Ali et al. *Boundary Value Problems* (2017) 2017:16
DOI 10.1186/s13661-017-0749-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Existence and stability of solution to a toppled systems of differential equations of non-integer order

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Abstract

The aim of this paper is developing conditions that guarantee the existence of a solution to a toppled system of differential equations of noninteger order with fractional integral boundary conditions where the nonlinear functions involved in the considered system are continuous and satisfy some growth conditions. We convert the system of differential equations to a system of fixed point problems for condensing mapping. With the help of techniques of the topological degree theory, we establish adequate conditions that ensure the existence and uniqueness of positive solutions to a toppled system under consideration. Moreover, some conditions are also developed for the Hyers-Ullam stability of the solution to the system under consideration. Finally, to demonstrate the obtained results, we provide an example.

MSC: 34A08; 35R11

Keywords: coupled systems; integral boundary conditions; non-integer order differential equations; Hyers-Ullam stability; topological degree method

1 Introduction

In last few decades, the area devoted to the study of classical and partial differential equations of arbitrary order has received much attention due to the fact that they exactly describe many nonlinear phenomenons in different disciplines of applied sciences like physics, chemistry, biology, viscoelasticity, control theory, fluid dynamics, hydrodynamics, aerodynamics, computer networking, signal and image processing, and so on. The interest in the study of differential equations of noninteger order is due to the fact that the models involve fractional-order derivatives are more realistic and accurate as compared to the models that involve classical derivatives. For the respective applications, we refer to [1–8]. Therefore, researchers study various aspects of the differential equations of arbitrary order. One of the most important area is devoted to the existence theory, which has been very well studied, and plenty of research articles and books are available on it; we refer, for example, to [9–15]. In all these articles, sufficient conditions for the existence of solutions to the corresponding problems are obtained by using classical fixed point theory. Using the classical fixed point theory often required stronger conditions, which shorten the applicability to a small number of applied problems and very particular systems of boundary value problems (BVPs). To use less restrictive conditions in order to extend tools to more classes of BVPs, researchers need to look for some other refined tools of functional analysis. One of the powerful tools is the degree approach of topology. The topological degree method proved to be a dominant tool in the study of many mathematical models that arise in applied nonlinear analysis. The concerned method, also called the prior estimate method, has been used by many researchers of mathematics to study the existence of solutions to both nonlinear ordinary and nonlinear partial differential equations. Mawhim [16] used the aforesaid method coupled with the degree theory of condensing mapping and showed that, under special assumptions, the problems

$$\begin{cases} u''(t) + \theta(t, u(t), u'(t)) = 0, & t \in I = [0, \pi], \\ u(t)|_{t=0} = u(t)|_{t=\pi} = 0, \end{cases}$$

and

$$\begin{cases} u''(t) + \theta(t, u(t)) = 0, & 0 \leq t \leq 1, \\ u(t)|_{t=0} = u(t)|_{t=1}, \end{cases}$$

admit solutions. Isaia [17], used the technique of degree method to develop some conditions that ensure the existence of solutions to the following nonlinear integral equations:

$$u(t) = \mathcal{H}\big(t, u(t)\big) + \int_a^b \mathcal{K}\big(t, s, u(s)\big)\, ds, \quad t \in [a, b],$$

where $\mathcal{H} : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{K} : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continues functions obeying some growth conditions.

In the same fashion, with the help of degree theory, Wang et al. [18] studied the existence and uniqueness of solutions to a class of nonlocal Cauchy problems given by

$$\begin{cases} D^q u(t) - \theta(t, u(t)) = 0, & 0 \leq t \leq T, \\ u(t)|_{t=0} + u_0 = g(u), \end{cases}$$

where D is the Caputo fractional derivative of order $q \in (0, 1]$, $u_0 \in \mathbb{R}$, and $\theta \in ([0, T] \times \mathbb{R}, \mathbb{R})$. The same problem was studied subject to multipoint boundary value problems by Khan and Shah [19]

$$\begin{cases} {}^cD^q u(t) - \theta(t, u(t)) = 0, & q \in (1, 2], 0 \leq t \leq 1, \\ u(t)|_{t=0} = g(u), \quad u(t)|_{t=1} = h(u) + \sum_{i=1}^{m-2} \lambda_i u(\eta_i), & \lambda_i, \eta_i \in (0, 1), \end{cases}$$

where $g, h : ([0, 1], \mathbb{R}) \to \mathbb{R}$ and $\theta : [0, 1] \times \mathbb{R} \to \mathbb{R}$. In the same line, by using topological degree theory Shah and Khan [20] studied the toppled system with nonlinear conditions provided as

$$\begin{cases} {}^cD^p u(t) - \theta_1(t, u(t), v(t)) = 0, \quad {}^cD^q v(t) - \theta_2(t, u(t), v(t)) = 0, \quad 0 \leq t \leq 1, \\ \lambda_1 u(t)|_{t=0} - \gamma_1 u(t)|_{t=\eta} - \mu_1 u(t)|_{t=1} - \phi(u) = 0, \\ \lambda_2 v(t)|_{t=0} - \gamma_2 v(t)|_{t=\xi} - \mu_2 v(t)|_{t=1} - \psi(v) = 0, \end{cases}$$

where $p, q, \eta, \xi \in (0,1)$ and $\theta_1, \theta_2 \in [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, and the nonlocal functions $\phi, \psi : [0,1] \times \mathbb{R} \to \mathbb{R}$. Inspired by the aforesaid work, we use the coincidence degree method to study the following toppled system with fractional integral boundary conditions of the form

$$\begin{cases} {}^cD^p u(t) = \theta_1(t, v(t)), & t \in J = [0,1], \\ {}^cD^q v(t) = \theta_2(t, u(t)), & t \in J = [0,1], \\ u(t)|_{t=0} = 0, & u(t)|_{t=1} = {}_0I_T^\gamma g(u) = \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g(u(s))\,ds, \\ v(t)|_{t=0} = 0, & v(t)|_{t=1} = {}_0I_T^\delta h(v) = \frac{1}{\Gamma(\delta)} \int_0^T (T-s)^{\delta-1} h(v(s))\,ds, \end{cases} \tag{1}$$

where $p, q, \delta, \gamma \in (1,2]$, ${}^cD$ denotes the Caputo fractional derivative, $g, h \in L[0,1]$ are boundary functions, and the nonlinear functions $\theta_1, \theta_2 : J \times \mathbb{R} \to \mathbb{R}$. We develop sufficient conditions for the existence and uniqueness and also study the Hyers-Ullam stability of the system. The study of stability for fractional-order system is quite recent. There are various methods available in the literature to study stability, one of which is the Lyapunov method that suffers from the difficulties of derivation of Lyapunov functions. Another important method to obtain stability for such a system is the Hyers-Ullam stability introduced by Ullam [21] in 1940, which was answered by Hyers [22] in 1941. Wang [23] was the first mathematician who investigated the Hyers-Ullam stability for the impulsive ordinary differential equations in 2012. In the same line, he also obtained the aforesaid stability for the evolution equations [24]. For more detail about the Hyers-Ullam stability, we refer to [25–27]. Urs [28], studied the Hyers-Ullam stability to the following system of periodic boundary value problems:

$$\begin{cases} u''(t) - \theta_1(t, u(t)) = \theta_2(t, v(t)), \\ v''(t) - \theta_1(t, v(t)) = \theta_2(t, u(t)), \\ u(t)|_{t=0} = u(t)_{t=T}, \qquad v(t)_{t=0} = v(t)_{t=T}. \end{cases}$$

We study the Hyers-Ullam stability to the toppled system (1) under consideration. We also give an example to verify the applicability of our results.

2 Background materials and auxiliary results

This section is concerned with some basic definitions, important theorems of general calculus, and topological degree theory; see [9, 10, 16, 29–31].

Definition 2.1 The noninteger-order integral of a function $\theta \in L^1([a,b], \mathbb{R})$ is provided as

$$I_{0+}^p \theta(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} \theta(s)\,ds.$$

Definition 2.2 The noninteger-order derivative in the Caputo sense of a function θ over the interval $[a,b]$ is defined as

$$D_{0+}^p \theta(t) = \frac{1}{\Gamma(m-p)} \int_a^t (t-s)^{m-p-1} \theta^{(m)}(s)\,ds,$$

where $m = [p] + 1$, and $[p]$ is the integer part of p.

Theorem 2.3 *The general solution to the differential equation of fractional order*

$$I^p\big[D^p\theta(t)\big] = y(t), \quad n-1 < q < n,$$

is given by

$$I^p\big[D^p\theta(t)\big] = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1}t^{m-1}$$

for arbitrary $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,m-1$.

The spaces for all continuous functions $u, v : [0,1] \to \mathbb{R}$ are denoted by $U = C([0,1],\mathbb{R})$, $V = C([0,1],\mathbb{R})$. Under the topological norms, they are Banach spaces. The respective norms can be defined as $\|u\| = \sup\{|u(t)| : 0 \leq t \leq 1\}$ and $\|v\| = \sup\{|v(t)| : 0 \leq t \leq 1\}$. Moreover, $\mathcal{E} = U \times V$ is also a Banach space under the norms $\|(u,v)\| = \|u\| + \|v\|$ and $|(u,v)| = \max\{\|u\|,\|v\|\}$.

Consider the class of all bounded set of $P(\mathcal{E})$ denoted by $\mathbb{B}$. Then, we review the following notions of [30].

Definition 2.4 The Kuratowski measure of noncompactness is the mapping $\varrho : \mathbb{B} \to (0,\infty)$ defined by

$$\varrho(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter} \leq d\}$$

for $B \in \mathbb{B}$.

Proposition 2.1 *The measure defined in Definition* 2.4 *for* ϱ *satisfies the following characteristics*:

(i) *For a relative compact B, the Kuratowski measure* $\varrho(B) = 0$;
(ii) ϱ *is a seminorm, that is,* $\varrho(\lambda B) = |\lambda|\varrho(B)$, $\lambda \in \mathbb{R}$, *and* $\varrho(B_1 + B_2) \leq \varrho(B_1) + \varrho(B_2)$;
(iii) $B_1 \subset B_2$ *yields* $\varrho(B_1) \leq \varrho(B_2)$; $\varrho(B_1 \cup B_2) = \sup\{\varrho(B_1), \varrho(B_2)\}$;
(iv) $\varrho(\operatorname{conv} B) = \varrho(B)$;
(v) $\varrho(\overline{B}) = \varrho(B)$.

Definition 2.5 Assume that $\mathbb{F} : \Theta \to U$ is a bounded and continuous map such that $\Theta \subset U$. Then $\mathbb{F}$ is ϱ-Lipschitz with $\mathbb{K} \geq 0$ such that

$$\varrho\big(\mathbb{F}(B)\big) \leq \mathbb{K}\varrho(B) \quad \text{for all bounded } B \subset \Theta.$$

Also, $\mathbb{F}$ is a strict ϱ-contraction if $\mathbb{K} < 1$.

Definition 2.6 A function $\mathbb{F}$ is ϱ-condensing if

$$\varrho\big(\mathbb{F}(B)\big) < \varrho(B) \quad \text{for all bounded } B \subset \Theta \text{ with } \varrho(B) > 0.$$

In other words, $\varrho(\mathbb{F}(B)) \geq \varrho(B)$ yields $\varrho(B) = 0$.

We denote the family of all strict ϱ-contractions $\mathbb{F} : \Theta \to U$ by $\vartheta C_\varrho(\Theta)$ and the family of all ϱ-condensing mappings $\mathbb{F} : \Theta \to U$ by $C_\varrho(\Theta)$.

Remark 1 $\vartheta C_\varrho(\Theta) \subset C_\varrho(\Theta)$, and every $\mathbb{F} \in C_\varrho(\Theta)$ is ϱ-Lipschitz such that $\mathbb{K} = 1$.

Further, we recall that $\mathbb{F} : \Theta \to U$ is Lipschitz in the presence of $\mathbb{K} > 0$ such that

$$\left\|\mathbb{F}(u) - \mathbb{F}(\bar{u})\right\| \le \mathbb{K}\|u - \bar{u}\| \quad \text{for all } u, \bar{u} \in \Theta.$$

Also, under the condition $\mathbb{K} < 1$, $\mathbb{F}$ is a strict contraction. We recall the following propositions in [15].

Proposition 2.2 *The sum of the two operators* $\mathbb{F}, \mathbb{G} : \Theta \to U$ *is* ϱ*-Lipschitz with constants* $\mathbb{K}_1 + \mathbb{K}_2$ *if and only if* $\mathbb{F}, \mathbb{G} : \Theta \to U$ *are* ϱ*-Lipschitz with constants* $\mathbb{K}_1$, $\mathbb{K}_2$, *respectively.*

Proposition 2.3 *The mapping* $\mathbb{F}$ *is* ϱ*-Lipschitz with constant* $\mathbb{K} = 0$ *if and only if* $\mathbb{F} : \Theta \to U$ *is compact.*

Proposition 2.4 *The operator* $\mathbb{F}$ *is* ϱ*-Lipschitz with constant* $\mathbb{K}$ *if and only if* $\mathbb{F} : \Theta \to U$ *is Lipschitz with constant* $\mathbb{K}$.

The next theorem of Isaia [17] plays an important role in achieving our main result.

Theorem 2.7 *Let* $\mathbb{F} : \mathcal{E} \to \mathcal{E}$ *be a* ϱ*-contraction, and*

$$\Psi = \{z \in \mathcal{E} : \textit{there exists } 0 \le \lambda \le 1 \textit{ such that } z = \lambda\mathbb{F}z\}.$$

If $\Psi \subset \mathcal{E}$ *is a bounded set with* $r > 0$ *such that* $\Psi \subset B_r(0)$, *then the degree*

$$\deg\bigl(I - \lambda\mathbb{F}, B_r(0), 0\bigr) = 1 \textit{ for every } \lambda \in [0,1].$$

Therefore, $\mathbb{F}$ *has at least one fixed point* (*which is the corresponding solution of the considered system*), *and the set of the fixed points of* $\mathbb{F}$ *lies in* $B_r(0)$.

The following assumptions are needed in the sequel.

(C_1) The nonlocal functions g, h for $x, u, y, v, \in \mathbb{R}$ satisfy

$$\left|g(x) - g(u)\right| \le \mathbb{K}_g|x - u|, \qquad \left|h(y) - h(v)\right| \le \mathbb{K}_h|y - v| \quad \text{with } \mathbb{K}_g, \mathbb{K}_h \in [0,1).$$

(C_2) With the given constants $C_g, C_h, M_g, M_h > 0$, and $q_1 \in [0,1)$, for $u, v \in \mathbb{R}$, the nonlocal functions g, h satisfy the following growth conditions:

$$\left|g(u)\right| \le C_g|u|^{q_1} + M_g, \qquad \left|h(v)\right| \le C_h|v|^{q_1} + M_h.$$

(C_3) In the presence of constants a, b, M_{θ_1}, M_{θ_2}, and $q_2 \in [0,1)$, for $u, v \in \mathbb{R}$, the nonlinear functions θ_1, θ_2 satisfy the following growth conditions:

$$\left|\theta_1(t, v)\right| \le a|v|^{q_2} + M_{\theta_1},$$
$$\left|\theta_2(t, u)\right| \le b|u|^{q_2} + M_{\theta_2}.$$

(C_4) There exist positive constants L_{θ_1}, L_{θ_2} such that, for all $x, u, y, v \in \mathbb{R}$,

$$\begin{aligned} \bigl|\theta_1(t,y) - \theta_1(t,v)\bigr| &\leq L_{\theta_1}|y - v|, \\ \bigl|\theta_2(t,x) - \theta_2(t,u)\bigr| &\leq L_{\theta_2}|x - u|. \end{aligned}$$

3 Main results

The current section is concerned to establish adequate conditions to the toppled system (1).

Theorem 3.1 *Let $\theta : J \to \mathbb{R}$ be a p times integrable function. Then the solution of the linear boundary value problem*

$$\begin{aligned} &D^p u(t) = \theta(t), \quad t \in J = [0,1], p \in (1,2], \\ &u(t)|_{t=0} = 0, \qquad u(t)|_{t=1} = {}_0 I_T^{\gamma} g(u) = \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g\bigl(u(s)\bigr)\, ds, \end{aligned} \tag{2}$$

is provided by

$$u(t) = \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g\bigl(u(s)\bigr)\, ds + \int_0^1 G_p(t,s)\theta(s)\, ds, \quad t \in [0,1], \tag{3}$$

where $G_p(t,s)$ is the Green's function given by

$$G_p(t,s) = \frac{1}{\Gamma(p)} \begin{cases} (t-s)^{p-1} - t(1-s)^{p-1}, & 0 \leq s \leq t \leq 1, \\ t(1-s)^{p-1}, & 0 \leq s \leq t \leq 1. \end{cases} \tag{4}$$

Proof Applying I^p on $D^p u(t) = \theta(t)$, by Theorem 2.3 we get

$$u(t) = c_0 + c_1 t + I^p \theta(t), \tag{5}$$

where c_0, c_1 are real constants. The conditions $u(t)_{t=0} = 0$, $u(t)_{t=1} = {}_0 I_T^{\gamma} g(u)$ yield that $c_0 = 0$ and $c_1 = {}_0 I_T^{\gamma} g(u) - I^p \theta(1)$. Hence, we get

$$\begin{aligned} u(t) &= t\bigl[{}_0 I_T^{\gamma} g(u) - I^p \theta(1)\bigr] + I^p \theta(t), \\ &= \frac{t}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g\bigl(u(s)\bigr)\, ds - \frac{t}{\Gamma(p)} \int_0^1 (1-s)^{p-1} \theta(s)\, ds \\ &\quad + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \theta(s)\, ds, \end{aligned}$$

which after rearranging can be written as

$$u(t) = \frac{t}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g\bigl(u(s)\bigr)\, ds + \int_0^1 G_p(t,s)\theta(s)\, ds.$$

□

By Theorem 3.1 the corresponding toppled system of Hammerstein-type integral equations to the toppled systems (1) is provided by

$$\begin{cases} u(t) = \frac{t}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g(u(s))\,ds + \int_0^1 G_p(t,s)\theta_1(s,v(s))\,ds, \\ v(t) = \frac{t}{\Gamma(\delta)} \int_0^T (T-s)^{\delta-1} h(v(s))\,ds + \int_0^1 G_q(t,s)\theta_2(s,u(s))\,ds, \end{cases} \tag{6}$$

where $G_q(t,s)$ is defined by

$$G_q(t,s) = \frac{1}{\Gamma(q)} \begin{cases} (t-s)^{q-1} - t(1-s)^{q-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{q-1}, & 0 \le s \le t \le 1. \end{cases} \tag{7}$$

Clearly,

$$\max_{t\in J} \left|G_p(t,s)\right| = \frac{(1-s)^{p-1}}{\Gamma(p)}, \qquad \max_{t\in J} \left|G_q(t,s)\right| = \frac{(1-s)^{q-1}}{\Gamma(q)}, \quad s \in J. \tag{8}$$

Define the operators $\mathbb{F}_1 : U \to U$, $\mathbb{F}_2 : V \to V$ by

$$\mathbb{F}_1(u)(t) = \frac{t}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} g\bigl(u(s)\bigr)\,ds,$$
$$\mathbb{F}_2(v)(t) = \frac{t}{\Gamma(\delta)} \int_0^T (T-s)^{\delta-1} h\bigl(v(s)\bigr)\,ds$$

and $\mathbb{G}_1, \mathbb{G}_2 : \mathcal{E} \to \mathcal{E}$ by

$$\mathbb{G}_1(v)(t) = \int_0^1 G_p(t,s)\theta_1\bigl(s,v(s)\bigr)\,ds, \qquad \mathbb{G}_2(u)(t) = \int_0^1 \mathbb{G}_q(t,s)\theta_2\bigl(s,u(s)\bigr)\,ds.$$

Hence, we have $\mathbb{F}(u,v) = (\mathbb{F}_1, \mathbb{F}_2)(u,v)$, $\mathbb{G}(u,v) = (\mathbb{G}_1, \mathbb{G}_2)(u,v)$, and $\mathbb{T}(u,v) = \mathbb{F}(u,v) + \mathbb{G}(u,v)$. So, the equivalent operator equation to the toppled system of Hammerstein-type integral equations (6) is given by

$$(u,v) = \mathbb{T}(u,v) = \mathbb{F}(u,v) + \mathbb{G}(u,v). \tag{9}$$

Thus the solutions of system (6) are the fixed points of operator equation (9).

Theorem 3.2 *In view of hypotheses* (C_1) *and* (C_2), *the operator* $\mathbb{F}$ *is Lipschitz and satisfies the growth condition given by*

$$\left\|\mathbb{F}(u,v)\right\| \le C_{\mathbb{F}} \left\|(u,v)\right\|^{q_1} + M_{\mathbb{F}} \quad \textit{for all } (u,v) \in \mathcal{E}. \tag{10}$$

Proof Thanks to hypothesis (C_1), we obtain

$$\begin{aligned} \left|\mathbb{F}_1(u)(t) - \mathbb{F}_1(\bar{u})(t)\right| &= \left|\frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \bigl[g(u) - g(\bar{u})\bigr]\,ds\right| \\ &\le \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} \left|g(u) - g(\bar{u})\right|\,ds, \end{aligned}$$

which implies that $\|\mathbb{F}_1(u) - \mathbb{F}_1(\bar{u})\| \leq \overline{K}_g\|u - \overline{u}\|$,

where $\overline{K}_g = \dfrac{K_g T^\gamma}{\Gamma(\gamma+1)} \in [0,1)$.

To obtain growth condition, consider

$$|\mathbb{F}_1(u)(t)| = \left|\frac{1}{\Gamma(\gamma)}\int_0^T (T-s)^{\gamma-1} g(u(s))\,ds\right| \leq \frac{1}{\Gamma(\gamma)}\int_0^T (T-s)^{\gamma-1}|g(u(s))|\,ds, \tag{11}$$

which implies that

$$\|\mathbb{F}_1 u\|_c \leq \frac{T^\gamma}{\Gamma(\gamma+1)}\left[C_g\|u\|_c^{q_1} + M_g\right].$$

Similarly, we have

$$\|\mathbb{F}_2 u\|_c \leq \frac{T^\delta}{\Gamma(\delta+1)}\left[C_h\|v\|_c^{q_1} + M_h\right].$$

Now

$$\begin{aligned}\|\mathbb{F}(u,v)\|_c &\leq \frac{T^\gamma}{\Gamma(\gamma+1)}\left[C_g\|u\|_c^{q_1} + M_g\right] + \frac{T^\delta}{\Gamma(\delta+1)}\left[C_h\|v\|_c^{q_1} + M_h\right] \\ &\leq \left(\frac{T^\gamma}{\Gamma(\gamma+1)}C_g\|u\|_c^{q_1} + \frac{T^\delta}{\Gamma(\delta+1)}C_h\|v\|_c^{q_1}\right) + \left(\frac{T^\gamma M_g}{\Gamma(\gamma+1)} + \frac{T^\delta M_h}{\Gamma(\delta+1)}\right) \\ &\leq C_{\mathbb{F}}\left[\|u\|^{q_1} + \|v\|^{q_1}\right] + M_{\mathbb{F}} = C_{\mathbb{F}}\|(u,v)\|^{q_1} + M_{\mathbb{F}},\end{aligned}$$

where

$$\max\left\{\frac{T^\gamma}{\Gamma(\gamma+1)}C_g, \frac{T^\delta}{\Gamma(\delta+1)}C_h\right\} = C_{\mathbb{F}} \quad \text{and} \quad M_{\mathbb{F}} = \frac{T^\gamma M_g}{\Gamma(\gamma+1)} + \frac{T^\delta M_h}{\Gamma(\delta+1)}. \qquad \square$$

Theorem 3.3 *In view of hypothesis* (C_3), *the operator* $\mathbb{G}$ *is continuous and satisfies the growth condition given by*

$$\|\mathbb{G}(u,v)\| \leq \Omega\|(u,v)\|^{q_2} + \Upsilon \quad \textit{for each } (u,v) \in \mathcal{E}, \tag{12}$$

where $\Omega = \eta(a+b)$, $\eta = \max\{\frac{1}{\Gamma(p+1)}, \frac{1}{\Gamma(q+1)}\}$, $\Upsilon = \eta(M_{\theta_1} + M_{\theta_2})$.

Proof Consider the bounded set $\mathbb{B}_r = \{(u,v) \in \mathcal{E} : \|(u,v)\| \leq r\}$ with a sequence (u_n, v_n) converging to (u,v) in $\mathbb{B}_r$. We have to show that $\|\mathbb{G}(u_n,v_n) - \mathbb{G}(u,v)\| \to 0$ as $n \to \infty$. Let us take

$$\begin{aligned}|(\mathbb{G}_1(v_n) - \mathbb{G}_1(v))(t)| \leq \frac{t}{\Gamma(p)}\bigg[&\int_0^t (t-s)^{p-1}|\theta_1(s,v_n(s)) - \theta_1(s,v(s))|\,ds \\ &+ \int_0^1 (1-s)^{p-1}|\theta_1(s,v_n(s)) - \theta_1(s,v(s))|\,ds\bigg].\end{aligned}$$

The continuity of θ_1 yields that $\theta_1(s,v_n(s)) \to \theta_1(s,v(s))$ as $n \to \infty$. Thanks to the Lebesgue dominated convergence theorem, we have $\int_0^t (t-s)^{p-1}|\theta_1(s,v_n) - \theta_1(s,v)|\,ds \to 0$ as $n \to \infty$.

Similarly, the same result can be also proved for the other terms. This implies that

$$\|\mathbb{G}_1(v_n)(t) - \mathbb{G}_1(v)(t)\| \to 0 \quad \text{as } n \to \infty, \tag{13}$$

and in the same way, we can show that

$$\|\mathbb{G}_2(u_n)(t) - \mathbb{G}_2(u)(t)\| \to 0 \quad \text{as } n \to \infty. \tag{14}$$

Thus, from (13) and (14) it follows that

$$\|\mathbb{G}(u_n, v_n)(t) - \mathbb{G}(u,v)(t)\| \to 0 \quad \text{as } n \to \infty.$$

To compute (12) for $\mathbb{G}$, using hypothesis (C_3) and (8), we get

$$\left|\mathbb{G}_1(v)(t)\right| = \left|\int_0^1 \mathbb{G}_p(t,s)\theta_1\big(s, v(s)\big)\,ds\right| \le \frac{1}{\Gamma(p+1)}\big(a\|v\|^{q_2} + M_{\theta_1}\big)$$

and

$$\left|\mathbb{G}_2(u)(t)\right| = \left|\int_0^1 \mathbb{G}_q(t,s)\theta_2\big(s, u(s)\big)\,ds\right| \le \frac{1}{\Gamma(q+1)}\big(b\|u\|^{q_2} + M_{\theta_2}\big).$$

Therefore, we get

$$\begin{aligned}
\|\mathbb{G}(u,v)\| &= \|\mathbb{G}_1(v)\| + \|\mathbb{G}_2(u)\| \\
&\le \eta\big(a\|v\|^{q_2} + M_{\theta_1}\big) + \eta\big(b\|u\|^{q_2} + M_{\theta_2}\big) \\
&\le \eta(a+b)\big(\|v\|^{q_2} + \|u\|^{q_2}\big) + \eta(M_{\theta_1} + M_{\theta_2}) = \Omega\|(u,v)\| + \Upsilon.
\end{aligned}$$

□

Theorem 3.4 *The operator $G : \mathcal{E} \to \mathcal{E}$ is compact and ϱ-Lipschitz with constant zero.*

Proof Consider a bounded set D with a sequence $\{(u_n, v_n)\}$ such that $D \subset \mathbb{B}_r \subseteq \mathcal{E}$. Then, in view of (12), we have

$$\|\mathbb{G}(u_n, v_n)\| \le \Omega\|(u,v)\| + \Upsilon \quad \text{for each } (u,v) \in \mathcal{E},$$

which yields that $\mathbb{G}$ is bounded. For each $(u_n, v_n) \in D$, we claim that, for any $t_1, t_2 \in [0,1]$,

$$\begin{aligned}
\left|\mathbb{G}_1 v_n(t_1) - \mathbb{G}_1 v_n(t_2)\right| &= \left|\int_0^1 G_p(t_1,s)\theta_1\big(s,v(s)\big)\,ds - \int_0^1 G_p(t_2,s)\theta_1\big(s,v(s)\big)\,ds\right| \\
&\le \left[\int_0^1 G_p(t_1,s) - \int_0^1 G_p(t_2,s)\right]\left|\theta_1\big(s,v(s)\big)\right|\,ds \\
&\le \left[\frac{(t_1-t_2)}{\Gamma(p)}\int_0^1 (1-s)^{p-1}\,ds\right]\big(a\|u\|^{q_2} + M_{\theta_1}\big) \\
&\quad + \frac{1}{\Gamma(p)}\left[\int_0^{t_1}(t_1-s)^{p-1}\,ds - \int_0^{t_2}(t_2-s)^{p-1}\,ds\right]\big(a\|u\|^{q_2} + M_{\theta_1}\big).
\end{aligned}$$

By simplifying we have

$$\left|\mathbb{G}_1 v_n(t_1) - \mathbb{G}_1 v_n(t_2)\right| \leq \left[\frac{(t_1 - t_2)}{\Gamma(p+1)} + \frac{(t_1^p - t_2^p)}{\Gamma(p+1)}\right]\left(a\|u\|^{q_2} + M_{\theta_1}\right). \tag{15}$$

Similarly,

$$\left|G_2 u_n(t_1) - G_2 u_n(t_2)\right| \leq \left[\frac{(t_1 - t_2)}{\Gamma(q+1)} + \frac{(t_1^q - t_2^q)}{\Gamma(q+1)}\right]\left(b\|v\|^{q_2} + M_{\theta_2}\right). \tag{16}$$

Now, if $t_1 \to t_2$, the right-hand sides of both (15) and (16) tend to 0. Thus, $\mathbb{G}_1$, $\mathbb{G}_2$ are equicontinuous, and therefore $\mathbb{G} = (\mathbb{G}_1, \mathbb{G}_2)$ is equicontinuous on D.

Hence, thanks to the Arzelà-Ascoli theorem, $\mathbb{G}(D)$ is compact. Also, by Proposition 2.3, $\mathbb{G}$ is ϱ-Lipschitz with constant 0. □

Theorem 3.5 *Under hypotheses* (C_1)-(C_3) *with* $\Omega + C_{\mathbb{F}} \leq 1$, *the toppled system* (1) *has at least one solution* $(u,v) \in \mathcal{E}$. *Moreover, the set of the solutions* W_0 *is bounded in* $\mathcal{E}$.

Proof By Theorem 3.2, $\mathbb{F}$ is ϱ-Lipschitz with constant $C_{\mathbb{F}} \in [0,1)$, and by Theorem 3.4 $\mathbb{G}$ is ϱ-Lipschitz with constant 0. Therefore, by Proposition 2.2, $\mathbb{T}$ is strictly ϱ-condensing with constant $\mathbb{K}$. Define

$$\mathbb{B} = \left\{(u,v) \in \mathcal{E} : \text{there exists } \lambda \in [0,1] \text{ such that } (u,v) = \lambda\mathbb{T}(u,v)\right\}.$$

We have to prove that $\mathbb{B}$ is bounded in $\mathcal{E}$. Let us consider

$$\begin{aligned}
\left\|(u,v)\right\| &= \left\|\lambda\mathbb{T}(u,v)\right\| \leq \left\|\mathbb{T}(u,v)\right\| \leq \left(\left\|\mathbb{F}(u,v)\right\| + \left\|\mathbb{G}(u,v)\right\|\right) \\
&\leq C_{\mathbb{F}}\left\|(u,v)\right\|^{q_1} + M_{\mathbb{F}} + \Omega\left\|(u,v)\right\|^{q_2} + \Upsilon \\
&= (C_{\mathbb{F}} + \Omega)\left\|(u,v)\right\|^{q_3} + M_{\mathbb{F}} + \Upsilon, \quad \text{where } q_3 = \max\{q_1, q_2\}.
\end{aligned}$$

Clearly, $\|(u,v)\|$ is bounded. If not, choose $\|(u,v)\| = \mathcal{R}$ such that $\mathcal{R} \to \infty$ and $0 < q_3 < 1$. Then

$$\begin{aligned}
&\left\|(u,v)\right\| \leq (C_{\mathbb{F}} + \Omega)\left\|(u,v)\right\|^{q_3}\left\|(u,v)\right\| + M_{\mathbb{F}} + \Upsilon, \\
&1 \leq (C_{\mathbb{F}} + \Omega)\frac{\|(u,v)\|^{q_3}}{\|(u,v)\|} + \frac{M_{\mathbb{F}} + \Upsilon}{\|(u,v)\|}, \\
&1 \leq \frac{(C_{\mathbb{F}} + \Omega)\mathcal{R}^{q_3}}{\mathcal{R}} + \frac{M_{\mathbb{F}} + \Upsilon}{\mathcal{R}}, \\
&1 \leq \frac{(C_{\mathbb{F}} + \Omega)}{\mathcal{R}^{1-q_3}} + \frac{M_{\mathbb{F}} + \Upsilon}{\mathcal{R}} \to 0 \quad \text{as } \mathcal{R} \to \infty,
\end{aligned}$$

which is a contradiction. Thus, $\mathbb{B}$ is bounded. Therefore, by Theorem 2.7 the operator $\mathbb{T}$ has at least one fixed point, which is the corresponding solution to (1), and the set of the solutions is bounded in $\mathcal{E}$. □

Theorem 3.6 *Assume that hypotheses* (C_1)-(C_4) *hold. Then the toppled system* (1) *has a unique solution if and only if* $\Delta < 1$, *where*

$$\Delta = \frac{K_g T^\gamma}{\Gamma(\gamma+1)} + \frac{K_h T^\delta}{\Gamma(\delta+1)} + \frac{L_{\theta_1}}{\Gamma(p+1)} + \frac{L_{\theta_2}}{\Gamma(q+1)}.$$

Proof Let (u,v) and $(\bar{u},\bar{v}) \in \mathcal{E}$ be two solutions. Then

$$\begin{aligned} \left|\mathbb{T}(u,v) - \mathbb{T}(\bar{u},\bar{v})\right| &= \left|\left[\mathbb{F}(u,v) + \mathbb{G}(u,v)\right] - \left[\mathbb{F}(\bar{u},\bar{v}) + \mathbb{G}(\bar{u},\bar{v})\right]\right| \\ &\le \left|\mathbb{F}(u,v) - \mathbb{F}(\bar{u},\bar{v})\right| + \left|\mathbb{G}(u,v) - \mathbb{G}(\bar{u},\bar{v})\right|, \end{aligned}$$

which implies on simplification that

$$\begin{aligned} &\left\|\mathbb{T}(u,v) - \mathbb{T}(\bar{u},\bar{v})\right\| \\ &\quad \le \left(\frac{K_g T^\gamma}{\Gamma(\gamma+1)} + \frac{K_h T^\delta}{\Gamma(\delta+1)} + \frac{L_{\theta_1}}{\Gamma(p+1)} + \frac{L_{\theta_2}}{\Gamma(q+1)}\right)\left\|(u,v) - (\bar{u},\bar{v})\right\|, \\ &\quad\text{which in turn implies that } \left\|\mathbb{T}(u,v) - \mathbb{T}(\bar{u},\bar{v})\right\| \le \Delta\left\|(u,v) - (\bar{u},\bar{v})\right\|. \end{aligned} \tag{17}$$

Therefore, the operator $\mathbb{T}$ is a contraction. Hence, the uniqueness of a solution to the toppled system (1) follows by the Banach fixed point theorem. □

4 Hyers-Ullam stability of toppled system (1)

This section is devoted to the investigation of the Hyers-Ullam stability of our proposed system. We recall the definition of the Hyers-Ullam stability.

Definition 4.1 The system of Hammerstein-type integral equation

$$\begin{cases} u(t) = \frac{t}{\Gamma(\gamma)}\int_0^T (T-s)^{\gamma-1} g(u(s))\,ds + \int_0^1 G_p(t,s)\theta_1(s,v(s))\,ds, \\ v(t) = \frac{t}{\Gamma(\delta)}\int_0^T (T-s)^{\delta-1} h(v(s))\,ds + \int_0^1 G_q(t,s)\theta_2(s,u(s))\,ds, \end{cases} \tag{18}$$

is said to be Hyers-Ullam stable if there exist $D_i > 0$ $(i = 1,2,3,4)$ such that, for all $\lambda_1, \lambda_2 > 0$ and for every solution (u^*, v^*) to the system of inequations

$$\begin{cases} |u^*(t) - \frac{t}{\Gamma(\gamma)}\int_0^T (T-s)^{\gamma-1} g(u^*(s))\,ds + \int_0^1 G_p(t,s)\theta_1(s,v^*(s))\,ds| \le \lambda_1, \\ |v^*(t) - \frac{t}{\Gamma(\delta)}\int_0^T (T-s)^{\delta-1} h(v^*(s))\,ds + \int_0^1 G_q(t,s)\theta_2(s,u^*(s))\,ds| \le \lambda_2, \end{cases} \tag{19}$$

there exists a unique solution (x,y) of (18) such that

$$\begin{aligned} &\left|x(t) - u^*(t)\right| \le D_1\lambda_1 + D_2\lambda_2, \\ &\left|y(t) - v^*(t)\right| \le D_3\lambda_1 + D_4\lambda_2. \end{aligned} \tag{20}$$

Theorem 4.2 *Under assumptions* (C_1)-(C_4), *the toppled system* (1) *is Hyers-Ullam stable*.

Proof Thanks to Theorem 3.6 and Definition 4.1, let (x,y) be the exact solution, and (u^*, v^*) be any other solution of toppled system (6). Then from the first equation of (6) we have

$$\begin{aligned} |x(t) - u^*(t)| &\leq \left| \frac{t}{\Gamma(\gamma+1)} \int_0^T (T-s)^{\gamma-1} \big(g(x) - g(u^*)\big)\, ds \right. \\ &\quad \left. + \int_0^1 G_p(t,s)\big(\theta_1\big(s, y(s)\big) - \theta_1\big(s, v^*(s)\big)\big)\, ds \right| \\ &\leq \frac{T^\gamma K_g}{\Gamma(\gamma+1)} \|x - u^*\| + \frac{L_{\theta_1}}{\Gamma(p+1)} \|y - v^*\| \\ &\leq D_1\epsilon_1 + D_2\epsilon_2, \quad \text{where } D_1 = \frac{T^\gamma K_g}{\Gamma(\gamma+1)}, D_2 = \frac{L_{\theta_1}}{\Gamma(p+1)}. \end{aligned} \tag{21}$$

By the same method we can obtain that

$$|y(t) - v^*(t)| \leq D_3\epsilon_1 + D_4\epsilon_2, \quad \text{where } D_3 = \frac{T^\delta K_h}{\Gamma(\delta+1)}, D_4 = \frac{L_{\theta_2}}{\Gamma(q+1)}. \tag{22}$$

Hence, in view of (21) and (22), the toppled system of integral equations (6) is Hyers-Ullam stable, and, consequently, the toppled system (1) is Hyers-Ullam stable. □

5 Example

Example 1 Consider the following toppled system:

$$\begin{cases} {}^cD^{\frac{3}{2}} u(t) = \frac{e^{-t}}{5+t^2} \cos|v(t)|, & t \in [0,1], \\ {}^cD^{\frac{3}{2}} v(t) = \frac{e^{-\pi t}}{5+t} \sin|u(t)|, & t \in [0,1], \\ u(t)|_{t=0} = 0, & u(t)_{t=1} = \frac{1}{\Gamma(\frac{3}{2})} \int_0^1 \frac{(1-s)^{\frac{1}{2}} \sin(u)}{4}\, ds, \\ v(t)|_{t=0} = 0, & v(t)_{t=1} = \frac{1}{\Gamma(\frac{3}{2})} \int_0^1 \frac{(1-s)^{\frac{1}{2}} \sin(v)}{4}\, ds. \end{cases} \tag{23}$$

Then we obviously see that $C_g = C_h = K_g = K_h = \frac{1}{4}$, $M_g = M_h = M_{\theta_1} = M_{\theta_2} = 0$, $a = b = L_{\theta_1} = L_{\theta_2} = \frac{1}{5}$. It is easy to prove that $\Delta = \frac{18}{15\sqrt{\pi}} < 1$. Hence, by Theorem 3.6 the toppled system (23) has a unique solution. Further, it is also straightforward to prove the conditions of Theorem 3.5. Also, by Theorem 4.2 the solution of the toppled system (23) is Hyers-Ullam stable.

Competing interests
All authors declare that none of them has competing interests.

Authors' contributions
All authors have made the same contribution and finalized the current version of this manuscript.

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Acknowledgements
Bessem Samet extends his appreciation to Distinguished Scientist Fellowship Program (DSFP) at King Saud University (Saudi Arabia).

Received: 23 August 2016 Accepted: 16 January 2017 Published online: 25 January 2017

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