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Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity ☆

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ABSTRACT

We study the global bifurcation and exact multiplicity of positive solutions of

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0, \\ f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho, \end{cases}$$

where $\lambda, \varepsilon > 0$ are two bifurcation parameters, and $\sigma, \rho > 0$, $0 < \kappa \leqslant \sqrt{\sigma\rho}$ are constants. We prove the global bifurcation of bifurcation curves for varying $\varepsilon > 0$. More precisely, there exists $\tilde{\varepsilon} > 0$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve is S-shaped for $0 < \varepsilon < \tilde{\varepsilon}$ and is monotone increasing for $\varepsilon \geqslant \tilde{\varepsilon}$. Thus we are able to determine the exact number of positive solutions by the values of ε and λ. Our results extend those of Hung and Wang (K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, Trans. Amer. Math. Soc., in press) from $\kappa \leqslant 0$ to $\kappa \leqslant \sqrt{\sigma\rho}$.

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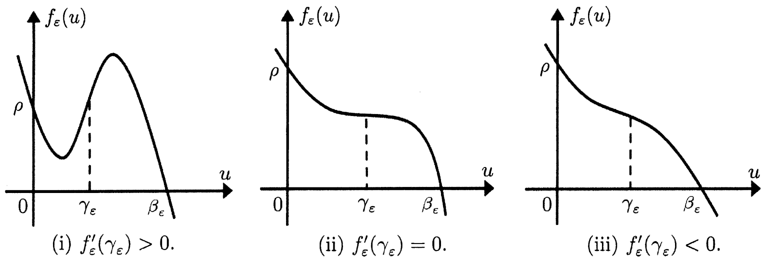

Fig. 1. Three possible graphs of $f_\varepsilon(u)$ satisfying (1.1)–(1.3).

1. Introduction

In this paper we study the global bifurcation and exact multiplicity of positive solutions of

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, \quad -1 < x < 1, \quad u(-1) = u(1) = 0, \\ f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho, \quad \lambda, \varepsilon > 0, \end{cases} \tag{1.1}$$

where λ, ε are two bifurcation parameters. Moreover, we mainly consider that

$$\sigma, \rho > 0, \tag{1.2}$$

and

$$0 < \kappa \leqslant \sqrt{\sigma\rho}. \tag{1.3}$$

If $f_\varepsilon(u)$ satisfies (1.1)–(1.3), for any $\varepsilon > 0$, it is easy to see that cubic polynomial $f_\varepsilon(u)$ has a unique inflection point at $\gamma_\varepsilon \equiv \sigma/(3\varepsilon) > 0$ and has a unique positive zero at some $\beta_\varepsilon > \gamma_\varepsilon$ such that f_ε satisfies

(i) $f_\varepsilon(0) = \rho > 0$ (positone), $f'_\varepsilon(0) = -\kappa < 0$, $f_\varepsilon(u) > 0$ on $(0, \beta_\varepsilon)$ and $f_\varepsilon(\beta_\varepsilon) = 0$,
(ii) $f_\varepsilon(u)$ is strictly convex on $(0, \gamma_\varepsilon)$ and is strictly concave on $(\gamma_\varepsilon, \infty)$. (So f_ε is convex–concave on $(0, \beta_\varepsilon)$.)

The proof the uniqueness of inflection point of cubic polynomial $f_\varepsilon(u)$ at $\gamma_\varepsilon = \sigma/(3\varepsilon)$ is trivial and the proof of the uniqueness of positive zero of $f_\varepsilon(u)$ at some $\beta_\varepsilon > \gamma_\varepsilon$ is given in Appendix A. Note that it is easy to see that β_ε is a continuous, strictly decreasing function of $\varepsilon > 0$. In addition, $\lim_{\varepsilon\to 0^+} \beta_\varepsilon = \infty$ and $\lim_{\varepsilon\to\infty} \beta_\varepsilon = 0$. Three possible graphs of $f_\varepsilon(u)$ satisfying (1.1)–(1.3) are depicted in Fig. 1.

For any $\varepsilon > 0$, on the $(\lambda, \|u\|_\infty)$-plane, we study the shape and structure of bifurcation curves S_ε of positive solutions of (1.1), defined by

$$S_\varepsilon \equiv \{(\lambda, \|u_\lambda\|_\infty):\ \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_ε is S-shaped if S_ε is a continuous curve and there exist two *positive* numbers $\lambda_* < \lambda^*$ such that S_ε has *exactly two* turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, and

(i) $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$,
(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_ε turns to the *left*,
(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_ε turns to the *right*.

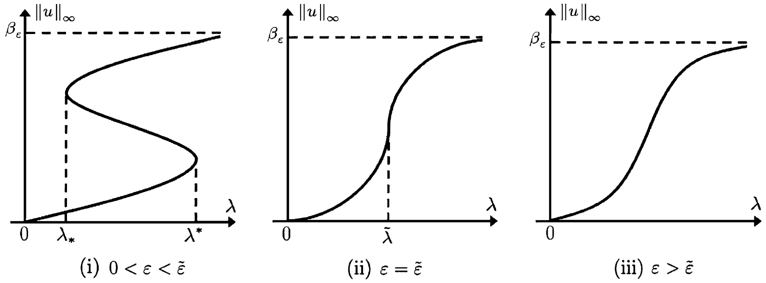

Fig. 2. Global bifurcation of bifurcation curves S_ε of (1.1), (1.2), and (either (1.3) or (1.4)) with varying $\varepsilon > 0$.

See Fig. 2(i) depicted above for example.

Problem (1.1) was first systematically studied by a celebrated paper by Smoller and Wasserman [8]. In particular, they considered (1.1) with $\varepsilon = 1$ and that cubic nonlinearity $f_{\varepsilon=1}(u)$ has *three real zeros* $a < b < c$. In this paper we discuss the general case with $\varepsilon > 0$ and $\sigma, \rho, \kappa \in \mathbb{R}$, so that $f_\varepsilon(u)$ may have exactly one positive zero, two distinct positive zeros or three distinct positive zeros. If ($\sigma \leqslant 0$, $\rho, \kappa \in \mathbb{R}$) or ($\rho \leqslant 0$, $\sigma, \kappa \in \mathbb{R}$), by applying the methods used in [8], we can prove that the structure of bifurcation curve S_ε of (1.1) is one of the following cases:

(i) The bifurcation curve S_ε of (1.1) is an empty set (that is, (1.1) has no positive solution for all $\lambda > 0$).
(ii) The bifurcation curve S_ε of (1.1) is a monotone curve on the $(\lambda, \|u\|_\infty)$-plane.
(iii) The bifurcation curve S_ε of (1.1) has exactly one turning point where the curve turns to the right on the $(\lambda, \|u\|_\infty)$-plane.

More precisely, we can give a classification of totally three qualitatively different bifurcation curves S_ε if ($\sigma \leqslant 0$, $\rho, \kappa \in \mathbb{R}$) or ($\rho \leqslant 0$, $\sigma, \kappa \in \mathbb{R}$). In these cases, (1.1) has *at most two* positive solutions for each $\lambda > 0$. So we mainly consider the remaining case (1.1), (1.2). In this case, it is more difficult to determine precisely the shape of the bifurcation curve S_ε and the exact multiplicity of positive solutions of (1.1), (1.2) since S_ε may have *two* turning points and (1.1), (1.2) may have *three* positive solutions for a certain range of positive λ.

Hung and Wang [1] very recently developed some time-map techniques to study the shape of the bifurcation curve S_ε and the exact multiplicity of (1.1), (1.2) with

$$\kappa \leqslant 0. \tag{1.4}$$

For (1.1), (1.2), (1.4), they [1, Theorem 2.1] proved that there exists a positive number $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$ satisfying

$$\left(\frac{25}{32}\left(\frac{\sigma^3}{27\rho}\right)\right)^{1/2} < \tilde{\varepsilon} < \left(\frac{\sigma^3}{27\rho}\right)^{1/2}$$

such that, on the $(\lambda, \|u\|_\infty)$-plane:

(i) For $0 < \varepsilon < \tilde{\varepsilon}$, the bifurcation curve S_ε of (1.1), (1.2), (1.4) is S-shaped (see Fig. 2(i)).
(ii) For $\varepsilon = \tilde{\varepsilon}$, the bifurcation curve $S_{\tilde{\varepsilon}}$ of (1.1), (1.2), (1.4) is monotone increasing. Moreover, (1.1), (1.2), (1.4) has exactly one (cusp type) degenerate positive solution $u_{\tilde{\lambda}}$ (see Fig. 2(ii)).
(iii) For $\varepsilon > \tilde{\varepsilon}$, the bifurcation curve S_ε of (1.1), (1.2), (1.4) is monotone increasing. Moreover, all positive solutions u_λ of (1.1), (1.2), (1.4) are *nondegenerate* (see Fig. 2(iii)).

Our results in this paper are extensions of those of Hung and Wang [1] from $\kappa \leqslant 0$ to $\kappa \leqslant \sqrt{\sigma\rho}$. In Theorem 2.1 stated below for (1.1)–(1.3) with varying $\varepsilon > 0$, we prove the same global bifurcation

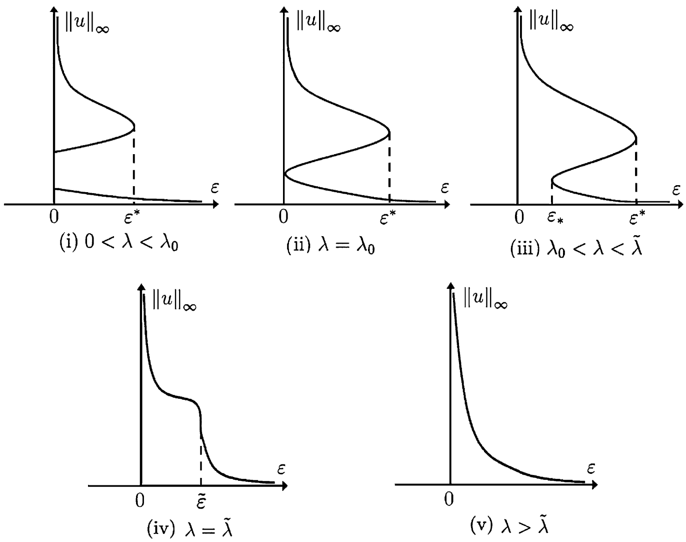

Fig. 3. Global bifurcation of bifurcation curves Σ_λ of (1.1), (1.2), and (either (1.3) or (1.4)) with varying $\lambda > 0$.

results of bifurcation curves S_ε. Hence we are able to determine the exact number of positive solutions by the values of ε and λ. In addition, we give lower and upper bounds of the critical bifurcation value $\tilde{\varepsilon}$. See Fig. 2.

While for any $\lambda > 0$, on the $(\varepsilon, \|u\|_\infty)$-plane, it is interesting to study the shape and structure of bifurcation curves Σ_λ of positive solutions of (1.1), defined by

$$\Sigma_\lambda \equiv \big\{ \big(\varepsilon, \|u_\varepsilon\|_\infty\big) \colon\ \varepsilon > 0 \text{ and } u_\varepsilon \text{ is a positive solution of (1.1)} \big\}.$$

(Note that we allow that bifurcation curve Σ_λ consists of two (or more) connected components.) We say that, on the $(\varepsilon, \|u\|_\infty)$-plane, the bifurcation curve Σ_λ is *reversed* S-shaped if Σ_λ is a continuous curve and there exist two numbers $\varepsilon_* < \varepsilon^*$ such that S_ε has *exactly two* turning points at some points $(\varepsilon_*, \|u_{\varepsilon_*}\|_\infty)$ and $(\varepsilon^*, \|u_{\varepsilon^*}\|_\infty)$, and

(i) $\varepsilon_* < \varepsilon^*$ and $\|u_{\varepsilon_*}\|_\infty < \|u_{\varepsilon^*}\|_\infty$,
(ii) at $(\varepsilon_*, \|u_{\varepsilon_*}\|_\infty)$ the bifurcation curve Σ_λ turns to the *right*,
(iii) at $(\varepsilon^*, \|u_{\varepsilon^*}\|_\infty)$ the bifurcation curve Σ_λ turns to the *left*.

See Fig. 3(iii) for example.

For (1.1), (1.2), (1.4), Hung and Wang [1, Theorem 2.3] proved that there exist two positive numbers $\lambda_0\ (= \lambda_0(\sigma, \kappa, \rho)) < \tilde{\lambda}\ (= \tilde{\lambda}(\sigma, \kappa, \rho))$ such that, on the $(\varepsilon, \|u\|_\infty)$-plane:

(i) For $0 < \lambda < \lambda_0$, the bifurcation curve Σ_λ of (1.1), (1.2), (1.4) has two disjoint connected components, the upper branch is $\supset$-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve (see Fig. 3(i)).
(ii) For $\lambda = \lambda_0$, the bifurcation curve Σ_{λ_0} of (1.1), (1.2), (1.4) has two disjoint connected components, the upper branch is $\supset$-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve (see Fig. 3(ii)).
(iii) For $\lambda_0 < \lambda < \tilde{\lambda}$, the bifurcation curve Σ_λ of (1.1), (1.2), (1.4) is reversed S-shaped (see Fig. 3(iii)).
(iv) For $\lambda = \tilde{\lambda}$, the bifurcation curve $\Sigma_{\tilde{\lambda}}$ of (1.1), (1.2), (1.4) is monotone decreasing. Moreover, (1.1), (1.2), (1.4) has exactly one (cusp type) degenerate positive solution $u_{\tilde{\varepsilon}}$ (see Fig. 3(iv)).
(v) For $\lambda > \tilde{\lambda}$, the bifurcation curve Σ_λ of (1.1), (1.2), (1.4) is monotone decreasing. Moreover, all positive solutions u_ε of (1.1), (1.2), (1.4) are *nondegenerate* (see Fig. 3(v)).

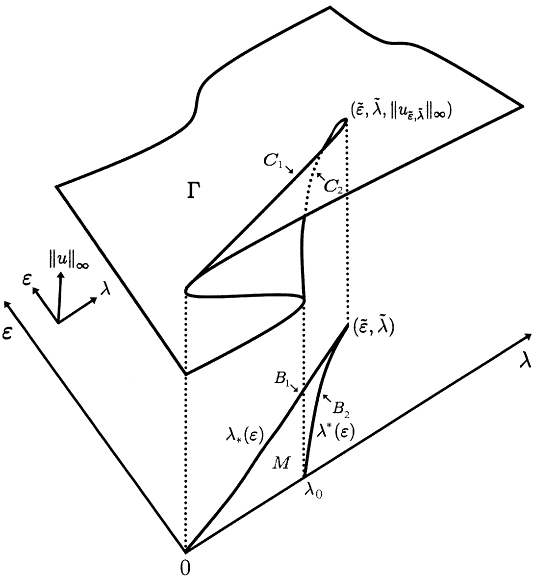

Fig. 4. The bifurcation surface Γ with the fold curve $C_\Gamma = C_1 \cup C_2$, and the projection of Γ onto F_q. $B_\Gamma = B_1 \cup B_2$ is the bifurcation set and $(\tilde{\varepsilon}, \tilde{\lambda})$ is the cusp point on F_q.

In Theorem 2.2 stated below for (1.1)–(1.3) with varying $\lambda > 0$, we prove the same global bifurcation results of bifurcation curve Σ_λ. Hence we are able to determine the exact number of positive solutions by the values of λ and ε. See Fig. 3.

We study, in the $(\varepsilon, \lambda, \|u\|_\infty)$-space, the shape and structure of the *bifurcation surface* Γ of positive solutions of (1.1), defined by

$$\Gamma \equiv \{(\varepsilon, \lambda, \|u_{\varepsilon,\lambda}\|_\infty):\ \varepsilon, \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a positive solution of } (1.1)\}$$

which has the appearance of a folded surface with the *fold curve*

$$C_\Gamma \equiv \{(\varepsilon, \lambda, \|u_{\varepsilon,\lambda}\|_\infty):\ \varepsilon, \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a } \textit{degenerate} \text{ positive solution of } (1.1)\}.$$

Let F_q denote the first quadrant of the (ε, λ)-parameter plane. We also study, on F_q, the *bifurcation set*

$$B_\Gamma \equiv \{(\varepsilon, \lambda):\ \varepsilon, \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a } \textit{degenerate} \text{ positive solution of } (1.1)\}$$

which is the projection of the fold curve C_Γ onto F_q. Let M denote the bounded, open connected subset of F_q, which is 'inside' B_Γ.

For (1.1), (1.2), (1.4), Hung and Wang [1, Theorem 2.4] proved that the following assertions (i)–(iv) (see Figs. 4 and 5):

(i) The fold curve C_Γ of (1.1), (1.2), (1.4) is a continuous curve in the $(\varepsilon, \lambda, \|u\|_\infty)$-space. Moreover, $C_\Gamma = C_1 \cup C_2$ where

$$C_1 \equiv \{(\varepsilon, \lambda_*(\varepsilon), \|u_{\varepsilon,\lambda_*(\varepsilon)}\|_\infty):\ 0 < \varepsilon \leqslant \tilde{\varepsilon}\} \quad \text{and} \quad C_2 \equiv \{(\varepsilon, \lambda^*(\varepsilon), \|u_{\varepsilon,\lambda^*(\varepsilon)}\|_\infty):\ 0 < \varepsilon \leqslant \tilde{\varepsilon}\}.$$

(ii) The bifurcation set B_Γ of (1.1), (1.2), (1.4) satisfies $B_\Gamma = B_1 \cup B_2$ where

$$B_1 \equiv \{(\varepsilon, \lambda_*(\varepsilon)):\ 0 < \varepsilon \leqslant \tilde{\varepsilon}\} \quad \text{and} \quad B_2 \equiv \{(\varepsilon, \lambda^*(\varepsilon)):\ 0 < \varepsilon \leqslant \tilde{\varepsilon}\}.$$

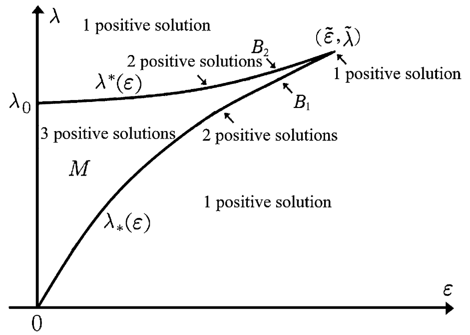

Fig. 5. The projection of the bifurcation surface Γ onto F_q. $B_\Gamma = B_1 \cup B_2$ is the bifurcation set and $(\tilde{\varepsilon}, \tilde{\lambda})$ is the cusp point on F_q.

(iii) $\lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$ are both continuous, strictly increasing on $(0, \tilde{\varepsilon}]$.
(iv) Problem (1.1), (1.2), (1.4) has exactly three positive solutions for $(\varepsilon, \lambda) \in M$, exactly two positive solutions for $(\varepsilon, \lambda) \in B_\Gamma \setminus \{(\tilde{\varepsilon}, \tilde{\lambda})\}$, and exactly one positive solution for $(\varepsilon, \lambda) \in (F_q \setminus (B_\Gamma \cup M)) \cup \{(\tilde{\varepsilon}, \tilde{\lambda})\}$.

In Theorem 2.3 stated below for (1.1)–(1.3), we prove the same structure of the bifurcation set B_Γ and the fold curve C_Γ. Hence we are able to determine the exact number of positive solutions of (1.1)–(1.3) by the values of ε and λ. See Figs. 4 and 5.

The paper is organized as follows. Section 2 contains statements of the main results: Theorems 2.1–2.3. Section 3 contains several lemmas needed to prove Theorems 2.1–2.3. Section 4 contains the proofs of Theorems 2.1–2.3. Finally, in Section 5, we give three conjectures on the shape of bifurcation curves S_ε of positive solutions of (1.1), (1.2) with evolution parameter $\kappa > \sqrt{\sigma\rho}$.

In this section, finally, we note that our main results (Theorems 2.1–2.3) in this paper extend those of Hung and Wang [1, Theorems 2.1, 2.3 and 2.4] from $\kappa \leqslant 0$ to $\kappa \leqslant \sqrt{\sigma\rho}$, and the proofs are more complicated. One of the main difficulties is that $f_\varepsilon(u)$ can initially decrease, but then increases to a peak before falling to zero on $(0, \beta_\varepsilon]$, see Fig. 1(i).

2. Main results

Theorem 2.1. *Consider* (1.1)–(1.3) *with varying* $\varepsilon > 0$. *There exists a positive number* $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$ *satisfying*

$$\left(\frac{25}{32}\left(\frac{\sigma^3}{27\rho}\right)\right)^{1/2} < \tilde{\varepsilon} < \left(\frac{\sigma^3}{27\rho}\right)^{1/2}$$

such that the following assertions (i)–(iii) *hold*:

(i) (*See Fig.* 2(i).) *For* $0 < \varepsilon < \tilde{\varepsilon}$, *the bifurcation curve* S_ε *is S-shaped on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover, there exist two positive numbers* $\lambda_* < \lambda^*$ *such that* (1.1)–(1.3) *has exactly one degenerate positive solution* u_{λ_*} *and* u_{λ^*} *for* $\lambda = \lambda_*$ *and* $\lambda = \lambda^*$, *respectively. More precisely,* (1.1)–(1.3) *has*:
  (a) *exactly three positive solutions* u_λ, v_λ, w_λ *with* $w_\lambda < u_\lambda < v_\lambda$ *for* $\lambda_* < \lambda < \lambda^*$,
  (b) *exactly two positive solutions* w_λ, u_λ *with* $w_\lambda < u_\lambda$ *for* $\lambda = \lambda_*$, *and exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda = \lambda^*$,
  (c) *exactly one positive solution* w_λ *for* $0 < \lambda < \lambda_*$, *and exactly one positive solution* v_λ *for* $\lambda > \lambda^*$. *Furthermore,*
  (d) $\lim_{\lambda \to 0^+} \|w_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \beta_\varepsilon$.
(ii) (*See Fig.* 2(ii).) *For* $\varepsilon = \tilde{\varepsilon}$, *the bifurcation curve* $S_{\tilde{\varepsilon}}$ *is monotone increasing on the* $(\lambda, \|u\|_\infty)$*-plane. Moreover,* (1.1)–(1.3) *has exactly one* (*cusp type*) *degenerate positive solution* $u_{\tilde{\lambda}}$. *More precisely, for all* $\lambda > 0$, (1.1)–(1.3) *has exactly one positive solution* u_λ *satisfying* $\lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = \beta_\varepsilon$.

(iii) (*See Fig.* 2(iii).) *For $\varepsilon > \tilde{\varepsilon}$, the bifurcation curve S_ε is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. Moreover, all positive solutions u_λ of* (1.1)–(1.3) *are nondegenerate. More precisely, for all $\lambda > 0$,* (1.1)–(1.3) *has exactly one positive solution u_λ satisfying $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \beta_\varepsilon$.*

Theorem 2.2. *Consider* (1.1)–(1.3) *with varying $\lambda > 0$. There exist two positive numbers λ_0 $(= \lambda_0(\sigma, \kappa, \rho))$ $< \tilde{\lambda}$ $(= \tilde{\lambda}(\sigma, \kappa, \rho))$ such that the following assertions* (i)–(v) *hold*:

(i) (*See Fig.* 3(i).) *For $0 < \lambda < \lambda_0$, on the $(\varepsilon, \|u\|_\infty)$-plane, the bifurcation curve Σ_λ has two disjoint connected components, the upper branch is $\supset$-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve. Moreover, there exists a positive number ε^* such that* (1.1)–(1.3) *has exactly one degenerate positive solution u_{ε^*} for $\varepsilon = \varepsilon^*$. More precisely,* (1.1)–(1.3) *has*:
  (a) *exactly three positive solutions u_ε, v_ε, w_ε with $w_\varepsilon < u_\varepsilon < v_\varepsilon$ for $0 < \varepsilon < \varepsilon^*$,*
  (b) *exactly two positive solutions w_ε, u_ε with $w_\varepsilon < u_\varepsilon$ for $\varepsilon = \varepsilon^*$,*
  (c) *exactly one positive solution w_ε for $\varepsilon > \varepsilon^*$.*
  Furthermore,
  (d) $0 = \lim_{\varepsilon\to\infty} \|w_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|w_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|v_\varepsilon\|_\infty = \infty$.

(ii) (*See Fig.* 3(ii).) *For $\lambda = \lambda_0$, on the $(\varepsilon, \|u\|_\infty)$-plane, the bifurcation curve Σ_{λ_0} has two disjoint connected components, the upper branch is $\supset$-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve. Moreover, there exists a positive number ε^* such that* (1.1)–(1.3) *has exactly one degenerate positive solution u_{ε^*} for $\varepsilon = \varepsilon^*$. More precisely,* (1.1)–(1.3) *has*:
  (a) *exactly three positive solutions u_ε, v_ε, w_ε with $w_\varepsilon < u_\varepsilon < v_\varepsilon$ for $0 < \varepsilon < \varepsilon^*$,*
  (b) *exactly two positive solutions w_ε, u_ε with $w_\varepsilon < u_\varepsilon$ for $\varepsilon = \varepsilon^*$,*
  (c) *exactly one positive solution w_ε for $\varepsilon > \varepsilon^*$.*
  Furthermore,
  (d) $0 = \lim_{\varepsilon\to\infty} \|w_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|w_\varepsilon\|_\infty = \lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|v_\varepsilon\|_\infty = \infty$.

(iii) (*See Fig.* 3(iii).) *For $\lambda_0 < \lambda < \tilde{\lambda}$, the bifurcation curve Σ_λ is reversed S-shaped on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover, there exist two positive numbers $\varepsilon_* < \varepsilon^*$ such that* (1.1)–(1.3) *has exactly one degenerate positive solution u_{ε_*} and u_{ε^*} for $\varepsilon = \varepsilon_*$ and $\varepsilon = \varepsilon^*$, respectively. More precisely,* (1.1)–(1.3) *has*:
  (a) *exactly three positive solutions u_ε, v_ε, w_ε with $w_\varepsilon < u_\varepsilon < v_\varepsilon$ for $\varepsilon_* < \varepsilon < \varepsilon^*$,*
  (b) *exactly two positive solutions u_ε, v_ε with $u_\varepsilon < v_\varepsilon$ for $\varepsilon = \varepsilon_*$, and exactly two positive solutions w_ε, u_ε with $w_\varepsilon < u_\varepsilon$ for $\varepsilon = \varepsilon^*$,*
  (c) *exactly one positive solution v_ε for $0 < \varepsilon < \varepsilon_*$, and exactly one positive solution w_ε for $\varepsilon > \varepsilon^*$.*
  Furthermore,
  (d) $\lim_{\varepsilon\to 0^+} \|v_\varepsilon\|_\infty = \infty$ *and* $\lim_{\varepsilon\to\infty} \|w_\varepsilon\|_\infty = 0$.

(iv) (*See Fig.* 3(iv).) *For $\lambda = \tilde{\lambda}$, the bifurcation curve $\Sigma_{\tilde{\lambda}}$ is monotone decreasing on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover,* (1.1)–(1.3) *has exactly one* (*cusp type*) *degenerate positive solution $u_{\tilde{\varepsilon}}$. More precisely, for all $\varepsilon > 0$,* (1.1)–(1.3) *has exactly one positive solution u_ε satisfying $\lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_\infty = \infty$ and $\lim_{\varepsilon\to\infty} \|u_\varepsilon\|_\infty = 0$.*

(v) (*See Fig.* 3(v).) *For $\lambda > \tilde{\lambda}$, the bifurcation curve Σ_λ is monotone decreasing on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover, all positive solutions u_ε of* (1.1)–(1.3) *are nondegenerate. More precisely, for all $\varepsilon > 0$,* (1.1)–(1.3) *has exactly one positive solution u_ε satisfying $\lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_\infty = \infty$ and $\lim_{\varepsilon\to\infty} \|u_\varepsilon\|_\infty = 0$.*

We give next remark to Theorem 2.2.

Remark 1. Considering (1.1)–(1.3) with $\varepsilon > 0$ generalized to $\varepsilon \in \mathbb{R}$, we define the bifurcation curve

$$\widetilde{\Sigma}_\lambda \equiv \bigl\{ (\varepsilon, \|u_\varepsilon\|_\infty) \colon \ \varepsilon \in \mathbb{R} \text{ and } u_\varepsilon \text{ is a positive solution of (1.1)} \bigr\}.$$

Actually, it can be easily proved that:

(i) For $0 < \lambda < \lambda_0$, the bifurcation curve $\widetilde{\Sigma}_\lambda$ is reversed S-shaped on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover, there exists $\varepsilon_* < 0$ such that (1.1)–(1.3) has exactly two positive solutions w_ε, u_ε with $w_\varepsilon < u_\varepsilon$ for $\varepsilon_* < \varepsilon \leqslant 0$, and exactly one positive solution u_ε for $\varepsilon = \varepsilon_*$, and no positive solution for $\varepsilon < \varepsilon_*$. See Fig. 6(i).

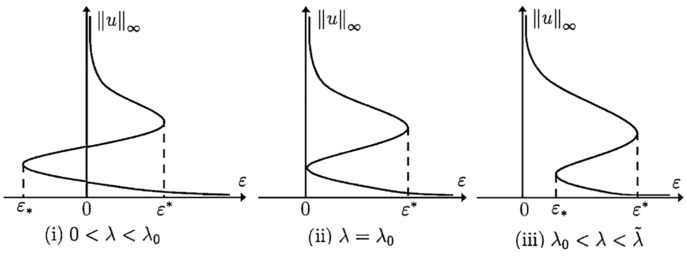

Fig. 6. Global bifurcation of bifurcation curves $\widetilde{\Sigma}_\lambda$ of (1.1)–(1.3) with $\varepsilon>0$ generalized to $\varepsilon\in\mathbb{R}$ and with varying $\lambda\in(0,\tilde{\lambda})$.

(ii) For $\lambda=\lambda_0$, the bifurcation curve $\widetilde{\Sigma}_{\lambda_0}$ is reversed S-shaped on the $(\varepsilon,\|u\|_\infty)$-plane. Moreover, (1.1)–(1.3) has exactly one positive solution u_ε for $\varepsilon=0$, and no positive solution for $\varepsilon<0$. See Fig. 6(ii).

Notice that, in Theorem 2.1, on the $(\lambda,\|u\|_\infty)$-plane, the bifurcation curve S_ε is S-shaped for $0<\varepsilon<\tilde{\varepsilon}$, see Fig. 2. While in Theorem 2.2 and Remark 1, on the $(\varepsilon,\|u\|_\infty)$ -plane, the bifurcation curve $\widetilde{\Sigma}_\lambda$ is *reversed* S-shaped for $0<\lambda<\tilde{\lambda}$, see Fig. 6.

Let $\tilde{\varepsilon}=\tilde{\varepsilon}(\sigma,\kappa,\rho)$, $\lambda_0=\lambda_0(\sigma,\kappa,\rho)$, $\tilde{\lambda}=\tilde{\lambda}(\sigma,\kappa,\rho)$, $\lambda_*=\lambda_*(\varepsilon)$, $\lambda^*=\lambda^*(\varepsilon)$, $\varepsilon_*=\varepsilon_*(\lambda)$ and $\varepsilon^*=\varepsilon^*(\lambda)$ be the values in Theorems 2.1 and 2.2 for (1.1)–(1.3). We study the structure of the bifurcation set B_Γ in the next theorem.

Theorem 2.3. *(See Fig. 5.) Consider* (1.1)–(1.3) *with* $(\varepsilon,\lambda)\in F_q$. *Then the bifurcation set* $B_\Gamma=B_1\cup B_2$ *where*

$$B_1\equiv\{(\varepsilon,\lambda_*(\varepsilon))\colon 0<\varepsilon\leqslant\tilde{\varepsilon}\}\quad\textit{and}\quad B_2\equiv\{(\varepsilon,\lambda^*(\varepsilon))\colon 0<\varepsilon\leqslant\tilde{\varepsilon}\}.$$

Moreover, (1.1)–(1.3) *has exactly three positive solutions for* $(\varepsilon,\lambda)\in M$, *exactly two positive solutions for* $(\varepsilon,\lambda)\in B_\Gamma\setminus\{(\tilde{\varepsilon},\tilde{\lambda})\}$, *and exactly one positive solution for* $(\varepsilon,\lambda)\in(F_q\setminus(B_\Gamma\cup M))\cup\{(\tilde{\varepsilon},\tilde{\lambda})\}$. *More precisely, the following assertions* (i) *and* (ii) *hold*:

(i) *Functions* $\lambda_*(\varepsilon)$ *and* $\lambda^*(\varepsilon)$ *are both continuous, strictly increasing on* $(0,\tilde{\varepsilon}]$ *and satisfy* $0=\lim_{\varepsilon\to0^+}\lambda_*(\varepsilon)<\lim_{\varepsilon\to0^+}\lambda^*(\varepsilon)=\lambda_0<\tilde{\lambda}=\lambda_*(\tilde{\varepsilon})=\lambda^*(\tilde{\varepsilon})$.
(ii) *Function* $\varepsilon^*(\lambda)$ *is continuous, strictly increasing on* $(0,\tilde{\lambda}]$ *and satisfies* $\lim_{\lambda\to0^+}\varepsilon^*(\lambda)=0$ *and* $\varepsilon^*(\tilde{\lambda})=\tilde{\varepsilon}$. *Function* $\varepsilon_*(\lambda)$ *is continuous, strictly increasing on* $(\lambda_0,\tilde{\lambda}]$ *and satisfies* $\lim_{\lambda\to\lambda_0^+}\varepsilon_*(\lambda)=0$ *and* $\varepsilon_*(\tilde{\lambda})=\tilde{\varepsilon}$.

In next remark, we give a precise characterization of the fold curve C_Γ in the $(\varepsilon,\lambda,\|u\|_\infty)$-space.

Remark 2. (See Fig. 4.) Consider (1.1)–(1.3). Then, by Theorem 2.3(i), the fold curve $C_\Gamma=C_1\cup C_2$ where

$$C_1\equiv\{(\varepsilon,\lambda_*(\varepsilon),\|u_{\varepsilon,\lambda_*(\varepsilon)}\|_\infty)\colon 0<\varepsilon\leqslant\tilde{\varepsilon}\}\quad\text{and}\quad C_2\equiv\{(\varepsilon,\lambda^*(\varepsilon),\|u_{\varepsilon,\lambda^*(\varepsilon)}\|_\infty)\colon 0<\varepsilon\leqslant\tilde{\varepsilon}\}.$$

Moreover, by applying (4.4)–(4.7) stated below, we are able to prove that:

(i) $\|u_{\varepsilon,\lambda_*(\varepsilon)}\|_\infty>\|u_{\varepsilon,\lambda^*(\varepsilon)}\|_\infty$ for $0<\varepsilon<\tilde{\varepsilon}$ and $\|u_{\tilde{\varepsilon},\lambda_*(\tilde{\varepsilon})}\|_\infty=\|u_{\tilde{\varepsilon},\lambda^*(\tilde{\varepsilon})}\|_\infty=\|u_{\tilde{\varepsilon},\tilde{\lambda}}\|_\infty$.
(ii) $\|u_{\varepsilon,\lambda_*(\varepsilon)}\|_\infty$ is a continuous, strictly decreasing function of $\varepsilon\in(0,\tilde{\varepsilon}]$ and $\|u_{\varepsilon,\lambda^*(\varepsilon)}\|_\infty$ is a continuous, strictly increasing function of $\varepsilon\in(0,\tilde{\varepsilon}]$.
(iii) C_Γ is a continuous curve in the $(\varepsilon,\lambda,\|u\|_\infty)$-space.

Observe that both $\lambda^*(\varepsilon)$ and $\lambda_*(\varepsilon)$ have continuous inverse functions on $(0,\tilde{\varepsilon}]$. Indeed, $\varepsilon_*(\lambda)$ is the inverse function of $\lambda^*(\varepsilon)$ on $(\lambda_0,\tilde{\lambda}]$ and $\varepsilon^*(\lambda)$ is the inverse function of $\lambda_*(\varepsilon)$ on $(0,\tilde{\lambda}]$.

3. Lemmas

To prove our results (Theorems 2.1–2.3), we need the following Lemmas 3.1–3.8 in which we develop new time-map techniques different from those developed in [1]. In particular, Lemma 3.3 is a key lemma in the proofs of Theorems 2.1–2.3. In Lemma 3.3, for any fixed $\varepsilon > 0$, we prove that the bifurcation curve S_ε is either monotone increasing or S-shaped on the $(\lambda, \|u\|_\infty)$-plane. To apply the time-map techniques for (1.1)–(1.3), in the following, we consider $\varepsilon \geqslant 0$. The time map formula which we apply to study (1.1)–(1.3) takes the form as follows

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha \left[F_\varepsilon(\alpha) - F_\varepsilon(u)\right]^{-1/2} du \equiv T_\varepsilon(\alpha) \quad \text{for } 0 < \alpha < \beta_\varepsilon \text{ and } \varepsilon \geqslant 0, \tag{3.1}$$

where $F_\varepsilon(u) \equiv \int_0^u f_\varepsilon(t)\,dt$ and β_ε the unique positive zero of cubic polynomial $f_\varepsilon(u)$ for $\varepsilon > 0$, and we let $\beta_{\varepsilon=0} \equiv \infty$. Observe that positive solutions $u_{\varepsilon,\lambda}$ for (1.1)–(1.3) correspond to

$$\|u_{\varepsilon,\lambda}\|_\infty = \alpha \quad \text{and} \quad T_\varepsilon(\alpha) = \sqrt{\lambda}. \tag{3.2}$$

Thus, studying of the exact number of positive solutions of (1.1)–(1.3) for fixed $\varepsilon \geqslant 0$ is equivalent to studying the shape of the time map $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$; and studying the exact number of positive solutions of (1.1)–(1.3) for fixed $\lambda > 0$ is equivalent to studying the number of roots of the equation $T_\varepsilon(\alpha) = \sqrt{\lambda}$ on $(0, \beta_\varepsilon)$ for varying $\varepsilon > 0$. Note that it can be proved that $T_\varepsilon(\alpha)$ is a thrice differentiable function of $\alpha \in (0, \beta_\varepsilon)$ for $\varepsilon \geqslant 0$. The proof is easy but tedious; we omit it.

We call a positive solution $u_{\varepsilon,\lambda}$ of (1.1)–(1.3) is *degenerate* if $T'_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and is *nondegenerate* if $T'_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$. So to find the degenerate positive solutions of (1.1)–(1.3), we only need to find the critical points of $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$. It is known that a *degenerate* positive solution $u_{\varepsilon,\lambda}$ of (1.1)–(1.3) is of *cusp type* if $T''_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and $T'''_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$, see Shi [6, p. 497] and [7, p. 214].

The main difficulty in proving our main results is to determine the exact number of critical points of the time map $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$ for all $\varepsilon > 0$. This question is partially answered in the following Lemmas 3.1 and 3.2. Lemma 3.1 follows from [5, Theorems 2.6, 2.9 and 3.2] and Lemma 3.2 mainly follows by applying [2, Theorem 2.1]; we omit the proofs.

Lemma 3.1. *Consider* (1.1)–(1.3)*. For any fixed* $\varepsilon > 0$, *the following assertions* (i) *and* (ii) *hold*:

(i) $\lim_{\alpha \to 0^+} T_\varepsilon(\alpha) = 0$ *and* $\lim_{\alpha \to \beta_\varepsilon^-} T_\varepsilon(\alpha) = \infty$.
(ii) *If* $T_\varepsilon(\alpha)$ *is not strictly increasing on* $(0, \gamma_\varepsilon)$, *then* $T_\varepsilon(\alpha)$ *is strictly increasing on* $(0, \tilde{\gamma}_\varepsilon)$ *and strictly decreasing on* $(\tilde{\gamma}_\varepsilon, \gamma_\varepsilon)$ *for some* $\tilde{\gamma}_\varepsilon \in (0, \gamma_\varepsilon)$.

Lemma 3.2. *Consider* (1.1)–(1.3)*. Then the following assertions* (i) *and* (ii) *hold*:

(i) *For any fixed* $\varepsilon \geqslant (\frac{\sigma^3}{27\rho})^{1/2}$, $T_\varepsilon(\alpha)$ *is a strictly increasing function on* $(0, \beta_\varepsilon)$.
(ii) *For any fixed positive* $\varepsilon \leqslant (\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}$, $T_\varepsilon(\alpha)$ *has exactly one local maximum and one local minimum on* $(0, \beta_\varepsilon)$.

However, there is a gap, what about the case where ε is between $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}$ and $(\frac{\sigma^3}{27\rho})^{1/2}$? First, in the next Lemma 3.3, we prove

Lemma 3.3. *Consider* (1.1)–(1.3)*. For any fixed* $\varepsilon > 0$, $T_\varepsilon(\alpha)$ *is either a strictly increasing function or has exactly two critical points, a local maximum and a local minimum, on* $(0, \beta_\varepsilon)$.

To prove Lemma 3.3, we develop some new time-map techniques. First, we define the auxiliary function

$$G_\varepsilon(\alpha) = 8\sqrt{2}\alpha^{\frac{5}{2}} T''_\varepsilon(\alpha). \tag{3.3}$$

Note that the auxiliary function $G_\varepsilon(\alpha) = 8\sqrt{2}\alpha^{\frac{5}{2}} T''_\varepsilon(\alpha)$ used in this paper is different from the auxiliary function $12\sqrt{2}T'_\varepsilon(\alpha) + 8\sqrt{2}\alpha T''_\varepsilon(\alpha)$ used in Hung and Wang [1]. Moreover, the techniques used in [1, Lemmas 3.4–3.5] for $\kappa \leqslant 0$ fails here under condition (1.3) $0 < \kappa \leqslant \sqrt{\sigma\rho}$, though it is expected that similar results hold. So we need to develop new techniques to obtain the following Lemma 3.4. The proof of Lemma 3.4 is rather long and technical, therefore we postpone it to Appendix B.

Lemma 3.4. *Consider* (1.1)–(1.3)*. For any fixed* $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$, $G'_\varepsilon(\alpha) > 0$ *for* $\alpha \in [\gamma_\varepsilon, \beta_\varepsilon)$.

For any fixed $\alpha > 0$, let

$$I_\alpha = \{\varepsilon > 0\colon\ \alpha \in (0, \beta_\varepsilon)\}.$$

Since β_ε is a continuous, strictly decreasing function of $\varepsilon > 0$, and $\lim_{\varepsilon\to 0^+} \beta_\varepsilon = \infty$ and $\lim_{\varepsilon\to\infty} \beta_\varepsilon = 0$, we obtain that $I_\alpha = (0, \varepsilon(\alpha))$ where $\alpha = \beta_{\varepsilon(\alpha)}$, and $\varepsilon(\alpha)$ is strictly decreasing in α.

Lemma 3.5. *Consider* (1.1)–(1.3)*. For any fixed* $\alpha > 0$, $T'_\varepsilon(\alpha)$ *is a continuously differentiable, strictly increasing function of* $\varepsilon \in I_\alpha \cup \{0\}$.

Proof. First, for any fixed $\alpha > 0$, it can be proved that $T'_\varepsilon(\alpha)$ is a continuously differentiable function of $\varepsilon \in I_\alpha \cup \{0\}$. The proof is easy but tedious; we omit it.

Secondly, since $f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho$, $F_\varepsilon(u) = \int_0^u f_\varepsilon(t)\,dt$ and by (3.1), we compute that

$$\begin{aligned}
T'_\varepsilon(\alpha) &= \frac{1}{\sqrt{2}} \int_0^1 \frac{1}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{1/2}}\,dv - \frac{\alpha}{2\sqrt{2}} \int_0^1 \frac{f_\varepsilon(\alpha) - f_\varepsilon(\alpha v)v}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{3/2}}\,dv \\
&= \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\varepsilon\frac{(\alpha^4-u^4)}{2} - \sigma\frac{(\alpha^3-u^3)}{3} + \rho(\alpha-u)}{[-\varepsilon\frac{(\alpha^4-u^4)}{4} + \sigma\frac{(\alpha^3-u^3)}{3} - \kappa\frac{(\alpha^2-u^2)}{2} + \rho(\alpha-u)]^{3/2}}\,du
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon} T'_\varepsilon(\alpha) &= \frac{1}{96\sqrt{2}\alpha} \int_0^\alpha \frac{(\alpha^4-u^4)[3\varepsilon(\alpha^4-u^4) + 2\sigma(\alpha^3-u^3) - 12\kappa(\alpha^2-u^2) + 42\rho(\alpha-u)]}{[-\varepsilon\frac{(\alpha^4-u^4)}{4} + \sigma\frac{(\alpha^3-u^3)}{3} - \kappa\frac{(\alpha^2-u^2)}{2} + \rho(\alpha-u)]^{5/2}}\,du \\
&> \frac{1}{48\sqrt{2}\alpha} \int_0^\alpha \frac{(\alpha^4-u^4)(\alpha-u)[\sigma(\alpha^2+\alpha u+u^2) - 6\kappa(\alpha+u) + 21\rho]}{[-\varepsilon\frac{(\alpha^4-u^4)}{4} + \sigma\frac{(\alpha^3-u^3)}{3} - \kappa\frac{(\alpha^2-u^2)}{2} + \rho(\alpha-u)]^{5/2}}\,du.
\end{aligned} \tag{3.4}$$

Let

$$\begin{aligned}
H(u) &\equiv \sigma(\alpha^2 + \alpha u + u^2) - 6\kappa(\alpha+u) + 21\rho \\
&= \sigma u^2 + (\sigma\alpha - 6\kappa)u + (\sigma\alpha^2 - 6\kappa\alpha + 21\rho).
\end{aligned}$$

Therefore, the proof is complete if we can prove that

$$H(u) > 0 \quad \text{for any given numbers } \sigma, \rho, \alpha > 0,\ 0 < \kappa \leqslant \sqrt{\sigma\rho}. \tag{3.5}$$

Note that the discriminant of quadratic polynomial $H(u)$ is $-3\sigma^2\alpha^2 + 12\sigma\kappa\alpha + (36\kappa^2 - 84\sigma\rho) \equiv \widetilde{H}(\alpha)$. By the assumption that $\kappa \leqslant \sqrt{\sigma\rho}$, the discriminant of quadratic polynomial $\widetilde{H}(\alpha)$ is $144\sigma^2(4\kappa^2 - 7\sigma\rho) < 0$. So $\widetilde{H}(\alpha) < 0$ for any given numbers $\sigma, \rho > 0$, $0 < \kappa \leqslant \sqrt{\sigma\rho}$. This implies that (3.5) holds. By (3.4) and (3.5), for any fixed $\alpha > 0$, $T'_\varepsilon(\alpha)$ is a strictly increasing function of $\varepsilon \in I_\alpha \cup \{0\}$.

The proof of Lemma 3.5 is complete. □

We are now in a position to prove Lemma 3.3.

Proof of Lemma 3.3. First, we prove that, for any fixed $\varepsilon > 0$, $T_\varepsilon(\alpha)$ is either a strictly increasing function or has a local maximum and a local minimum, on $(0, \beta_\varepsilon)$. By Lemma 3.2, we only need to consider the case $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2} < \varepsilon < (\frac{\sigma^3}{27\rho})^{1/2}$.

For any fixed $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2} < \varepsilon < (\frac{\sigma^3}{27\rho})^{1/2}$, by Lemma 3.1(ii) (resp. Lemma 3.4), we know that all (possible) critical points of $T_\varepsilon(\alpha)$ on $(0, \gamma_\varepsilon]$ (resp. on $[\gamma_\varepsilon, \beta_\varepsilon)$) are discrete. Moreover, since $\lim_{\alpha\to 0^+} T_\varepsilon(\alpha) = 0$ and $\lim_{\alpha\to\beta_\varepsilon^-} T_\varepsilon(\alpha) = \infty$ and by Lemma 3.1(i), we obtain that $T'_\varepsilon(\alpha)$ changes sign an even number of times or infinitely times. Assume that $T_\varepsilon(\alpha)$ is neither a strictly increasing function nor does it have exactly one local maximum and one local minimum on $(0, \beta_\varepsilon)$. Then there exist three numbers $\alpha_1, \alpha_2, \alpha_3 \in (0, \beta_\varepsilon)$ such that $\alpha_1 < \alpha_2 < \alpha_3$ are critical points of $T_\varepsilon(\alpha)$, α_1, α_3 are local maxima, and α_2 is a local minimum. Thus $T''_\varepsilon(\alpha_1), T''_\varepsilon(\alpha_3) \leqslant 0$ and $T''_\varepsilon(\alpha_2) \geqslant 0$.

By Lemma 3.4, for any fixed $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2} < \varepsilon < (\frac{\sigma^3}{27\rho})^{1/2}$, $G_\varepsilon(\alpha) = 8\sqrt{2}\alpha^{\frac{5}{2}} T''_\varepsilon(\alpha)$ is a strictly increasing function on $[\gamma_\varepsilon, \beta_\varepsilon)$. Since $\alpha_2 \geqslant \gamma_\varepsilon$ by Lemma 3.1(ii), we obtain that

$$8\sqrt{2}\alpha_3^{\frac{5}{2}} T''_\varepsilon(\alpha_3) = G_\varepsilon(\alpha_3) > G_\varepsilon(\alpha_2) = 8\sqrt{2}\alpha_2^{\frac{5}{2}} T''_\varepsilon(\alpha_2) \geqslant 0.$$

Therefore $T''_\varepsilon(\alpha_3) > 0$. This contradicts to that $T''_\varepsilon(\alpha_3) \leqslant 0$. So $T_\varepsilon(\alpha)$ is either a strictly increasing function or has exactly one local maximum and one local minimum on $(0, \beta_\varepsilon)$.

Next, suppose that $T_\varepsilon(\alpha)$ has exactly a local maximum α_M and a local minimum α_m for some fixed $\varepsilon > 0$. Then $0 < \alpha_M < \alpha_m < \beta_\varepsilon$ by Lemma 3.1(i). We can prove that $T_\varepsilon(\alpha)$ has exactly two critical points α_M, α_m on $(0, \beta_\varepsilon)$ by applying Lemma 3.5 and similar arguments used in the proof of [1, Lemma 3.3]; we omit it.

The proof of Lemma 3.3 is complete. □

Let

$$E = \left\{ \begin{array}{l} \varepsilon > 0\colon\ T_\varepsilon(\alpha) \text{ has exactly two critical points,} \\ \text{a local maximum and a local minimum, on } (0, \beta_\varepsilon) \end{array} \right\}.$$

By Lemma 3.3, for any $\varepsilon > 0$, $T_\varepsilon(\alpha)$ is either a strictly increasing function or has exactly two critical points, a local maximum and a local minimum, on $(0, \beta_\varepsilon)$. Thus

$$E = \left\{ \varepsilon > 0\colon\ T'_\varepsilon(\alpha) < 0 \text{ for some } \alpha \in (0, \beta_\varepsilon) \right\}. \tag{3.6}$$

We obtain the following two lemmas by modifying the same arguments used in the proof of [1, Lemmas 3.7–3.8]; we omit the proofs.

Lemma 3.6. *The set E is open and connected.*

Lemma 3.7. $(0, (\frac{25}{32}(\frac{\sigma^3}{27\rho}))^{1/2}] \subset E$.

The following Lemma 3.8(i) determine the shape of $T_{\varepsilon=0}(\alpha)$ on $(0,\infty)$, and Lemma 3.8(ii) is a basic comparison theorem for the time map formula. Lemma 3.8(i) follows from [5, Theorem 3.2] and Lemma 3.8(ii) by modifying [5, Theorems 2.3 and 2.4]; we omit the proofs.

Lemma 3.8. *Consider* (1.1)–(1.3). *The following assertions* (i) *and* (ii) *hold*:

(i) $T_{\varepsilon=0}(\alpha)$ *has exactly one critical point at some* α_0, *a maximum, on* $(0,\infty)$. *Moreover,* $\lim_{\alpha\to 0^+} T_{\varepsilon=0}(\alpha) = \lim_{\alpha\to\infty} T_{\varepsilon=0}(\alpha) = 0$.
(ii) *For any fixed* $\alpha > 0$, $T_\varepsilon(\alpha)$ *is a continuous, strictly increasing function of* $\varepsilon \in I_\alpha \cup \{0\}$.

4. Proofs of the main results

We first recall that a positive solution $u_{\varepsilon,\lambda}$ of (1.1) is *degenerate* if $T'_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and is *non-degenerate* if $T'_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$. Also, a *degenerate* positive solution $u_{\varepsilon,\lambda}$ of (1.1) is of *cusp type* if $T''_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and $T'''_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$.

Proof of Theorem 2.1. To prove Theorem 2.1, by (3.1) and Lemma 3.1(i), it suffices to prove that there exists a positive number $\tilde\varepsilon = \tilde\varepsilon(\sigma,\kappa,\rho)$ such that the following parts (I)–(III) hold:

(I) For $0 < \varepsilon < \tilde\varepsilon$, on $(0,\beta_\varepsilon)$, $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some α_ε^- and a local minimum at some α_ε^+ $(>\alpha_\varepsilon^-)$, satisfying $\lambda^* = (T_\varepsilon(\alpha_\varepsilon^-))^2$ and $\lambda_* = (T_\varepsilon(\alpha_\varepsilon^+))^2$.
(II) For $\varepsilon = \tilde\varepsilon$, $T_{\tilde\varepsilon}(\alpha)$ is a strictly increasing function and has exactly one critical point, at some $\tilde\alpha$, on $(0,\beta_{\tilde\varepsilon})$. Moreover, $T'_{\tilde\varepsilon}(\tilde\alpha) = 0$, $T'_{\tilde\varepsilon}(\alpha) > 0$ for $\alpha \in (0,\beta_{\tilde\varepsilon}) \setminus \{\tilde\alpha\}$, $T''_{\tilde\varepsilon}(\tilde\alpha) = 0$ and $T'''_{\tilde\varepsilon}(\tilde\alpha) \neq 0$. (So (1.1)–(1.3) has exactly one (cusp type) degenerate positive solution $u_{\tilde\lambda}$ with $\tilde\lambda \equiv (T_{\tilde\varepsilon}(\tilde\alpha))^2$ and $\tilde\alpha = \|u_{\tilde\lambda}\|_\infty$.)
(III) For $\varepsilon > \tilde\varepsilon$, $T_\varepsilon(\alpha)$ is a strictly increasing function and has no critical point on $(0,\beta_\varepsilon)$. Moreover, $T'_\varepsilon(\alpha) > 0$ for $\alpha \in (0,\beta_\varepsilon)$.

Note that, by (3.2) and the above parts (I)–(III), we obtain immediately the exact multiplicity result of positive solutions of (1.1)–(1.3) for $0 < \varepsilon < \tilde\varepsilon$ and the uniqueness result of positive solution of (1.1)–(1.3) for $\varepsilon \geqslant \tilde\varepsilon$. Moreover, ordering properties and asymptotic behaviors of positive solutions of (1.1)–(1.3) in parts (I)–(III) can be obtained easily. We then prove parts (I)–(III) as follows.

By Lemmas 3.2, 3.6 and 3.7, we obtain that $E = (0,\tilde\varepsilon)$ where $\tilde\varepsilon = \sup E$ satisfies $(\frac{25}{32}(\frac{\sigma^3}{27\rho}))^{1/2} < \tilde\varepsilon < (\frac{\sigma^3}{27\rho})^{1/2}$. So, for $0 < \varepsilon < \tilde\varepsilon$, on $(0,\beta_\varepsilon)$, $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some α_ε^- and a local minimum at some α_ε^+ $(>\alpha_\varepsilon^-)$, satisfying $\lambda^* = (T_\varepsilon(\alpha_\varepsilon^-))^2$ and $\lambda_* = (T_\varepsilon(\alpha_\varepsilon^+))^2$. So part (I) holds.

For $\varepsilon > \tilde\varepsilon$, by Lemma 3.5 and (3.6), we obtain that

$$T'_\varepsilon(\alpha) > T'_{\tilde\varepsilon}(\alpha) \geqslant 0 \quad \text{for } \alpha \in (0,\beta_\varepsilon) \subset (0,\beta_{\tilde\varepsilon}),$$

and hence $T_\varepsilon(\alpha)$ has no critical point on $(0,\beta_\varepsilon)$. So part (III) holds.

We prove the remaining part (II). For $\varepsilon = \tilde\varepsilon$, we know that

$$T'_{\tilde\varepsilon}(\alpha) \geqslant 0 \quad \text{on } (0,\beta_{\tilde\varepsilon}). \tag{4.1}$$

We first prove the existence of a critical point of $T_{\tilde\varepsilon}(\alpha)$ on $(0,\beta_{\tilde\varepsilon})$. Choose a sequence $\{\varepsilon_n\} \subset E = (0,\tilde\varepsilon)$ such that $\varepsilon_n \nearrow \tilde\varepsilon$ as $n\to\infty$. Let $\alpha_{\varepsilon_n}^- < \alpha_{\varepsilon_n}^+$ be two critical points of $T_{\varepsilon_n}(\alpha)$ on $(0,\beta_{\varepsilon_n})$ for each $n \in \mathbb{N}$ (see Fig. 7). Then by Lemma 3.5 again, we obtain that

$$T'_{\varepsilon_n}(\alpha_{\varepsilon_{n+1}}^-) < T'_{\varepsilon_{n+1}}(\alpha_{\varepsilon_{n+1}}^-) = 0 \quad \text{and} \quad T'_{\varepsilon_n}(\alpha_{\varepsilon_{n+1}}^+) < T'_{\varepsilon_{n+1}}(\alpha_{\varepsilon_{n+1}}^+) = 0.$$

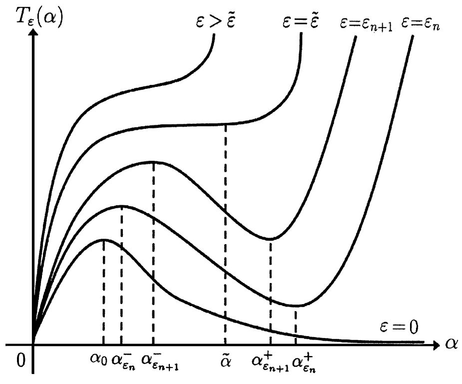

Fig. 7. Graphs of $T_\varepsilon(\alpha)$ for $\alpha \in (0, \beta_\varepsilon)$ with varying $\varepsilon \geqslant 0$.

Hence $\alpha^-_{\varepsilon_n} < \alpha^-_{\varepsilon_{n+1}} < \alpha^+_{\varepsilon_{n+1}} < \alpha^+_{\varepsilon_n}$ and

$$\alpha^-_{\varepsilon_n} < \tilde{\alpha}^- \equiv \lim_{n\to\infty} \alpha^-_{\varepsilon_n} \leqslant \tilde{\alpha}^+ \equiv \lim_{n\to\infty} \alpha^+_{\varepsilon_n} < \alpha^+_{\varepsilon_n} \quad \text{for all } n \in \mathbb{N}.$$

These imply that

$$T'_{\varepsilon_n}\big(\tilde{\alpha}^-\big), T'_{\varepsilon_n}\big(\tilde{\alpha}^+\big) < 0 \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 3.5, we obtain that $T'_\varepsilon(\alpha)$ is a continuous function of $\varepsilon \in I_\alpha$. Thus

$$T'_{\tilde{\varepsilon}}\big(\tilde{\alpha}^-\big) = \lim_{n\to\infty} T'_{\varepsilon_n}\big(\tilde{\alpha}^-\big) \leqslant 0 \quad \text{and} \quad T'_{\tilde{\varepsilon}}\big(\tilde{\alpha}^+\big) = \lim_{n\to\infty} T'_{\varepsilon_n}\big(\tilde{\alpha}^+\big) \leqslant 0. \tag{4.2}$$

So $T'_{\tilde{\varepsilon}}(\tilde{\alpha}^-) = T'_{\tilde{\varepsilon}}(\tilde{\alpha}^+) = 0$ by (4.1) and (4.2), and hence $T_{\tilde{\varepsilon}}(\alpha)$ has critical points at $\tilde{\alpha}^-, \tilde{\alpha}^+$ on $(0, \beta_{\tilde{\varepsilon}})$.

We then prove the uniqueness of critical point of $T_{\tilde{\varepsilon}}(\alpha)$ on $(0, \beta_{\tilde{\varepsilon}})$. That is, we prove that $\tilde{\alpha} \equiv \tilde{\alpha}^- = \tilde{\alpha}^+$ is the unique critical point of $T_{\tilde{\varepsilon}}(\alpha)$ on $(0, \beta_{\tilde{\varepsilon}})$. Suppose that $\hat{\alpha} < \bar{\alpha}$ are two critical points of $T_{\tilde{\varepsilon}}(\alpha)$ on $(0, \beta_{\tilde{\varepsilon}})$. We know that all (possible) critical points of $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$ are discrete as in the proof of Lemma 3.3. Hence there exist positive numbers $\alpha_1 < \hat{\alpha} < \alpha_2 < \bar{\alpha}$ such that

$$T'_{\tilde{\varepsilon}}(\alpha_1), T'_{\tilde{\varepsilon}}(\alpha_2) > 0.$$

By Lemma 3.5, we obtain that $T'_\varepsilon(\alpha)$ is a continuous, strictly increasing function of $\varepsilon \in I_\alpha$. Hence there exists a positive $\hat{\varepsilon} < \tilde{\varepsilon}$ such that

$$T'_{\hat{\varepsilon}}(\alpha_1) > 0, \qquad T'_{\hat{\varepsilon}}(\hat{\alpha}) < 0, \qquad T'_{\hat{\varepsilon}}(\alpha_2) > 0, \qquad T'_{\hat{\varepsilon}}(\bar{\alpha}) < 0.$$

Thus $T_{\hat{\varepsilon}}(\alpha)$ has at least two local maxima on $(0, \beta_{\hat{\varepsilon}})$, which contradicts to the facts that $\hat{\varepsilon} \in E$ and $T_{\hat{\varepsilon}}(\alpha)$ has exactly one local maximum on $(0, \beta_{\hat{\varepsilon}})$. So $T_{\tilde{\varepsilon}}(\alpha)$ has at most one critical point on $(0, \beta_{\tilde{\varepsilon}})$. By the above analysis,

$$T'_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0 \quad \text{and} \quad T'_{\tilde{\varepsilon}}(\alpha) > 0 \quad \text{for } \alpha \in (0, \beta_{\tilde{\varepsilon}}) \setminus \{\tilde{\alpha}\}. \tag{4.3}$$

Next, if $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) > 0$ (resp. $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) < 0$), then $T_{\tilde{\varepsilon}}(\alpha)$ has a local minimum (resp. a local maximum) at $\tilde{\alpha}$, which contradicts to (4.3). So $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0$. By Lemma 3.1(ii), we have

$$\alpha_{\varepsilon_n}^{+} \geqslant \gamma_{\varepsilon_n} > \gamma_{\tilde{\varepsilon}} \quad \text{for all } n \in \mathbb{N},$$

and hence $\tilde{\alpha} = \lim_{n\to\infty} \alpha_{\varepsilon_n}^{+} \geqslant \gamma_{\tilde{\varepsilon}}$. By Lemma 3.4, $G'_{\tilde{\varepsilon}}(\alpha) > 0$ for all $\alpha \in [\gamma_{\tilde{\varepsilon}}, \beta_{\tilde{\varepsilon}})$. So

$$G'_{\tilde{\varepsilon}}(\tilde{\alpha}) = \tilde{\alpha}^{\frac{3}{2}}\left[20\sqrt{2}T''_{\tilde{\varepsilon}}(\tilde{\alpha}) + 8\sqrt{2}\tilde{\alpha}T'''_{\tilde{\varepsilon}}(\tilde{\alpha})\right] > 0.$$

Therefore $T'''_{\tilde{\varepsilon}}(\tilde{\alpha}) > 0$ since $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0$. This completes the proof of part (II).

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. Recall (3.1) with $\varepsilon \geqslant 0$,

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\alpha} \left[F_\varepsilon(\alpha) - F_\varepsilon(u)\right]^{-1/2} du \equiv T_\varepsilon(\alpha) \quad \text{for } 0 < \alpha < \beta_\varepsilon,$$

where β_ε the unique positive zero of cubic polynomial $f_\varepsilon(u)$ for $\varepsilon > 0$ and $\beta_{\varepsilon=0} = \infty$. Thus, studying the exact number of positive solutions of (1.1)–(1.3) for fixed $\lambda > 0$ is equivalent to studying the number of roots of the equation $T_\varepsilon(\alpha) = \sqrt{\lambda}$ on $(0, \beta_\varepsilon)$ for varying $\varepsilon > 0$. Since we have studied the behaviors of $T_\varepsilon(\alpha)$ for all varying $\varepsilon \geqslant 0$ (see the proofs of Theorem 2.1 and Lemma 3.8(i) and Fig. 7), there exist two positive numbers $\lambda_0\ (= \lambda_0(\sigma, \kappa, \rho)) < \tilde{\lambda}\ (= \tilde{\lambda}(\sigma, \kappa, \rho))$ such that the following parts (I)–(III) hold:

(I) For $0 < \lambda \leqslant \lambda_0$, there exists a positive number $\varepsilon^* = \varepsilon^*(\lambda)$ such that the equation $T_\varepsilon(\alpha) = \sqrt{\lambda}$ has exactly three roots on $(0, \beta_\varepsilon)$ for $0 < \varepsilon < \varepsilon^*$, exactly two roots on $(0, \beta_\varepsilon)$ for $\varepsilon = \varepsilon^*$, and exactly one root on $(0, \beta_\varepsilon)$ for $\varepsilon > \varepsilon^*$.

(II) For $\lambda_0 < \lambda < \tilde{\lambda}$, there exist two positive numbers $\varepsilon_*\ (= \varepsilon_*(\lambda)) < \varepsilon^*\ (= \varepsilon^*(\lambda))$ such that the equation $T_\varepsilon(\alpha) = \sqrt{\lambda}$ has exactly three roots on $(0, \beta_\varepsilon)$ for $\varepsilon_* < \varepsilon < \varepsilon^*$, exactly two roots on $(0, \beta_\varepsilon)$ for $\varepsilon = \varepsilon_*$ and $\varepsilon = \varepsilon^*$, and exactly one root on $(0, \beta_\varepsilon)$ for $0 < \varepsilon < \varepsilon_*$ and $\varepsilon > \varepsilon^*$.

(III) For $\lambda \geqslant \tilde{\lambda}$, the equation $T_\varepsilon(\alpha) = \sqrt{\lambda}$ has exactly one root on $(0, \beta_\varepsilon)$ for all $\varepsilon > 0$.

Notice that $\lambda_0 = (T_{\varepsilon=0}(\alpha_0))^2$ and $\tilde{\lambda} = (T_{\tilde{\varepsilon}}(\tilde{\alpha}))^2$, where α_0 is the unique critical point of $T_{\varepsilon=0}(\alpha)$ and $\tilde{\alpha}$ be the unique critical point of $T_{\tilde{\varepsilon}}(\alpha)$. Hence (3.2) and the above parts (I)–(III) imply immediately the exact multiplicity result of positive solutions of (1.1)–(1.3) for $\lambda \in (0, \tilde{\lambda})$ and the uniqueness result of positive solution of (1.1)–(1.3) for $\lambda \geqslant \tilde{\lambda}$. Moreover, ordering properties and asymptotic behaviors of positive solutions of (1.1)–(1.3) in parts (I)–(III) can be obtained easily.

The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3. By Theorem 2.1, for any $\varepsilon \geqslant \tilde{\varepsilon}$, we obtain that (1.1)–(1.3) has exactly one positive solution for all $\lambda > 0$. In addition, for any $\varepsilon \in (0, \tilde{\varepsilon})$, there exist two positive numbers $\lambda_*(\varepsilon) < \lambda^*(\varepsilon)$ such that (1.1)–(1.3) has exactly three positive solutions for $\lambda_*(\varepsilon) < \lambda < \lambda^*(\varepsilon)$, exactly two positive solutions for $\lambda = \lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$, and exactly one positive solution for $0 < \lambda < \lambda_*(\varepsilon)$ and $\lambda > \lambda^*(\varepsilon)$, where $\lambda_*(\varepsilon) = (T_\varepsilon(\alpha_\varepsilon^+))^2$ and $\lambda^*(\varepsilon) = (T_\varepsilon(\alpha_\varepsilon^-))^2$ in which $\alpha_\varepsilon^- < \alpha_\varepsilon^+$ are two critical points of $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$.

First, letting $\alpha_{\tilde{\varepsilon}}^- = \alpha_{\tilde{\varepsilon}}^+ \equiv \tilde{\alpha}$, we prove that α_ε^- (resp. α_ε^+) is a continuous, strictly increasing (resp. strictly decreasing) function on $(0, \tilde{\varepsilon}]$ and $\lim_{\varepsilon\to 0^+} \alpha_\varepsilon^- = \alpha_0$ (resp. $\lim_{\varepsilon\to 0^+} \alpha_\varepsilon^+ = \infty$) as follows (cf. Fig. 7.) By similar arguments in the proof of Theorem 2.1, we obtain that α_ε^- (resp. α_ε^+) is a strictly increasing (resp. strictly decreasing) function on $(0, \tilde{\varepsilon}]$. For any fixed $\alpha \in (\alpha_0, \tilde{\alpha})$, by Theorem 2.1(ii) and Lemma 3.8(i), we obtain that

$$T'_{\varepsilon=0}(\alpha) < 0 \quad \text{and} \quad T'_{\tilde{\varepsilon}}(\alpha) > 0.$$

Then by Lemma 3.5, $T'_\varepsilon(\alpha)$ is a continuously differentiable, strictly increasing function of $\varepsilon \in [0, \tilde{\varepsilon}]$. This implies that there exists a unique $\varepsilon \in (0, \tilde{\varepsilon})$ such that $T'_\varepsilon(\alpha) = 0$. So

$$\alpha_\varepsilon^- : (0, \tilde{\varepsilon}] \to (\alpha_0, \tilde{\alpha}] \text{ is a strictly increasing, surjective function,} \tag{4.4}$$

and hence α_ε^- is a continuous function on $(0, \tilde{\varepsilon}]$ and $\lim_{\varepsilon \to 0^+} \alpha_\varepsilon^- = \alpha_0$. Similarly, we can prove that

$$\alpha_\varepsilon^+ : (0, \tilde{\varepsilon}] \to [\tilde{\alpha}, \infty) \text{ is a strictly decreasing, surjective function,} \tag{4.5}$$

and hence α_ε^+ is also a continuous function on $(0, \tilde{\varepsilon}]$ and $\lim_{\varepsilon \to 0^+} \alpha_\varepsilon^+ = \infty$.

Secondly, let

$$\lambda_*(0) \equiv 0, \qquad \lambda^*(0) \equiv \lambda_0 = \big(T_{\varepsilon=0}(\alpha_0)\big)^2, \quad \text{and} \quad \lambda_*(\tilde{\varepsilon}) = \lambda^*(\tilde{\varepsilon}) \equiv \tilde{\lambda} = \big(T_{\tilde{\varepsilon}}(\tilde{\alpha})\big)^2.$$

By (4.4), (4.5), Lemma 3.5 and Lemma 3.8(ii), it can be proved that $\lambda^* = (T_\varepsilon(\alpha_\varepsilon^-))^2$ and $\lambda_* = (T_\varepsilon(\alpha_\varepsilon^+))^2$ satisfy

$$\lambda^*(\varepsilon) : [0, \tilde{\varepsilon}] \to [\lambda_0, \tilde{\lambda}] \text{ is a continuous, strictly increasing function} \tag{4.6}$$

and

$$\lambda_*(\varepsilon) : [0, \tilde{\varepsilon}] \to [0, \tilde{\lambda}] \text{ is a continuous, strictly increasing function.} \tag{4.7}$$

Moreover,

$$\lim_{\varepsilon \to 0^+} \lambda^*(\varepsilon) = \lambda_0, \qquad \lim_{\varepsilon \to 0^+} \lambda_*(\varepsilon) = 0, \quad \text{and} \quad \lambda_*(\tilde{\varepsilon}) = \lambda^*(\tilde{\varepsilon}) = \tilde{\lambda}. \tag{4.8}$$

The proofs are easy but tedious and hence we omit them.

Finally, by (4.6)–(4.8), $\lambda^*(\varepsilon)$ and $\lambda_*(\varepsilon)$ both have continuous inverse functions on $(0, \tilde{\varepsilon}]$. Indeed, by Theorem 2.2 and (3.1), $\varepsilon_*(\lambda) = (\lambda^*)^{-1}(\varepsilon)$ on $(\lambda_0, \tilde{\lambda}]$ and $\varepsilon^*(\lambda) = (\lambda_*)^{-1}(\varepsilon)$ on $(0, \tilde{\lambda}]$ where $\varepsilon_*(\tilde{\lambda}) = \varepsilon^*(\tilde{\lambda}) \equiv \tilde{\varepsilon}$. So we obtain that

$$\varepsilon^*(\lambda) : (0, \tilde{\lambda}] \to (0, \tilde{\varepsilon}] \text{ is a continuous, strictly increasing function}$$

and

$$\varepsilon_*(\lambda) : (\lambda_0, \tilde{\lambda}] \to (0, \tilde{\varepsilon}] \text{ is a continuous, strictly increasing function.}$$

Moreover,

$$\lim_{\lambda \to 0^+} \varepsilon^*(\lambda) = \lim_{\lambda \to \lambda_0^+} \varepsilon_*(\lambda) = 0.$$

The proof of Theorem 2.3 is complete. □

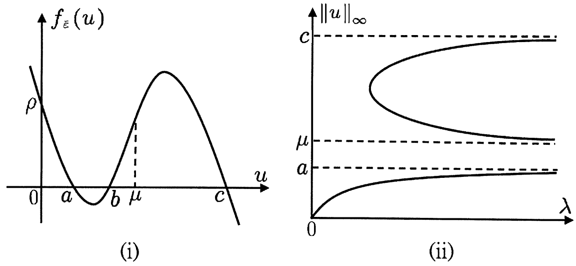

Fig. 8. (i) The graph of $f_{\breve{\varepsilon}}(u)$ in (5.2). (ii) The conjectured bifurcation curve of (5.2).

5. Conjectures

In this section, we analyze (1.1), (1.2) more precisely. First, if

$$\kappa \leqslant \sqrt{\sigma\rho},$$

the exact multiplicity results of positive solutions for (1.1), (1.2) was determine precisely by Theorem 2.1 and [1, Theorem 2.1]. By some numerical simulations, we give next three conjectures on the shape of bifurcation curves S_ε of positive solutions of (1.1), (1.2) with $\kappa > \sqrt{\sigma\rho}$.

Conjecture 5.1. *Consider* (1.1), (1.2) *where*

$$\sqrt{\sigma\rho} < \kappa \leqslant \sqrt{3\sigma\rho}.$$

Then there exists a positive number $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$ *satisfying*

$$\left(\frac{25}{32}\left(\frac{\sigma^3}{27\rho}\right)\right)^{1/2} < \tilde{\varepsilon} < \left(\frac{\sigma^3}{27\rho}\right)^{1/2}$$

such that all results in Theorem 2.1(i)–(iii) *hold.*

While

$$\kappa > \sqrt{3\sigma\rho}, \tag{5.1}$$

we remark that there exists some $\breve{\varepsilon} > 0$ such that cubic nonlinearity $f_{\breve{\varepsilon}}(u)$ has *three* positive zeros $a < b < c$ and $\int_a^c f_{\breve{\varepsilon}}(t)\,dt > 0$ (see Fig. 8(i).) For which $f_{\breve{\varepsilon}}(u)$, it is easy to check that $a + c > 2b$ and there exists $\mu \in (b, c)$ such that $\int_a^\mu f_{\breve{\varepsilon}}(t)\,dt = 0$. So problem (1.1), (1.2), (5.1) can be written as

$$\begin{cases} u''(x) + \lambda\breve{\varepsilon}(u-a)(u-b)(c-u) = 0, & -1 < x < 1, \quad u(-1) = u(1) = 0, \\ \lambda, \breve{\varepsilon} > 0, \quad 0 < a < b < c, & a + c > 2b. \end{cases} \tag{5.2}$$

It was conjectured that the bifurcation curve of positive solution of (5.2) is broken S-shaped (see Fig. 8(ii)) on the $(\lambda, \|u\|_\infty)$-plane. A proof was claimed by Smoller and Wasserman [8, Theorem 2.1], but their proof has a gap. Assuming additional different conditions on constants a, b and c, Wang [9] and Korman, Li and Ouyang [3] gave partial proofs of the above conjecture independently. For this conjecture, Korman, Li and Ouyang [4] gave a computer-assisted proof. Further investigation on this long-standing conjecture is needed. We give next two conjectures for (1.1), (1.2), (5.1).

Conjecture 5.2. *Consider* (1.1), (1.2) *where*

$$\sqrt{3\sigma\rho} < \kappa < 2\sqrt{\sigma\rho}. \tag{5.3}$$

Then there exist two positive numbers $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\sigma, \kappa, \rho) < \varepsilon_0 = \varepsilon_0(\sigma, \kappa, \rho)$ such that the following assertions (i)–(iii) *hold*:

(i) (*See Fig.* 2(i).) *If $0 < \varepsilon < \tilde{\varepsilon}_0$, then the bifurcation curve S_ε is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, the exact multiplicity results of positive solutions in Theorem* 2.1(i) *hold.*
(ii) (*See Fig.* 8(ii).) *If $\tilde{\varepsilon}_0 \leqslant \varepsilon < \varepsilon_0$, then the bifurcation curve S_ε is broken S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist $\lambda^* > 0$ such that* (1.1), (1.2), (5.3) *has exactly three positive solutions for $\lambda > \lambda^*$, exactly two positive solutions for $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda^*$.*
(iii) (*See Fig.* 2(iii).) *If $\varepsilon \geqslant \varepsilon_0$, then the bifurcation curve S_ε is a monotone curve on the $(\lambda, \|u\|_\infty)$-plane. Moreover,* (1.1), (1.2), (5.3) *has exactly one positive solution for all $\lambda > 0$.*

Conjecture 5.3. *Consider* (1.1), (1.2) *where*

$$\kappa \geqslant 2\sqrt{\sigma\rho}. \tag{5.4}$$

Then there exists a positive number $\varepsilon_0 = \varepsilon_0(\sigma, \kappa, \rho)$ such that the following assertions (i) *and* (ii) *hold*:

(i) (*See Fig.* 8(ii).) *If $0 < \varepsilon < \varepsilon_0$, then the bifurcation curve S_ε is broken S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist $\lambda^* > 0$ such that* (1.1), (1.2), (5.4) *has exactly three positive solutions for $\lambda > \lambda^*$, exactly two positive solutions for $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda^*$.*
(ii) (*See Fig.* 2(iii).) *If $\varepsilon \geqslant \varepsilon_0$, then the bifurcation curve S_ε is a monotone curve on the $(\lambda, \|u\|_\infty)$-plane. Moreover,* (1.1), (1.2), (5.4) *has exactly one positive solution for all $\lambda > 0$.*

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Appendix A

If $f_\varepsilon(u)$ satisfies (1.1)–(1.3), for any $\varepsilon > 0$, we prove that $f_\varepsilon(u)$ has a unique positive zero at some $\beta_\varepsilon > \gamma_\varepsilon = \sigma/(3\varepsilon)$. We first compute that

$$f'_\varepsilon(u) = -3\varepsilon u^2 + 2\sigma u - \kappa. \tag{A.1}$$

(i) (See Fig. 1(i).) If $\sigma^2 > 3\varepsilon\kappa$, then $f_\varepsilon(u)$ has exactly two critical points, a local minimum at $p_1 = \frac{\sigma - \sqrt{\sigma^2 - 3\varepsilon\kappa}}{3\varepsilon} \in (0, \gamma_\varepsilon)$ and a local maximum at $p_2 = \frac{\sigma + \sqrt{\sigma^2 - 3\varepsilon\kappa}}{3\varepsilon} \in (\gamma_\varepsilon, \infty)$. We compute that

$$\begin{aligned} f_\varepsilon(p_1) &= \left(-\varepsilon u^3 + \sigma u^2 - \kappa u + \rho\right)\Big|_{u=p_1} \\ &= \left(\frac{1}{3}\sigma u^2 - \frac{2}{3}\kappa u + \rho\right)\Bigg|_{u=p_1} \end{aligned}$$

since $f'_\varepsilon(p_1) = -3\varepsilon p_1^2 + 2\sigma p_1 - \kappa = 0$. Let $Q_1(u) \equiv \frac{1}{3}\sigma u^2 - \frac{2}{3}\kappa u + \rho$, then the discriminant of quadratic polynomial $Q_1(u)$ is $\frac{4}{9}(\kappa^2 - 3\sigma\rho) < 0$ by (1.3). So $f_\varepsilon(p_1) = Q_1(p_1) > 0$, and hence $f_\varepsilon(\gamma_\varepsilon) > 0$ and $f'_\varepsilon(\gamma_\varepsilon) > 0$. So $f_\varepsilon(u)$ has a unique positive zero at some $\beta_\varepsilon > \gamma_\varepsilon$.

(ii) (See Fig. 1(ii).) If $\sigma^2 = 3\varepsilon\kappa$, then $f'_\varepsilon(\gamma_\varepsilon) = 0$ and $f'_\varepsilon(u) < 0$ on $(0, \gamma_\varepsilon) \cup (\gamma_\varepsilon, \infty)$ by (A.1). We compute that

$$f_\varepsilon(\gamma_\varepsilon) = \left(-\varepsilon u^3 + \sigma u^2 - \kappa u + \rho\right)\Big|_{u=\gamma_\varepsilon} = \left(\frac{2}{3}\sigma u^2 - \kappa u + \rho\right)\Big|_{u=\gamma_\varepsilon}$$

since $\gamma_\varepsilon = \sigma/(3\varepsilon)$. Let $Q_2(u) \equiv \frac{2}{3}\sigma u^2 - \kappa u + \rho$, then the discriminant of quadratic polynomial $Q_2(u)$ is $\kappa^2 - \frac{8}{3}\sigma\rho < 0$ by (1.3). So $f_\varepsilon(\gamma_\varepsilon) = Q_2(\gamma_\varepsilon) > 0$. So $f_\varepsilon(u)$ has a unique positive zero at some $\beta_\varepsilon > \gamma_\varepsilon$.

(iii) (See Fig. 1(iii).) If $\sigma^2 < 3\varepsilon\kappa$, then $f'_\varepsilon(\gamma_\varepsilon) < 0$ and $f'_\varepsilon(u) < 0$ on $(0, \infty)$ by (A.1). We obtain $f_\varepsilon(\gamma_\varepsilon) > 0$ by the same argument in part (ii). So $f_\varepsilon(u)$ has a unique positive zero at some $\beta_\varepsilon > \gamma_\varepsilon$.

So by above (i)–(iii), for any $\varepsilon > 0$, $f_\varepsilon(u)$ has a unique positive zero at some $\beta_\varepsilon > \gamma_\varepsilon$.

Appendix B

Proof of Lemma 3.4. The proof of Lemma 3.4 is rather technical, so we divide the proof into next Steps 1–5.

Step 1. We compute $G'_\varepsilon(\alpha)$.

By (3.1) and (3.3), we compute that

$$\begin{aligned} G_\varepsilon(\alpha) &= 8\sqrt{2}\alpha^{\frac{5}{2}} T''_\varepsilon(\alpha) \\ &= -8\alpha^{\frac{5}{2}} \int_0^1 \frac{f_\varepsilon(\alpha) - f_\varepsilon(\alpha v)v}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{3/2}}\,dv - 4\alpha^{\frac{7}{2}} \int_0^1 \frac{f'_\varepsilon(\alpha) - f'_\varepsilon(\alpha v)v^2}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{3/2}}\,dv \\ &\quad + 6\alpha^{\frac{7}{2}} \int_0^1 \frac{[f_\varepsilon(\alpha) - f_\varepsilon(\alpha v)v]^2}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{5/2}}\,dv \end{aligned}$$

and

$$G'_\varepsilon(\alpha) = \frac{1}{\sqrt{\alpha}} \int_0^\alpha \frac{K_\varepsilon(\alpha, u)}{[\triangle F_\varepsilon]^{7/2}}\,du, \tag{B.1}$$

where

$$\begin{aligned} K_\varepsilon(\alpha, u) &= -20(\triangle F_\varepsilon)^2(\triangle f_\varepsilon) - 22(\triangle F_\varepsilon)^2(\triangle \tilde{f}_\varepsilon) - 4(\triangle F_\varepsilon)^2(\triangle \hat{f}_\varepsilon) \\ &\quad + 33(\triangle F_\varepsilon)(\triangle f_\varepsilon)^2 + 18(\triangle F_\varepsilon)(\triangle f_\varepsilon)(\triangle \tilde{f}_\varepsilon) - 15(\triangle f_\varepsilon)^3, \end{aligned} \tag{B.2}$$

$$\triangle F_\varepsilon = F_\varepsilon(\alpha) - F_\varepsilon(u), \tag{B.3}$$

$$\triangle f_\varepsilon = \alpha f_\varepsilon(\alpha) - u f_\varepsilon(u), \tag{B.4}$$

$$\triangle \tilde{f}_\varepsilon = \alpha^2 f'_\varepsilon(\alpha) - u^2 f'_\varepsilon(u), \tag{B.5}$$

$$\triangle \hat{f}_\varepsilon = \alpha^3 f''_\varepsilon(\alpha) - u^3 f''_\varepsilon(u). \tag{B.6}$$

For $0 < u < \alpha$, we let $A \equiv \varepsilon(\alpha^4 - u^4)$, $B \equiv \sigma(\alpha^3 - u^3)$, $C \equiv \kappa(\alpha^2 - u^2)$, $D \equiv \rho(\alpha - u)$. Then $A, B, C, D > 0$. Since $f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho$ and by (B.3)–(B.6), we obtain that

$$\triangle F_\varepsilon = -A/4 + B/3 - C/2 + D, \tag{B.7}$$

$$\triangle f_\varepsilon = -A + B - C + D, \tag{B.8}$$

$$\triangle \tilde{f}_\varepsilon = -3A + 2B - C, \tag{B.9}$$

$$\triangle \hat{f}_\varepsilon = -6A + 2B. \tag{B.10}$$

Substituting (B.7)–(B.10) into (B.2), we have that

$$\begin{aligned} K_\varepsilon(\alpha, u) = \frac{1}{72}\big(&168ABC - 1356ABD - 504ACD - 168BCD + 9A^3 - 144D^3 - 2AB^2 \\ &+ 12A^2B + 90AC^2 - 207A^2C - 60B^2C + 2646AD^2 + 1134A^2D - 1248BD^2 \\ &+ 560B^2D + 468CD^2 + 72C^2D\big). \end{aligned} \tag{B.11}$$

So Lemma 3.4 holds if we can prove that $K_\varepsilon(\alpha, u) > 0$ for any fixed $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$, $0 \leqslant \kappa \leqslant \sqrt{\sigma\rho}$, $\alpha \in [\gamma_\varepsilon, \beta_\varepsilon)$ and $0 < u < \alpha$.

Step 2. We make a transformation for $K_\varepsilon(\alpha, u)$.

Although both $T_\varepsilon(\alpha)$ and $G_\varepsilon(\alpha)$ are only defined for $\alpha \in (0, \beta_\varepsilon)$, $K_\varepsilon(\alpha, u)$ is well defined for all $\alpha \in \mathbb{R}$. So Lemma 3.4 holds if we can prove that $K_\varepsilon(\alpha, u) > 0$ for any fixed $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$, $0 \leqslant \kappa \leqslant \sqrt{\sigma\rho}$, $\alpha \geqslant \gamma_\varepsilon$ and $0 < u < \alpha$. Since $\gamma_\varepsilon = \frac{\sigma}{3\varepsilon}$, we consider $K_\varepsilon(\alpha, u)$ when $\alpha \geqslant \gamma_\varepsilon$, $0 \leqslant \kappa \leqslant \sqrt{3}\varepsilon\gamma_\varepsilon^2$, $\frac{7}{10}\varepsilon\gamma_\varepsilon^3 \leqslant \rho \leqslant \varepsilon\gamma_\varepsilon^3$, and $0 < u < \alpha$. Let

$$\begin{aligned} \alpha &= (r+1)\gamma_\varepsilon, \quad r \in [0, \infty), \\ \kappa &= s\varepsilon\gamma_\varepsilon^2, \quad s \in [0, \sqrt{3}\,], \\ \rho &= t\varepsilon\gamma_\varepsilon^3, \quad t \in \left[\frac{7}{10}, 1\right], \\ u &= y\gamma_\varepsilon, \quad y \in (0, r+1). \end{aligned}$$

Thus

$$A = \varepsilon\big(\alpha^4 - u^4\big) = \varepsilon\gamma_\varepsilon^4\big[(r+1)^4 - y^4\big], \tag{B.12}$$

$$B = \sigma\big(\alpha^3 - u^3\big) = 3\varepsilon\gamma_\varepsilon^4\big[(r+1)^3 - y^3\big], \tag{B.13}$$

$$C = \kappa\big(\alpha^2 - u^2\big) = s\varepsilon\gamma_\varepsilon^4\big[(r+1)^2 - y^2\big], \tag{B.14}$$

$$D = \rho(\alpha - u) = t\varepsilon\gamma_\varepsilon^4(r+1-y). \tag{B.15}$$

Substituting (B.12)–(B.15) into (B.11), we obtain that

$$K_\varepsilon(\alpha,u)=\frac{1}{8}\varepsilon^3\gamma_\varepsilon^{12}(r+1-y)^3\widetilde{K}_\varepsilon(r,s,t,y), \tag{B.16}$$

where

$$\widetilde{K}_\varepsilon(r,s,t,y)=\sum_{j=0}^{9}k_j(r,s,t)y^j \tag{B.17}$$

with

$$\begin{aligned}
k_0(r,s,t)={}&\bigl(3-122t^2+10s^2-16t^3+8s^2t+234t-27s+52st^2-112st\bigr)\\
&+\bigl(-392st+16s^2t+50t^2+736t+50s^2+27-125s+52st^2\bigr)r\\
&+\bigl(100s^2+466t^2+8s^2t-243s+730t-504st+106\bigr)r^2\\
&+\bigl(-280st+238+294t^2+240t+100s^2-285s\bigr)r^3\\
&+\bigl(-265s-56st+50s^2+336+190t\bigr)r^4+\bigl(-207s+304t+308+10s^2\bigr)r^5\\
&+(126t-105s+182)r^6+(66-23s)r^7+13r^8+r^9,\\
k_1(r,s,t)={}&\bigl(9+52st^2+468t+30s^2-81s-224st-122t^2+16s^2t\bigr)\\
&+\bigl(16s^2t+172t^2-560st+72-294s+1004t+120s^2\bigr)r\\
&+\bigl(294t^2+180s^2-435s-448st+456t+246\bigr)r^2\\
&+\bigl(-112st+24t+468+120s^2-420s\bigr)r^3+\bigl(30s^2+356t+540-375s\bigr)r^4\\
&+(252t+384-246s)r^5+(162-69s)r^6+36r^7+3r^8,\\
k_2(r,s,t)={}&\bigl(18+40s^2-135s+702t-224st+8s^2t-122t^2\bigr)\\
&+\bigl(126+294t^2+804t+120s^2-355s-336st\bigr)r\\
&+\bigl(-370s-120t+366-112st+120s^2\bigr)r^2+\bigl(40s^2+156t-330s+570\bigr)r^3\\
&+(-295s+378t+510)r^4+(258-115s)r^5+66r^6+6r^7,\\
k_3(r,s,t)={}&\bigl(-125s+40s^2+268t-168st+30+294t^2\bigr)\\
&+\bigl(-236s-80t+80s^2+176-112st\bigr)r\\
&+\bigl(-258s+156t+40s^2+414\bigr)r^2+(496-308s+504t)r^3\\
&+(314-161s)r^4+96r^5+10r^6,\\
k_4(r,s,t)={}&\bigl(36-61s-56st+34t+30s^2\bigr)+\bigl(172-148t-103s+30s^2\bigr)r\\
&+(-203s+378t+312)r^2+(264-161s)r^3+100r^4+12r^5,\\
k_5(r,s,t)={}&\bigl(-7s+10s^2+36-200t\bigr)+(252t+132-62s)r+(168-115s)r^2+84r^3+12r^4,\\
k_6(r,s,t)={}&(-13s+28+126t)+(72-69s)r+54r^2+10r^3,
\end{aligned}$$

$$k_7(r,s,t) = 16 - 23s + 24r + 6r^2,$$

$$k_8(r,s,t) = 7 + 3r,$$

$$k_9(r,s,t) = 1.$$

So Lemma 3.4 holds if we can prove $\widetilde{K}_\varepsilon(r,s,t,y) > 0$ for any fixed $y \in (0, r+1)$, $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10}, 1]$ and $s \in [0, \sqrt{3}\,]$.

Step 3. For any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10}, 1]$, we show that $\widetilde{K}_\varepsilon(r,s,t,y)$ is strictly decreasing with respect to s on $[0, \sqrt{3}\,]$.

By (B.11), we compute that

$$\begin{aligned} 72\frac{\partial K_\varepsilon}{\partial C} &= -207A^2 - 60B^2 - 504AD - 168BD + 180AC \\ &\quad + 468D^2 + 144CD + 168AB \end{aligned} \tag{B.18}$$

and

$$72\frac{\partial^2 K_\varepsilon}{\partial C^2} = 180A + 144D > 0.$$

By (B.12)–(B.16), we compute that

$$\begin{aligned} \frac{\partial^2 \widetilde{K}_\varepsilon}{\partial s^2} &= \frac{8}{\varepsilon^3 \gamma_\varepsilon^{12}(r+1-y)^3} \frac{\partial^2 K_\varepsilon}{\partial s^2} \\ &= \frac{8}{\varepsilon^3 \gamma_\varepsilon^{12}(r+1-y)^3} \left[\frac{\partial^2 K_\varepsilon}{\partial C^2} \left(\frac{\partial C}{\partial s} \right)^2 + \frac{\partial K_\varepsilon}{\partial C} \frac{\partial^2 C}{\partial s^2} \right] \\ &= \frac{1}{9\varepsilon^3 \gamma_\varepsilon^{12}(r+1-y)^3} (180A + 144D) \left[\varepsilon \gamma_\varepsilon^4 \left((r+1)^2 - y^2 \right) \right]^2 \\ &= \left\{ 20\left[(r+1)^3 + (r+1)^2 y + (r+1) y^2 + y^3 \right] + 18t \right\} (r+1+y)^2 > 0. \end{aligned}$$

This implies that, for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega$, $\widetilde{K}_\varepsilon(r,s,t,y)$ is concave up as a function of $s \in [0, \sqrt{3}\,]$. Hence $\widetilde{K}_\varepsilon(r,s,t,y)$ is strictly decreasing with respect to s on $[0, \sqrt{3}\,]$ if we can prove that

$$\frac{\partial \widetilde{K}_\varepsilon}{\partial s}(r, \sqrt{3}, t, y) < 0 \quad \text{for any } y \in (0, r+1),\ (r,t) \in \Omega. \tag{B.19}$$

By (B.12)–(B.15) and (B.18), we compute that

$$\begin{aligned} \frac{\partial \widetilde{K}_\varepsilon}{\partial s} &= \frac{8}{\varepsilon^3 \gamma_\varepsilon^{12}(r+1-y)^3} \frac{\partial K_\varepsilon}{\partial C} \frac{\partial C}{\partial s} \\ &= \frac{[-207A^2 - 60B^2 - 504AD - 168BD + 180AC + 468D^2 + 144CD + 168AB](r+1+y)}{9\varepsilon^2 \gamma_\varepsilon^8 (r+1-y)^2}, \end{aligned}$$

and

$$-\frac{\partial \widetilde{K}_\varepsilon}{\partial s}(r, \sqrt{3}, t, y) = (r+1+y)\sum_{j=0}^{6} g_j(r,t)y^j, \tag{B.20}$$

where

$$\begin{aligned}
g_0(r,t) &= 23r^6 + 82r^5 + (125 - 20\sqrt{3})r^4 + (140 - 80\sqrt{3} + 56t)r^3 + (145 - 120\sqrt{3} + 224t)r^2 \\
&\quad + (98 - 80\sqrt{3} + 280t - 16\sqrt{3}t)r + \big(27 - 20\sqrt{3} + 112t - 16\sqrt{3}t - 52t^2\big), \\
g_1(r,t) &= 46r^5 + 118r^4 + (132 - 40\sqrt{3})r^3 + (148 - 120\sqrt{3} + 56t)r^2 \\
&\quad + (142 - 120\sqrt{3} + 168t)r + (54 - 40\sqrt{3} + 112t - 16\sqrt{3}t), \\
g_2(r,t) &= 69r^4 + 108r^3 + (90 - 40\sqrt{3})r^2 + (132 - 80\sqrt{3} + 56t)r + (81 - 40\sqrt{3} + 112t), \\
g_3(r,t) &= 92r^3 + 108r^2 + (60 - 40\sqrt{3})r + (44 - 40\sqrt{3} + 56t), \\
g_4(r,t) &= 69r^2 + 26r + (17 - 20\sqrt{3}), \\
g_5(r,t) &= 46r - 10, \\
g_6(r,t) &= 23.
\end{aligned}$$

To prove (B.19), we claim that, for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega$,

$$\sum_{j=0}^{n} g_j(r,t)y^j \geqslant \left(\frac{y}{r+1}\right)^n \tilde{g}_n(r,t) > 0, \quad n = 0,1,2,3,4,5,6, \tag{B.21}$$

where

$$\tilde{g}_n(r,t) = \sum_{j=0}^{n} (r+1)^j g_j(r,t), \quad n = 0,1,2,3,4,5,6.$$

First, we compute that, $\tilde{g}_0(r,t) = g_0(r,t)$,

$$\begin{aligned}
\tilde{g}_1(r,t) &= 69r^6 + 246r^5 + (375 - 60\sqrt{3})r^4 + (420 - 240\sqrt{3} + 112t)r^3 \\
&\quad + (435 - 360\sqrt{3} + 448t)r^2 + (294 - 240\sqrt{3} + 560t - 32\sqrt{3}t)r \\
&\quad + \big(81 - 60\sqrt{3} + 224t - 32\sqrt{3}t - 52t^2\big), \\
\tilde{g}_2(r,t) &= 138r^6 + 492r^5 + (750 - 100\sqrt{3})r^4 + (840 - 400\sqrt{3} + 168t)r^3 \\
&\quad + (870 - 600\sqrt{3} + 672t)r^2 + (588 - 400\sqrt{3} + 840t - 32\sqrt{3}t)r \\
&\quad + \big(162 - 100\sqrt{3} + 336t - 32\sqrt{3}t - 52t^2\big), \\
\tilde{g}_3(r,t) &= 230r^6 + 876r^5 + (1410 - 140\sqrt{3})r^4 + (1480 - 560\sqrt{3} + 224t)r^3 \\
&\quad + (1290 - 840\sqrt{3} + 840t)r^2 + (780 - 560\sqrt{3} + 1008t - 32\sqrt{3}t)r \\
&\quad + \big(206 - 140\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2\big),
\end{aligned}$$

$$\begin{aligned}\tilde{g}_4(r,t) &= 299r^6 + 1178r^5 + (1945 - 160\sqrt{3})r^4 + (1980 - 640\sqrt{3} + 224t)r^3 \\ &\quad + (1565 - 960\sqrt{3} + 840t)r^2 + (874 - 640\sqrt{3} + 1008t - 32\sqrt{3}t)r \\ &\quad + \big(223 - 160\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2\big), \\ \tilde{g}_5(r,t) &= 345r^6 + 1398r^5 + (2355 - 160\sqrt{3})r^4 + (2340 - 640\sqrt{3} + 224t)r^3 \\ &\quad + (1695 - 960\sqrt{3} + 840t)r^2 + (870 - 640\sqrt{3} + 1008t - 32\sqrt{3}t)r \\ &\quad + \big(213 - 160\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2\big), \\ \tilde{g}_6(r,t) &= 368r^6 + 1536r^5 + (2700 - 160\sqrt{3})r^4 + (2800 - 640\sqrt{3} + 224t)r^3 \\ &\quad + (2040 - 960\sqrt{3} + 840t)r^2 + (1008 - 640\sqrt{3} + 1008t - 32\sqrt{3}t)r \\ &\quad + \big(236 - 160\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2\big).\end{aligned}$$

As regarded as a polynomial of r, the coefficients of $\tilde{g}_n(r,t)$ are all positive for $n \in \{0,1,2,3,4,5,6\}$, where $t \in [\frac{7}{10}, 1]$. So for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega$,

$$\tilde{g}_n(r,t) > 0, \quad n = 0,1,2,3,4,5,6. \tag{B.22}$$

Suppose (B.21) holds for $n = l$ where $l \in \{0,1,2,3,4,5\}$. By (B.22) and since $0 < y < r+1$, we have

$$\begin{aligned}\sum_{j=0}^{l+1} g_j(r,t)y^j &= \sum_{j=0}^{l} g_j(r,t)y^j + g_{l+1}(r,t)y^{l+1} \\ &\geqslant \left[\left(\frac{y}{r+1}\right)^l \tilde{g}_l(r,t)\right]\left(\frac{y}{r+1}\right) + g_{l+1}(r,t)y^{l+1} \\ &= \left(\frac{y}{r+1}\right)^{l+1}\left[\sum_{j=0}^{l}(r+1)^j g_j(r,t)\right] + \left(\frac{y}{r+1}\right)^{l+1}(r+1)^{l+1}g_{l+1}(r,t) \\ &= \left(\frac{y}{r+1}\right)^{l+1}\sum_{j=0}^{l+1}(r+1)^j g_j(r,t) \\ &= \left(\frac{y}{r+1}\right)^{l+1}\tilde{g}_{l+1}(r,t).\end{aligned} \tag{B.23}$$

So (B.21) holds for $n = l+1$ where $l \in \{0,1,2,3,4,5\}$. By (B.20)–(B.23), we obtain that

$$\frac{\partial \widetilde{K}_\varepsilon}{\partial s}(r, \sqrt{3}, t, y) = -(r+1+y)\sum_{j=0}^{6} g_j(r,t)y^j < 0 \quad \text{for any } y \in (0, r+1),\ (r,t) \in \Omega,$$

and hence (B.19) holds.

So for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10}, 1]$, $\widetilde{K}_\varepsilon(r,s,t,y)$ is strictly decreasing with respect to s on $[0, \sqrt{3}]$.

Step 4. We show that $\widetilde{K}_\varepsilon(r,s,t,y) > 0$ for any fixed $y \in (0, r+1)$, $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10}, 1]$ and $s \in [0, \sqrt{3}]$.

By Step 3, Step 4 holds if we can prove that

$$\widetilde{K}_\varepsilon(r, \sqrt{3}, t, y) > 0 \quad \text{for any } y \in (0, r+1),\ (r, t) \in \Omega. \tag{B.24}$$

By (B.17),

$$\widetilde{K}_\varepsilon(r, \sqrt{3}, t, y) = \sum_{j=0}^{9} h_j(r, t) y^j, \tag{B.25}$$

where

$$\begin{aligned}
h_0(r,t) = {} & r^9 + 13r^8 + (66 - 23\sqrt{3})r^7 + (182 - 105\sqrt{3} + 126t)r^6 + (338 - 207\sqrt{3} + 304t)r^5 \\
& + (486 - 265\sqrt{3} + 190t - 56\sqrt{3}t)r^4 + \big(538 - 285\sqrt{3} + 240t - 280\sqrt{3}t + 294t^2\big)r^3 \\
& + \big(406 - 243\sqrt{3} + 754t - 504\sqrt{3}t + 466t^2\big)r^2 + \big(177 - 125\sqrt{3} + 784t - 392\sqrt{3}t \\
& + 50t^2 + 52\sqrt{3}t^2\big)r + \big(33 - 27\sqrt{3} + 258t - 112\sqrt{3}t + 52\sqrt{3}t^2 - 122t^2 - 16t^3\big), \\
h_1(r,t) = {} & 3r^8 + 36r^7 + (162 - 69\sqrt{3})r^6 + (384 - 246\sqrt{3} + 252t)r^5 \\
& + (630 - 375\sqrt{3} + 356t)r^4 + (828 - 420\sqrt{3} + 24t - 112\sqrt{3}t)r^3 \\
& + \big(786 - 435\sqrt{3} + 456t - 448\sqrt{3}t + 294t^2\big)r^2 + \big(432 - 294\sqrt{3} + 1052t \\
& - 560\sqrt{3}t + 172t^2\big)r + \big(99 - 81\sqrt{3} + 516t - 224\sqrt{3}t + 52\sqrt{3}t^2 - 122t^2\big), \\
h_2(r,t) = {} & 6r^7 + 66r^6 + (258 - 115\sqrt{3})r^5 + (510 - 295\sqrt{3} + 378t)r^4 + (690 - 330\sqrt{3} + 156t)r^3 \\
& + (726 - 370\sqrt{3} - 120t - 112\sqrt{3}t)r^2 + \big(486 - 355\sqrt{3} + 804t - 336\sqrt{3}t + 294t^2\big)r \\
& + \big(138 - 135\sqrt{3} + 726t - 224\sqrt{3}t - 122t^2\big), \\
h_3(r,t) = {} & 10r^6 + 96r^5 + (314 - 161\sqrt{3})r^4 + (496 - 308\sqrt{3} + 504t)r^3 \\
& + (534 - 258\sqrt{3} + 156t)r^2 + (416 - 236\sqrt{3} - 112\sqrt{3}t - 80t)r \\
& + \big(150 - 125\sqrt{3} + 268t - 168\sqrt{3}t + 294t^2\big), \\
h_4(r,t) = {} & 12r^5 + 100r^4 + (264 - 161\sqrt{3})r^3 + (312 - 203\sqrt{3} + 378t)r^2 \\
& + (262 - 103\sqrt{3} - 148t)r + (126 - 61\sqrt{3} + 34t - 56\sqrt{3}t), \\
h_5(r,t) = {} & 12r^4 + 84r^3 + (168 - 115\sqrt{3})r^2 + (132 - 62\sqrt{3} + 252t)r + (66 - 7\sqrt{3} - 200t), \\
h_6(r,t) = {} & 10r^3 + 54r^2 + (72 - 69\sqrt{3})r + (28 - 13\sqrt{3} + 126t), \\
h_7(r,t) = {} & 6r^2 + 24r + (16 - 23\sqrt{3}), \\
h_8(r,t) = {} & 3r + 7, \\
h_9(r,t) = {} & 1.
\end{aligned}$$

By applying the same arguments used in Step 3, we can prove that, for any fixed $y \in (0, r+1)$ and $(r, t) \in \Omega$,

$$\sum_{j=0}^{n} h_j(r,t) y^j \geqslant \left(\frac{y}{r+1}\right)^n \tilde{h}_n(r,t) > 0, \quad n = 0, 1, 2, \ldots, 9, \tag{B.26}$$

where

$$\tilde{h}_n(r,t)=\sum_{j=0}^{n}(r+1)^j h_j(r,t),\quad n=0,1,2,\ldots,9.$$

By (B.25) and (B.26), we obtain that

$$\widetilde{K}_\varepsilon(r,\sqrt{3},t,y)=\sum_{j=0}^{9}h_j(r,t)y^j>0\quad\text{for any } y\in(0,r+1),\ (r,t)\in\Omega,$$

and hence (B.24) holds.

By Step 3 and (B.24), $\widetilde{K}_\varepsilon(r,s,t,y)>0$ for any fixed $y\in(0,r+1)$, $(r,t)\in\Omega\equiv[0,\infty)\times[\frac{7}{10},1]$ and $s\in[0,\sqrt{3}\,]$.

Step 5. Finally, we complete the proof of this lemma by the above analyzes.

By Step 4 and (B.16), we have $K_\varepsilon(\alpha,u)>0$ for any fixed $\varepsilon\in[(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2},(\frac{\sigma^3}{27\rho})^{1/2}]$, $\alpha\in[\gamma_\varepsilon,\beta_\varepsilon)$ and $0<u<\alpha$. So by (B.1), we obtain that, for any fixed $\varepsilon\in[(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2},(\frac{\sigma^3}{27\rho})^{1/2}]$,

$$G'_\varepsilon(\alpha)=\frac{1}{\sqrt{\alpha}}\int_0^\alpha\frac{K_\varepsilon(\alpha,u)}{[\triangle F_\varepsilon]^{7/2}}\,du>0\quad\text{on }[\gamma_\varepsilon,\beta_\varepsilon).$$

The proof of Lemma 3.4 is complete. □

References

[1] K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, Trans. Amer. Math. Soc., in press.
[2] K.-C. Hung, S.-H. Wang, A theorem on S-shaped bifurcation curve for a positone problem with convex–concave nonlinearity and its applications to the perturbed Gelfand problem, J. Differential Equations 251 (2011) 223–237.
[3] P. Korman, Y. Li, T. Ouyang, Exact multiplicity results for boundary value problems with nonlinearities generalizing cubic, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996) 599–616.
[4] P. Korman, Y. Li, T. Ouyang, Computing the location and the direction of bifurcation, Int. Math. Res. Lett. 12 (2005) 933–944.
[5] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J. 20 (1970) 1–13.
[6] J. Shi, Persistence and bifurcation of degenerate solutions, J. Funct. Anal. 169 (1999) 494–531.
[7] J. Shi, Multi-parameter bifurcation and applications, in: H. Brezis, K.C. Chang, S.J. Li, P. Rabinowitz (Eds.), ICM2002 Satellite Conference on Nonlinear Functional Analysis: Topological Methods, Variational Methods and Their Applications, World Scientific, Singapore, 2003, pp. 211–222.
[8] J. Smoller, A. Wasserman, Global bifurcation of steady-state solutions, J. Differential Equations 39 (1981) 269–290.
[9] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, J. Differential Equations 77 (1989) 199–202.