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Asymptotical behavior of one class of p-adic singular Fourier integrals

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ABSTRACT

We study the asymptotical behavior of the p-adic singular Fourier integrals

$$J_{\pi_\alpha,m;\varphi}(t)=\langle f_{\pi_\alpha;m}(x)\chi_p(xt),\varphi(x)\rangle=F[f_{\pi_\alpha;m}\varphi](t),\quad |t|_p\to\infty,\ t\in\mathbb{Q}_p,$$

where $f_{\pi_\alpha;m}\in\mathcal{D}'(\mathbb{Q}_p)$ is a *quasi associated homogeneous* distribution (generalized function) of degree $\pi_\alpha(x)=|x|_p^{\alpha-1}\pi_1(x)$ and order m, $\pi_\alpha(x)$, $\pi_1(x)$, and $\chi_p(x)$ are a multiplicative, a normed multiplicative, and an additive characters of the field $\mathbb{Q}_p$ of p-adic numbers, respectively, $\varphi\in\mathcal{D}(\mathbb{Q}_p)$ is a test function, $m=0,1,2,\dots$, $\alpha\in\mathbb{C}$. If $\operatorname{Re}\alpha>0$ the constructed asymptotics constitute a p-adic version of the well-known Erdélyi lemma. Theorems which give asymptotic expansions of singular Fourier integrals are the Abelian type theorems. In contrast to the real case, all constructed asymptotics have the *stabilization* property.

1. Introduction

1.1. p-Adic mathematical physics

According to the well-known Ostrovsky theorem, *any nontrivial valuation on the field of the rational numbers $\mathbb{Q}$ is equivalent either to the real valuation $|\cdot|$ or to one of the p-adic valuations $|\cdot|_p$*, where p is any prime number. This p-adic norm $|\cdot|_p$ is defined as follows: if an arbitrary rational number $x\neq 0$ is represented as $x=p^\gamma\frac{m}{n}$, where $\gamma=\gamma(x)\in\mathbb{Z}$ and the integers m, n are not divisible by p, then

$$|x|_p=p^{-\gamma},\quad x\neq 0,\qquad |0|_p=0.$$

The norm $|\cdot|_p$ satisfies the *strong triangle inequality*

$$|x+y|_p\leqslant\max(|x|_p,|y|_p)\tag{1.1}$$

and is non-Archimedean. Consequently, it is possible to construct a completion of $\mathbb{Q}$ only with respect to the real valuation $|\cdot|$ or to one of the p-adic valuations $|\cdot|_p$. The field $\mathbb{Q}_p$ of p-adic numbers is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p$.

Thus there are two equal in rights universes: the "real universe" and the "p-adic one." The latter universe is non-Archimedean, and in consequence of this has some specific and surprising properties. This leads to interesting deviations from the classical "real universe."

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As is well known, during a few hundred years theoretical physics has been developed on the basis of real (and later also complex) numbers. However, in the last 20 years the field of p-adic numbers $\mathbb{Q}_p$ (as well as its algebraic extensions) has been intensively used in theoretical and mathematical physics, stochastics, psychology, cognitive and social sciences, biology, image analysis (see [5,20–22,24,31–34] and references therein). Thus, notwithstanding the fact that the p-adic numbers were discovered by K. Hensel around the end of the nineteenth century, the theory of p-adic numbers has already penetrated into several areas of mathematics and applied researches.

Since p-adic analysis and p-adic mathematical physics are young areas there are many unsolved problems, which have been solved in standard real setting. Since "p-adic universe" is in a sense dual to the "real universe," solving such type of problems is important.

Recall that in the usual ($\mathbb{R}$) case there is a theory of so-called *oscillating integrals*, which have the form $\int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)\,d^nx$. These integrals frequently occur in applied and mathematical physics. The classical problem related to *oscillating integrals* is to investigate their asymptotical behavior when the parameter t tends to infinity ([6,14], [18, 7.8]). In the p-adic setting oscillating integrals were studied in [17,36].

In particular, there are many problems where solutions are obtained as Fourier integrals which cannot be evaluated exactly. Nevertheless, these solutions are not less important because it is often possible to study the asymptotic behavior of these integrals [19, 9]. The problem of the asymptotical behavior of the *Fourier integrals* is related to the well-known Erdélyi lemma [11,12]. In the one-dimensional case this lemma describes the asymptotics of the Fourier transforms of functions $f(x)$ defined on $\mathbb{R}$ and having singularities of the type $x_{\pm}^{\alpha-1}\log^m x_{\pm}\varphi(x)$, where $\alpha>0$ and $\varphi(x)$ is sufficiently smooth [14, Ch. III, §1]. There are multidimensional generalizations of this lemma ([14, Ch. III], [6, Ch. II, §7], [35]). The above-mentioned problems are close to the problem of constructing asymptotics of the Fourier transform of distributions

$$(x\pm i0)^{\alpha-1}\log^m(x\pm i0)\varphi(x), \quad m=0,1,2\ldots,\ \alpha\in\mathbb{C},$$

where $\varphi(x)\in\mathcal{D}(\mathbb{R})$ and $(x\pm i0)^{\alpha-1}\log^m(x\pm i0)$ are *quasi associated homogeneous distributions* of degree $\alpha-1$ and order m (see [16, Ch. I, §4], and Remark 4.1). These asymptotics were constructed in [7,9,10] (see also [14, Ch. III, §1.6, §8]). In these papers the following asymptotical formulas were derived:

$$(x\pm i0)^{\alpha-1}\log^m(x\pm i0)e^{itx}\approx\delta(x)2\pi\sum_{k=0}^{m}\frac{d^{m-k}}{d\alpha^{m-k}}\left(\frac{e^{\pm i\frac{\pi(\alpha-1)}{2}}}{\Gamma(-\alpha+1)}\right)\frac{\log^k|t|}{|t|^{\alpha}}, \quad t\to\mp\infty, \tag{1.2}$$

$$(x\pm i0)^{\alpha-1}\log^m(x\pm i0)e^{itx}=o\big(|t|^{-N}\big), \quad t\to\pm\infty, \tag{1.3}$$

for any $N\in\mathbb{N}$, where $\alpha\notin\mathbb{N}$. Some particular cases of these formulas were studied in [19, 9]. In [26], the asymptotic behavior of singular Fourier integrals of pseudo-functions having power and logarithmic singularities are studied.

In p-adic analysis the last problem have not been studied so far. However, taking into account that p-adic mathematical physics is intensively developed, studying these type of problems in the p-adic setting is very important.

1.2. Contents of the paper

In this paper the asymptotical behavior of the *p-adic singular Fourier integrals*

$$J_{\pi_\alpha,m;\varphi}(t)=\big\langle f_{\pi_\alpha;m}(x)\chi_p(xt),\varphi(x)\big\rangle=F[f_{\pi_\alpha;m}\varphi](t), \quad |t|_p\to\infty, \tag{1.4}$$

is studied, where $f_{\pi_\alpha;m}\in\mathcal{D}'(\mathbb{Q}_p)$ is a *quasi associated homogeneous distribution* of degree $\pi_\alpha(x)=|x|_p^{\alpha-1}\pi_1(x)$ and order m, $m=0,1,2\ldots$, $\alpha\in\mathbb{C}$ (see Definitions 2.1, 2.2 and Theorem 2.2), $\varphi\in\mathcal{D}(\mathbb{Q}_p)$, F is the Fourier transform; $\pi_\alpha(x)$, $\pi_1(x)$, and $\chi_p(x)$ are a multiplicative (2.4), a normed multiplicative (2.5), and an additive characters of the field $\mathbb{Q}_p$, respectively.

Remark 1.1. (i) Let us note that the linear span of set of distributions mentioned above

$$\begin{aligned}\mathcal{AH}_0(\mathbb{R})&=\operatorname{span}\big\{(x\pm i0)^{\alpha}\log^m(x\pm i0):\ \alpha\in\mathbb{C},\ m\in\mathbb{N}_0\big\}\\&=\operatorname{span}\big\{x_{\pm}^{\alpha}\log^k x_{\pm},\ P\big(x_{\pm}^{-n}\log^{m-1}x_{\pm}\big):\ \alpha\in\mathbb{C},\ \alpha\neq-1,-2,\ldots,-n,\ldots;\ n,m\in\mathbb{N},\ k\in\mathbb{N}_0\big\}\subset\mathcal{D}'(\mathbb{R})\end{aligned}$$

constitutes a class important for application in mathematical physics, $\mathbb{N}_0=\{0\}\cup\mathbb{N}$. Recall that this class was first introduced and studied in the book [16, Ch. I, §4] as a class of the so-called *associated homogeneous distributions*. Later *associated homogeneous distributions* were studied in the book [13]. Unfortunately, results on *associated homogeneous distribution* from the books [13,16] are not quite consistent and have self-contradictory (for details, see [27]). The problems of introducing of the concept of *associated homogeneous distribution* for $\mathcal{D}'(\mathbb{R}^n)$ and relating mathematics were studied in [27]. According to [27], *direct transfer* of the notion of an *associated eigenvector* to the case of distributions *is impossible for the order* $m\geqslant2$. Thus there exist only *associated homogeneous distributions* of order $m=0$, i.e., *homogeneous distributions* (see Definition [16, Ch. I, §3.11, (1)], [18, 3.2]) and of order $m=1$ (see Definition [16, Ch. I, §4.1, (1), (2)]). Moreover, in [27], a definition of *quasi associated homogeneous distribution* which is a natural generalization of the notion of *associated homogeneous distribution* was introduced and a mathematical description of all quasi associated homogeneous distributions was given. It was proved in [27] that the class of *quasi associated homogeneous distributions* coincides with the class of distributions $\mathcal{AH}_0(\mathbb{R})$ introduced in [16, Ch. I, §4] as the class of *associated homogeneous distributions*.

By adaptation of definitions from [27] to the case of the field $\mathbb{Q}_p$ (instead of the real field), a notion of the *p-adic quasi associated homogeneous distribution* was introduced in [1,2] by Definition 2.2. (In [1,2] these new distributions were named as *associated homogeneous distributions*.) In [1,2] a mathematical description of all *p-adic quasi associated homogeneous distributions* $f_{\pi_\alpha;m}$ was given (see Theorem 2.2 and formulas (2.9), (2.12)).

(ii) Note that associated homogeneous distributions from $\mathcal{D}'(\mathbb{R})$ are parametrized by $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}_0$, while associated homogeneous distributions from $\mathcal{D}'(\mathbb{Q}_p)$ are parametrized by $\alpha \in \mathbb{C}$, $\pi_1(x)$, and $m \in \mathbb{N}_0$ (cf. [27, Definition 3.3] and Definition 2.2).

In Section 2, some facts from p-adic theory of distributions are presented. In particular, the results on *quasi associated homogeneous distributions* [1,2] are given in Section 2.2. In Section 3, a definition of the *stable* p-adic asymptotical expansion is introduced. This concept is relevant for p-adic asymptotic analysis (see below). In Sections 4, 5, we prove Theorems 4.1, 4.2, 5.1 which describe the asymptotical behavior of the p-adic singular Fourier integrals (1.4). Here the sequence

$$\big\{|t|_p^{-\alpha}\pi_1^{-1}(t)\log_p^{m-k}|t|_p\colon\ k=0,1,2,\dots,m\big\},\quad |t|_p\to\infty$$

is an *asymptotic sequence*, and any coefficient of the asymptotic expansion is proportional to the Dirac delta function. Here we note that, although statement of Theorem 4.1 can be obtained as a corollary of Theorem 5.1, we prove these theorems separately to demonstrate different methods for calculating p-adic asymptotics. In Section 6, by Corollary 6.1 a p-adic version of the well-known Erdélyi lemma is given. This lemma is a direct consequence of Theorems 4.1, 5.1 for the case $\operatorname{Re}\alpha > 0$. In Section 7, auxiliary lemmas are proved.

The asymptotical formulas (4.1)–(4.4), (4.15)–(4.18), (5.1)–(5.4) obtained by Theorems 4.1, 4.2, 5.1 are p-adic analogs of formulas (1.2), (1.3). However, in contrast to (1.2), (1.3), p-adic asymptotical formulas (4.1)–(4.4), (4.15)–(4.18), (5.1)–(5.4) have a *specific property of the stabilization*. Namely, the left- and right-hand sides of these formulas are *exact equalities* for sufficiently big $|t|_p > s(\varphi)$, where $s(\varphi)$ is the stabilization parameter (see Definition 3.3 and Remark 4.1). The stabilization parameter $s(\varphi)$ depends on the *parameter of constancy* of the function φ (see (2.1)) and the rank of the character $\pi_1(x)$ (see (2.6)). Asymptotics of this type we call *stable* asymptotical expansions (see Definitions 3.1–3.3). This p-adic phenomenon is quite different from the "real asymptotic properties." It was first discovered in our paper [3, Theorem 5.1], where some weak asymptotics were calculated.

This *asymptotic stabilization property* is similar to another p-adic phenomenon: *if* $\lim_{n\to\infty} x_n = x$, $x_n, x \in \mathbb{Q}_p$, $|x|_p \neq 0$, *then* $\lim_{n\to\infty}|x_n|_p = |x|_p$ *and the sequence of norm* $\{|x_n|_p\colon n\in\mathbb{N}\}$ *must be* stabilize *for sufficiently large* n. Indeed, since $|x_n - x|_p < |x|_p$ for sufficiently large n, according to the *strong triangle inequality* (1.1), we have

$$|x_n|_p = \big|(x_n - x) + x\big|_p = \max\big(|x_n - x|_p, |x|_p\big) = |x|_p \quad \text{for sufficiently large } n.$$

It may well be that *stabilization* is a typical property of p-adic asymptotics.

It remains to note that Theorems 4.1, 4.2, 5.1 are the *Abelian type theorems*. Theorems of this type are inverse to the Tauberian theorems (see [30] and the references therein). For the p-adic case Tauberian theorems for distributions were first proved in [4,23]. In this paper we study the asymptotical behavior of the singular Fourier integrals $J_{\pi_\alpha,m;\varphi}(t) = F[g(x)](t)$, where the functions $g(x) = |x|_p^{\alpha-1}\log_p^m|x|_p\pi_1(x)\varphi(x)$ admit the estimate $g(x) = O(|x|_p^{\alpha-1}\log_p^m|x|_p)$, $|x|_p \to 0$. If $\alpha \neq 0$, according to Theorems 4.1, 5.1, we have

$$J_{\pi_\alpha,m;\varphi}(t) = O\big(|t|_p^{-\alpha}\log_p^m|t|_p\big),\quad |t|_p\to\infty.$$

This connection between asymptotical behavior of $g(x)$ and $J_{\pi_\alpha,m;\varphi}(t)$ is a typical Abelian type theorem.

The results of this paper allow a development of an area of p-adic harmonic analysis which has not been studied so far. In addition, a new technique of constructing p-adic weak asymptotics is developed. Moreover, a new effect of the p-adic *asymptotic stabilization* is observed.

Since the asymptotical formulas for the Fourier transform of quasi associated homogeneous distributions from $\mathcal{D}'(\mathbb{R})$ have many applications (see, for example [10,28,35]), we hope that their p-adic versions may be also useful in the p-adic mathematical physics.

2. Preliminary results in *p*-adic analysis

2.1. *p-Adic functions and distributions*

We shall use intensively the notations and results from [31]. Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ the sets of positive integers, integers, complex numbers, respectively, and set $\mathbb{N}_0 = 0 \cup \mathbb{N}$. Denote by $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ the multiplicative group of the field $\mathbb{Q}_p$.

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}_p\colon\ |x-a|_p \leqslant p^\gamma\}$ the ball of radius p^γ with center at a point $a \in \mathbb{Q}_p$ and by $S_\gamma(a) = \{x \in \mathbb{Q}_p\colon\ |x-a|_p = p^\gamma\} = B_\gamma(a)\setminus B_{\gamma-1}(a)$ its boundary (sphere), $\gamma \in \mathbb{Z}$. For $a = 0$ we set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$.

On $\mathbb{Q}_p$ one can define the Haar measure, i.e., a positive measure dx which is invariant with respect to shifts, $d(x+a) = dx$, and normalized by the equality $\int_{|\xi|_p\leqslant 1} dx = 1$.

A complex-valued function f defined on $\mathbb{Q}_p$ is called *locally-constant* if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$f(x+x') = f(x), \quad x' \in B_{l(x)}.$$

Let $\mathcal{E}(\mathbb{Q}_p)$ and $\mathcal{D}(\mathbb{Q}_p)$ be the linear spaces of locally-constant $\mathbb{C}$-valued functions on $\mathbb{Q}_p$ and locally-constant $\mathbb{C}$-valued functions with compact supports (so-called test functions), respectively. According to Lemma 1 from [31, VI.1], for any $\varphi \in \mathcal{D}(\mathbb{Q}_p)$ there exists $l \in \mathbb{Z}$, such that

$$\varphi(x+x') = \varphi(x), \quad x' \in B_l, \ x \in \mathbb{Q}_p. \tag{2.1}$$

The largest number $l = l(\varphi)$ for which the last relation holds is called the *parameter of constancy* of the function φ. Let us denote by $\mathcal{D}^l_N(\mathbb{Q}_p)$ the space of test functions from $\mathcal{D}(\mathbb{Q}_p)$ with supports in the disc B_N and with parameter of constancy $\geqslant l$. The following embedding holds: $\mathcal{D}^l_N(\mathbb{Q}_p) \subset \mathcal{D}^{l'}_{N'}(\mathbb{Q}_p)$, $N \leqslant N'$, $l \geqslant l'$. Here $\mathcal{D}(\mathbb{Q}_p) = \lim_{N\to\infty} \operatorname{ind} \mathcal{D}_N(\mathbb{Q}_p)$, $\mathcal{D}_N(\mathbb{Q}_p) = \lim_{l\to-\infty} \operatorname{ind} \mathcal{D}^l_N(\mathbb{Q}_p)$. Denote by $\mathcal{D}'(\mathbb{Q}_p)$ the set of all linear functionals on $\mathcal{D}(\mathbb{Q}_p)$.

Denote by $\Delta_k(x) \stackrel{\text{def}}{=} \Omega(p^{-k}|x|_p)$ the characteristic function of the ball B_k, $k \in \mathbb{Z}$, $x \in \mathbb{Q}_p$, where

$$\Omega(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 1, \\ 0, & t > 1. \end{cases}$$

If $f \in \mathcal{D}'(\mathbb{Q}_p)$, $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, then the convolution $f * \varphi \in \mathcal{E}(\mathbb{Q}_p)$ and [31, VII, (1.7)]

$$(f * \varphi)(x) = \langle f(\xi), \varphi(x-\xi) \rangle. \tag{2.2}$$

The Fourier transform of a test function $\varphi \in \mathcal{D}(\mathbb{Q}_p)$ is defined by the formula

$$F[\varphi](\xi) = \int \chi_p(\xi x)\varphi(x)\,dx, \quad \xi \in \mathbb{Q}_p,$$

where the function $\chi_p(\xi x) = e^{2\pi i\{\xi x\}_p}$ for every fixed $\xi \in \mathbb{Q}_p$ is an *additive character* of the field $\mathbb{Q}_p$, $\{\xi x\}_p$ is the fractional part of the number ξx [31, VII.2, VII.3].

Lemma 2.1. *(See [29, III, (3.2)], [31, VII.2].) Fourier transform is a linear isomorphism $\mathcal{D}(\mathbb{Q}_p)$ into $\mathcal{D}(\mathbb{Q}_p)$. Moreover,*

$$\varphi \in \mathcal{D}^l_N(\mathbb{Q}_p) \quad \textit{iff} \quad F[\varphi] \in \mathcal{D}^{-N}_{-l}(\mathbb{Q}_p). \tag{2.3}$$

We define the Fourier transform $F[f]$ of a distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ by the relation [31, VII.3], $\langle F[f], \varphi \rangle = \langle f, F[\varphi] \rangle$ for all $\varphi \in \mathcal{D}(\mathbb{Q}_p)$.

Any *multiplicative character* (see [31, III.2]) π of the field $\mathbb{Q}_p$ can be represented as

$$\pi(x) \stackrel{\text{def}}{=} \pi_\alpha(x) = |x|_p^{\alpha-1}\pi_1(x), \quad x \in \mathbb{Q}_p, \tag{2.4}$$

where $\pi(p) = p^{1-\alpha}$ and $\pi_1(x)$ is a *normed multiplicative character* such that

$$\pi_1(x) = \pi_1\big(|x|_p x\big), \qquad \pi_1(p) = \pi_1(1) = 1, \qquad \big|\pi_1(x)\big| = 1. \tag{2.5}$$

We denote $\pi_0 = |x|_p^{-1}$.

Lemma 2.2. *(See [29, I.7], [31, III, (2.2)].) Let $A_0 = S_0 = \{x \in \mathbb{Q}_p\colon |x|_p = 1\}$, $A_k = B_{-k}(1) = \{x \in \mathbb{Q}_p\colon |x-1|_p \leqslant p^{-k}\}$, $k \in \mathbb{N}$. If π_1 is a normed multiplicative character* (2.5), *then there exists $k \in \mathbb{N}_0$ such that*

$$\pi_1(x) \equiv 1, \quad x \in A_k. \tag{2.6}$$

The smallest $k_0 \in \mathbb{N}_0$ for which the equality (2.6) holds is called the *rank of the normed multiplicative character* $\pi_1(x)$. There is only one *zero rank character*, namely, $\pi_1(x) \equiv 1$.

Let us introduce the p-adic Γ-*functions* (see [31, VIII, (2.2), (2.17)]):

$$\Gamma_p(\alpha) \stackrel{\text{def}}{=} \Gamma_p\big(|x|_p^{\alpha-1}\big) = \int_{\mathbb{Q}_p} |x|_p^{\alpha-1}\chi_p(x)\,dx = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}, \tag{2.7}$$

$$\Gamma_p(\pi_\alpha) \stackrel{\text{def}}{=} F[\pi_\alpha](1) = \int_{\mathbb{Q}_p} |x|_p^{\alpha-1}\pi_1(x)\chi_p(x)\,dx. \tag{2.8}$$

Here the integrals in the right-hand sides of (2.7), (2.8) are defined by means of analytic continuation with respect to α. According to [29, III, Theorem (4.2)], Γ-function (2.8) can be also defined as improper integral $\lim_{k\to\infty} \int_{p^{-k} \leqslant |x|_p \leqslant p^k} \cdot\, dx$.

2.2. Homogeneous and quasi associated homogeneous distributions

Let us recall some facts on p-adic homogeneous and quasi associated homogeneous distributions.

Definition 2.1. (See [15, Ch. II, §2.3], [31, VIII.1].) Let π_α be a multiplicative character (2.4) of the field $\mathbb{Q}_p$. A distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ is called *homogeneous* of degree π_α if for all $\varphi \in \mathcal{D}(\mathbb{Q}_p)$ we have

$$\left\langle f, \varphi\left(\frac{x}{t}\right)\right\rangle = \pi_\alpha(t)|t|_p\langle f, \varphi\rangle, \quad \forall t \in \mathbb{Q}_p^*,$$

i.e., $f(tx) = \pi_\alpha(t) f(x)$, $t \in \mathbb{Q}_p^*$.

The following theorem gives a description of all *homogeneous distributions*.

Theorem 2.1. *(See [15, Ch. II, §2.3], [31, VIII.1].) Every homogeneous distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ of degree π_α has the form*

(a) $C\pi_\alpha$ *if* $\pi_\alpha \ne \pi_0 = |x|_p^{-1}$;
(b) $C\delta$ *if* $\pi_\alpha = \pi_0 = |x|_p^{-1}$, *where C is a constant.*

Definition 2.2. (See [1,2].) A distribution $f_m \in \mathcal{D}'(\mathbb{Q}_p)$ is said to be *quasi associated homogeneous* of degree π_α and order m, $m \in \mathbb{N}_0$, if for all $\varphi \in \mathcal{D}(\mathbb{Q}_p)$ we have

$$\left\langle f_m, \varphi\left(\frac{x}{t}\right)\right\rangle = \pi_\alpha(t)|t|_p\langle f_m, \varphi\rangle + \sum_{j=1}^{m} \pi_\alpha(t)|t|_p \log_p^j |t|_p \langle f_{m-j}, \varphi\rangle, \quad \forall t \in \mathbb{Q}_p^*,$$

where $f_{m-j} \in \mathcal{D}'(\mathbb{Q}_p)$ is an associated homogeneous distribution of degree π_α and order $m-j$, $j = 1, 2, \dots, m$, i.e.,

$$f_m(tx) = \pi_\alpha(t) f_m(x) + \sum_{j=1}^{m} \pi_\alpha(t) \log_p^j |t|_p f_{m-j}(x).$$

If $m = 0$ we set that the above sum is empty.

The class of *quasi associated homogeneous* distributions of order $m = 0$ coincides with the class of *homogeneous* distributions.

Theorem 2.2. *(See [1,2].) Every associated homogeneous distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ of degree $\pi_\alpha(x)$ and order $m \in \mathbb{N}$ (with accuracy up to an associated homogeneous distribution of order $\leqslant m-1$) has the form*

(a) $C\pi_\alpha(x) \log_p^m |x|_p$ *if* $\pi_\alpha \ne \pi_0 = |x|_p^{-1}$;
(b) $CP(|x|_p^{-1} \log_p^{m-1} |x|_p)$ *if* $\pi_\alpha = \pi_0 = |x|_p^{-1}$, *where C is a constant.*

According to the papers [1,2], an *associated homogeneous distribution* of degree $\pi_\alpha(x) = |x|_p^{\alpha-1}\pi_1(x) \ne \pi_0(x) = |x|_p^{-1}$ and order $m \in \mathbb{N}$ is defined as

$$\begin{aligned}\bigl\langle \pi_\alpha(x) \log_p^m |x|_p, \varphi(x)\bigr\rangle = {} & \int_{B_0} |x|_p^{\alpha-1}\pi_1(x) \log_p^m |x|_p \bigl(\varphi(x) - \varphi(0)\bigr)\, dx + \int_{\mathbb{Q}_p \setminus B_0} |x|_p^{\alpha-1}\pi_1(x) \log_p^m |x|_p \varphi(x)\, dx \\ & + \varphi(0) \int_{B_0} |x|_p^{\alpha-1}\pi_1(x) \log_p^m |x|_p\, dx, \end{aligned} \tag{2.9}$$

for all $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, where

$$I_0(\alpha; m) = \int_{B_0} |x|_p^{\alpha-1}\pi_1(x) \log_p^m |x|_p\, dx = \log_p^m e \frac{d^m I_0(\alpha)}{d\alpha^m} = \log_p^m e \begin{cases} 0, & \pi_1(x) \not\equiv 1, \\ \frac{d^m}{d\alpha^m}\left(\frac{1-p^{-1}}{1-p^{-\alpha}}\right), & \pi_1(x) \equiv 1, \end{cases} \tag{2.10}$$

where the integral

$$I_0(\alpha) = \int_{B_0} |x|_p^{\alpha-1}\pi_1(x)\, dx = \begin{cases} 0, & \pi_1(x) \not\equiv 1, \\ \frac{1-p^{-1}}{1-p^{-\alpha}}, & \pi_1(x) \equiv 1, \end{cases} \tag{2.11}$$

is well defined for $\operatorname{Re}\alpha > 0$, and for $\alpha \ne \alpha_j = \frac{2\pi i}{\ln p} j$, $j \in \mathbb{Z}$ (2.11) is defined by means of analytic continuation.

According to the same papers, an *associated homogeneous distribution* of degree $\pi_0(x) = |x|_p^{-1}$ and order $m \in \mathbb{N}$ is defined as

$$\left\langle P\left(\frac{\log_p^{m-1}|x|_p}{|x|_p}\right), \varphi\right\rangle = \int_{B_0} \frac{\log_p^{m-1}|x|_p}{|x|_p}\bigl(\varphi(x) - \varphi(0)\bigr)\,dx + \int_{\mathbb{Q}_p \setminus B_0} \frac{\log_p^{m-1}|x|_p}{|x|_p}\varphi(x)\,dx, \tag{2.12}$$

for all $\varphi \in \mathcal{D}(\mathbb{Q}_p)$.

3. p-Adic stable distributional asymptotics

Let us introduce a definition of the distributional asymptotics [8] adapted to the case of $\mathbb{Q}_p$.

Definition 3.1. A sequence of continuous complex-valued functions $\psi_k(t)$ on the multiplicative group $\mathbb{Q}_p^*$ is called an asymptotic sequence, as $|t|_p \to \infty$ if $\psi_{k+1}(t) = o(\psi_k(t))$, $|t|_p \to \infty$ for all $k = 1, 2, \ldots$.

Definition 3.2. Let $f(x,t) \in \mathcal{D}'(\mathbb{Q}_p)$ be a distribution depending on t as a parameter, and $C_k(x) \in \mathcal{D}'(\mathbb{Q}_p)$ be distributions, $k = 1, 2, \ldots$. We say that the relation

$$f(x,t) \approx \sum_{k=1}^{\infty} C_k(x)\psi_k(t), \quad |t|_p \to \infty, \tag{3.1}$$

is an asymptotical expansion of the distribution $f(x,t)$, as $|t|_p \to \infty$, with respect to an asymptotic sequence $\{\psi_k(t)\}$ if

$$\bigl\langle f(x,t), \varphi(x)\bigr\rangle \approx \sum_{k=1}^{\infty} \bigl\langle C_k(x), \varphi(x)\bigr\rangle \psi_k(t), \quad |t|_p \to \infty, \tag{3.2}$$

for any $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, i.e.,

$$\bigl\langle f(x,t), \varphi(x)\bigr\rangle - \sum_{k=1}^{N} \bigl\langle C_k(x), \varphi(x)\bigr\rangle \psi_k(t) = o\bigl(\psi_N(t)\bigr), \quad |t|_p \to \infty,$$

for any N.

Definition 3.3. Suppose that a distribution $f(x,t) \in \mathcal{D}'(\mathbb{Q}_p)$ has the asymptotical expansion (3.1). If for any test function $\varphi(x) \in \mathcal{D}(\mathbb{Q}_p)$ there exists a number $s(\varphi)$ depending on φ such that for all $|t|_p > s(\varphi)$ relation (3.2) is an *exact equality*, we say that the asymptotical expansion (3.1) is *stable* and write

$$f(x,t) = \sum_{k=1}^{\infty} C_k(x)\psi_k(t), \quad |t|_p \to \infty. \tag{3.3}$$

A number $s(\varphi)$ is called the *stabilization parameter* of the asymptotical expansion (3.1).

4. Asymptotic formulas for singular Fourier integrals (the case $\pi_1(x) \equiv 1$)

4.1. The case $f_{\pi_\alpha;m}(x) = |x|_p^{\alpha-1}\log_p^m |x|_p$, $\alpha \ne 0$, $m = 0, 1, 2, \ldots$

Theorem 4.1. *Let $\varphi \in \mathcal{D}_N^l(\mathbb{Q}_p)$. Then the functional $J_{\pi_\alpha,m;\varphi}(t)$ has the following asymptotical behavior:*

(a) *If $m = 0$, then*

$$J_{\pi_\alpha,0;\varphi}(t) = \bigl\langle |x|_p^{\alpha-1}\chi_p(xt), \varphi(x)\bigr\rangle = \varphi(0)\frac{\Gamma_p(\alpha)}{|t|_p^{\alpha}}, \quad |t|_p > p^{-l}, \tag{4.1}$$

the Γ-function $\Gamma_p(\alpha)$ is given by (2.7), *i.e., in the weak sense*

$$|x|_p^{\alpha-1}\chi_p(xt) = \delta(x)\frac{\Gamma_p(\alpha)}{|t|_p^{\alpha}}, \quad |t|_p \to \infty. \tag{4.2}$$

(b) *If $m = 1, 2, \ldots,$ then*

$$J_{\pi_\alpha,m;\varphi}(t) = \bigl\langle |x|_p^{\alpha-1}\log_p^m |x|_p\,\chi_p(xt), \varphi(x)\bigr\rangle = \varphi(0)\sum_{k=0}^{m} C_m^k \log_p^k e\,\frac{d^k\Gamma_p(\alpha)}{d\alpha^k}\,\frac{\log_p^{m-k}|t|_p}{|t|_p^{\alpha}}, \quad |t|_p > p^{-l}, \tag{4.3}$$

i.e., in the weak sense

$$|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)=\delta(x)\sum_{k=0}^{m}C_m^k\log_p^k e\,\frac{d^k\Gamma_p(\alpha)}{d\alpha^k}\frac{\log_p^{m-k}|t|_p}{|t|_p^{\alpha}},\quad |t|_p\to\infty,\tag{4.4}$$

with respect to an asymptotic sequence $\{|t|_p^{-\alpha}\log_p^{m-k}|t|_p:\ k=0,1,\dots,m\}$.

Thus for any $\varphi\in\mathcal{D}(\mathbb{Q}_p)$, *relations* (4.1), (4.3) *are exact equalities for sufficiently big* $|t|_p>p^{-l}$, *i.e., these asymptotical expansions are stable with the stabilization parameter* $s(\varphi)=p^{-l}$.

Proof. Let $\operatorname{Re}\alpha>0$. In this case $|x|_p^{\alpha-1}\log_p^m|x|_p\varphi(x)\in\mathcal{L}^1(\mathbb{Q}_p)$, and the integral

$$J_{\pi_\alpha,m;\varphi}(t)=\langle|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt),\varphi(x)\rangle=\int_{\mathbb{Q}_p}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\varphi(x)\,dx$$

converges absolutely. Hence, according to the Riemann–Lebesque theorem [31, VII.3], $J_{\pi_\alpha,m;\varphi}(t)\to0$, as $|t|_p\to\infty$. More precisely, since $\varphi(x)\in\mathcal{D}_N^l(\mathbb{Q}_p)$ then, in view of Lemmas 7.1, 7.2,

$$J_{\pi_\alpha,m;\varphi}(t)=\varphi(0)\frac{1}{|t|_p^{\alpha}}\sum_{k=0}^{m}C_m^k\log_p^k e\,\frac{d^k\Gamma_p(\alpha)}{d\alpha^k}\log_p^{m-k}|t|_p,\quad\forall\,|t|_p>p^{-l}.\tag{4.5}$$

Thus relations (4.1)–(4.4) hold.

Let $\operatorname{Re}\alpha<0$. In this case we define the functional $J_{\pi_\alpha,m;\varphi}(t)$ by the analytical continuation with respect to α. According to (2.9), (2.10):

$$\begin{aligned}J_{\pi_\alpha,m;\varphi}(t)&=\langle|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt),\varphi(x)\rangle\\&=\int_{B_0}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\big(\varphi(x)-\varphi(0)\big)\,dx+\int_{\mathbb{Q}_p\setminus B_0}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\varphi(x)\,dx\\&\quad+\varphi(0)\int_{B_0}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\,dx,\end{aligned}\tag{4.6}$$

for all $\varphi\in\mathcal{D}(\mathbb{Q}_p)$, where the last integral in (4.6) is defined by means of analytic continuation with respect to α.

Since $\varphi\in\mathcal{D}_N^l(\mathbb{Q}_p)$, it is *natural* to rewrite functional (4.6) as the following sum:

$$J_{\pi_\alpha,m;\varphi}(t)=J^1_{\pi_\alpha,m;\varphi}(t)+J^2_{\pi_\alpha,m;\varphi}(t)+\varphi(0)J^0_{\pi_\alpha,m}(t),\tag{4.7}$$

where

$$J^1_{\pi_\alpha,m;\varphi}(t)=\int_{B_l}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\big(\varphi(x)-\varphi(0)\big)\,dx,\tag{4.8}$$

$$J^2_{\pi_\alpha,m;\varphi}(t)=\int_{\mathbb{Q}_p\setminus B_l}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\varphi(x)\,dx,\tag{4.9}$$

$$J^0_{\pi_\alpha,m}(t)=\int_{B_l}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\,dx.\tag{4.10}$$

Here integral (4.10) is defined by means of analytic continuation with respect to α.

For $\operatorname{Re}\alpha>0$ and $m=0$, according to (7.4), integral (4.10) is equal to

$$J^0_{\pi_\alpha,0}(t)=F\big[|x|_p^{\alpha-1}\Delta_l(x)\big](t)=\int_{B_l}\chi_p(tx)|x|_p^{\alpha-1}\,dx=\frac{1-p^{-1}}{1-p^{-\alpha}}p^{\alpha l}\Delta_{-l}(t)+\frac{\Gamma_p(\alpha)}{|t|_p^{\alpha}}\big(1-\Delta_{-l}(t)\big).\tag{4.11}$$

For any $\alpha\neq\alpha_j=\frac{2\pi i}{\ln p}j$, $j\in\mathbb{Z}$ we define $J^0_{\pi_\alpha,0}(t)$ by means of analytic continuation with respect to α.

Differentiating relation (4.11) with respect to α, we obtain

$$\begin{aligned}J^0_{\pi_\alpha,m}(t)&=F\big[|x|_p^{\alpha-1}\log_p^m|x|_p\Delta_l(x)\big](t)=\int_{B_l}\chi_p(tx)|x|_p^{\alpha-1}\log_p^m|x|_p\,dx=\log_p^m e\frac{d^m}{d\alpha^m}J^0_{\pi_\alpha,0}(t)\\&=\Delta_{-l}(t)\big(1-p^{-1}\big)\frac{d^m}{d\alpha^m}\left(\frac{p^{\alpha l}}{1-p^{-\alpha}}\right)\log_p^m e+\big(1-\Delta_{-l}(t)\big)\frac{1}{|t|_p^{\alpha}}\sum_{k=0}^{m}C_m^k\log_p^k e\,\frac{d^k\Gamma_p(\alpha)}{d\alpha^k}\log_p^{m-k}|t|_p.\end{aligned}\tag{4.12}$$

Note that by using formulas from [15, Ch. II, §2.2], [31, IV], relation (4.12) can be calculated explicitly.

According to (4.12), we have

$$J^0_{\pi_\alpha,m}(t)=\frac{1}{|t|_p^\alpha}\sum_{k=0}^{m}C_m^k\frac{d^k\Gamma_p(\alpha)}{d\alpha^k}\log_p^{m-k}|t|_p,\quad |t|_p>p^{-l}. \tag{4.13}$$

Since $\varphi\in\mathcal{D}^l_N(\mathbb{Q}_p)$, it is clear that the functions

$$|x|_p^{\alpha-1}\log_p^m|x|_p\big(\varphi(x)-\varphi(0)\big)\Delta_l(x)=0,$$
$$|x|_p^{\alpha-1}\log_p^m|x|_p\varphi(x)\big(1-\Delta_l(x)\big)\in\mathcal{D}^l_N(\mathbb{Q}_p).$$

Thus for their Fourier transforms, according to (2.3), we have

$$J^1_{\pi_\alpha,m;\varphi}(t)=\int_{B_l}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\big(\varphi(x)-\varphi(0)\big)\,dx=0,$$
$$J^2_{\pi_\alpha,m;\varphi}(t)=\int_{\mathbb{Q}_p\setminus B_l}|x|_p^{\alpha-1}\log_p^m|x|_p\chi_p(xt)\varphi(x)\,dx=0, \tag{4.14}$$

for all $|t|_p>p^{-l}$. Thus for $\operatorname{Re}\alpha<0$ relations (4.7), (4.14), (4.13) imply (4.1)–(4.4). □

4.2. The case $f_{\pi_0;m}(x)=P(\frac{\log_p^m|x|_p}{|x|_p})$, $m=0,1,2,\ldots$

Theorem 4.2. *Let $\varphi\in\mathcal{D}^l_N(\mathbb{Q}_p)$. Then the functional $J_{\pi_\alpha,m;\varphi}(t)$ has the following asymptotical behavior:*

(a) *If $m=0$, then*

$$J_{\pi_0,0;\varphi}(t)=\left\langle P\left(\frac{1}{|x|_p}\right)\chi_p(xt),\varphi(x)\right\rangle=\varphi(0)\left(-\frac{1}{p}-\left(1-\frac{1}{p}\right)\log_p\left(\frac{|t|_p}{p^{-l}}\right)\right),\quad |t|_p>p^{-l}, \tag{4.15}$$

i.e., in the weak sense

$$P\left(\frac{1}{|x|_p}\right)\chi_p(xt)=\delta(x)\left(-\frac{1}{p}-\left(1-\frac{1}{p}\right)\log_p\left(\frac{|t|_p}{p^{-l}}\right)\right),\quad |t|_p\to\infty. \tag{4.16}$$

(b) *If $m=1,2,\ldots$, then*

$$\begin{aligned}J_{\pi_0,m;\varphi}(t)&=\left\langle P\left(\frac{\log_p^m|x|_p}{|x|_p}\right)\chi_p(xt),\varphi(x)\right\rangle\\&=\varphi(0)\Bigg\{\frac{1}{p}(-1)^{m+1}\big(\log_p|t|_p-1\big)^m\\&\quad+\left(1-\frac{1}{p}\right)\frac{1}{m+1}\Bigg((-1)^{m+1}\big(\log_p^{m+1}|t|_p-(-l)^{m+1}\big)-(-1)^mC^1_{m+1}\mathbf{B}_1\big(\log_p^m|t|_p-(-l)^m\big)\\&\quad+\sum_{r=2}^{m}(-1)^{m+1-r}C^r_{m+1}\mathbf{B}_r\big(\log_p^{m+1-r}|t|_p-(-l)^{m+1-r}\big)\Bigg)\Bigg\},\quad |t|_p>p^{-l},\end{aligned} \tag{4.17}$$

where the Bernoulli numbers $\mathbf{B}_r$, $r=0,1,\ldots,m$ are defined by (7.7), i.e., in the weak sense,

$$\begin{aligned}P\left(\frac{\log_p^m|x|_p}{|x|_p}\right)\chi_p(xt)=\delta(x)\Bigg\{&-\frac{1}{p}\big(-\log_p|t|_p+1\big)^m\\&+\left(1-\frac{1}{p}\right)\frac{1}{m+1}\Bigg((-1)^{m+1}\big(\log_p^{m+1}|t|_p-(-l)^{m+1}\big)-(-1)^mC^1_{m+1}\mathbf{B}_1\big(\log_p^m|t|_p-(-l)^m\big)\\&+\sum_{r=2}^{m}(-1)^{m+1-r}C^r_{m+1}\mathbf{B}_r\big(\log_p^{m+1-r}|t|_p-(-l)^{m+1-r}\big)\Bigg)\Bigg\},\quad |t|_p\to\infty,\end{aligned} \tag{4.18}$$

with respect to an asymptotic sequence $\{\log_p^{m+1-k}|t|_p\colon k=0,1,\ldots,m+1\}$.

Thus for any $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, relations (4.15), (4.17) are exact equalities for sufficiently big $|t|_p > p^{-l}$, i.e., these asymptotical expansions are stable with the stabilization parameter $s(\varphi) = p^{-l}$.

Proof. According to (2.12), we have

$$J_{\pi_0,m;\varphi}(t) = \left\langle P\left(\frac{\log_p^m |x|_p}{|x|_p}\right)\chi_p(xt), \varphi(x)\right\rangle = \int_{B_0} \frac{\log_p^m |x|_p}{|x|_p}\big(\varphi(x)\chi_p(xt) - \varphi(0)\big)\,dx + \int_{\mathbb{Q}_p\setminus B_0} \frac{\log_p^m |x|_p}{|x|_p}\chi_p(xt)\varphi(x)\,dx, \tag{4.19}$$

for all $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, $m = 0, 1, 2, \ldots$.

Since $\varphi \in \mathcal{D}^l_N(\mathbb{Q}_p)$, it is *natural* to rewrite the functional $J_{\pi_0,m;\varphi}(t)$ in the form of the sum of integrals:

$$J_{\pi_0,m;\varphi}(t) = J^1_{\pi_0,m;\varphi}(t) + J^2_{\pi_0,m;\varphi}(t) + \varphi(0)J^0_{\pi_0,m}(t), \tag{4.20}$$

where

$$J^1_{\pi_0,m;\varphi}(t) = \int_{B_l} \frac{\log_p^m |x|_p}{|x|_p}\chi_p(xt)\big(\varphi(x) - \varphi(0)\big)\,dx, \tag{4.21}$$

$$J^2_{\pi_0,m;\varphi}(t) = \int_{\mathbb{Q}_p\setminus B_l} \frac{\log_p^m |x|_p}{|x|_p}\chi_p(xt)\varphi(x)\,dx, \tag{4.22}$$

$$J^0_{\pi_0,m}(t) = \int_{B_l} \frac{\log_p^m |x|_p}{|x|_p}\big(\chi_p(xt) - 1\big)\,dx. \tag{4.23}$$

Since $\varphi \in \mathcal{D}^l_N(\mathbb{Q}_p)$, it is clear that

$$\frac{\log_p^m |x|_p}{|x|_p}\big(\varphi(x) - \varphi(0)\big)\Delta_l(x) = 0,$$
$$\frac{\log_p^m |x|_p}{|x|_p}\varphi(x)\big(1 - \Delta_l(x)\big) \in \mathcal{D}^l_N(\mathbb{Q}_p).$$

Thus as above, according to (2.3), for their Fourier transforms (4.21), (4.22) we have

$$J^1_{\pi_0,m;\varphi}(t) = J^2_{\pi_0,m;\varphi}(t) = 0, \quad \forall |t|_p > p^{-l}. \tag{4.24}$$

Let us calculate integral (4.23). Suppose that $|t|_p = p^M$, $M > -l$.

We start with the case $m = 0$. Taking into account that $-M + 1 \leqslant l$, according to formulas from [15, Ch. II, §2.2], [31, IV], we have

$$\begin{aligned} J^0_{\pi_0,0}(t) &= \int_{B_l} \frac{\chi_p(xt) - 1}{|x|_p}\,dx = \sum_{\gamma=-\infty}^{l} p^{-\gamma}\int_{S_\gamma}\big(\chi_p(xt) - 1\big)\,dx \\ &= -p^{-(-M+1)}p^{-M+1-1} - \sum_{\gamma=-M+1}^{l} p^{-\gamma}\left(1 - \frac{1}{p}\right)p^\gamma \\ &= -\frac{1}{p} - \left(1 - \frac{1}{p}\right)(l + M) = -\frac{1}{p} - \left(1 - \frac{1}{p}\right)\log_p\left(\frac{|t|_p}{p^{-l}}\right). \end{aligned} \tag{4.25}$$

Relations (4.24) and (4.25) imply that

$$J_{\pi_0,0;\varphi}(t) = \varphi(0)\left(-\frac{1}{p} - \left(1 - \frac{1}{p}\right)\log_p\left(\frac{|t|_p}{p^{-l}}\right)\right), \quad |t|_p > p^{-l}. \tag{4.26}$$

Note that the last relation can also be proved if we use the representation of functional (4.19) in the form of convolution $J_{\pi_0,m;\varphi}(t) = F[P(\frac{1}{|x|_p})](t) * F[\varphi(x)](t)$, and formula [31, IX, (2.8)]:

$$F\big[\big(1 - p^{-1}\big)\log_p |x|_p\big](t) = -P\left(\frac{1}{|t|_p}\right) - p^{-1}\delta(t).$$

In the case $m = 1, 2, \ldots$, for $|t|_p = p^M$, $M > -l$, using formulas from [15, Ch. II, §2.2], [31, IV], we obtain

$$\begin{aligned}
J^0_{\pi_0,m}(t) &= \int\limits_{B_l} \frac{\log_p^m |x|_p}{|x|_p}\big(\chi_p(xt) - 1\big)\,dx \\
&= \sum_{\gamma=-\infty}^{l} p^{-\gamma}\gamma^m \int\limits_{S_\gamma}\big(\chi_p(xt) - 1\big)\,dx \\
&= -p^{-(-M+1)}(-M+1)^m p^{-M+1-1} - \sum_{\gamma=-M+1}^{l} p^{-\gamma}\gamma^m\left(1 - \frac{1}{p}\right)p^\gamma \\
&= -\frac{1}{p}(-M+1)^m - \left(1 - \frac{1}{p}\right)\sum_{\gamma=-M+1}^{l}\gamma^m. \qquad (4.27)
\end{aligned}$$

Next, using formulas (7.8), (7.9), relation (4.27) can be easily transformed to the following form

$$\begin{aligned}
J^0_{\pi_0,m}(t) &= -\frac{1}{p}(-M+1)^m - \left(1 - \frac{1}{p}\right)\big(\mathbf{S}_m(l) - \mathbf{S}_m(-M)\big) \\
&= -\frac{1}{p}(-M+1)^m \\
&\quad + \left(1 - \frac{1}{p}\right)\frac{1}{m+1}\left((-M)^{m+1} - l^{m+1} - C^1_{m+1}\mathbf{B}_1\big((-M)^m - l^m\big) + \sum_{r=2}^{m} C^r_{m+1}\mathbf{B}_r\big((-M)^{m+1-r} - l^{m+1-r}\big)\right) \\
&= -\frac{1}{p}(-1)^m(\log_p|t|_p - 1)^m + \left(1 - \frac{1}{p}\right)\frac{1}{m+1}\left((-1)^{m+1}\big(\log_p^{m+1}|t|_p - \log_p^{m+1}p^{-l}\big)\right. \\
&\quad \left. - (-1)^m C^1_{m+1}\mathbf{B}_1\big(\log_p^m|t|_p - \log_p^m p^{-l}\big) + \sum_{r=2}^{m}(-1)^{m+1-r}C^r_{m+1}\mathbf{B}_r\big(\log_p^{m+1-r}|t|_p - \log_p^{m+1-r}p^{-l}\big)\right), \qquad (4.28)
\end{aligned}$$

where the Bernoulli numbers $\mathbf{B}_r$, $r = 0, 1, \ldots, m$ are defined by (7.7), the polynomial $\mathbf{S}_m(\gamma_0)$ is given by (7.8).

Relations (4.20), (4.24), (4.28) imply

$$\begin{aligned}
J_{\pi_0,m;\varphi}(t) = \varphi(0)\Bigg\{&\frac{1}{p}(-1)^{m+1}(\log_p|t|_p - 1)^m \\
&+ \left(1 - \frac{1}{p}\right)\frac{1}{m+1}\left((-1)^{m+1}\big(\log_p^{m+1}|t|_p - \log_p^{m+1}p^{-l}\big) - (-1)^m C^1_{m+1}\mathbf{B}_1\big(\log_p^m|t|_p - \log_p^m p^{-l}\big)\right. \\
&\left. + \sum_{r=2}^{m}(-1)^{m+1-r}C^r_{m+1}\mathbf{B}_r\big(\log_p^{m+1-r}|t|_p - \log_p^{m+1-r}p^{-l}\big)\right)\Bigg\}, \qquad (4.29)
\end{aligned}$$

for all $|t|_p > p^{-l}$.

Thus relations (4.17), (4.18) hold. $\square$

Corollary 4.1. *If* $\alpha = 1$, *then relations* (4.1), (2.7) *imply the statement* (2.3) *of Lemma* [31, VII.2].

Remark 4.1. The asymptotical expansion (4.4) can be represented in the form

$$|x|_p^{\alpha-1}\log_p^m|x|_p\,\chi_p(xt) = \delta(x)\frac{\log_p^m|t|_p}{|t|_p^\alpha}\left(\sum_{k=0}^{N} A_k(\alpha)\log_p^{-k}|t|_p + o\big(\log_p^{-N}|t|_p\big)\right), \quad |t|_p \to \infty, \qquad (4.30)$$

where $A_k(\alpha)$ is an explicit computable constant, $k = 0, 1, \ldots$. Here a *stabilization property* is expressed be the following assertion: *for $N \geqslant m$ and for enough large $|t|_p$ the remainder disappears and the asymptotic expansion turns to an exact equality*.[2]

The same remark is also true for the case of asymptotical expansions (4.18).

[2] We emphasize the representation (4.30) after a remark of the anonymous referee of this paper.

Remark 4.2. Since $\varphi \in \mathcal{D}^l_N(\mathbb{Q}_p)$, to calculate asymptotics of the functionals $J_{\pi_\alpha,m;\varphi}(t)$ and $J_{\pi_0,m;\varphi}(t)$ (for the case $\pi_1(x) \equiv 1$, $\alpha \neq 0$), it is *natural* to represent these functionals as the sums of integrals (4.7) and (4.20), respectively. However, we can represent these functionals as the sums of integrals

$$\widetilde{J}^1_{\pi_\alpha,m;\varphi}(t) = \int_{B_{l_0}} |x|_p^{\alpha-1}(x) \log_p^m |x|_p \chi_p(xt)\big(\varphi(x) - \varphi(0)\big)\,dx, \tag{4.31}$$

$$\widetilde{J}^2_{\pi_\alpha,m;\varphi}(t) = \int_{\mathbb{Q}_p\setminus B_{l_0}} |x|_p^{\alpha-1} \log_p^m |x|_p \chi_p(xt)\varphi(x)\,dx, \tag{4.32}$$

$$\widetilde{J}^0_{\pi_\alpha,m}(t) = \int_{B_{l_0}} |x|_p^{\alpha-1} \log_p^m |x|_p \chi_p(xt)\,dx, \tag{4.33}$$

where $l_0 \in \mathbb{Z}$. For example, we can choose $l_0 = 0$, as in the standard representations (4.6) and (4.19). In this case

$$|x|_p^{\alpha-1} \log_p^m |x|_p \big(\varphi(x) - \varphi(0)\big)\Delta_{l_0}(x) \in \mathcal{D}_N^{\min(l,l_0)}(\mathbb{Q}_p),$$
$$|x|_p^{\alpha-1} \log_p^m |x|_p \varphi(x)\big(1 - \Delta_{l_0}(x)\big) \in \mathcal{D}_N^{\min(l,l_0)}(\mathbb{Q}_p).$$

and, as above, according to (2.3), $\widetilde{J}^1_{\pi_\alpha,m;\varphi}(t) = \widetilde{J}^2_{\pi_\alpha,m;\varphi}(t) = 0$ for all $|t|_p > p^{\max(-l,-l_0)}$. Thus repeating the above calculations almost word for word, we obtain the asymptotic formulas from Theorem 4.1. However, in this case the minimal stabilization parameter is equal to $s(\varphi) = p^{\max(-l,-l_0)}$.

The same remark is also true for the case of Theorems 4.2.

5. Asymptotic formulas for singular Fourier integrals (the case $\pi_1(x) \not\equiv 1$)

Now we consider the case of distributions $f_{\pi_\alpha;m}(x) = |x|_p^{\alpha-1}\pi_1(x) \log_p^m |x|_p$, $m = 0, 1, 2, \ldots$.

Theorem 5.1. *Let $\varphi \in \mathcal{D}^l_N(\mathbb{Q}_p)$, and let $k_0 > 0$ be the rank of the character $\pi_1(x)$. Then the functional $J_{\pi_\alpha,m;\varphi}(t)$ has the following asymptotical behavior*:

(a) *If $m = 0$, then*

$$J_{\pi_\alpha,0;\varphi}(t) = \big\langle |x|_p^{\alpha-1}\pi_1(x)\chi_p(xt), \varphi(x)\big\rangle = \varphi(0)\frac{\Gamma_p(\pi_\alpha)}{|t|_p^\alpha \pi_1(t)}, \quad |t|_p > p^{-l+k_0}, \tag{5.1}$$

for all $\varphi \in \mathcal{D}^l_N(\mathbb{Q}_p)$, where the Γ-function $\Gamma_p(\pi_\alpha)$ is given by (2.8), *i.e., in the weak sense*

$$|x|_p^{\alpha-1}\pi_1(x)\chi_p(xt) = \delta(x)\frac{\Gamma_p(\pi_\alpha)}{|t|_p^\alpha \pi_1(t)}, \quad |t|_p \to \infty. \tag{5.2}$$

(b) *If $m = 1, 2, \ldots$, then*

$$\begin{aligned} J_{\pi_\alpha,m;\varphi}(t) &= \big\langle |x|_p^{\alpha-1}\pi_1(x) \log_p^m |x|_p \chi_p(xt), \varphi(x)\big\rangle \\ &= \varphi(0)\sum_{k=0}^m C_m^k \log_p^k e \frac{d^k\Gamma_p(\pi_\alpha)}{d\alpha^k}\frac{\log_p^{m-k}|t|_p}{|t|_p^\alpha \pi_1(t)}, \quad |t|_p > p^{-l+k_0}, \end{aligned} \tag{5.3}$$

for all $\varphi \in \mathcal{D}^l_N(\mathbb{Q}_p)$, i.e., in the weak sense

$$|x|_p^{\alpha-1}\pi_1(x) \log_p^m |x|_p \chi_p(xt) = \delta(x)\sum_{k=0}^m C_m^k \log_p^k e\,\frac{d^k\Gamma_p(\pi_\alpha)}{d\alpha^k}\frac{\log_p^{m-k}|t|_p}{|t|_p^\alpha \pi_1(t)}, \quad |t|_p \to \infty, \tag{5.4}$$

with respect to an asymptotic sequence $\{\pi_{\alpha+1}^{-1}(t)\log_p^{m-k}|t|_p\colon k = 0, 1, \ldots, m\}$.

Thus for any $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, relations (5.1), (5.3) *are exact equalities for sufficiently big $|t|_p > p^{-l+k_0}$, i.e., these asymptotical expansions are stable with the stabilization parameter $s(\varphi) = p^{-l+k_0}$.*

Proof. Let $m = 0$. Taking into account formulas [31, VII, (3.10), (5.4)], the functional $J_{\pi_\alpha,0;\varphi}(t)$ can be rewritten as a convolution*:*

$$J_{\pi_\alpha,0;\varphi}(t) = \big\langle |x|_p^{\alpha-1}\pi_1(x)\chi_p(xt), \varphi(x)\big\rangle = F\big[|x|_p^{\alpha-1}\pi_1(x)\varphi(x)\big](t) = F\big[|x|_p^{\alpha-1}\pi_1(x)\big](t) * \psi(t), \tag{5.5}$$

where $\psi(\xi)=F[\varphi(x)](\xi)$ and according to [31, VIII, (2.1)],

$$F\big[|x|_p^{\alpha-1}\pi_1(x)\big](t)=\Gamma_p(\pi_\alpha)|t|_p^{-\alpha}\pi_1^{-1}(t). \tag{5.6}$$

Since $\varphi(x)\in\mathcal{D}^l_N(\mathbb{Q}_p)$ then, in view of (2.3), $\psi(\xi)\in\mathcal{D}^{-N}_{-l}(\mathbb{Q}_p)$. If $|t|_p>p^{-l}$, according to (5.6), (2.2), relation (5.5) can be rewritten as

$$J_{\pi_\alpha,0;\varphi}(t)=\Gamma_p(\pi_\alpha)\int\limits_{\mathbb{Q}_p}|t-\xi|_p^{-\alpha}\pi_1^{-1}(t-\xi)\psi(\xi)\,d\xi. \tag{5.7}$$

Since $|t|_p>p^{-l}$ and $\xi\in B_{-l}$ the last integral is well defined for any α. Moreover, we have $|t-\xi|_p=|t|_p$ for $|t|_p>p^{-l}$, $\xi\in B_{-l}$. Thus relation (5.7) can be transformed to the form

$$J_{\pi_\alpha,0;\varphi}(t)=\Gamma_p(\pi_\alpha)|t|_p^{-\alpha}\pi_1^{-1}(t)\Psi(t),\quad |t|_p>p^{-l}, \tag{5.8}$$

where

$$\Psi(t)=\int\limits_{B_{-l}}\pi_1^{-1}\left(1-\frac{\xi}{t}\right)\psi(\xi)\,d\xi. \tag{5.9}$$

Let k_0 be the rank of the character $\pi_1(x)\not\equiv 1$. In is clear that if $|t|_p>p^{-l+k_0}$, then the inequality $|\frac{\xi}{t}|_p\leqslant p^{-k_0}$ holds for all $\xi\in B_{-l}$. Thus in view of (2.6), we see that $\pi_1^{-1}(1-\frac{\xi}{t})\equiv 1$ for all $\xi\in B_{-l}$ and $|t|_p>p^{-l+k_0}$. Next, applying an analog of the Lebesque theorem to limiting passage under the sign of an integral to (5.9), and taking into account that $|\pi_1(x)|=1$ and

$$\int\limits_{B_{-l}}\psi(\xi)\,d\xi=\int\limits_{\mathbb{Q}_p}\psi(\xi)\,d\xi=\varphi(0),$$

we see that (5.8), (5.9) imply relations (5.1), (5.2) for all $|t|_p>p^{-l+k_0}$.

If $m=1,2,\dots$, differentiating (5.8) with respect to α, we obtain

$$J_{\pi_\alpha,m;\varphi}(t)=\log_p^m\frac{d^m}{d\alpha^m}J_{\pi_\alpha,0;\varphi}(t)=\sum_{k=0}^m C_m^k\log_p^k e\,\frac{d^k\Gamma_p(\pi_\alpha)}{d\alpha^k}|t|_p^{-\alpha}\log_p^{m-k}|t|_p\pi_1^{-1}(t)\Psi(t),\quad |t|_p>p^{-l}. \tag{5.10}$$

Just as above, since $\Psi(t)=\varphi(0)$ for $|t|_p>p^{-l+k_0}$, relation (5.10) implies (5.3), (5.4). □

The analogues of Remarks 4.1, 4.2 are also true for the case of Theorem 5.1.

6. *p*-Adic version of the Erdélyi lemma

Theorems 4.1, 5.1 for $\operatorname{Re}\alpha>0$ imply the following p-adic version of the well-known Erdélyi lemma.

Corollary 6.1. *Let k_0 be the rank of the character π_1, and $\varphi\in\mathcal{D}^l_N(\mathbb{Q}_p)$. Then for* $\operatorname{Re}\alpha>0$, $m=0,1,2,\dots$, *we have*

$$\int\limits_{\mathbb{Q}_p}|x|_p^{\alpha-1}\pi_1(x)\log_p^m|x|_p\,\chi_p(xt)\varphi(x)\,dx=\varphi(0)\sum_{k=0}^m C_m^k\log_p^k e\,\frac{d^k\Gamma_p(\pi_\alpha)}{d\alpha^k}\frac{\log_p^{m-k}|t|_p}{|t|_p^\alpha\pi_1(t)},\quad |t|_p>p^{-l+k_0}.$$

Moreover, for any $\varphi\in\mathcal{D}(\mathbb{Q}_p)$, the last relation is a stable asymptotical expansion.

7. Some auxiliary lemmas

Lemma 7.1. *Let $\varphi(x)\in\mathcal{D}^l_N(\mathbb{Q}_p)$ and $\psi(t)=F[|x|_p^\alpha\varphi(x)](t)$,* $\operatorname{Re}\alpha>-1$. *Then*

$$\psi(t)=\begin{cases}\in\mathcal{D}^{-N}_{-l}(\mathbb{Q}_p), & |t|_p\leqslant p^{-l},\\ \varphi(0)\Gamma_p(\alpha+1)\frac{1}{|t|_p^{\alpha+1}}, & |t|_p>p^{-l},\end{cases} \tag{7.1}$$

where $\Gamma_p(\alpha)$ is given by formula (2.7).

Proof. Since $\operatorname{Re}\alpha > -1$ the integral $\psi(t)$ converges absolutely. We rewrite it as the sum $\psi(t) = \psi_1(t) + \psi_2(t)$, where

$$\psi_1(t) = \int\limits_{B_l} \chi_p(tx)|x|_p^\alpha \varphi(x)\,dx, \qquad \psi_2(t) = \int\limits_{\mathbb{Q}_p \setminus B_l} \chi_p(tx)|x|_p^\alpha \varphi(x)\,dx. \tag{7.2}$$

If $x \in \mathbb{Q}_p \setminus B_l$ the function $|x|_p^\alpha$ has a parameter of constancy $\geqslant l$, i.e., $|x|_p^\alpha \varphi(x) \in \mathcal{D}_N^l$. Hence according to (2.3),

$$\psi_2(t) = F\big[|x|_p^\alpha \big(1 - \Delta_l(x)\big)\varphi(x)\big](t) \in \mathcal{D}_{-l}^{-N}, \tag{7.3}$$

i.e. $\psi_2(t) = 0$ if $|t|_p > p^{-l}$.

Since $\varphi(x) \in \mathcal{D}_N^l(\mathbb{Q}_p)$, the function $\psi_1(t)$ can be rewritten as

$$\psi_1(t) = \int\limits_{B_l} \chi_p(tx)|x|_p^\alpha\, \varphi(x)\,dx = \varphi(0) \int\limits_{B_l} \chi_p(tx)|x|_p^\alpha\,dx.$$

Next, according to [31, VII.2, Example 9] and (2.7), for $\operatorname{Re}\alpha > -1$ we have

$$F\big[|x|_p^\alpha \Delta_l(x)\big](t) = \int\limits_{B_l} \chi_p(tx)|x|_p^\alpha\,dx = \frac{1-p^{-1}}{1-p^{-(\alpha+1)}} p^{l(\alpha+1)} \Delta_{-l}(t) + \frac{\Gamma_p(\alpha+1)}{|t|_p^{\alpha+1}} \big(1 - \Delta_{-l}(t)\big). \tag{7.4}$$

To complete the proof of the lemma, it remains to use (7.2)–(7.4). □

Lemma 7.2. *Let $\varphi(x) \in \mathcal{D}_N^l(\mathbb{Q}_p)$, $\psi(t) = F[|x|_p^\alpha \log_p^m |x|_p \varphi(x)](t)$, $\operatorname{Re}\alpha > -1$, $m = 1, 2, \ldots$. Then*

$$\psi(t) = \begin{cases} \in \mathcal{D}_{-l}^{-N}(\mathbb{Q}_p), & |t|_p \leqslant p^{-l}, \\ \varphi(0) \sum_{k=0}^m C_m^k \log_p^{m-k} e \frac{d^{m-k}\Gamma_p(\alpha+1)}{d\alpha^{m-k}} \frac{\log_p^k |t|_p}{|t|_p^{\alpha+1}}, & |t|_p > p^{-l}. \end{cases} \tag{7.5}$$

Proof. Since $\operatorname{Re}\alpha > -1$, by differentiating the identity (7.4) with respect to α, we derive the following identity:

$$\begin{aligned} F\big[|x|_p^\alpha \log_p^m |x|_p \Delta_l(x)\big](t) &= \int\limits_{B_l} \chi_p(tx)|x|_p^\alpha \log_p^m |x|_p\,dx \\ &= \big(1-p^{-1}\big)\log_p^m e \frac{d^m}{d\alpha^m}\left(\frac{p^{l(\alpha+1)}}{1-p^{-(\alpha+1)}}\right)\Delta_{-l}(t) \\ &\quad + \big(1 - \Delta_{-l}(t)\big) \sum_{k=0}^m C_m^k \log_p^{m-k} e \frac{d^{m-k}\Gamma_p(\alpha+1)}{d\alpha^{m-k}} \frac{\log_p^k |t|_p}{|t|_p^{\alpha+1}}, \end{aligned} \tag{7.6}$$

where C_m^k are binomial coefficients, $\Gamma_p(\alpha)$ is given by formula (2.7).

Next, repeating the constructions of Lemma 7.1 practically word for word, we obtain the proof of Lemma 7.2. □

Now we introduce the well-known relation, which we shall need to calculate some integrals.

Recall that the Bernoulli numbers are defined by the following recurrence relation

$$\mathbf{B}_0 = 1, \qquad \sum_{r=0}^{\gamma-1} C_\gamma^r \mathbf{B}_r = 0. \tag{7.7}$$

In particular, $\mathbf{B}_1 = -\frac{1}{2}$, $\mathbf{B}_{2j-1} = 0$, $j = 2, 3, \ldots$, $\mathbf{B}_2 = \frac{1}{6}$, $\mathbf{B}_4 = -\frac{1}{30}$.

Proposition 7.1. *(See [25].)*

$$\begin{aligned} \mathbf{S}_s(\gamma_0) &= \sum_{\gamma=1}^{\gamma_0} \gamma^s = \frac{1}{s+1} \sum_{r=0}^{s} C_{s+1}^r \mathbf{B}_r \gamma_0^{s+1-r} + \gamma_0^s \\ &= \frac{1}{s+1}\big(\gamma_0^{s+1} - C_{s+1}^1 \mathbf{B}_1 \gamma_0^s + C_{s+1}^2 \mathbf{B}_2 \gamma_0^{s-1} + \cdots + C_{s+1}^s \mathbf{B}_s \gamma_0\big), \quad \gamma_0 \geqslant 1, \end{aligned} \tag{7.8}$$

where $\mathbf{B}_r$ are the Bernoulli numbers, $r = 0, 1, \ldots, s$, $s = 0, 1, 2, \ldots$.

One can consider the right-hand side of $\mathbf{S}_s(\gamma_0)$ as a polynomial with respect to γ_0.

Lemma 7.3. *(See [1,3].) If we consider $\mathbf{S}_s(\gamma_0)$ as a polynomial with respect to γ_0 then for $\gamma_0 \leqslant -1$, $s = 0, 1, 2, \ldots,$ we have*

$$\sum_{\gamma=\gamma_0+1}^{0} \gamma^s = -\mathbf{S}_s(\gamma_0). \tag{7.9}$$

Proof. To prove the lemma we rewrite the last sum by using relation (7.8) as

$$\sum_{\gamma=\gamma_0+1}^{0} \gamma^s = (-1)^s \sum_{\gamma=1}^{-\gamma_0-1} \gamma^s$$
$$= -\frac{1}{s+1}\big((\gamma_0+1)^{s+1} - (-1)C_{s+1}^1 \mathbf{B}_1(\gamma_0+1)^s + C_{s+1}^2 \mathbf{B}_2(\gamma_0+1)^{s-1} + \cdots + (-1)^s C_{s+1}^s \mathbf{B}_s(\gamma_0+1)\big). \tag{7.10}$$

Using (7.7), it is easy to see that the coefficients of γ_0^{s+1}, γ_0^s, γ_0^{s-1} in the last sum are equal to 1, $C_{s+1}^1 + C_{s+1}^1\mathbf{B}_1 = -C_{s+1}^1\mathbf{B}_1$, and $C_{s+1}^2 + C_{s+1}^1\mathbf{B}_1 C_s^1 + C_{s+1}^2\mathbf{B}_2 = C_{s+1}^2\mathbf{B}_2$, respectively. Taking into account relation (7.8) and the relation $\mathbf{B}_{2k-1} = 0$, $j = 2, 3, \ldots,$ we calculate the coefficient of γ_0^{s-j}:

$$C_{s+1}^{s-j} + C_{s+1}^1\mathbf{B}_1 C_s^{s-j} + C_{s+1}^2\mathbf{B}_2 C_{s-1}^{s-j} - C_{s+1}^3\mathbf{B}_3 C_{s-2}^{s-j} + \ldots + (-1)^j C_{s+1}^j \mathbf{B}_j C_{s-j}^{s+1-j} + (-1)^{j+1} C_{s+1}^{j+1}\mathbf{B}_{j+1}$$
$$= C_{s+1}^{j+1} \sum_{r=0}^{j} C_{j+1}^r \mathbf{B}_r + C_{s+1}^{j+1}\mathbf{B}_{j+1} = C_{s+1}^{j+1}\mathbf{B}_{j+1},$$

$j = 2, 3, \ldots, s-1$. The coefficient of γ_0^0 is equal to $\sum_{r=0}^{s} C_{s+1}^r \mathbf{B}_r = 0$. The lemma is thus proved. □

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