O(g)$; for the second we have

$$\begin{aligned}\text{l.h.s.} &= \langle (f \upharpoonright g); \varphi, (g \upharpoonright f); \psi \rangle; ! = \langle (f; \varphi) \upharpoonright g, (g; \psi) \upharpoonright f \rangle \\ &= (f; \varphi) \cap (g; \psi),\end{aligned}$$

noticing $\mathrm{Dom}((g; \psi) \upharpoonright f) \leqslant \mathrm{Dom}(g)$, and using the property which follows. ▮

Here are other useful properties:

FACT. — $(f \upharpoonright g) \upharpoonright h = f \upharpoonright (\mathrm{Dom}(g) \cap \mathrm{Dom}(h))$
— $\langle f \upharpoonright g, h \rangle = \langle f, h \rangle \upharpoonright g$
— $\langle f \upharpoonright h, g \upharpoonright h \rangle = \langle f, g \rangle \upharpoonright h$
— $\times$ *is functorial.*

The reader will find more identities in Section 8. Now we define ranges.

DEFINITION 3.3. Call *range* of $f: a \to b$ the minimum arrow $\psi: b \to t$, if any, s.t. $f; \psi = f; !$. We shall write $\psi = \mathrm{Ran}(f)$. If all the arrows have ranges we say that $\mathbb{C}$ *has ranges.*

Arrows do not necessarily have ranges. An example is the category of topological spaces and continuous partial functions defined on open subsets. Then a constant function has no range, in general, since singletons are not open, in general. Clearly when f has a range then

FACT. — $f; (g \upharpoonright \mathrm{Ran}(f)) = f; g$
— $h \upharpoonright \mathrm{Ran}(f) = k \upharpoonright \mathrm{Ran}(f) \Rightarrow f; h = f; k.$

A natural and desirable property of ranges is that $\Leftarrow$ also holds.

DEFINITION 3.4. If $f: a \to b$ has a range and satisfies, for any $g, h: b \to c$,

$$f; h = f; k \Rightarrow h \upharpoonright \mathrm{Ran}(f) = k \upharpoonright \mathrm{Ran}(f),$$

we say that f is a *partial epi.*

The definition of partial epi is slightly redundant: that $\mathrm{Ran}(f)$ is the minimum ψ s.t. $f; \psi = f; !$ is a consequence of

$$f; \psi = f; \mathrm{Ran}(f) \Rightarrow \psi \upharpoonright \mathrm{Ran}(f) = \mathrm{Ran}(f) \upharpoonright \mathrm{Ran}(f).$$

Again if an arrow has a range it need not be a partial epi. An example is the category of partial orders and partial monotone functions f, i.e., s.t. if

$x \in A$, $x \leqslant y$, then $y \in A$ and $f(x) \leqslant f(y)$, where A is the domain of definition of f. Take for instance $f = \mathrm{Id}\colon \{x\} \to \{x, y\}$, where $x \leqslant y$. Then $\mathrm{Ran}(f) = \{x, y\}$ and if $f; h = f; k$ h, k can very well disagree on y.

FACT. *Let f, g be partial epis s.t.* $\mathrm{Ran}(f) = \mathrm{Dom}(g)$. *Then $f; g$ is a partial epi and* $\mathrm{Ran}(f; g) = \mathrm{Ran}(g)$. *For any $\varphi : a \to t$,* $\mathrm{Id} \upharpoonright \varphi$ *is a partial epi and* $\mathrm{Ran}(\mathrm{Id} \upharpoonright \varphi) = \varphi$.

Proof. We check $f; g; \mathrm{Ran}(g) = f; \mathrm{Ran}(f) = \mathrm{Dom}(f)$. If $f; g; h = f; g; k$, we use that f is a partial epi, and get $(g; h) \upharpoonright \mathrm{Ran}(f) = (g; k) \upharpoonright \mathrm{Ran}(f)$, i.e., $g; h = g; k$, since $\mathrm{Ran}(f) = \mathrm{Dom}(g)$, whence $h \upharpoonright \mathrm{Ran}(g) = k \upharpoonright \mathrm{Ran}(g)$. The last assertion is immediate. ∎

The reader will find more properties of ranges after Definition 5.1. A stronger property is that of having a partial inverse.

DEFINITION 3.5. We say that $f : a \to b$ has *partial inverse* $g : b \to a$ (which we shall write f^{-1}) when

$$f; g = \mathrm{Id} \upharpoonright f \qquad \text{and} \qquad g; f = \mathrm{Id} \upharpoonright g.$$

In the next two facts we assume that f has partial inverse g.

FACT. $\mathrm{Ran}(f) = g; !$ *and f is a partial epi.*

Proof. $f; g; ! = (\mathrm{Id} \upharpoonright \mathrm{Dom}(f)); ! = \mathrm{Dom}(f)$. Suppose $f; \psi = f; !$, $f; h = f; k$; then left composing by g yields $(g; !) \cap \psi = g; !$, $h \upharpoonright \mathrm{Dom}(g) = k \upharpoonright \mathrm{Dom}(g)$. ∎

FACT. *g is minimum s.t.* $f; g = \mathrm{Id} \upharpoonright f$.

Proof. If $f; h = \mathrm{Id} \upharpoonright f$, then

$$h \upharpoonright g = g; f; h = g \upharpoonright g; f; ! = g \upharpoonright g = g. \quad ∎$$

This ensures the uniqueness of the partial inverse.

DEFINITION 3.6. We say that $\mathbb{C}$ has *internal equality* when $\langle \mathrm{Id}_a, \mathrm{Id}_a \rangle$ has a partial inverse $\equiv_a$ for any a. We shall use the following abbreviation:

$$f \equiv g = \langle f, g \rangle; \mathrm{Ran}(\langle \mathrm{Id}, \mathrm{Id} \rangle) = \langle f, g \rangle; \equiv; !$$

for $f, g : b \to a$.

We list some properties of the internal equality:

FACT. — $\equiv_a = \mathrm{Snd}_{a,a} \restriction \mathrm{Ran}(\langle \mathrm{Id}, \mathrm{Id} \rangle) = \mathrm{Fst}_{a,a} \restriction \mathrm{Ran}(\langle \mathrm{Id}, \mathrm{Id} \rangle)$

— $\equiv \leqslant \mathrm{Fst}, \mathrm{Snd}$

— $\langle \mathrm{Snd}, \mathrm{Fst} \rangle; \equiv \; = \; \equiv$ (*hint*: $\langle \mathrm{Snd}, \mathrm{Fst} \rangle$ *is its own inverse*)

— $f \equiv g \leqslant f; !, f \equiv g \leqslant g; !$

— $f \equiv f = f; !$

— $h; (f \equiv g) = (h; f) \equiv (h; g)$. ▮

The following property characterizes the binary internal equality:

FACT. $\mu \leqslant f \equiv g \Leftrightarrow f \restriction \mu = g \restriction \mu$ *and* $\mu \leqslant f; !, g; !$.

Proof. For $\Rightarrow$, notice

$$f \restriction (f \equiv g) = g \restriction (f \equiv g) = \langle f, g \rangle; \equiv.$$

For $\Leftarrow$, one has

$$(f \equiv g) \cap \mu = (\langle f, g \rangle \restriction \mu); \equiv; ! = \langle f \restriction \mu, g \restriction \mu \rangle; \equiv; ! = (f \restriction \mu); !. \quad ▮$$

In a category with internal equality all homsets have finite lower bounds:

FACT. $\langle f, g \rangle; \equiv \; = f \cap g$.

Proof. If $h \leqslant f, g$, use $h = \langle h, h \rangle; \equiv$. ▮

The binary internal equality is symmetric and "transitive":

FACT. — $f \equiv g = g \equiv f$

— $(f \equiv g) \cap (g \equiv h) \leqslant f \equiv h$.

Proof. For the first property notice $\langle g, f \rangle; \langle \mathrm{Snd}, \mathrm{Fst} \rangle = \langle f, g \rangle$; for the second, set $\mu = (f \equiv g) \cap (g \equiv h)$; then $\mu \leqslant f \equiv g$ implies $\mu \leqslant f; !$, $f \restriction \mu = g \restriction \mu$; we conclude from the similar statements about g, h and the above characterization. ▮

These properties justify the notation

$$f \equiv g \equiv h = (f \equiv g) \cap (g \equiv h) = (\sigma(f) \equiv \sigma(g)) \cap (\sigma(g) \equiv \sigma(h)),$$

where σ is any permutation of $\{f, g, h\}$.

Our last observation about internal equality is that it is co-lax natural (the inequality is reversed).

FACT. $\equiv; f \leqslant (f \times f); \equiv$.

Proof. We have

$$\equiv ; f \leqslant \mathrm{Fst}; f$$

$$(f \times f); \equiv \leqslant (f \times f); \mathrm{Fst} \leqslant \mathrm{Fst}; f$$

thus it is enough to show $\equiv ; f; ! \leqslant (f \times f); \equiv ; !$. Using the above characterization, we are left to check

$$(\mathrm{Fst}; f) \upharpoonright (\equiv ; f) = \equiv ; f = (\mathrm{Snd}; f) \upharpoonright (\equiv ; f)$$

(notice $\equiv ; f \leqslant \mathrm{Fst}; f$). ∎

Finally we introduce a notion similar for domains to that of partial epi for ranges.

DEFINITION 3.7. We say that $f: a \to b$ is a *partial mono* when for any $g, h: c \to a$,

$$g; f; ! = g; !, \quad h; f; ! = h; !, \quad g; f = h; f \;\Rightarrow\; g = h.$$

To get some intuition on the definition, one may restrict to the case where g, h have ranges; then the property means

$$\mathrm{Ran}(g), \mathrm{Ran}(h) \leqslant \mathrm{Dom}(f), \quad g; f = h; f \;\Rightarrow\; g = h.$$

Remark. For total arrows, being partial mono reduces to being mono.

The following definition will be justified in Section 5.

DEFINITION 3.8. A *partial topos* is a pCCC (see 4.1), where all partial monos have partial inverses.

Notice that the definition implies internal equality, since it is easily checked that $\langle \mathrm{Id}, \mathrm{Id} \rangle$ is a partial mono. Clearly we do not expect that all arrows are partial monos, while we shall see that if enough structure is given, all arrows are partial epis (see 7.3).

The categorical notions of ranges and partial epis, monos were defined in Di Paola and Heller (1984) in pointed (dominical, see Section 8) categories; they also appear in Rosolini (1986), where they are called images, range maps, monos in the category of domains (see Section 5), respectively.

4. PARTIAL CARTESIAN CLOSED STRUCTURE

Now we define the partial counterpart of the exponential.

DEFINITION 4.1. A pC is called *partially cartesian closed* or *pCCC* if

— there exists a binary operation $\Rightarrow$ (or $a \Rightarrow_p b$ to stress partiality) of *partial exponentiation* on objects s.t. for all objects a, b, c there exist bijections $\lambda_{a,b,c}$ from $a \times b \to c$ onto $a \to_T b \Rightarrow c$ which are lax natural, i.e., for all $f: d \to a,\ g: a \times b \to c$,

$$f; \lambda(g) \leqslant \lambda(\langle \mathrm{Fst}; f, \mathrm{Snd} \rangle; g)$$

(as for the lifting, the equality holds when f is maximal).

FACT. *As in the case of lifting, one may define an extension* λ' *of* λ^{-1} *s.t.* $g \leqslant \lambda(\lambda'(g))$ *for all arrows* $g: c \to a \Rightarrow b$.

Proof. Take $\mathrm{App}_{a,b} = \lambda^{-1}(\mathrm{Id})$ and $\lambda'(g) = \langle \mathrm{Fst}; g, \mathrm{Snd} \rangle; \mathrm{App}$. ▮

Hence any arrow into $a \Rightarrow b$ has a total extension, i.e., $a \Rightarrow b$ is a complete object. Here are some properties of a pCCC:

FACT. — $\langle \mathrm{Fst}; \lambda(f), \mathrm{Snd} \rangle; \mathrm{App} = f$
— $\langle \lambda(f), g \rangle; \mathrm{App} = \langle \mathrm{Id}, g \rangle; f$.

Proof. For the first notice that the l.h.s. is $\lambda^{-1}(\lambda(f))$; for the second,

$$\begin{aligned}\langle \mathrm{Id}, g \rangle; f &= \langle \mathrm{Id}, g \rangle; \langle \mathrm{Fst}; \lambda(f), \mathrm{Snd} \rangle; \mathrm{App}\\ &= \langle\langle \mathrm{Id}, g \rangle; \mathrm{Fst}; \lambda(f), \langle \mathrm{Id}, g \rangle; \mathrm{Snd} \rangle; \mathrm{App}.\end{aligned}$$

Then notice $\langle \mathrm{Id}, g \rangle; \mathrm{Snd} = g$, and

$$\langle \mathrm{Id}, g \rangle; \mathrm{Fst}; \lambda(f) = \mathrm{Dom}(g); \lambda(f) = \lambda(f) \upharpoonright \mathrm{Dom}(g) = \lambda(f) \upharpoonright g. \quad ▮$$

FACT. $f; \lambda(g) = f; \lambda(h) \Leftrightarrow \langle \mathrm{Fst}; f, \mathrm{Snd} \rangle; g = \langle \mathrm{Fst}; f, \mathrm{Snd} \rangle; h$.

Proof. $\Leftarrow$ is clear from

$$f; \lambda(g) = \lambda(\langle \mathrm{Fst}; f, \mathrm{Snd} \rangle; g) \upharpoonright f.$$

If $f; \lambda(g) = f; \lambda(h)$, then also

$$\langle \mathrm{Fst}; f; \lambda(g), \mathrm{Snd} \rangle; \mathrm{App} = \langle \mathrm{Fst}; f; \lambda(h), \mathrm{Snd} \rangle; \mathrm{App}$$

and we use $\langle \mathrm{Fst}; f; \lambda(g), \mathrm{Snd} \rangle = \langle \mathrm{Fst}; f, \mathrm{Snd} \rangle; \langle \mathrm{Fst}; \lambda(g), \mathrm{Snd} \rangle$, by the functoriality of $\times$. ▮

To get some intuition about the last equivalence, we can restrict our attention to domains, and we get easily as a consequence: for any $\mu: a \to t$,

$$\lambda(g) \upharpoonright \mu = \lambda(h) \upharpoonright \mu \Leftrightarrow g \upharpoonright (\mathrm{Fst}; \mu) = h \upharpoonright (\mathrm{Fst}; \mu).$$

The rest of the section investigates some properties of pCCCs, mostly inspired by Longo and Moggi (1984) and Asperti and Longo (1985), where, again, apparently only a looser definition (adjunction) appears in the pointed category setting. Again, too, another case where the definition reduces to an adjunction is when coreflexives split (see next section).

PROPOSITION 4.2. *A pCCC has liftings.*

Proof. We take $a^{\uparrow} = t \Rightarrow a$, and use that $b \rightarrow_T t \Rightarrow a$ is equipotent to $b \times t \rightarrow a$ which is equipotent to $b \rightarrow a$. Indeed $\mathrm{Fst}_{b,t}$, $\langle \mathrm{Id}_b, !_b \rangle$ are inverse isomorphisms between $b \times t$ and b (use (p5) and some of the properties proved above). Hence we set for $g: b \rightarrow a$, $\tau_b(g) = \lambda(\mathrm{Fst}_{b,t}; g)$; we are only left to prove lax naturality:

$$f; \tau_b(g) = f; \lambda(\mathrm{Fst}; g) = \lambda(\langle \mathrm{Fst}; f, \mathrm{Snd} \rangle; \mathrm{Fst}; g) = \lambda(\mathrm{Fst}; f; g) = \tau_b(f; g)$$

(we have used that Snd is total). ∎

PROPOSITION 4.3. *If the (total) exponential $a \Rightarrow_T b$ exists in $\mathbb{C}_T$ and b is complete, then $a \Rightarrow_T b$ is a total retract of $a \Rightarrow b$ and is complete.*

Proof. The second part of the statement follows from the completeness of $a \Rightarrow b$, and a remark made above. The first part internalizes the characterization of complete objects. If $b \lhd_T b^{\uparrow}$ through (in, out), a total retraction is obtained through

$$(\lambda(\lambda_T^{-1}(\mathrm{Id}_{a \Rightarrow_T b})), \lambda_T(\tau(\lambda'(\mathrm{Id}_{a \Rightarrow b})); \mathrm{out}))$$

(use naturality of λ_T, of $\tau_{(a \Rightarrow b) \times a, b}$ restricted to total arrows, and finally that $f \leqslant \tau(f)$; out, hence $f = \tau(f)$; out when f is total). ∎

The following remark is due to E. Moggi, who corrected an error in an early draft: if $\mathbb{C}$ is a pCCC, then the full subcategory of $\mathbb{C}_T$ which has as objects finite products of partial exponentials is a CCC (hint: define $a \Rightarrow_T (b \Rightarrow c) = (a \times b) \Rightarrow c$).

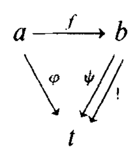
5. FROM PARTIAL TO TOTAL

In this section we show how to obtain total structures out of partial ones. The main point of the section is that the total counterpart of a pCCC is more than a CCC; it is a CCC with (some) pullbacks, endowed with a distinguished class of monos and a classifier for that class. This may be viewed as a weak notion of topos. Moreover, a simple condition on a pCCC (that partial monos have partial inverses) ensures that the class of monos in the associated total category is the class of all monos (turning the total category into a topos).

The main drawback of $\mathbb{C}_T$ is that it has not enough objects to yield a total exponential for any objects a, b. The following construction enriches considerably the collection of "objects." It is a variant of the classical slice category constructions.

DEFINITION 5.1. Let $\mathbb{C}$ be a pCC. $\mathbb{C}^\wedge$ is the category whose objects are arrows $\varphi: a \to t$ of $\mathbb{C}$, for any a, and whose arrows $f: \varphi \to^\wedge \psi$ are arrows $f: a \to b$ of $\mathbb{C}$ s.t.

— $f; ! = f; \psi \leqslant \varphi$ (we have $\psi: b \to t$) (as shown).



That we have a category, with the composition of $\mathbb{C}$, is a routine verification. Notice that $\mathrm{Id}_\varphi = \mathrm{Id} \upharpoonright \varphi$. Another obvious observation is that for $f: \varphi \to^\wedge \psi$, we have $f \upharpoonright \varphi = f$.

There is a more symmetrical formulation of the same construction, based on the other notion of domain. Take as objects domains, i.e., arrows $\alpha \leqslant \mathrm{Id}$, and as arrows $f: \alpha \to \beta$, arrows f of $\mathbb{C}$ s.t.

$$\alpha; f; \beta = f.$$

This is the well-known idempotent splitting construction (named such after Freyd's allegories (see forthcoming Freyd and Scedrov, (1988)), which is also refered to by others as the Karoubi envelope. Notice that here we do not split all idempotents, only the coreflexive ones, i.e., those less than or equal to the identity.

FACT. *The two categories are isomorphic.*

Proof. Observe

— $f; \psi = f; ! \Leftrightarrow f; ! \leqslant f; \psi \Leftrightarrow f \upharpoonright (f; \psi) = f \Leftrightarrow f; \mathrm{Dom}(\psi) = f$

— $f; ! \leqslant \varphi \Leftrightarrow f \upharpoonright \varphi = f \Leftrightarrow \mathrm{Dom}(\varphi); f = f.$ ∎

We shall use whichever definition seems most convenient in each case.

Our main line is to prove that partial notions on $\mathbb{C}$ infer corresponding total notions on $\mathbb{C}_T^\wedge$. This goal splits into two rather natural subgoals, as we learned from Carboni (1985), who independently (and more elegantly) proved similar results. First prove that $\mathbb{C}^\wedge$ inherits the properties of $\mathbb{C}$; second prove that if $\mathbb{C}$ enjoys the coreflexives splitting property, then $\mathbb{C}_T$ has the total structure corresponding to the partial structure of $\mathbb{C}$.

THEOREM 5.2. *If $\mathbb{C}$ is a pCC (has internal equality) (is a pCCC) (all partial monos have partial inverses), then $\mathbb{C}^\wedge$ inherits that property.*

Proof. We do not give all the details. First the pC-structure on $\mathbb{C}^\wedge$ is given by $f \leqslant g$ iff $f \leqslant g$ as arrows of $\mathbb{C}$. Total arrows are those $f\colon \varphi \to^\wedge \psi$ s.t. $f; ! = \varphi$. The domain classifier is given by

— $\mathrm{Id}: t \to t$, $!_\varphi = \varphi$.

Partial products are defined by

— $\alpha \times^\wedge \beta = \alpha \times \beta$

— $\mathrm{Fst}^\wedge = \mathrm{Fst} \upharpoonright \alpha \times \beta$, $\mathrm{Snd}^\wedge = \mathrm{Snd} \upharpoonright \alpha \times \beta$

— $\langle f, g \rangle^\wedge = \langle f, g \rangle$.

The internal equality is

— $\equiv_\alpha = \equiv_a \upharpoonright \alpha \times \alpha$.

Yielding the pCCC structure requires more imagination. The trick is the internalization of the transformation of any map into a map from, say α to β. We set

$$\alpha \Rightarrow \beta = \lambda((\mathrm{Id} \times \alpha); \mathrm{App}; \beta) \equiv \mathrm{Id}.$$

Finally the partial monos of $\mathbb{C}^\wedge$ are exactly the partial monos of $\mathbb{C}$: observe that for $f\colon \varphi \to \psi$

$$f; !_\psi = f; \psi = f; !. \quad \blacksquare$$

Before proceeding to the second half of our goal, we need some preliminary observations and definitions. First we check that indeed coreflexives split in $\mathbb{C}^\wedge$. Recall that an arrow f *splits* when there exist r, s s.t.

— $r; s = f$, $\quad s; r = \mathrm{Id}$.

FACT. *Splittings of the same arrow are isomorphic.*

Proof. Let $r; s = r'; s'$, $s; r = \mathrm{Id} = s'; r'$. Then $s; r'$ and $s'; r$ are inverse: indeed

$$s; r'; s'; r = s; r; s; r. \quad \blacksquare$$

A pair of arrows r, s s.t. $s; r = \mathrm{Id}$ is called a splitting pair (of $r; s$).

FACT. *s is total and mono.*

Proof. Notice $\mathrm{Id} \upharpoonright s \geqslant \mathrm{Id} \upharpoonright s; r$. $\blacksquare$

FACT. *In* $\mathbb{C}^\wedge$ *all arrows* $f \leqslant \mathrm{Id}$ *split.*

Proof. It is easily seen that such arrows have the form $\mathrm{Id} \restriction \psi : \varphi \to \varphi$, where $\psi \leqslant \varphi$. Take $r = \mathrm{Id} \restriction \psi : \varphi \to \psi$, $s = \mathrm{Id} \restriction \psi : \psi \to \varphi$. ▮

Remark. One can prove more: the construction ^ is the free splitting construction w.r.t. the class of arrows to be split. But we shall not need that.

Now we look at the case where $\mathbb{C}$ is a pC of all coreflexives which split. This gives rise to an interesting class of monos.

Notation. $\mathcal{M}(\mathbb{C})$ is the class of arrows s for which there exists r s.t. r, s is a splitting pair.

FACT. *Such an* r *is necessarily unique.*

Proof. Notice $r' \leqslant r; s; r' = r$ if r, r' are two of them. ▮

The main property of $\mathcal{M}(\mathbb{C})$ is the following.

FACT. *There exists a bijection between* $a \to b$ *and (equivalence classes) of arrows of the form* $\langle s, f \rangle$ *into* $a \times b$, *where* s *is in* $\mathcal{M}(\mathbb{C})$ *and* f *is total.*

Proof. Let $f: a \to b$ be given, let r, s be a splitting pair of $\mathrm{Dom}(f)$. Then $\langle s, s; f \rangle$ has the desired type, since $s; f; ! = s;\ r;\ s;\ ! = s; ! = !$. Conversely with $\langle s, g \rangle$ we associate $r; g$. We check that the transformations are inverse: for one way,

$$r; s; f = \mathrm{Dom}(f); f = f,$$

while for the other,

$$\mathrm{Dom}(r;\, g) = \mathrm{Dom}(r) \qquad \text{and} \qquad s; r; g = g. \quad ▮$$

This fact suggests the anticipation of the inverse investigation "from total to partial" of Section 6. Given a cartesian category $\mathbb{C}$, the classical way of representing partial functions in $\mathbb{C}$ from, say a to b, is by equivalence classes of arrows $\langle j, f \rangle$ into $a \times b$, where j is mono, w.r.t. the equivalence

$$\langle j, f \rangle \equiv \langle m, g \rangle \Leftrightarrow \langle j, f \rangle = i;\ \langle m, g \rangle \qquad \text{for some iso } i.$$

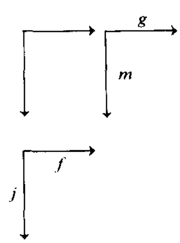
One requires classically that $\mathbb{C}$ have finite limits, in order to be able to compose partial arrows. All of this may be relativized to a fixed class of monos, closed under identities, composition, and pullbacks (thus $\mathbb{C}$ is required to have pullbacks of all these distinguished monos); such a class is called a *dominion* by P. Rosolini. Given a dominion $\mathcal{M}$ on a cartesian category, the category $\mathbb{C}_{P(\mathcal{M})}$ is defined below.

Notation. If $h; f = k; g$ is a pullback square, we write $h = f^{-1}(g)$, $k = g^{-1}(f)$, and $h; f = k; g = f \therefore g$.

DEFINITION 5.3. Let $\mathbb{C}$ be a category with finite limits. We define the category $\mathbb{C}_{P(\mathcal{M})}$ to have the same objects as $\mathbb{C}$, and as arrows from a to b the equivalence classes of arrows $\langle j, f \rangle$ into $a \times b$, where j is in $\mathcal{M}$. Those classes $[\langle j, f \rangle]$ will be written simply $\langle j, f \rangle$ when there is no risk of confusion. Identities and composition are given by

— $\mathrm{Id}_P = \langle \mathrm{Id}_a, \mathrm{Id}_a \rangle : a \to_P a$

— $\langle j, f \rangle ;_P \langle m, g \rangle = \langle f^{-1}(m); j, m^{-1}(f); g \rangle$.



That we have obtained a category is a routine verification. Notice that $\mathbb{C}$ is embedded in $\mathbb{C}_{P(\mathcal{M})}$ via the faithful functor Γ taking f into $\langle \mathrm{Id}, f \rangle$. Those will often be identified in the sequel. Notice

$$\langle j, f \rangle ;_P g = \langle j, f; g \rangle, \qquad h ;_P \langle j, f \rangle = \langle j, h; f \rangle,$$

where $\langle j, f \rangle ;_P g$ stands for $\langle j, f \rangle ;_P \langle \mathrm{Id}, g \rangle$.

Notation. We shall write $\mathbb{C}_{\_,\_}$ for $(\mathbb{C}_{\_})_{\_}$.

FACT. *$\mathbb{C}_{P(\mathcal{M})}$ has a pC structure. $\mathbb{C}_{P(\mathcal{M}),T}$ is isomorphic to $\mathbb{C}$.*

Proof. Define the order on arrows by

$$\langle j, f \rangle \leqslant \langle m, g \rangle \Leftrightarrow \langle j, f \rangle = h; \langle m, g \rangle \qquad \text{for some } h.$$

Take as total arrows those of the form $\langle i, f \rangle$ where i is iso. That total arrows are maximal is the consequence of the following easy fact on isos and monos: if $i = n; m$, i is iso and m is mono, then n is iso. Hence we may identify the total arrows of $\mathbb{C}_P$ and the arrows in $\mathbb{C}$.

Compositions of total arrows are total because isos are stable under composition and pullbacks. Finally, monotonicity of composition is tedious but easy: we check it in the left argument. So suppose $\langle j, f \rangle = h; \langle j', f' \rangle$ and set $A = f'^{-1}(m)$:

$$\begin{aligned}\langle j, f \rangle ;_P \langle m, g \rangle &= \langle h^{-1}(A); h; j', A^{-1}(h); m^{-1}(f'); g \rangle \\ &= A^{-1}(h); \langle A; j', m^{-1}(f'); g \rangle. \quad \blacksquare\end{aligned}$$

Now we are in a position to state the definition of our candidates to be a total counterpart of a pCCC. This notion has been suggested by one of the referees.

DEFINITION 5.4. A CC $\mathbb{C}$ with a dominion $\mathcal{M}$ has a $\mathcal{M}$-classifier iff the embedding functor $\Gamma: \mathbb{C} \to \mathbb{C}_{P(\mathcal{M})}$ has a right adjoint (denoted $\tilde{}$).

Before we can state the main result of the section we need some simplifications arising from the splitting hypothesis.

FACT. *In a pCC, where all coreflexives split, partial monos have partial inverses iff total monos have partial inverses. The definition of liftings and pCCC structure can be merely phrased in terms of adjunction.*

Proof. Let f be a partial mono, and r, s split $\mathrm{Dom}(f)$. We show that $s; f$ is mono. Let $g; s; f = h; s; f$. Since $(g; s); f; ! = g; s; !(= g; !)$ we get $g; s = h; s$, whence $g = h$. Let f' be the partial inverse of $s; f$. We show that $g; s$ is the partial inverse of f. For one way, observe that $\mathrm{Id} \upharpoonright g = \mathrm{Id} \upharpoonright g; s$; for the other, from $s; f; g = \mathrm{Id}$ we get $r; s; f; g = f; g = r$.

We show how to recover lax naturality from the adjunction property in the case of the pCCC structure. Suppose that r, s split $\mathrm{Dom}(f)$. Then

$$r; s; \lambda((f \times \mathrm{Id}); g) = r; \lambda(((s; f) \times \mathrm{Id}); g) \qquad \text{by naturality, since } s \text{ is total}$$

$$= r; (s; f); \lambda(g) \qquad \text{by naturality, since } s; f \text{ is total}$$

$$= \mathrm{Dom}(f); f; \lambda(g) = f; \lambda(g). \quad \blacksquare$$

Now we can prove the theorem of the section.

THEOREM 5.5. *In the whole statement $\mathbb{C}$ is a pCC, where all coreflexives split. The following properties hold:*

(i) *$\mathbb{C}_T$ is cartesian and $\mathcal{M}(\mathbb{C})$ is a dominion on $\mathbb{C}_T$ s.t. $\mathbb{C}_{T, P(\mathcal{M}(\mathbb{C}))}$ is isomorphic to $\mathbb{C}$.*

(ii) *If $\mathbb{C}$ has internal equality, then $\mathbb{C}_T$ has finite limits.*

(iii) *If $\mathbb{C}$ is a pCCC, then $\mathbb{C}_T$ has a $\mathcal{M}(\mathbb{C})$-classifier.*

(iv) *If $\mathbb{C}$ is a pCCC and has internal equality, then $\mathbb{C}_T$ is also cartesian closed.*

(v) *If $\mathbb{C}$ is a partial topos, then $\mathbb{C}_T$ is a topos.*

Proof. We do not need the splitting hypothesis to get that $\mathbb{C}_T$ is cartesian (cf. Section 2). We show that $\mathcal{M}(\mathbb{C})$ is a dominion. Clearly Id, Id is a splitting pair. If r, s and r', s' split coreflexives and s', s are composable, then $r; r', s'; s$ also split a coreflexive. Now if $f: a \to b$ is total and r', s' split $\alpha' \leqslant \mathrm{Id}$, then we show that

$$— \quad (s; f; r'); s' = s; f$$

is a pullback, where r, s split $\mathrm{Id} \upharpoonright f; \alpha'$. We first prove the commutation. Since r is epi, it is enough to show $\alpha; f = \alpha; f; \alpha'$, i.e., $f \upharpoonright f; \alpha' = f; \alpha' \upharpoonright f; \alpha'$, which is done by checking that the two sides have the same domain. Suppose now that $g; s' = h; f$. We show that $h; r$ is mediating (uniqueness is obvious since s is mono):

$$h; r; s = h; (\mathrm{Id} \upharpoonright f; \alpha') = h \upharpoonright (g; s'; \alpha') = h \upharpoonright g; s' = h,$$

since g, s' are total. For the other triangle it is enough to check

$$h; r; (s; f; r'); s' = h; \alpha; f; \alpha' = h; \alpha; f = h; f = g; s'.$$

The isomorphism has already been proved in a fact above.

We come to the proof of (ii). Let $h: a \to b, k: c \to b$, and set

$$A = (\mathrm{Fst}; h) \equiv (\mathrm{Snd}; k).$$

A pullback diagram is obtained through A; Fst, A; Snd.

As for (iii), set $a^{\sim} = t \Rightarrow a$ $(= a^{\dagger}$, cf. 4.2). And, indeed, classifying $\mathcal{M}(\mathbb{C})$ reduces to the statement that $\mathbb{C}$ has liftings, since modulo the isomorphism of categories $\subseteq: \mathbb{C}_T \to \mathbb{C}$ becomes $\Gamma: \mathbb{C}_T \to \mathbb{C}_{T, P(\mathcal{M}(\mathbb{C}))}$.

For (iv) we define two arrows $a \Rightarrow b \to a \Rightarrow b^{\dagger}$: one is meant as the identity, the other as the transformation from a partial arrow to a total one. We define $a \Rightarrow_T b$ as the source of

$$\lambda(\mathrm{App}; \mathrm{in}_b) \equiv \lambda(\tau(\mathrm{App}))$$

(for notations we refer to Section 1 on liftings).

For (v) we only need to check that starting from a partial topos the class $\mathcal{M}(\mathbb{C})$ is the class of all monos. This is precisely what the hypothesis that any mono in $\mathbb{C}_T$ has a partial inverse says. ∎

6. From Total to Partial

As in Section 5, we show that $\mathbb{C}_P$ has the expected partial properties corresponding to those of $\mathbb{C}$.

THEOREM 6.1. *If $\mathbb{C}$ is a cartesian category with a dominion $\mathcal{M}$, $\mathbb{C}_{P(\mathcal{M})}$ is a pCC, where all coreflexives split. If, moreover, $\mathbb{C}$ is cartesian closed and has a $\mathcal{M}$-classifier, then $\mathbb{C}_{P(\mathcal{M})}$ is a pCCC. If $\mathbb{C}$ has finite limits, then $\mathbb{C}_P$ has internal equality. If $\mathbb{C}$ is a topos, then $\mathbb{C}_P$ is a partial topos.*

Proof. Establishing the pCC structure is routine. Splitting follows from the observation that $\langle m, f\rangle \leqslant \langle \mathrm{Id}, \mathrm{Id}\rangle$ iff $m = f$, and then $\langle m, \mathrm{Id}\rangle$, $\langle \mathrm{Id}, m\rangle$ split $\langle m, m\rangle$.

For the pCCC structure, take $a \Rightarrow_T b = a \Rightarrow b^{\sim}$, and remember that naturality is enough (cf. last section). For the internal equality take

$$\equiv\ = \langle\langle \mathrm{Id}, \mathrm{Id}\rangle, \mathrm{Id}\rangle : a \times a \rightarrow_P a.$$

Finally we check that partial monos in $\mathbb{C}_P$ have partial inverses. First notice that $\langle j, f\rangle$ is a partial mono iff f is mono. Indeed it is easy to check

$$\langle m, g\rangle ;_P \langle j, f\rangle ;_P ! = \langle m, g\rangle ;_P ! \Leftrightarrow g^{-1}(j) = \mathrm{Id}$$

so that $\langle j, f\rangle$ is a partial mono iff

$$\begin{aligned} &(g^{-1}(j) = h^{-1}(j) = \mathrm{Id}, j^{-1}(g); f = j^{-1}(h); g \Rightarrow g = h) \\ &\Leftrightarrow (g = g'; j, h = h'; j, g'; f = h'; f \Rightarrow g = h) \\ &\Leftrightarrow (g'; f = h'; f \Rightarrow g' = h'). \end{aligned}$$

Now it is very easy to check that $\langle j, f\rangle$ has $\langle f, j\rangle$ as partial inverse. ∎

The theorem has an important consequence: indeed there exist partial toposes; build $\mathbb{C}_P$ out of your favorite topos $\mathbb{C}$. Notice that partial continuous functions do not yield a partial topos, but only a pCCC. Following Plotkin (1985) we build a category PCont taking as objects the partial orders, where all increasing sequences have lubs, and as arrows the *partial continuous* functions, i.e., the partial monotone (cf. Section 3) functions f with domain of definition A s.t.

— if (x_n) is an increasing sequence and $\bigcup x_n \in A$, then $x_m \in A$ for some m and $f(\bigcup x_n) = \bigcup \{f(x_n) \mid n \geqslant m\}$.

FACT. PCont *is a pCCC; it is also a pointed category whose induced pC structure is the same as the one induced by partial products.*

Total arrows are just ordinary continuous functions, defined everywhere. Notice that the partial order $\leqslant_f$ on arrows (i.e., in, say, $D \rightarrow E$) is distinct from the partial order $\leqslant_e$ in the structure of $D \Rightarrow E$:

$f \leqslant_f g$ iff $g(x)$ is defined and $g(x) = f(x)$ whenever $f(x)$ is defined

$f \leqslant_e g$ iff $g(x)$ is defined and $f(x) \leqslant g(x)$ whenever $f(x)$ is defined.

These orders are often called flat, extensional (or pointwise), respectively, whence the subscripts above.

FACT. *In* PCont *all coreflexives split.*

We recall the following fact:

FACT. *The category* Cont *of* cpo's *and continuous functions is not balanced* (epi + mono ≠ iso).

Proof. The trouble is with monotonicity. Take $D = \{x, y\}$, two partial orders $\leqslant_1$, $\leqslant_2$ defined by

$$a \leqslant_1 a, b \leqslant_1 b, \qquad a \leqslant_2 a, b \leqslant_2 b, a \leqslant_2 b.$$

Then $\mathrm{id}: (D, \leqslant_1) \to (D, \leqslant_2)$ is mono and epi, but not iso since its inverse is not even monotone. ∎

FACT. PCont *is not a partial topos.*

Proof. By the two properties above, noticing $\mathrm{PCont}_T = \mathrm{Cont}$. ∎

7. Relating the Constructions

In this section we first sum up the relationships between the constructions of Sections 5 and 6. This will help us to prove more properties of partial toposes (7.3, 7.5) and to give a characterization of toposes (7.4).

The first proposition of this section relates the two constructions of total categories, the direct one and the indirect one.

PROPOSITION 7.1. *When $\mathbb{C}$ is a pCC, there is a full and faithful functor from $\mathbb{C}_T$ to $\mathbb{C}_T^{\wedge}$, which preserves the product structure, and, if $\mathbb{C}$ is a pCCC, its restriction to the full subCCC of $\mathbb{C}_T$ (cf. remark after* 4.3) *preserves the cartesian closed structure.*

Proof. Define $F(a) = !_a$, $F(f) = f$. Notice $!_a \times_T !_b = \mathrm{Fst}; !_a \cap \mathrm{Snd}; !_b = !_{a \times b}$. The last part of the statement amounts to a check that for any a, b,

$$\lambda(\mathrm{App}; \langle \mathrm{Id}, ! \rangle; \mathrm{App}): ((t \Rightarrow a) \Rightarrow (t \Rightarrow b)) \to ((t \Rightarrow a) \Rightarrow b)$$

is iso in $\mathbb{C}_T$ from $!_{t \Rightarrow a} \Rightarrow_T !_{t \Rightarrow b}$ onto $!_{(t \Rightarrow a) \Rightarrow b}$, which is tedious but easy. ∎

The following proposition sums up how far the $_P$ and $_T$ constructions are inverse.

PROPOSITION 7.2. *If $\mathbb{C}$ is a CC with a dominion $\mathcal{M}$, $\mathbb{C}$ and $\mathbb{C}_{P(\mathcal{M}),T}$ are isomorphic. There exists a full and faithful structure-preserving functor from $\mathbb{C}$ to $\mathbb{C}_{T,P(\mathcal{M}(\mathbb{C}))}^{\wedge} \approx \mathbb{C}^{\wedge}$, if $\mathbb{C}$ is a pCC. This embedding becomes an equivalence when coreflexives split in $\mathbb{C}$.*

Hence there is a dissymetry: the equivalence does not hold, in general, in the ${}_{T,P}$ direction. This can be repaired by weakening the requirements on a dominion. Let us call *weak dominion* a class $\mathcal{M}$ of monos on a $CC\ \mathbb{C}$ s.t.

— for any $f: a \to b$, $m: b' \to b \in \mathcal{M}$ $f^{-1}(m)$ exists, and for any $n: a \to a' \in \mathcal{M}$ $f^{-}(m)$; $n \in \mathcal{M}$.

— $\mathrm{Id}_t \in \mathcal{M}$ (Id_a, $\mathrm{Id}_b \in \mathcal{M} \Rightarrow \mathrm{Id}_{a\times b} \in \mathcal{M}$).

Clearly a weak dominion is a dominion iff it contains all identities. Given a weak dominion on $\mathbb{C}$, $\mathbb{C}_{P(\mathcal{M})}$ is defined as in the case of a dominion, except that the objects are only those objects a of $\mathbb{C}$ s.t. $\mathrm{Id}_a \in \mathcal{M}$.

FACT. *$\mathbb{C}_{P(\mathcal{M})}$ is a pCC, where $\mathcal{M}$ is a weak dominion.*

Notice that this does not extend well to CCCs: the candidate for being a partial exponential is not necessarily and object of $\mathbb{C}_{P(\mathcal{M})}$. Now we have to put a weak dominion structure on $\mathbb{C}^{\wedge}_{T}$ for a pCC $\mathbb{C}$.

FACT. *The class $\omega\mathcal{M}(\mathbb{C})$ of arrows of the form* $\mathrm{Id} \upharpoonright \varphi : \varphi \to !_a$ *satisfies the conditions listed above.*

FACT. $\mathbb{C} \approx \mathbb{C}^{\wedge}_{T,P(\omega\mathcal{M}(\mathbb{C}))}$, *where $\mathbb{C}$ is a pCC and $\mathcal{M}$ is as in the above fact.*

But even if we do not have, in general, an equivalence, the arrows of $\mathbb{C}^{\wedge} \approx \mathbb{C}^{\wedge}_{T,P(\mathcal{M}(\mathbb{C}))}$ are arrows of $\mathbb{C}$ (with more information). In some cases, this will allow us to prove a property by first proving that it holds for all pC's with a given structure whenever it holds for all $\mathbb{C}_P$'s, where $\mathbb{C}$ ranges over all categories with the matching structure. The rest of the section investigates applications of this method.

Remark. The correspondence in 7.2 can be lifted to an adjunction

— $\lambda\mathbb{C}.\mathbb{C}^{\wedge}_{T} \dashv \lambda\mathbb{C}.\mathbb{C}_P$

with the slight difficulty that we have to deal with $(\mathbb{C}^{\wedge}_{P})_T$, which is not as simple as (but is equivalent to) $\mathbb{C}_{P,T}$ (we refer to Obtulowicz, 1982, for details).

In the next proposition we use a property of toposes (preservation of epis by pullbacks) to prove a property of partial toposes.

PROPOSITION 7.3. *In a partial topos $\mathbb{C}$, all morphisms are partial epis.*

Proof. It is easy to see that $f: a \to b$ is a partial epi iff $f: !_a \to !_b$ is a partial epi in $\mathbb{C}^{\wedge}$. So we are reduced to prove that for any topos $\mathbb{C}$, $\mathbb{C}_P$ has the property of the statement. Notice that the range of any $\langle j, f\rangle: a \to_P b$, if any, is the minimum m s.t. $f^{-1}(m) = \mathrm{Id}$, i.e., the minimum m s.t. $f = g; m$

for some g. A basic property of toposes is that $\mathrm{Im}(f)$ satisfies precisely this property (where $f = f^*; \mathrm{Im}(f)$ is the epi-mono factorization of f), so that

— $\mathrm{Ran}(\langle j, f\rangle) = \mathrm{Im}(f)$.

Now we show that $\langle j, f\rangle$ is a partial epi:

$$\langle j, f\rangle ;_P \langle m, g\rangle$$
$$= \langle j, f\rangle ;_P \langle n, h\rangle \Leftrightarrow f^{-1}(m) = f^{-1}(n), m^{-1}(f); g = n^{-1}(f); h$$

We want to prove

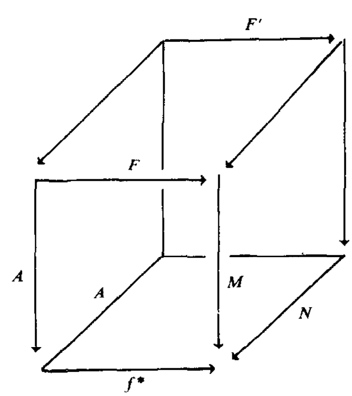
$$\langle m, g\rangle \upharpoonright \mathrm{Im}(f) = \langle n, h\rangle \upharpoonright \mathrm{Im}(f),$$

i.e.,

— $m \therefore \mathrm{Im}(f) = n \therefore \mathrm{Im}(f)$

— $m^{-1}(\mathrm{Im}(f)); g = n^{-1}(\mathrm{Im}(f)); h$.

We use the decomposition $f = f^*; \mathrm{Im}(f)$. For the first we show that $M = \mathrm{Im}(f)^{-1}(m)$ and $N = \mathrm{Im}(f)^{-1}(n)$ are equivalent. We use $f^{*-1}(M) = f^{*-1}(N) = A$ and that f^* is epi. It is enough to prove that $M^{-1}(N)$ (hence symmetrically $N^{-1}(M)$) is epi (hence iso since it is mono). We work on the cube made out of f^*, M, N, and write $F = m^{-1}(f^*)$:



We have

$$F^{-1}(M^{-1}(N)) = A^{-1}(f^{*-1}(N)) = A^{-1}(A) = \mathrm{Id},$$

i.e., $F = F'; M^{-1}(N)$ for some F', so that $M^{-1}(N)$ is epi. For the second equality we may now suppose

$$M^{-1}(f^*) = N^{-1}(f^*) = B.$$

We use $m^{-1}(f) = B; m^{-1}(\mathrm{Im}(f))$ and that B is epi:

$$m^{-1}(f); g = n^{-1}(f); h \Rightarrow m^{-1}(\mathrm{Im}(f)); g = n^{-1}(\mathrm{Im}(f)); h. \quad \blacksquare$$

The next result provides a characterization of toposes.

THEOREM 7.4. *A cartesian closed category $\mathbb{C}$ which has pullbacks is a topos iff $\mathbb{C}_P$ is a pCCC.*

Proof. If $\mathbb{C}$ is a topos, then $\mathbb{C}_P$ is a pCCC by Theorem 6.1. Reciprocally we suppose that $\mathbb{C}$ is a CCC and $\mathbb{C}_P$ is a pCCC. Since $\mathbb{C}_{P,T}$ is isomorphic to $\mathbb{C}$, it is enough, by 5.5, to prove that $\mathbb{C}_P$ is a partial topos, which is easy by noting that we just use that $\mathbb{C}$ has finite limits in the proof that partial monos have partial inverses. ∎

The final result of the section uses that toposes have initial objects and coproducts. We shall need the following fact.

FACT. *In a category with binary products and an initial object* 0 *s.t.* $a \times 0 \approx 0$ *for all objects* a, *then for any object* b

$$b \to 0 \neq \varnothing \Rightarrow b \approx 0.$$

Proof. Let $f: b \to 0$ and set $f' = \langle \mathrm{Id}, f \rangle; i_{b\times 0}^{-1}: b \to 0$, where $i_{b\times 0}^{-1}$ is the inverse of the unique $i_{b\times 0}: 0 \to b \times 0$. Then

$$i_b; f' = i_0 = \mathrm{Id}, \qquad f'; i_b = f'; i_{b\times 0}; \mathrm{Fst} = \mathrm{Id}. \quad \blacksquare$$

PROPOSITION 7.5. *Every partial topos has zeros making it into a pointed category. Those zeros are minimum. Moreover, for all pairs of arrows $f, g: a \to b$ s.t. $(f; !) \cap (g; !) = O$, f, g have a lub $f \cup g$.*

Proof. Let $\mathbb{C}$ be a partial topos. If we know that the statement holds for $\mathbb{C}^\wedge$, then it holds for $\mathbb{C}$, noticing that the preservation of products implies the preservaton of the order. So we are left to prove the statement for $\mathbb{C}_P$, where $\mathbb{C}$ is a topos. We use that toposes have initial objects. Then using cartesian closedness we notice that, if 0 is the initial object, for any objects a, b,

$$0 \times a \to b \approx 0 \to (a \Rightarrow b) = \{i_{a\Rightarrow b}\};$$

hence $0 \times a$ is initial, i.e., $0 \times a \approx 0$.

We take in $\mathbb{C}_P$ $O_{a,b} = i_{a\times b} = \langle i_a, i_b \rangle$. That i_a is mono is an immediate consequence of the fact quoted before the proposition. Another consequence is that for any $f: a \to b$, i_a, Id_0 provide a pullback for f, i_b, so that $O_{a,b}; P\langle j, f \rangle = O$, $\langle j, f \rangle;_P O_{a,b} = O$ for any $\langle j, f \rangle$ with suitable source, target. For any $\langle j, f\rangle: a \to_P b$, $\langle j, f\rangle \restriction O = \langle\!\langle j, f\rangle, O\rangle P; P\,\mathrm{Fst}$, and we check easily $\langle\!\langle j, f\rangle, O\rangle_P = \langle i, i\rangle$. Finally, suppose $(\langle j: c \to a, f\rangle;_P!) \cap (\langle m: d \to a, g\rangle;_P!) = O$, i.e., $j \therefore m = O$, i.e., O_c, O_d form a pullback of j, m. This is known as the property of being disjoint. Then, by a simple application of the preservation of coproducts by pullbacks, we get that $[j, m]: c + d \to a$ is mono (see Goldblatt, 1984, p. 119, for details). We take

— $\langle j, f\rangle \cup \langle m, g\rangle = \langle [j, m], [f, g]\rangle$.

Indeed $\langle j, f\rangle = \mathrm{in}\,1;\ \langle [j, m], [f, g]\rangle$, where $\mathrm{in}\,1: c \to c + d$ is the left injection of the coproduct, and, if $\langle j, f\rangle = p;\ \langle n, h\rangle$, $\langle m, g\rangle = q;\ \langle n, h\rangle$, then

$$[j, m] = [p; n, q; n] = [p, q]; n. \quad \blacksquare$$

Warning. We have not proved that the pC structure induced by the zeros is the same as the pC structure determined by the products, and we may guess from the funny pC structures on pointed categories exhibited in Section 1 that they are different, in general. But the two preorders have the same minimums (namely the zeros), and total arrows, i.e., s.t. $\mathrm{Id} \restriction f = \mathrm{Id}$, are total as defined with zeros, i.e., $g; f = O \Rightarrow g = O$ (see Section 8).

8. Equational Presentation of Partial Toposes

The main result of the section is that partial toposes admit an equational characterization (Theorem 8.7). This is a rather simple observation up to the partial cartesian closed structure, but one needs more imagination for partial toposes: we use apparently stronger equations than the partial inversibility axiom. Before coming to this main result (8.7), we spend some time on various systems of equations (as well as other nonequational descriptions) of the pCC structure. This variety shows that the pCC structure is rich enough to enjoy quite different definitions, and stable enough to ensure that all the approaches developed independently to capture it indeed agree. The reader not interested in these meanders can skip directly to 8.7 after 8.1.

PROPOSITION 8.1. *Let $\mathbb{C}$ be a category equipped with operations on objects and arrows as in 2.1. $\mathbb{C}$ is a pCC iff the following identities hold (greek letters denote arrows into t, $f \restriction g$ is an abbreviation for $\langle f, g\rangle$;* Fst):

(1) $f \restriction f = f$

(2) $f \restriction (g \restriction h) = f \restriction (h \restriction g)$

(3) $f \restriction (g \restriction h) = (f \restriction g) \restriction h$

(4) $f \upharpoonright (g \upharpoonright h) = f \upharpoonright \langle g, h \rangle$
(5) $(f; g) \upharpoonright h = (f \upharpoonright h); g$
(6) $\langle f, g \rangle \upharpoonright h = \langle f \upharpoonright h, g \rangle$
(7) $\langle f, g \rangle \upharpoonright h = \langle f, g \upharpoonright h \rangle$
(8) $f; \langle g, h \rangle = \langle f; g, f; h \rangle$
(9) $\langle \mathrm{Fst}, \mathrm{Snd} \rangle = \mathrm{Id}$
(10) $\langle f, g \rangle; \mathrm{Fst} = \langle g, f \rangle; \mathrm{Snd}$
(11) $\mathrm{Id} \upharpoonright ! = \mathrm{Id}$
(12) $! \upharpoonright \varphi = \varphi$.

Proof. Checking the equations is an easy exercise, using the same tricks as in Sections 2 and 3. We check the other direction.

We observe that the two equations about ! guarantee that the isomorphism α between $a \to t$ and the set of arrows $f: a \to a$ s.t. $f \leqslant \mathrm{Id}$ holds. First notice

— $h \upharpoonright (f; g) = h \upharpoonright (f; (\mathrm{Id} \upharpoonright g))$
— $f \upharpoonright g = f \upharpoonright (g; !)$.

Indeed

$$h \upharpoonright (f; (\mathrm{Id} \upharpoonright g)) = h \upharpoonright (f \upharpoonright (f; g)) = h \upharpoonright ((f; g) \upharpoonright f)$$
$$= h \upharpoonright ((f \upharpoonright f); g) = h \upharpoonright (f; g)$$
$$f \upharpoonright (g; !) = f \upharpoonright (g; (\mathrm{Id} \upharpoonright !)) = f \upharpoonright g.$$

Remember $\alpha(\varphi) = \mathrm{Id} \upharpoonright \varphi$, $\alpha'(f) = f; !$,

$$\alpha(\alpha'(f)) = \mathrm{Id} \upharpoonright (f; !) = \mathrm{Id} \upharpoonright f = f$$
$$\alpha'(\alpha(\varphi)) = (\mathrm{Id} \upharpoonright \varphi); ! = ! \upharpoonright \varphi = \varphi.$$

That α is monotonic will result from the proof of monotonicity of ";" below. Also

— $\mathrm{Id} \upharpoonright f = \mathrm{Id} \Leftrightarrow f; ! = !$.

Indeed if $\mathrm{Id} \upharpoonright f = \mathrm{Id}$,

$$! = (\mathrm{Id} \upharpoonright f); ! = ! \upharpoonright f = ! \upharpoonright (f; !) = f; !$$

and if $f; ! = !$,

$$\mathrm{Id} \upharpoonright f = \mathrm{Id} \upharpoonright (f; !) = \mathrm{Id} \upharpoonright ! = \mathrm{Id}.$$

We show that $f \leqslant g$ given by $g \restriction f = f$, total arrows f given by $\mathrm{Id} \restriction f = \mathrm{Id}$ yield a pC. $\leqslant$ is reflexive, antisymmetric, transitive, using respectively (1), (2), (3); for the right monotonicity, use the following easy consequence of (8):

— $(h; g) \restriction (h; f) = h; (g \restriction f)$.

For the left monotonicity, suppose $f \leqslant g$; then

$$f; h = (g \restriction f); h = (g; h) \restriction f \leqslant g; h.$$

The other axioms of a pC will follow from the proof of the axioms about t (cf. Section 2). (t1) is just (12).

(t3) Assume $g \restriction f = f$, $g \restriction f' = f'$, $(f'; !) \restriction (f; !) = f; !$. First we notice

$$(f'; !) \restriction (f; !) = (f'; !) \restriction f = (f' \restriction f); !.$$

Then

$$f' \restriction f = g \restriction (f' \restriction f) = g \restriction f = f.$$

(p1) Consequence of (9); indeed, using $\mathrm{Id} \restriction \langle \mathrm{Fst}, \mathrm{Snd} \rangle = \mathrm{Id}$,

$$\mathrm{Id} \restriction \mathrm{Fst} = \mathrm{Id} \restriction \mathrm{Fst} \restriction \mathrm{Snd} \restriction \mathrm{Fst} = \mathrm{Id} \restriction \mathrm{Fst} \restriction \mathrm{Snd} = \mathrm{Id}.$$

(t4) We can use the "maximal implies maximum" argument to get from (p1)

— $\mathrm{Fst}_{t,t} = \mathrm{Snd}_{t,t} = !_{t \times t}$.

Now assume $h' \restriction h = h$, $(h'; !) \restriction \psi = \psi = (h'; \varphi) \restriction \psi$; we deduce, using (10),

$$(h; \varphi) \restriction \psi = ((h'; \varphi) \restriction h) \restriction \psi = \psi \restriction h = \psi \restriction (h; !) = (h; !) \restriction \psi = \psi.$$

(p2) Use (4), (6), (7).

(p4) Use (3), (1).

(p6) By remarks above (p6) is equivalent to

— $\mathrm{Id} \restriction \langle f, g \rangle = (\mathrm{Id} \restriction f) \cap (\mathrm{Id} \restriction g)$.

Indeed $\mathrm{Id} \restriction \langle f, g \rangle = \mathrm{Id} \restriction f \restriction g \leqslant \mathrm{Id} \restriction f$ by (p4); suppose $\mathrm{Id} \restriction f \restriction k = k = \mathrm{Id} \restriction g \restriction k$; then remark $\mathrm{Id} \restriction f \restriction g \restriction k = k$. ∎

We can make some variations on the set of axioms. The following set stresses on domains as endomorphisms (in the context of this section we abbreviate $\mathrm{Dom}^D(f) = \mathrm{Id} \restriction f$ into $\mathrm{Dom}(f)$).

PROPOSITION 8.2. *The presentation* (2)–(5) *is equivalent to* (2′)–(5′) *below, where* $\mathrm{Dom}(f)$ *stands for* $\langle \mathrm{Id}, f \rangle$; Fst:

(2′) $f \upharpoonright g = \mathrm{Dom}(g); f$

(3′) $\mathrm{Dom}(f; g) = \mathrm{Dom}(f; \mathrm{Dom}(g))$

(4′) $\mathrm{Dom}(f); \mathrm{Dom}(g) = \mathrm{Dom}(g); \mathrm{Dom}(f)$

(5′) $\mathrm{Dom}(\langle f, g\rangle) = \mathrm{Dom}(f); \mathrm{Dom}(g)$.

Proof. First we prove (2′)–(5′):

$$\mathrm{Dom}(g); f = (\mathrm{Id} \upharpoonright g); f =_5 f \upharpoonright g.$$

(3′) is an instance of the first equality checked in the proof of 8.1. For (4′) we first notice

— $f \upharpoonright \mathrm{Id} = f$ by (2′) and (1),

$$(\mathrm{Id} \upharpoonright f); (\mathrm{Id} \upharpoonright g) =_5 (\mathrm{Id} \upharpoonright g) \upharpoonright f =_2 (\mathrm{Id} \upharpoonright f) \upharpoonright g = (\mathrm{Id} \upharpoonright g); (\mathrm{Id} \upharpoonright f)$$

$$\mathrm{Id} \upharpoonright \langle f, g\rangle =_{4,2,3} (\mathrm{Id} \upharpoonright g) \upharpoonright f = (\mathrm{Id}; (\mathrm{Id} \upharpoonright g)) \upharpoonright f =_5 (\mathrm{Id} \upharpoonright g); (\mathrm{Id} \upharpoonright f).$$

Reciprocally we prove (2)–(5) from (2′)–(5′). First we notice

— $\mathrm{Dom}(\mathrm{Dom}(g)) = \mathrm{Dom}(g)$ as an instance of (3′) for $f = \mathrm{Id}$

— $\mathrm{Dom}(\mathrm{Dom}(f); \mathrm{Dom}(g)) = \mathrm{Dom}(f);\ \mathrm{Dom}(g)$ from (5′) and the previous equality

$$\begin{aligned} f \upharpoonright (g \upharpoonright h) &=_{2'} \mathrm{Dom}(\mathrm{Dom}(h); g); f =_{3'} \mathrm{Dom}(\mathrm{Dom}(h); \mathrm{Dom}(g)); f \\ &= \mathrm{Dom}(h); \mathrm{Dom}(g); f =_{2'} (f \upharpoonright g) \upharpoonright h \\ &=_{4'} \mathrm{Dom}(g); \mathrm{Dom}(h); f = (f \upharpoonright h) \upharpoonright g \end{aligned}$$

$$f \upharpoonright \langle g, h\rangle =_{2'} \mathrm{Dom}(\langle g, h\rangle); f =_{5'} \mathrm{Dom}(g); \mathrm{Dom}(h); f$$

$$(f; g) \upharpoonright h = \mathrm{Dom}(h); f; g = (f \upharpoonright h); g. \quad \blacksquare$$

Here is yet another presentation of the same theory (less what regards the domain classifier), due to P. Rosolini (1986), where the structures are called *p-categories*.

PROPOSITION 8.3. *The presentation* (1)–(10) *augmented by*

— $\Delta = \langle \mathrm{Id}, \mathrm{Id}\rangle$

— $f \times g = \langle \mathrm{Fst}; f, \mathrm{Snd}; g\rangle$

is equivalent to the following presentation where (1″), (2″) *express functoriality,* (3″)–(5″) *express naturality of the diagonal and the projections,* (6″)–(12″) *express some kind of coherence conditions between those functors, and* (13″)–(14″) *express the naturality of the commutativity, associativity*:

(1″) $\mathrm{Id} \times \mathrm{Id} = \mathrm{Id}$

(2″) $(f; g) \times (h; k) = f \times h; g \times k$

(3″) $\Delta; f \times f = f; \Delta$

(4″) $\mathrm{Fst}; f = f \times \mathrm{Id}; \mathrm{Fst}$

(5″) $\mathrm{Snd}; f = \mathrm{Id} \times f; \mathrm{Snd}$

(6″) $\Delta; \mathrm{Fst} = \mathrm{Id}$

(7″) $\Delta; \mathrm{Snd} = \mathrm{Id}$

(8″) $\Delta; \mathrm{Fst} \times \mathrm{Snd} = \mathrm{Id}$

(9″) $\mathrm{Id} \times \mathrm{Fst}; \mathrm{Fst} = \mathrm{Fst}$

(10″) $\mathrm{Id} \times \mathrm{Snd}; \mathrm{Fst} = \mathrm{Fst}$

(11″) $\mathrm{Fst} \times \mathrm{Id}; \mathrm{Snd} = \mathrm{Snd}$

(12″) $\mathrm{Snd} \times \mathrm{Id}; \mathrm{Snd} = \mathrm{Snd}$

(13″) $\beta; g \times f = f \times g; \beta$, *where* $\beta = \langle \mathrm{Snd}, \mathrm{Fst} \rangle$

(14″) $\alpha; (f \times g) \times h = f \times (g \times h); \alpha$, *where* $\alpha = \langle\langle \mathrm{Fst}, \mathrm{Snd}; \mathrm{Fst} \rangle, \mathrm{Snd}; \mathrm{Snd} \rangle$ *augmented by*

— $\langle f, g \rangle = \Delta; f \times g$

(*we have omitted types, i.e., objects, which are clear from the context, and we assume that* $\times$ *has highest precedence*).

Remarks. (1) The definition stresses that, in the partial case, only $\mathrm{Fst}: \lambda a.a \times b \to \lambda a.a$, not $\mathrm{Fst}: \lambda(a, b).\, a \times b \to \lambda(a, b).a$ is natural.

(2) It should be clear from what follows that α, β are actually natural equivalences, and, although we do not have a formal proof of it, that a "semi-groupoidal" structure holds, i.e., coherence of commutativity and associativity follows from the other axioms.

(3) (5″) (or (4″)) is redundant, i.e., the naturality of one projection follows from the naturality of the other projection, in the context of the other axioms.

Proof of 8.3. The proof is very tedious. We only give samples. Here is the proof of (1″) (we use $\langle f \upharpoonright h, g \rangle = \langle f, g \upharpoonright h \rangle$):

$$\begin{aligned}
&\langle \mathrm{Fst}; f, \mathrm{Snd}; h \rangle; \langle \mathrm{Fst}; g, \mathrm{Snd}; k \rangle \\
&\quad = \langle (\mathrm{Fst}; f; g) \upharpoonright (\mathrm{Snd}; h), (\mathrm{Snd}; h; k) \upharpoonright (\mathrm{Fst}; f) \rangle \\
&\quad = \langle \mathrm{Fst}; f; g, ((\mathrm{Snd}; h; k) \upharpoonright (\mathrm{Snd}; h)) \upharpoonright (\mathrm{Fst}; f) \rangle \\
&\quad = \langle \mathrm{Fst}; f; g, (\mathrm{Snd}; h; k) \upharpoonright (\mathrm{Fst}; f) \rangle \\
&\quad = \langle \mathrm{Fst}; f; g, \mathrm{Snd}; h; k \rangle.
\end{aligned}$$

Reciprocally (1), (8), (9), (2′) are easy. (10) is proved in the same way as the redundancy above. For (3′) we first prove

— $\mathrm{Id} \times f; \mathrm{Fst} = \mathrm{Id} \times \mathrm{Dom}(f); \mathrm{Fst}$,

$$\begin{aligned}\mathrm{Id} \times \mathrm{Dom}(f); \mathrm{Fst} &= \mathrm{Id} \times \Delta; (\mathrm{Id} \times f); \mathrm{Id} \times \mathrm{Fst}; \mathrm{Fst}\\ &= \mathrm{Id} \times \Delta; \mathrm{Id} \times (\mathrm{Id} \times f); \mathrm{Id} \times \mathrm{Snd}; \mathrm{Fst}\\ &= \mathrm{Id} \times \Delta; \mathrm{Id} \times (\mathrm{Id} \times f; \mathrm{Snd}); \mathrm{Fst}\\ &= \mathrm{Id} \times \Delta; \mathrm{Id} \times \mathrm{Snd}; \mathrm{Id} \times f; \mathrm{Fst}\\ &= \mathrm{Id} \times (\Delta; \mathrm{Snd}); \mathrm{Id} \times f; \mathrm{Fst} = \mathrm{Id} \times f; \mathrm{Fst}.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{Dom}(f; g) &= \Delta; \mathrm{Id} \times f; \mathrm{Id} \times g; \mathrm{Fst} = \Delta; \mathrm{Id} \times f; \mathrm{Id} \times \mathrm{Dom}(g); \mathrm{Fst}\\ &= \mathrm{Dom}(f; \mathrm{Dom}(g)).\end{aligned}$$

We sketch (7):

$$\langle f, g\rangle \upharpoonright h = \Delta; \Delta \times \mathrm{Id}; (f \times g) \times h; \mathrm{Fst}$$

$$\langle f, g \upharpoonright h\rangle = \Delta; \mathrm{Id} \times \Delta; f \times (g \times h); \mathrm{Id} \times \mathrm{Fst}.$$

We get (7) if we prove

— $\Delta; \Delta \times \mathrm{Id} = \Delta; \mathrm{Id} \times \Delta; \alpha$

— $\alpha; \mathrm{Fst} = \mathrm{Id} \times \mathrm{Fst}$.

For the first, we use

— $\Delta; \mathrm{Id} \times \Delta; \mathrm{Snd}; \mathrm{Snd} = \mathrm{Id}$

— $\Delta; \mathrm{Id} \times \Delta; \mathrm{Snd}; \mathrm{Fst} = \mathrm{Id}$

— $\mathrm{Id} \times \Delta; \mathrm{Fst} = \mathrm{Fst}$ (use (9″)),

$$\begin{aligned}\Delta; \mathrm{Id} \times \Delta; \alpha = \Delta; (\Delta; \mathrm{Id} \times \Delta) \times (\Delta; \mathrm{Id} \times \Delta);\\ (\Delta; \mathrm{Fst} \times (\mathrm{Snd}; \mathrm{Fst})) \times (\mathrm{Snd}; \mathrm{Snd}) = \Delta; \gamma \times \mathrm{Id},\end{aligned}$$

where $\gamma = \Delta;\ \mathrm{Id} \times \Delta; \Delta; \mathrm{Fst} \times (\mathrm{Snd}; \mathrm{Fst}))$. The same kind of surgery yields

$$\gamma = \Delta; (\mathrm{Id} \times \mathrm{Id}) = \Delta.$$

For the second we use (2′), (3′), and

— $\mathrm{Dom}(\mathrm{Snd}) = \mathrm{Id}$ (use (10″)),

$$\begin{aligned}\alpha; \mathrm{Fst} &= (\Delta; \mathrm{Fst} \times (\mathrm{Snd}; \mathrm{Fst})) \upharpoonright (\mathrm{Snd}; \mathrm{Snd}) = \Delta; \mathrm{Fst} \times (\mathrm{Snd}; \mathrm{Fst})\\ &= \Delta; \mathrm{Fst} \times \mathrm{Snd}; \mathrm{Id} \times \mathrm{Fst} = \mathrm{Id} \times \mathrm{Fst}.\end{aligned}$$

(4′), (5′), and (6) are proved in a similar, more tedious way. ∎

Proposition 8.1 suggests that p-categories are essentially pCCs less the domain classifier; this is stressed by the following proposition (proof left to the reader).

PROPOSITION 8.4. *The* p-categories, *i.e., the categories equipped with an operation* $\times$ *on objects and operations* $\langle\ \rangle$, Fst, Snd *on arrows* (*of arity* 2, 0, 0) *and satisfying* (1)–(10) (*or the equivalent presentations of* 8.2, 8.3) *are exactly those pC categories satisfying* (p1)–(p5) (*cf.* 2.1), (5′), (3′), *as well as the following, where* $\mathrm{Dom}(f) = \langle \mathrm{Id}, f\rangle; \mathrm{Fst}$:

(p7) *the total arrows are exactly those arrows* $f: a \to b$ *s.t.* $\mathrm{Dom}(f) = \mathrm{Id}$

(p8) *if* $f, f', g: a \to b$ *are s.t.* $f, f' \leqslant g$ *and* $\mathrm{Dom}(f) \leqslant \mathrm{Dom}(f')$, *then* $f \leqslant f'$

(p9) $\mathrm{Dom}(f); \mathrm{Dom}(f) = \mathrm{Dom}(f)$.

(*The reader should compare* (5′) *with* (p6) *and* (3′), (p7)–(p9) *with* (t1)–(t4).) ∎

The last proposition allows us to use more natural arguments than solely explicit equational reasoning for proving properties of p-categories. As an example we show a last variation on (1)–(12), which has been suggested by Robinson and Rosolini (1986). Of course equational completeness guarantees that we could have produced a complete equational proof (Exercise: do it for the next proposition!).

PROPOSITION 8.5. *In* 8.1 (12) *can be replaced by* (12′) *or* (13′) *below*

(12′) $\mathrm{Fst}; \langle \mathrm{Id}, !\rangle = \mathrm{Id}$

(13′) $\mathrm{Fst}; ! = \mathrm{Snd}$,

where the typing is left to the reader. The pCCs are exactly the p-categories $\mathbb{C}$ *such that* $\mathbb{C}_T$ *has a terminal object.*

Proof. We first prove (13′), which implies (12′) by (8), (9). We know that Fst is total, i.e., $\mathrm{Fst}; ! = !$, and, remember from Section 2, $\mathrm{Snd} = !$ by the "maximum–maximal" argument. We show $(12') \Rightarrow (13') \Rightarrow (12)$, by nonequational arguments. From $\langle \mathrm{Fst}, \mathrm{Fst}; !\rangle = \mathrm{Id}$ we get $\langle \mathrm{Fst}, \mathrm{Fst}; !\rangle; \mathrm{Snd} = \mathrm{Snd}$, and using that Fst is total we reduce the l.h.s. to $\mathrm{Fst}; !$. For (12) we compose by $\langle \mathrm{Id}, \varphi\rangle$:

$$\varphi = \langle \mathrm{Id}, \varphi\rangle; \mathrm{Snd} = \langle \mathrm{Id}, \varphi\rangle; \mathrm{Fst}; ! \leqslant \mathrm{Id}; ! = !.$$

Now we prove the second assertion. We already know one direction from Section 2 and the propositions above. Suppose that t, $!_x$ define a terminal object in $\mathbb{C}_T$. Then (11) holds since $!_x$ is total, being an arrow of $\mathbb{C}_T$; (13′) holds because both sides are total, thus equal to $!$. ▌

Remark. One more formulation is, putting together (11) and (12′): the pCCs are exactly the p-categories with an object 1 and arrows $!_a : a \to 1$ s.t. $\langle \mathrm{Id}_a, !_a \rangle$ and $\mathrm{Fst}_{a,1}$ are inverse isos between a and $a \times 1$, for every object a.

We describe now yet another way of looking at the same concept of partial cartesian structure, starting from a symmetric monoidal structure. Indeed the pioneering work in the field of categorical partiality was done by H.-J. Hoehnke (1977), in this setting. We also integrate in the next exposition results of J. Schreckenberger (1984). Finally, still another recent approach to categorical partiality has been proposed by A. Carboni (1985). All these definitions agree with ours and Rosolini's, as we show below.

Let us first recall that a symmetric monoidal category can be described equationally as a category $\mathbb{C}$ equipped with a functor $\times : \mathbb{C} \to \mathbb{C}$, a distinguished object 1, and natural equivalences

$$\mathrm{Ass}: (\lambda(a, b, c).a \times (b \times c)) \to (\lambda(a, b, c).(a \times b) \times c)$$

$$\mathrm{Com}: \lambda(a, b).a \times b \to \lambda(a, b).b \times a$$

$$\mathrm{IdL}: (\lambda a.1 \times a) \to (\lambda a.a)$$

satisfying

$$\mathrm{Ass}; \mathrm{Ass} = (\mathrm{Id} \times \mathrm{Ass}); \mathrm{Ass}; (\mathrm{Ass} \times \mathrm{Id})$$

$$\mathrm{IdL} = \mathrm{Ass}; (\mathrm{IdL} \times \mathrm{Id})$$

$$\mathrm{Com}(a, b); \mathrm{Com}(b, a) = \mathrm{Id}$$

$$\mathrm{Ass}; \mathrm{Com}; \mathrm{Ass} = (\mathrm{Id} \times \mathrm{Com}); \mathrm{Ass}; (\mathrm{Com} \times \mathrm{Id}).$$

We do not write the equations which state the functoriality of $\times$, the naturality and inversibility of Ass, Com, IdL. Together with the four coherence axioms above they yield an equational presentation of symmetric monoidal categories. As is well known the main interest of the above presentation is that these properties are enough to ensure coherence.

COHERENCE THEOREM. *Given two linear objects S, T (i.e., S, T are terms built with* 1, $\times$, *and variables, and a variable may appear at most once in the term), all terms between S, T are equal in the theory of symmetric monoidal categories.*

Proof. MacLane (1971). ▌

Remark. Notice that the restriction to linear objects is not explicit in MacLane (1971), which takes a more abstract view, not of arrows in the category $\mathbb{C}$, but of arrows in $\mathbb{C}^n \to \mathbb{C}$ (natural transformations): in this view there is no risk of confusing $\mathrm{Com}: a \times a \to a \times a$ and $\mathrm{Id}: a \times a \to a \times a$, since they are instances of natural transformations which do not have the same functors as targets.

Remark. In the presentation above the right identity IdR: $a \times 1 \to a$ is not primitive; it is defined as IdR = Com; IdL.

The partial cartesian structure on a symmetric monoidal category is imposed by axioms involving arrows $\Delta_a : a \to a \times a$, $!_a : a \to 1$.

PROPOSITION 8.6. *Let* $\mathbb{C}$ *be a symmetric monoidal category equipped with a natural transformation* $\Delta : \lambda a.a \to \lambda a.a \times a$ *and an arrow* $!_a : a \to 1$ *for each object a. The following are equivalent*:

I (Hoehnke). *The set of properties*

(i) *for any closed term* f; $! = !$

(ii) Δ; $(! \times \mathrm{Id})$; IdL = Id

(iii) Δ; $((\mathrm{Id} \times !) \times (! \times \mathrm{Id}))$; $(\mathrm{IdR} \times \mathrm{IdL}) = \mathrm{Id}$.

II (Schreckenberger). *The equational properties* (ii), (iii), *and*

(iv) $(!_a \times !_b)$; IdR $= !_{a \times b}$.

III. *Any object has a unique structure of commutative comonoid, which is given by* Δ, !, *where the comonoid structure is described by* (ii) *and*

(coass) Δ; $(\mathrm{Id} \times \Delta)$; Ass $= \Delta$; $\Delta \times \mathrm{Id}$
(cocom) Δ; Com $= \Delta$.

Moreover, any of these definitions is equivalent to the assertion that $\mathbb{C}$ is a pCC. Here are the codings:

$$\mathrm{Snd} = (! \times \mathrm{Id});\ \mathrm{IdL}$$

$$\mathrm{Fst} = (\mathrm{Id} \times !);\ \mathrm{IdR}\ (= \mathrm{Com};\ \mathrm{Snd})$$

$$\langle f, g \rangle = \Delta;\ (f \times g).$$

Conversely, for a pCC

$$\mathrm{Ass} = \langle\langle \mathrm{Fst}, \mathrm{Snd}; \mathrm{Fst} \rangle, \mathrm{Snd}; \mathrm{Snd} \rangle \qquad (\mathrm{Ass}^{-1} \text{ left to the reader})$$

$$\mathrm{Com} = \mathrm{Snd} \times \mathrm{Fst}$$

$$\mathrm{IdL} = \mathrm{Snd},\ \mathrm{IdL}^{-1} = \langle !, \mathrm{Id} \rangle$$

$$\Delta = \langle \mathrm{Id}, \mathrm{Id} \rangle.$$

Remarks. By definition of IdR and naturality of Com we get from (ii)

(ii′) Δ; $(\mathrm{Id} \times !)$; IdR = Id.

Notice also that III is not an equational presentation.

Before proving the proposition we state some consequences and variants of definitions I, II.

FACT. — $(f; g=\mathrm{Id}) \Rightarrow f: ! = !$

— for any closed term f into 1, $f = !$,

(v) $!_1 = \mathrm{Id}_1$.

Proof. Clearly the second assertion follows from the last and (i). The first and last properties are derivable from (ii) and the naturality of Δ. Here is the first:

$$f; ! = \Delta; (\mathrm{Id} \times !); \mathbf{IdR; f; !} = \Delta; ((f; !) \times !); \mathbf{IdR} = \Delta; (f \times \mathbf{Id}); (! \times !); \mathrm{IdL}$$
$$= f; \Delta; (\mathrm{Id} \times g); (! \times !); \mathrm{IdL} = f; \Delta; (! \times \mathrm{Id}); \mathbf{(Id \times (g; !)); IdL}$$
$$= f; \mathbf{\Delta; (! \times Id); IdL;}\ g; ! = f; g; ! = !.$$

For the last we go through the steps:

— $\langle f, \mathrm{Id} \rangle; \mathrm{Fst} = f$.

We get

$$\Delta; (f \times \mathrm{Id}); (\mathrm{Id} \times !); \mathrm{IdR} = \Delta; (\mathrm{Id} \times !); \mathbf{(f \times Id); IdR} = \Delta; (\mathrm{Id} \times !); \mathrm{IdR}; f = f.$$

From $\mathrm{IdL}(1) = \mathrm{IdR}(1)$, which holds by coherence, and (iii) we deduce

— $\langle \mathrm{Id}, ! \rangle = \langle !, \mathrm{Id} \rangle$.

Then we establish

— $!; ! = !$.

Indeed we compute

$$! = \Delta; (\mathrm{Id} \times !); \mathrm{IdR}; ! = \mathbf{\Delta; (Id \times !); (! \times Id);}\ \mathrm{IdR} = !; \Delta; \mathrm{IdR}$$
$$\mathrm{Id} = \Delta; (\mathrm{Id} \times !); \mathrm{IdR} = \Delta; (! \times \mathrm{Id}); \mathrm{IdR} = \Delta; \mathrm{IdR}; !$$

and

$$! = !; \Delta, \mathrm{IdR}; ! = !; !.$$

Now we can prove

$$! = \langle !, \mathrm{Id} \rangle; \mathrm{Fst} = \langle \mathrm{Id}, ! \rangle; \mathrm{Fst} = \Delta; (\mathrm{Id} \times !); (\mathrm{Id} \times !); \mathrm{IdR}$$
$$= \Delta; (\mathrm{Id} \times !); \mathrm{IdR} = \mathrm{Id}. \quad \blacksquare$$

Remark. As quoted by Schreckenberger the original definition of

Hoehnke was more redundant than I: it contained the two first properties of the above fact.

FACT. *In the presence of* (ii) *and the naturality of* Δ, (iv) *is equivalent to*

— (iv′) $(!_a \times !_b); !_{1\times 1} = !_{a\times b}$.

Proof. Notice that since IdR is iso one has by the last fact,

$$\mathrm{IdR}(1); !_1 = !_{1\times 1}$$

which again by the last fact reduces to $\mathrm{IdR}(1) = !_{1\times 1}$. ∎

Remark. Indeed Schreckenberger proposed (iv′) rather than (iv).

Proof of 8.6. Setting IV for the property of being a pCC, according to one of the characterizations given before the statement, the plan of the proof is:

$$\mathrm{IV} \Rightarrow \mathrm{I} \Rightarrow \mathrm{II} \Rightarrow \mathrm{IV} \Rightarrow \mathrm{III} \Rightarrow \mathrm{II}.$$

IV ⇒ I. If $\mathbb{C}$ is a pCC, let Ass, Com, IdL, and their inverses be defined as above. We know from a remark above that IdL, IdL^{-1} are indeed inverses, and from (14″) that Ass is natural. We check the naturality of IdL:

$$(\mathrm{Id} \times f); \mathrm{IdL} = \langle \mathrm{Fst}, \mathrm{Snd}; f\rangle; \mathrm{Snd} = \mathrm{Snd}; f.$$

The inversibility of Ass as well as the coherence equations are easily checked, using systematically $\langle f, g\rangle; \mathrm{Fst} = f$ when g is total. The naturality of Δ is (3″). If $f; g = \mathrm{Id}$, then $! = f; g; ! \leqslant f; !$, thus $f; ! = !$. For (i) we have to check that total arrows compose (which is part of the definition of a pC), and that if f, g are total, so is $f \times g$: observe

$$\langle \mathrm{Fst}; f, \mathrm{Snd}; g\rangle; ! = (\mathrm{Fst}; f; !) \cap (\mathrm{Snd}; g; !).$$

(ii), (iii) are just (7″), (8″), noticing

$$(! \times \mathrm{Id}); \mathrm{IdL} = \langle \mathrm{Fst}; !, \mathrm{Snd}\rangle; \mathrm{Snd} = \mathrm{Snd}.$$

I ⇒ II. Obvious: (iv) is an instance of a property quoted in a fact above.
II ⇒ IV. Proving (1″)–(12″) is almost routine; we show (4″), (9″) only:

(4″) $(\mathrm{Id} \times !); \mathbf{IdR; f} = (\mathrm{Id} \times !); (\mathrm{f} \times \mathrm{Id}); \mathrm{IdR} = (f \times \mathrm{Id}); \mathrm{Fst}$

(9″) $(\mathrm{Id} \times \mathrm{Fst}); \mathrm{Fst} = (\mathrm{Id} \times A); \mathrm{IdR}$, where

$$A = (\mathrm{Id} \times !); \mathbf{IdR; ! = (! \times !); IdR} = !,$$

using IdR = IdL by coherence.

(11) is (ii), using $\mathrm{IdL}(1) = \mathrm{Snd}$ by (v), and (12′) follows since IdL is iso. We are left with (13″) and (14″); it is enough to show

$$\mathrm{Com} = \langle \mathrm{Snd}, \mathrm{Fst} \rangle$$

$$\mathrm{Ass} = \langle\langle \mathrm{Fst}, \mathrm{Snd}; \mathrm{Fst} \rangle, \mathrm{Snd}; \mathrm{Snd} \rangle.$$

It is enough to prove, following Hoehnke, $\mathrm{II} \Rightarrow \mathrm{III}'$, where III′ is III without uniqueness. Indeed,

$$\mathrm{Com} = \Delta; \mathbf{(Fst \times Snd); Com} = \mathbf{\Delta; Com;} (\mathrm{Snd} \times \mathrm{Fst})$$

$$= \langle \mathrm{Snd}, \mathrm{Fst} \rangle$$

$$\langle\langle \mathrm{Fst}, \mathrm{Snd}; \mathrm{Fst} \rangle, \mathrm{Snd}; \mathrm{Snd} \rangle$$

$$= \mathbf{\Delta; (\Delta \times Id);} ((\mathrm{Fst} \times (\mathrm{Snd}; \mathrm{Fst})) \times (\mathrm{Snd}; \mathrm{Snd}))$$

$$= \Delta; \mathbf{(Id \times \Delta); (Fst \times ((Snd; Fst) \times (Snd; Snd)));} \mathrm{Ass}$$

$$= \Delta; (\mathrm{Fst} \times (\mathbf{\Delta; (Snd \times Snd);} (\mathrm{Fst} \times \mathrm{Snd})); \mathrm{Ass} = \mathrm{Ass}.$$

So we are left with III′: cocom is proved by first observing

$$\mathrm{Fst} = \mathbf{(Id \times !); Com;} \mathrm{IdL} = \mathrm{Com}; (! \times \mathrm{Id}); \mathrm{IdL} = \mathrm{Com}; \mathrm{Snd}.$$

We get

$$\Delta; \mathrm{Com} = \mathbf{\Delta; Com; \Delta;} (\mathrm{Fst} \times \mathrm{Snd}) = \Delta; ((\Delta; \mathrm{Com}; \mathrm{Fst}) \times (\Delta; \mathrm{Com}; \mathrm{Snd}),$$

where $\Delta; \mathrm{Com}; \mathrm{Fst} = \Delta; \mathrm{Snd} = \mathrm{Id}$.

coass requires an explicit use of coherence. From (ii), (iii) we can reduce the proof of $f = g$ to $f; \mathrm{Fst} = g; \mathrm{Fst}$ and $f; \mathrm{Snd} = g; \mathrm{Snd}$. Indeed,

$$f = \mathbf{f; \Delta;} (\mathrm{Fst} \times \mathrm{Snd}) = \Delta; ((f; \mathrm{Fst}) \times (f; \mathrm{Snd})).$$

Call $A = \Delta; (\Delta \times \mathrm{Id})$, $B = \Delta; (\mathrm{Id} \times \Delta); \mathrm{Ass}$:

$$A; \mathrm{Fst} = \Delta; \mathbf{(\Delta \times Id); (Id \times !);} \mathrm{IdR} = \Delta; (\mathrm{Id} \times !); \mathbf{(\Delta \times Id); IdR}$$

$$= \Delta; (\mathrm{Id} \times !); \mathrm{IdR}; \Delta = \Delta.$$

For computing $B; \mathrm{Fst}$ we use the coherence $\mathrm{Ass}; \mathrm{IdR} = \mathrm{Id} \times \mathrm{IdR}$:

$$B; \mathrm{Fst} = \Delta; (\mathrm{Id} \times \Delta); \mathbf{Ass; (Id \times !);} \mathrm{IdR} = \Delta; (\mathrm{Id} \times \Delta); (\mathrm{Id} \times (\mathrm{Id} \times !)); \mathbf{Ass; IdR}$$

$$= \Delta; (\mathrm{Id} \times \mathbf{\Delta}); (\mathrm{Id} \times \mathbf{(Id \times !)}); (\mathrm{Id} \times \mathbf{IdR}) = \Delta.$$

For computing $A; \mathrm{Snd}$ we use $!_1 = \mathrm{Id}_1$, whence $\Delta; \mathrm{IdL} = \mathrm{Id}$:

$$A; \mathrm{Snd} = \Delta; ((\Delta; !) \times \mathrm{Id}); \mathrm{IdL}$$

and $\Delta; ! = \Delta; (! \times !); \mathrm{IdL} = !; \Delta; \mathrm{IdL} = !.$

For computing B; Snd we use the coherence Ass; $(\mathrm{IdL} \times \mathrm{Id}) = \mathrm{IdL}$:

$$\begin{aligned} B; \mathrm{Snd} &= \Delta; (\mathrm{Id} \times \Delta); \mathrm{Ass}; (! \times \mathrm{Id}); \mathrm{IdL} \\ &= \Delta; (\mathrm{Id} \times \Delta); \mathbf{Ass; ((! \times !) \times Id);} (\mathrm{IdL} \times \mathrm{Id}); \mathrm{IdL} \\ &= \Delta; (\mathrm{Id} \times \Delta); (! \times (! \times \mathrm{Id})); \mathbf{Ass; (IdL \times Id);} \mathrm{IdL} \\ &= \Delta; (\mathrm{Id} \times \Delta); (! \times (! \times \mathrm{Id})); \mathbf{IdL; IdL} \\ &= \Delta; \mathbf{(Id \times \Delta); (! \times (! \times Id));} (\mathrm{Id} \times \mathrm{IdL}); \mathrm{IdL} \\ &= \Delta; (! \times \mathrm{Id}); (\mathrm{Id} \times \Delta); (\mathrm{Id} \times \mathbf{(! \times Id)}); (\mathrm{Id} \times \mathbf{IdL}); \mathrm{IdL} \\ &= \Delta; (! \times \mathrm{Id}); \mathrm{IdL} = \mathrm{Id}. \end{aligned}$$

IV $\Rightarrow$ III. From IV $\Rightarrow$ I we already know that the category is symmetric monoidal. IV $\Rightarrow$ III′ is established much in the same way, using again systematically $\langle f, g \rangle; \mathrm{Fst} = f$ when g is total. For the uniqueness, we observe that if Δ'_a, $!'_a$ satisfy

$$\Delta'; (\mathrm{Id} \times !'); \mathrm{IdR} = \mathrm{Id}$$

$$\Delta' = \Delta'; (\mathrm{Snd} \times \mathrm{Fst}).$$

We get, setting $A = \Delta'; \mathrm{Fst} = \Delta'; \mathrm{Snd}$

$$\mathrm{Id} = \langle A, A \rangle; (\mathrm{Id} \times !'); \mathrm{Fst} = \langle A, A; !' \rangle; \mathrm{Fst} \leqslant A.$$

Thus $A = \mathrm{Id}$, and the equality is now $\langle \mathrm{Id}, !' \rangle; \mathrm{Fst} = \mathrm{Id}$, i.e., $!'$ is total, whence $!' = !$.

III $\Rightarrow$ II see Robinson and Rosolini (1986). Only (iii) and (iv) have to be proved, which is done by using coherence and uniqueness. ∎

Remarks. Definition I is Hoehnke's definition of *pre-dht-symmetric categories* (dht-symmetric categories have, moreover, a zero object 0 satisfying $a \times 0 = 0$ (not only $a \times 0 \approx 0$) for all a); definition II is Schreckenberger's equational characterization of pre-dht-symmetric categories, and definition III is an adaptation of Carboni's notion of a bicategory of partial maps, where we forget about the additional locally ordered bicategory structure, which among others allows Carboni to axiomatize orders which may be different from the canonical pC structure on a pCC (cf. the discussion on "flat" and "extensional" in Section 6).

The main drawback of this bunch of definitions is that they have an inherent redundancy: the canonical natural equivalences are both primitive and defined in terms of Fst, Snd, i.e., Δ, !. The redundancy is visible from the proof of 8.6, where only part of the axioms of coherence are used to prove II $\Rightarrow$ IV. Thus the equational presentation of Schreckenberger is surely not minimal.

Among the three definitions, III looks the most elegant and the least arbitrary. It suggests looking at the ways of extracting a pCC from a symmetric monoidal category $\mathbb{C}$. It should be true that, taking as objects structures $(a, \Delta, !)$, where a is an object of $\mathbb{C}$ and $(\Delta, !)$ is a commutative comonoid structure on it, as arrows between $(a, \Delta, !)$ and $(a', \Delta', !')$ those $f: a \to_{\mathbb{C}} a'$ s.t. $f; \Delta' = \Delta; (f \times f)$ in $\mathbb{C}$, we should get a pCC, where binary products are defined as in the proof of III $\Rightarrow$ II above.

Summarizing, here are all the equivalent definitions of a pCC that we have exhibited:

EQUATIONAL DEFINITIONS. — (1)–(12) and the variants obtained by replacing (2)–(5) by (2′)–(5′), or (12) by (12′) or (13′), or (1)–(10) by (1″)–(14″)

— Schreckenberger's equations of a pre-dht-symmetric category.

— p-category with an object 1 and arrows into it s.t. the obvious arrows between $a \times 1$ and a for any a are inverse isos

NONEQUATIONAL DEFINITIONS. — p-category $\mathbb{C}$ with a terminal object in $\mathbb{C}_T$

— Hoehnke's original definition of a pre-dht-symmetric category

— symmetric monoidal category s.t. every object carries a unique commutative comonoid structure, and s.t. the multiplications are natural.

To end this discussion of partial products, we point out that the dominical categories of Di Paola and Heller (1984) are the pointed p-categories where the two induced pC structures coincide, and where, moreover, for all appropriate f, g, 0,

$$\langle f, g \rangle = 0 \Rightarrow f = 0 \text{ or } g = 0.$$

Now we axiomatize partial toposes equationally.

THEOREM 8.7. *Let $\mathbb{C}$ be a pCC. If, moreover, $\mathbb{C}$ is equipped with the operations on objects and arrows of* 4.1, *then $\mathbb{C}$ is a pCCC iff*

(13) $\lambda(f); ! = !$

(14) $\langle \mathrm{Fst}; \lambda(f), \mathrm{Snd} \rangle; \mathrm{App} = f$

(15) $\lambda(\langle \mathrm{Fst}; f, \mathrm{Snd} \rangle; \mathrm{App}) \restriction f = f.$

A pCC has internal equality iff it is equipped with arrows $\equiv : a \times a \to a$, for all objects a, s.t.

(16) $\langle \mathrm{Id}, \mathrm{Id} \rangle; \equiv\; = \mathrm{Id}$

(17) $\equiv; \langle \mathrm{Id}, \mathrm{Id} \rangle = \mathrm{Id} \restriction \equiv.$

A pCCC with internal equality is a partial topos if it has arrows $\sigma_a : (a \Rightarrow t) \to a$, for all objects a, and satisfies

(18) $\lambda(\equiv ; !); \sigma = \mathrm{Id}$

(19) $f^{-}; f = \mathrm{Id} \upharpoonright (f^{-})$ *for all* $f: a \to b$, *where* $f^{-} = \lambda(\mathrm{Fst} \equiv (\mathrm{Snd}; f)); \sigma_a$.

Proof. Only the last statement requires a proof. Suppose that the identities hold. We show that all partial monos have partial inverses. First we prove

— $(\mathrm{Fst}; h) \equiv (\mathrm{Snd}; h) = \ \equiv; h; !$, when h is a partial mono.

$\geqslant$ does not require the hypothesis on h. We just have to notice

$$(\mathrm{Fst}; h) \equiv (\mathrm{Snd}; h) = ((\mathrm{Fst}; h) \cap (\mathrm{Snd}; h)); !.$$

Now we use that h is a partial mono, i.e., mono in $\mathbb{C}_T^{\wedge}$, so that the following diagrams are pullback diagrams in $\mathbb{C}_T^{\wedge}$, setting $A = (\mathrm{Fst}; h) \equiv (\mathrm{Snd}; h)$:

$$\begin{array}{ccc} h; ! & \xrightarrow{\mathrm{Id} \upharpoonright h} & h; ! \\ {\scriptstyle \mathrm{Id} \upharpoonright h}\downarrow & & \downarrow{\scriptstyle h} \\ h; ! & \xrightarrow[h]{} & ! \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\mathrm{Snd} \upharpoonright A} & h; ! \\ {\scriptstyle \mathrm{Fst} \upharpoonright A}\downarrow & & \downarrow{\scriptstyle h} \\ h; ! & \xrightarrow[h]{} & ! \end{array}$$

Let $k: A \to h; !$ be the iso identifying the two diagrams, i.e.,

$$k; ! = k; h; ! = A, \qquad k = \mathrm{Fst} \upharpoonright A = \mathrm{Snd} \upharpoonright A,$$

noticing $k; (\mathrm{Id} \upharpoonright h) = k \upharpoonright k; h = k \upharpoonright k = k$. We are left to show

— $A \leqslant \ \equiv ; !, \mathrm{Fst}; h; !,$

since by (t4) $(\equiv ; !) \cap (\mathrm{Fst}; h; !) = \ \equiv ; h; !$.

For the first use $\mathrm{Fst} \upharpoonright A = \mathrm{Snd} \upharpoonright A$ and notice $\equiv ; ! = \mathrm{Fst} \equiv \mathrm{Snd}$; for the second, $A = k; h; ! \leqslant \mathrm{Fst}; h; !$.

Now we prove that h has a partial inverse. (19) is half of the statement; by easy calculations we get

$$h; \lambda(\mathrm{Fst} \equiv (\mathrm{Snd}; h)) = \lambda(A) \upharpoonright h = \lambda((\equiv ; !) \upharpoonright (\mathrm{Fst}; h; !)) \upharpoonright h = \lambda(\equiv ; !) \upharpoonright h.$$

Hence $h; h^{-} = (\lambda(\equiv ; !) \upharpoonright h); \sigma = \mathrm{Id} \upharpoonright h$.

Now we show that if $\mathbb{C}$ is a partial topos, then arrows σ_a can be defined s.t. (18), (19) hold. By Section 7, it is enough to work in $\mathbb{C}_P$, where $\mathbb{C}$ is a topos. We know that $\equiv \ = \langle\!\langle \mathrm{Id}, \mathrm{Id} \rangle, \mathrm{Id} \rangle$, hence

— $\lambda_P(\equiv ; !) = \langle \mathrm{Id}, K \rangle$, where $K = \lambda(\rho(\langle\!\langle \mathrm{Id}, \mathrm{Id} \rangle, ! \rangle))$,

where ρ is the natural bijection of 5.4.

We only have to check that K is mono, because then we shall set

— $\sigma = \langle K, \mathrm{Id} \rangle$

and (18) will hold as an instance of partial inversibility. We have, setting $K' = \langle\!\langle \mathrm{Id}, \mathrm{Id} \rangle, ! \rangle$,

$$g; K = h; K \Rightarrow (g \times \mathrm{Id}); \rho(K') = (h \times \mathrm{Id}); \rho(K')$$

$$\Rightarrow \langle\!\langle \mathrm{Id}, g \rangle, ! \rangle = \langle\!\langle \mathrm{Id}, h \rangle, ! \rangle \Rightarrow g = h,$$

using $(g \times \mathrm{Id})^{-1}(\langle \mathrm{Id}, \mathrm{Id} \rangle) = \langle \mathrm{Id}, g \rangle$.

Now we prove (19). By easy calculations we get

— $\mathrm{Fst} \equiv (\mathrm{Snd};_P \langle j, f \rangle) = \langle\!\langle f, j \rangle, ! \rangle$,

so that, setting $L = \lambda(\rho(\langle\!\langle f, j \rangle, ! \rangle))$,

— $\langle j, f \rangle^{-} = \langle \mathrm{Id}, L \rangle;_P \langle K, \mathrm{Id} \rangle = \langle L^{-1}(K), K^{-1}(L) \rangle$.

We investigate the pullback of K, L. We have, setting $L' = \langle\!\langle f, j \rangle, ! \rangle$,

$$g; K = h; L \Rightarrow \langle \mathrm{Id}, g \rangle = \langle h^{-1}(f), f^{-1}(h); j \rangle,$$

using $(h \times \mathrm{Id})^{-1}(\langle f, j \rangle) = \langle h^{-1}(f), f^{-1}(h); j \rangle$; so

— $g = f^{-1}(h); j, \ h = f^{-1}(h); f$.

In particular, for $g = K^{-1}(L)$, $h = L^{-1}(K)$, setting $B = f^{-1}(L^{-1}(K))$,

— $K^{-1}(L) = B; j, \ L^{-1}(K) = B; f$.

Hence $\langle j, f \rangle^{-} = \langle B; f, B; j \rangle$, so that

$$\langle j, f \rangle^{-};_P \langle j, f \rangle = \langle B; f, B; f \rangle = \mathrm{Id} \restriction (B; f) = \mathrm{Id} \restriction \langle j, f \rangle^{-}. \quad \blacksquare$$

As a final remark we would like to mention the work of E. Moggi (1985), which throws some light on the deductive power of the equational systems of this section. Moggi establishes precise links between call by value and partiality. It is clear that (1)–(15) of 8.1, 8.5 can be viewed as a weak theory of CCCs (they are consequences of the equations describing CCCs (see, for example, Curien, 1986). Now this weakening is, in a sense made precise by Moggi (1985), the counterpart of the weakening of β-equality which forbids stating

$$(\lambda x . M)N = M[x \leftarrow N]$$

unless N is a value, where a value is any variable, any abstraction, or any application $N_1 N_2$, where N_1, N_2 are values and N_1 is not an abstraction. Moggi shows that, using the well-known interpretation of λ-calculus into categories, we only use the weaker equations of pCCCs to validate the call by value equality. Moreover, he also gets completeness of the interpretation by appropriately internalizing the equality in the translation, as we briefly suggest now. First the statement above, which we restrict to the case $M = x$, is, formally

$$E(N) \vdash (\lambda x.x)N = N,$$

where $E(\_)$ is D. Scott's (1979) existence predicate.

We refer to Curien (1986) for definition of the categorical translation of a λ-expression. Here, if $[\![N]\!] = B$, we get $[\![(\lambda x.x)N]\!] = \langle \lambda(\mathrm{Snd}), B\rangle; \mathrm{App}$.

We need an intepretation of $E(\_)$ and $\_ = \_$. We set, following Moggi (1985),

— $[\![E(N)]\!] = [\![N]\!];!$
— $[\![M = N]\!] = [\![M]\!] \equiv [\![N]\!]$

and we translate a statement $P_1, ..., P_n \vdash P$ into

$$[\![P]\!] \upharpoonright ([\![P_1]\!] \cap \cdots \cap [\![P_n]\!]) = [\![P_1]\!] \cap \cdots \cap [\![P_n]\!].$$

We check the validity on our example. We use the second equality stated after 4.2, called Beta in Curien (1986) because it allows us to start simulating a β-reduction. We get

$$\langle \lambda(\mathrm{Snd}), B\rangle; \mathrm{App} = \langle \mathrm{Id}, B\rangle; \mathrm{Snd} = B \qquad (\mathrm{Id\ is\ total!}),$$

whence

$$[\![(\lambda x.x)N = N]\!] = B \equiv B = B;! \qquad (\text{cf. after 3.6})$$

and the translation $(B \equiv B) \upharpoonright B;! = (B;!) \upharpoonright (B;!)$ is proved. As a hint for believing in completeness, we stress that the translation of

$$\vdash (\lambda x.x)N = N$$

cannot be proved. Indeed we should have

$$(B \equiv B) \upharpoonright ! = !, \qquad \text{i.e.,} \quad B;! = B$$

which does not hold, in general.

This suggests further investigation of the equational presentations of this section as a combinatory version of call-by-value evaluation. Due to their operational significance, it would be worthwhile to get decision procedures for them.

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REFERENCES

ASPERTI, A., AND LONGO, G. (1985), Categories of partial morphisms and the relations between type structures, invited lecture, Banach semester 1985, Nota Scientifica S-7-85, University of Pisa (revised version, March 1986).

BURRONI, A. (1981), Algèbres graphiques, *Cahiers Topologie Géom. Différentielle* **22**, 3.

CARBONI, A. (1985), Bicategories of partial maps, Universita di Milano, preprint.

COUSINEAU, G., CURIEN, P. L., AND MAUNY, M. (1987), The categorical abstract machine, *Sci. Comput. Programming* **8**, 173–202.

CURIEN, P-L. (1986), "Categorical Combinators, Sequential Algorithms and Functional Programming," Research Notes in Theoretical Computer Science, Pitman, New York.

DI PAOLA, R., AND HELLER, A. (1984), Dominical categories, preprint, City University of New York, New York.

FREYD, P., AND SCEDROV, A. (1988), "Geometric Logic," to appear.

GOLDBLATT, R. (1984), "Topoi, the Categorial Analysis of Logic," 2nd ed., North-Holland, Amsterdam.

HOEHNKE, H-J. (1977), On partial algebras, *in* "Universal Algebra," Colloq. Math. Soc. János Bolyai Vol. 29, North-Holland, Amsterdam.

JOHNSTONE, P. (1977), "Topos Theory," Academic Press, New York/London.

KOCK, A., AND WRAITH, G. (1971), "Elementary Toposes," Aarhus University Lecture Notes Series 30, Aarhus, Denmark.

LONGO, G., AND MOGGI, E. (1984), Cartesian closed categories of enumerations for effective type structures (Part I), *in* "Semantics of Data Types," Lecture Notes in Comput. Sci. Vol. 173, Springer-Verlag, New York/Berlin.

MAC LANE, S. (1971), "Categories for the Working Mathematician," Springer-Verlag, New York/Berlin.

MOGGI, E. (1985), Categories of partial morphisms and the λ_p-calculus, *in* "Proceedings, Workshop on Category Theory and Computer Programming, Guildford," Lecture Notes in Computer Science Vol. 240, Springer-Verlag, New York/Berlin.

OBTULOWICZ, A. (1982), "The Logic of Categories of Partial Functions and Its Applications," thesis; *Dissertationes Math.* **241**.

PINO, R. (1987), Le λ-calcul partiel, Rapport LIENS-INRIA.

PLOTKIN G. (1985), Types and partial functions, course notes, University of Edinburgh.

ROBINSON, E., AND ROSOLINI, G. (1986), Categories of partial maps, Internal Report 18 (86-87), Universita di Parma.

ROSOLINI, G. (1986), "Continuity and Effectiveness in Topoi," Ph. D. thesis, Carnegie Mellon University and Merton College.

SCOTT, D. (1979), Identity and existence in intuitionistic logic, *in* "Applications of Sheaves" (M. Fourman, C. Mulvey, and D. Scott, Eds.), Lecture Notes in Mathematics Vol. 753, Springer-Verlag, New York/Berlin.

SCHRECKENBERGER, J. (1984), "Zur Theorie der dht-symmetrischen Kategorien," dissertation, Pädagogische Hochschule, Potsdam.

VOGEL, H-J. (1984), "Eine Kategorientheoretische Sprache zur Beschreibung von Birkhoff-Algebren," Report of the Academy of Sciences of the DDR R-MATH-06/84, Berlin.

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