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Letter to the Editor

Two new asymptotic expansions of the ratio of two gamma functions

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Abstract

Formal expansions, giving as particular cases semiasymptotic expansions, of the ratio of two gamma functions are obtained.

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The ratio of two gamma functions of (not very different) large arguments admits a well-known asymptotic expansion,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim z^{\alpha-\beta} \sum_{n=0}^{\infty} (-1)^n \frac{(\beta-\alpha)_n}{n!} B_n^{(\alpha-\beta+1)}(\alpha) \frac{1}{z^n} \quad \text{as } z \to \infty, \tag{1}$$

where the symbols $B_n^{(\sigma)}(x)$ stand for the generalized Bernoulli polynomials [5,7]. A modification of that expansion, tending to improve its computational efficiency, was suggested by Fields [3], who adopted a related large parameter, namely $w=z+(\alpha+\beta-1)/2$, to obtain an expansion [5, Section 2.11, Eq. (14)] [7, Eq. (3.32)] that contains only even negative powers of the large parameter. Here, with a quite different purpose, we present two expansions, namely

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = w^{\beta-\alpha} \sum_{n=0}^{\infty} \frac{(\beta-\alpha)_n}{n!} {}_2F_0(-n, z+\beta;\,; w), \tag{2}$$

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$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = w^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(\alpha-\beta)_n}{n!} {}_1F_1(-n; z+\alpha; w) \tag{3}$$

that are purely formal for arbitrary w but become semiasymptotic (a term to be specified below) if one takes $w = 1/z$ in (2) and $w = z$ in (3) to obtain

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim z^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(\beta-\alpha)_n}{n!} {}_2F_0(-n, z+\beta;\,; 1/z) \quad \text{as } z \to \infty, \tag{4}$$

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim z^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(\alpha-\beta)_n}{n!} {}_1F_1(-n; z+\alpha; z) \quad \text{as } z \to \infty. \tag{5}$$

Expansion (2) is obtained by making use of the definition of the beta function [5, Section 2.6, Eq. (3)],

$$B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \tag{6}$$

and of its contour integral representation [5, Section 2.7, Eq. (11)],

$$B(x, y) = \frac{\csc(\pi y)}{2\mathrm{i}} \int_{-\infty \mathrm{e}^{\mathrm{i}\delta}}^{(0+)} \mathrm{e}^{vx}(\mathrm{e}^{v}-1)^{y-1}\,\mathrm{d}v, \quad -\pi/2 < \delta < \pi/2,$$

$$\delta - \pi < \arg(\mathrm{e}^{v}-1) \leqslant \delta + \pi, \quad \Re(x\mathrm{e}^{\mathrm{i}\delta}) > 0, \quad y \neq 1, 2, 3, \ldots, \tag{7}$$

to give, by choosing in (7) $\delta = 0$,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{\csc(\pi(\beta-\alpha))}{\Gamma(\beta-\alpha)2\mathrm{i}} \int_{-\infty}^{(0+)} \mathrm{e}^{v(z+\alpha)}(\mathrm{e}^{v}-1)^{\beta-\alpha-1}\,\mathrm{d}v,$$

$$-\pi < \arg(\mathrm{e}^{v}-1) \leqslant \pi, \quad \Re(z+\alpha) > 0, \quad (\beta-\alpha) \neq 1, 2, 3, \ldots. \tag{8}$$

By changing the integration variable from v to

$$u = (1-\mathrm{e}^{-v})/w, \tag{9}$$

w being an arbitrary positive parameter in order to have the same integration contour, one obtains

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{\csc(\pi(\beta-\alpha))}{\Gamma(\beta-\alpha)2\mathrm{i}} \int_{-\infty}^{(0+)} (1-uw)^{-z-\beta}(uw)^{\beta-\alpha-1} w\,\mathrm{d}u. \tag{10}$$

Now we may introduce in the integrand the Maclaurin expansion in powers of u, convergent for $|uw| < 1$,

$$(1-uw)^{-\gamma}\mathrm{e}^{-u} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} {}_2F_0(-n, \gamma;\,; w) u^n, \tag{11}$$

that reflects a known generating relation [4, Chapter 6, Eq. (9)] for the generalized Bessel polynomials. By formally interchanging sum and integration, (10) becomes

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{w^{\beta-\alpha}\csc(\pi(\beta-\alpha))}{\Gamma(\beta-\alpha)2\mathrm{i}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} {}_2F_0(-n, z+\beta;\,; w) \int_{-\infty}^{(0+)} \mathrm{e}^{u} u^{\beta-\alpha+n-1}\,\mathrm{d}u. \tag{12}$$

Now it is immediate to recognize in the last integral a representation of the gamma function given in [5, Section 2.7, Eq. (2)] and to obtain, in this way, the formal expansion (2).

Expansion (3) can be deduced from a representation of the ratio of gamma functions as a Mellin transform [6, Eq. 8.4.47.2],

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{1}{\Gamma(\alpha-\beta)} \int_0^\infty x^{\beta-\alpha-1} {}_0F_1(z+\alpha; -1/x)\, \mathrm{d}x,$$
$$-(1+2\Re(z+\alpha))/4 < \Re(\beta-\alpha) < 0. \tag{13}$$

Let us use a new integration variable,

$$t = 1/wx, \tag{14}$$

w being an arbitrary positive parameter, and substitute the Maclaurin expansion in powers of t

$${}_0F_1(z+\alpha; -wt)\mathrm{e}^t = \sum_{n=0}^{\infty} \frac{1}{n!} {}_1F_1(-n; z+\alpha; w) t^n, \tag{15}$$

closely related to an expression [1, Eq. 22.9.16] showing that the generating function of the Laguerre polynomials is essentially a Bessel function. By interchanging, formally, integration and summation, we obtain from (13)

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{w^{\alpha-\beta}}{\Gamma(\alpha-\beta)} \sum_{n=0}^{\infty} \frac{1}{n!} {}_1F_1(-n; z+\alpha; w) \int_0^\infty t^{\alpha-\beta+n-1} \mathrm{e}^{-t}\, \mathrm{d}t. \tag{16}$$

The last integral is a representation of the gamma function, provided the restriction in (13) is satisfied. One obtains in this way the formal expansion (3). A striking feature of both expansions (2) and (3) is the presence in their right-hand sides of an arbitrary parameter w, not appearing in the left-hand sides. In fact, the formal derivatives with respect to w of the right-hand sides cancel out identically. The restriction of w to positive values, needed to obtain the expansions, is therefore unnecessary. Moreover, the principle of analytic continuation can be invoked to relax the restrictions in Eqs. (7) and (13). But the series in the right-hand sides of (2) and (3) are not convergent, hence the formal nature of the expansions.

Now, let us show that the particular forms (4) and (5) are semiasymptotic expansions. We say that a sequence of functions $\{\psi_n(t)\}$, $n = 0, 1, 2, \ldots$, of the complex variable t is a semiasymptotic one for $t \to t_0$ if, for each n,

$$\psi_{n+1}(t) = \text{either } \mathrm{O}(\psi_n(t)) \text{ or } \mathrm{o}(\psi_n(t)), \quad \psi_{n+2}(t) = \mathrm{o}(\psi_n(t)) \quad \text{as } t \to t_0.$$

Also, in analogy with the definition of asymptotic expansion [2, Section 1.3], we say that, given a semiasymptotic sequence, $\{\psi_n(t)\}$, $n=0, 1, 2, \ldots$, the (formal) series $\sum_{n=0}^{\infty} a_n \psi(t)$ is a semiasymptotic expansion to $N+1$ terms of $f(t)$ as $t \to t_0$ if

$$f(t) = \sum_{n=0}^{N} a_n \psi(t) + \mathrm{O}(\psi_{N+1}) \quad \text{as } t \to t_0.$$

A semiasymptotic expansion to any number of terms, i.e., with $N = \infty$, will be called a semiasymptotic expansion. It is not difficult to see that sequence

$$\psi_n(z) \equiv \frac{(\beta-\alpha)_n}{n!} {}_2F_0\left(-n, z+\beta; ; \frac{1}{z}\right), \quad n = 0, 1, 2, \ldots \tag{17}$$

is a semiasymptotic one as $z \to \infty$. In fact, as it can be easily deduced from

$$ {}_2F_0\left(-0, z+\beta; ; \frac{1}{z}\right) = 1, \quad {}_2F_0\left(-1, z+\beta; ; \frac{1}{z}\right) = -\beta\frac{1}{z} $$

and the recurrence

$$ z\,{}_2F_0\left(-(n+1), z+\beta; ; \frac{1}{z}\right) = -(n+\beta)\,{}_2F_0\left(-n, z+\beta; ; \frac{1}{z}\right) + n\,{}_2F_0\left(-(n-1), z+\beta; ; \frac{1}{z}\right), \tag{18} $$

the hypergeometric sum ${}_2F_0(-n, z+\beta; ; 1/z)$ is a polynomial in the variable $1/z$ the highest order term being $(\beta)_n(-1/z)^n$ and the lowest one being of order $[(n+1)/2]$. Moreover, a relation can be established among the elements of the semiasymptotic sequence $\{\psi_n(z)\}$ and those of the asymptotic one

$$ \phi_n(z) \equiv (-1)^n \frac{(\beta-\alpha)_n}{n!} B_n^{(\alpha-\beta+1)}(\alpha)\frac{1}{z^n}. $$

Such relation reads

$$ \psi_n(z) = \mathrm{O}(\phi_{[(n+1)/2]}(z)) \quad \text{as } z \to \infty. $$

Therefore, since the right-hand side of (4) is a rearrangement of that of (1)

$$ \sum_{n=0}^{N} \psi_n(z) = \sum_{n=0}^{[N/2]} \phi_n(z) + \mathrm{O}(\psi_{N+1}(z)) \quad \text{as } z \to \infty. $$

In consequence,

$$ \begin{aligned} \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} - \sum_{n=0}^{N} \psi_n(z) &= \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} - \sum_{n=0}^{[N/2]} \phi_n(z) + \mathrm{O}(\psi_{N+1}(z)) \\ &= \mathrm{O}(\phi_{[N/2]+1}(z)) + \mathrm{O}(\psi_{N+1}(z)) \\ &= \mathrm{O}(\psi_{N+1}(z)) \quad \text{as } z \to \infty, \end{aligned} $$

that proves that expansion (4) is semiasymptotic. The procedure to demonstrate that (5) is also a semiasymptotic expansion is analogous to the preceding one, the semiasymptotic nature of the sequence $\{{}_1F_1(-n; z+\alpha; z)\}$, $n = 0, 1, 2, \ldots$, for $z \to \infty$ following from

$$ {}_1F_1(-0; z+\alpha; z) = 1, \quad {}_1F_1(-1; z+\alpha; z) = \alpha/(z+\alpha) $$

and the recurrence

$$ (z+\alpha+n)\,{}_1F_1(-(n+1); z+\alpha; z) = (2n+\alpha)\,{}_1F_1(-n; z+\alpha; z) - n\,{}_1F_1(-(n-1); z+\alpha; z). \tag{19} $$

Our expansions (4) and (5) could be used in the computation of the ratio of two gamma functions. Of course, they have no advantage of (1), in terms of accuracy, for that purpose. Nevertheless, the successive terms in the right-hand sides of (4) and (5) are more easily calculable than those in (1) or in the Field's expansion. Besides this, our expansions may be useful for an algebraic manipulation of the ratio of gamma functions.

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