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Applications of Koszul homology to numbers of generators and syzygies

Peter Schenzel

Martin-Luther-Univ. Halle-Wittenberg, Fachb. Math. und Informatik, D–06099 Halle, Germany

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Abstract

Several spectral sequence techniques are used in order to derive information about the structure of finite free resolutions of graded modules. These results cover estimates of the minimal number of generators of defining ideals of projective varieties. In fact there are generalizations of a classical result of Dubreil. On the other hand there are investigations about the shifts and the dimension of Betti numbers. To this end there is a local analogue of Green's considerations developed in [5].

1. Introduction

Let $X \subset \mathbb{P}^n_K$ be an algebraic variety, K an algebraically closed field. Let $\mathscr{I}_X$ denote the ideal sheaf of X. Then $\mathscr{I}_X$ admits a finite minimal resolution

$$\mathscr{F}_\bullet : 0 \to \mathscr{F}_s \to \cdots \to \mathscr{F}_i \to \cdots \to \mathscr{F}_1 \to \mathscr{I}_X \to 0,$$

where $\mathscr{F}_i \simeq \bigoplus_{j\in\mathbb{Z}} \mathcal{O}^{i_j}(-j)$. Here only finitely many i_j are non-zero.

The resolution $\mathscr{F}_\bullet$ reflects several geometric and arithmetic properties of X. For instance, the length s of $\mathscr{F}_\bullet$ satisfies $s \geq \operatorname{codim} X$. The equality characterizes when X is arithmetically Cohen–Macaulay. On the other hand

$$\operatorname{reg} M := \max\{j - i \mid j \in \mathbb{Z} \text{ and } i_j \neq 0\}$$

is called the Castelnuovo–Mumford regularity of X. It is determined by the vanishing of the cohomology $H^i(X, \mathscr{I}_X(n))$, see e.g., [9]. The rank of $\mathscr{F}_1$ is determined by the minimal number of generators $\mu(I_X)$ of I_X, the saturated ideal of X in $R = K[x_0, \ldots, x_n]$.

* E-mail: schenzel@mathematik.uni-halle.de.

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There is a classical result by P. Dubreil, see [3], that for X a set of points in the projective plane $\mathbb{P}^2_K$ it follows that

$$\mu(I_X) \le a(I_X) + 1,$$

where $a(I_X)$ denotes the smallest degree of a hypersurface that contains X. There is an extension of this result to the case of X arithmetically Cohen–Macaulay of codimension two, see [2] or [12]. More recently Martin and Migliore [7, Theorem 2.5] extended Dubreil's Theorem to X a locally Cohen–Macaulay scheme. One of the main points of the present paper is an extension of their result to an arbitrary scheme $X \subset \mathbb{P}^n_K$. In fact it turns out that

$$\mu(I_X) \le a(I_X) + 1 + C(I_X)$$

for a certain correcting term $C(I_X)$, see Theorem 4.1 and Corollary 4.3.

The integer $C(I_X)$ is determined by the dimensions of the Koszul homology modules of $H^i_*(X, \mathcal{O}_X)$ with respect to a certain system of linear forms. These kinds of invariants have been considered by Green in his fundamental paper [5]. In fact we develop a complete local analogue of these modules for any finitely generated graded R-module M. These invariants

$$H_i(\underline{l}; H^j_{\mathfrak{m}}(M)), \quad i,j \in \mathbb{Z},$$

$\underline{l} = l_1, \ldots, l_r$, a system of generic linear forms, are graded R-modules of finite length. We call them Green modules of M with respect to $\underline{l}$. As indicated in Green's paper, see [5], its graded components play an important rôle in getting information about the minimal free resolution of M over R. Under additional assumptions on X resp. M there are explicit geometric interpretations of the Green modules. So e.g. in the case of X an arithmetically Cohen–Macaulay scheme of codimension three it follows that $\mu(I_X) \le a(I_X) + 1 + \deg X$.

On the other hand the Green modules are intimately related to the structure of the minimal free resolution of $\mathcal{F}_\bullet$ of M. In fact they are the ingredients of a spectral sequence for computing $\operatorname{Tor}_i^R(K, M)$, $i \in \mathbb{Z}$. That is, they describe in a certain sense the Betti numbers and their shifts. This is worked out in more detail in Section 5, where it is shown that

$$\operatorname{reg} X = \max\{j - i \mid j \ge \operatorname{codim} X \text{ and } i_j \neq 0\},$$

see Theorem 5.2. That is, the regularity of X is completely determined by the tail of $\mathcal{F}_\bullet$. More precisely, under certain additional assumptions there is an explicit computation of i_j in terms of the graded components of the Green modules, see Theorem 5.5 for the precise statement. It turns out that this is a generalization of Green's duality theorem, see [5, Section 2].

On the other hand Theorem 5.5 is a far reaching generalization of Rao's observation on how much the resolution of the Hartshorne–Rao module $M(C)$ of a curve $C \subset \mathbb{P}^3_K$ determines the resolution of $\mathcal{J}_C$ at the tail, see [11, Eq. (2.5)]. One application of this type concerns the resolution of certain curves $C \subset \mathbb{P}^n_K$ of arithmetic genus $g_a(C) = 0$, see Example 5.7.

While the applications of our results are motivated by geometric questions we formulate and prove them in terms of graded modules over R and its local cohomology modules $H^j_{\mathfrak{m}}(M)$. To this end we fix a few homological preliminaries in Section 2. These concern Koszul homology, local cohomology and some basic facts about spectral sequences. The spectral sequence related to a double complex is in several variations one of the basic tools of our investigations. In Section 3 we summarize the details about the Green modules. The most important result is Theorem 3.4. It proves the finite length of $H_i(\underline{l}; H^j_{\mathfrak{m}}(M))$ for a generic system of linear forms, i.e. for almost all $t \in \mathbb{Z}$ it follows that

$$H_i(\underline{l}; H^j_{\mathfrak{m}}(M))_{i+t} = 0 \quad \text{for all } i, j \in \mathbb{Z}.$$

Section 4 is devoted to the estimates of the number of generators of I_X, i.e. to the desired variations of Dubreil's theorem. The final Section 5 concerns the relation of the syzygies of the modules of 'deficiencies' $H^j_{\mathfrak{m}}(M)$ to those of M. In particular it yields the new characterization of $\operatorname{reg} M$ resp. the generalization of Green's duality theorem.

2. Koszul homology and local cohomology

First fix a few notation and conventions. Let $A = \bigoplus_{n \geq 0} A_n$ denote a graded Noetherian ring such that $A_0 = K$ is an infinite field and $A = K[A_1]$. Then A is an epimorphic image of the polynomial ring $R = K[x_1, \ldots, x_r]$ in the variables $x_1, \ldots, x_r$, $r = \dim_K A_1$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ denote a graded A-module. For $k \in \mathbb{Z}$ let $M(k)$ denote the module M with the grading given by $[M(k)]_n = M_{k+n}$, $n \in \mathbb{Z}$. Mostly we consider a graded A-module as a module over R. For more details about graded modules and rings see [1, Section 1.5].

Let $X^\bullet, Y^\bullet$ denote two complexes of graded A-modules. Let $Z^\bullet$ denote the single complex associated to the double complex $X^\bullet \otimes_A Y^\bullet$. Then there are the following two spectral sequences:

$$\begin{aligned} E_2^{ij} &= H^i(X^\bullet \otimes_A H^j(Y^\bullet)) \Rightarrow E^{i+j} = H^{i+j}(Z^\bullet) \\ {}'E_2^{ij} &= H^i(H^j(X^\bullet) \otimes_A Y^\bullet) \Rightarrow {}'E^{i+j} = H^{i+j}(Z^\bullet). \end{aligned}$$

See e.g. [4, Appendix A3] or [15, Section 5.6] for an introduction and the basic results concerning spectral sequences. Here we remark that by the definitions all the homomorphisms are homogeneous of degree zero.

Let $\underline{f} = f_1, \ldots, f_s$ denote a system of homogeneous elements. Then $K_\bullet(\underline{f}; A)$ denotes the Koszul complex with respect to $\underline{f}$. Fix the following definitions:

$$\begin{aligned} K_\bullet(\underline{f}; M) &:= K_\bullet(\underline{f}; A) \otimes_A M, & H_i(\underline{f}; M) &:= H_i(K_\bullet(\underline{f}; M)), \\ K^\bullet(\underline{f}; M) &:= \operatorname{Hom}_A(K_\bullet(\underline{f}; A), M), & H^i(\underline{f}; M) &:= H^i(K^\bullet(\underline{f}; M)), \end{aligned}$$

where $i \in \mathbb{Z}$, see e.g. [1, Section 1.6] or [4, Section 17]. Note that all the modules resp. complexes are graded. The homomorphims are homogeneous of degree zero. In particular let $\underline{\mathfrak{m}} = x_1, \ldots, x_r$ denote a generating set of $\mathfrak{m}$, the ideal generated by all forms of positive degree. Then $K_\bullet(\underline{x}; R)$ provides a finite free resolution of K, the residue field. Therefore

$$H_i(\mathfrak{m}; M) \simeq \operatorname{Tor}_i^R(K, M) \quad \text{and} \quad H^i(\mathfrak{m}; M) \simeq \operatorname{Ext}_R^i(K, M), \quad i \in \mathbb{Z}.$$

In the following split $\mathfrak{m}$ into two subsets $\underline{x}$ and $\underline{y}$. Then we compare their Koszul homologies.

Lemma 2.1. *Put* $\underline{x} = x_1, \ldots, x_s$ *and* $\underline{y} = x_{s+1}, \ldots, x_r$ *for an integer* $1 \le s < r$. *Then*

$$\dim_K H_n(\mathfrak{m}; M) \le \sum_{i=\max\{0, n-(r-s)\}}^{\min\{s,n\}} \dim_K H_i(\underline{x}; H_{n-i}(\underline{y}; M)), \quad n \in \mathbb{N},$$

for any finitely generated graded R-module M.

Proof. First note that

$$K_\bullet(\mathfrak{m}; M) \simeq K_\bullet(\underline{x}; R) \otimes_R K_\bullet(\underline{y}; M)$$

as follows by view of the construction of the Koszul complex. But then there is the following spectral sequence:

$$E_{ij}^2 = H_i(\underline{x}; H_j(\underline{y}; M)) \Rightarrow E_{i+j} = H_{i+j}(\mathfrak{m}; M).$$

Moreover note that all the E_{ij}^2-terms are finite dimensional K-vector spaces. This follows because all of them are annihilated by $\mathfrak{m}$. The subsequent terms E_{ij}^n are subquotients of E_{ij}^2. So they are also finite dimensional and

$$\dim_K E_{ij}^n \le \dim_K E_{ij}^2 \quad \text{for all } n \ge 2.$$

Now for large n one has $E_{ij}^\infty = E_{ij}^n$. Furthermore it is known that

$$E_{i+j} = H_{i+j}(\mathfrak{m}; M)$$

admits a finite filtration whose quotients are $E_{i,n-i}^\infty$, $i = 0, 1, \ldots, n$. This implies that

$$\dim_K H_n(\mathfrak{m}; M) = \sum_{i=0}^{n} \dim_K E_{i,n-i}^\infty.$$

So the claim follows because of the above estimates. □

For further investigations the case of $n = 1$ is of a particular interest. To this end formulate it as a separate corollary.

Corollary 2.2. *Let $\underline{x}, \underline{y}$, and M as in Lemma 2.1. Then*

$$\dim_K H_1(\mathfrak{m}; M) \leq \dim_K H_0(\underline{x}; H_1(\underline{y}; M)) + \dim_K H_1(\underline{x}; H_0(\underline{y}; M)).$$

The general idea behind Lemma 2.1 is a bound of the Betti numbers

$$b_n(M) := \dim_K \operatorname{Tor}_n^R(K, M).$$

For $n = 1$ and $M = R/I, I$ a homogeneous ideal of R, this yields a bound for the minimal number of generators $\mu(I)$ of I, see Theorem 4.1.

For several investigations we need the local cohomology modules $H^i_{\mathfrak{m}}(M), i \in \mathbb{Z}$, of M with respect to $\mathfrak{m}$. To this end denote by $K^\bullet_f(A)$ the complex $0 \to A \to A_f \to 0$, where f denotes a homogeneous element and A_f is the localization with respect to f. The middle homomorphism denotes the canonical map into the localization. For $\underline{\mathfrak{m}} = x_1 \ldots, x_r$ define

$$K^\bullet := \bigotimes_{i=1}^{r} K^\bullet_{x_i} \quad \text{and} \quad K^\bullet(M) := K^\bullet \otimes_A M$$

the Čech complex of A and M. Then there are canonical isomorphisms

$$H^i_{\mathfrak{m}}(M) \simeq H^i(K^\bullet \otimes_A M) \quad \text{for all } i \in \mathbb{Z}.$$

For the details of this facts see e.g. [6] or [1, Section 3.5].

3. The Green modules

Let M denote a finitely generated graded R-module. In this section we introduce certain invariants related to the local cohomology and the Koszul homology of M. To this end we need the notion of a generic system of elements.

Definition 3.1. A system of linear elements $\underline{l} = l_1, \ldots, l_s$ is said to be a generic linear system of elements with respect to M provided

$$l_i \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Ass}_R(M/(l_1, \ldots, l_{i-1})M) \setminus \{\mathfrak{m}\}.$$

Here $\mathfrak{m}$ denotes the ideal generated by the variables $x_1, \ldots, x_r$ in R.

Note that $\underline{l}$ is a generic system of linear elements if and only if the following quotients

$$((l_1, \ldots, l_{i-1})M :_M l_i)/(l_1, \ldots, l_{i-1})M, \quad i = 1, \ldots, s,$$

are graded R-modules of finite length. This observation is helpful in order to check whether a given $\underline{l}$ is a generic linear system. The most important property of a general linear system is related to a certain finiteness property of $H_i(\underline{l}; H^j_{\mathfrak{m}}(M))$ which we shall prove in this section. In order to do that we need another auxiliary statement.

Lemma 3.2. *Let $\underline{l} = l_1, \dots, l_s$ denote a generic linear system with respect to M. Then $H^i(\underline{l}; M)$ is an R-module of finite length in the following two cases:*
(a) $i < s$,
(b) *for all $i \in \mathbb{Z}$, provided $s \geq \dim_R M$.*

Proof. First prove the claim in (b). To this end note that

$$(\underline{l}, \operatorname{Ann}_R M) H^i(\underline{l}; M) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

Therefore $\operatorname{Supp}_R H^i(\underline{l}; M) \subseteq \operatorname{Supp}_R M/\underline{l}M$. Since $s \geq \dim_R M$ it follows that $\operatorname{Supp}_R M\underline{l}M \subseteq V(\mathfrak{m})$. Recall that $\underline{l}$ is generically chosen. This proves (b) since $H^i(\underline{l}; M)$ is a finitely generated R-module.

The statement in (a) will be shown by an induction on $d := \dim_R M$. First note that the case $d = 0$ is covered by the claim proved in (b). So let $d > 0$. First suppose that $\operatorname{depth}_R M > 0$. Then $l = l_1$ is an M-regular element. The short exact sequence

$$0 \to M(-1) \to M \to M/lM \to 0$$

induces short exact sequences

$$0 \to H^i(\underline{l}; M) \to H^i(\underline{l}; M/lM) \to H^{i+1}(\underline{l}; M)(-1) \to 0$$

for all $i \in \mathbb{Z}$. Note that $lH^i(\underline{l}; M) = 0$. Hence the induced maps on the Koszul cohomology are trivial. Now put $\underline{l}' = l_2, \dots, l_s$. Then

$$H^i(\underline{l}; M/lM) \simeq H^i(\underline{l}'; M/lM) \oplus H^{i-1}(\underline{l}'; M/lM)(-1), \quad i \in \mathbb{Z},$$

see [1, Section 1.6]. Note that l acts trivially by multiplication on M/lM, Now by induction hypothesis $H^i(\underline{l}'; M/lM)$ is of finite length for all $i < s-1$. Therefore $H^i(\underline{l}; M/lM)$ is of finite length for all $i < s-1$. Whence the above short exact sequence proves the claim.

Finally let $\operatorname{depth}_R M = 0$. Then $N := \bigcup_{n \geq 1} (0 :_M l^n)$ is an R-module of finite length as follows by the definition of $\underline{l}$. Then $\operatorname{depth}_R M/N > 0$. By the first part of the inductive step and $\dim_R M/N = d$ the claim is true for M/N. Note that $\underline{l}$ forms a generic system of linear forms with respect to M/N as easily seen by a localization argument with respect to non-maximal prime ideals. By (b) the claim is true for N and all $i \geq 0$ since $\dim_R N = 0$. So the final statement for M follows from the induced long exact Koszul cohomology sequence derived from $0 \to N \to M \to M/N \to 0$. $\square$

In the following consider the functor $\operatorname{Hom}_K(\square, K) = \square^\vee$ on the category of graded R-modules. By the graded version of the Local Duality Theorem, see [1, Section 3.6.19], it turns out that there is a natural graded isomorphism of degree zero

$$H^i_{\mathfrak{m}}(M) \simeq (\operatorname{Ext}_R^{r-i}(M, R(-r)))^\vee, \quad i \in \mathbb{Z}.$$

Put $K_M^i := \operatorname{Ext}_R^{r-i}(M, R(-r))$, $i \in \mathbb{Z}$. Then $K_M^i = 0$ for $i < 0$ and $i > \dim_R M := d$. In particular $K_M := K_M^d$ is called the canonical module of M. These modules are studied in a systematic way in [13, Section 3]. Here we mention only that

$$\dim_R K_M^i \leq i, \text{ for } 0 \leq i < d \quad \text{and} \quad \dim_R K_M = d,$$

see [13, Section 3.1] for the details.

For the next results we need another definition of genericity. It is related to the modules of 'deficiency' K_M^i.

Definition 3.3. A generic system of linear elements $\underline{l} = l_1, \ldots, l_s$ is called a strongly generic linear system of elements with respect to M provided it is a generic linear system of elements for all K_M^i, $i = 0, \ldots, d$.

Because K is an infinite field it is clear that strongly generic linear systems of elements with respect to M always exist. Their construction is just an application of prime avoidance arguments.

Theorem 3.4. *Suppose that $\underline{l} = l_1, \ldots, l_s$ is a strongly generic linear system of elements with respect to M. Then $H_i(\underline{l}; H_{\mathfrak{m}}^j(M))$ is a graded R-module of finite length in the following two cases:*

(a) *$i < s$, and*

(b) *for all $i \in \mathbb{Z}$ provided $s \geq j$.*

Proof. First observe that there are canonical isomorphisms

$$H_i(\underline{l}; N^\vee) \simeq (H^i(\underline{l}; N))^\vee, \quad i \in \mathbb{Z}.$$

This follows because of the isomorphism of complexes

$$K_\bullet(\underline{l}; R) \otimes_R (N)^\vee \simeq (\operatorname{Hom}_R(K_\bullet(\underline{l}; R), N))^\vee,$$

which is well known. Here N denotes an arbitrary graded R-module. Put $N = K_M^i$. Then it follows that

$$H_i(\underline{l}; H_{\mathfrak{m}}^j(M)) \simeq (H^i(\underline{l}; K_M^j))^\vee \quad \text{for all } i, j \in \mathbb{Z}.$$

By Lemma 3.2 it is known that $H^i(\underline{l}; K_M^j)$ is an R-module of finite length for $i < s$ resp. for all $i \in \mathbb{Z}$ provided $s \geq \dim_R K_M^j$. Because of $\dim_R K_M^j \leq j$ this finishes the proof. □

In his paper [5] Green considered the following situation. Let $\mathscr{F}$ denote a coherent sheaf on X, a compact complex manifold. Then he considered the vector spaces $\mathscr{K}_{p,q}^i(X, \mathscr{F})$. Let $i \geq 1$. Then it is easy to see, see [5], that

$$\mathscr{K}_{p,q}^i(X, \mathscr{F}) \simeq H_p(\mathfrak{m}; H_{\mathfrak{m}}^{i+1}(M))_{p+q},$$

where M denotes the associated graded module to $\mathscr{F}$. So in an obvious way the Koszul homology modules $H_p(\mathfrak{m}; H^{i+1}_{\mathfrak{m}}(M))$ are graded analogues of the invariants introduced by Green. As an application of Theorem 3.4 it turns out that $\mathscr{K}^i_{p,q}(X, \mathscr{F}) = 0$ for all $q \ll 0$ resp. for all $q \gg 0$ provided $\underline{l}$ is strongly generically chosen. For the numerical influence of these finitely many non-vanishing $\mathscr{K}^i_{p,q}(X, \mathscr{F})$ on free resolutions see the results in Section 5.

The most important feature of $H_p(\mathfrak{m}; H^{i+1}_{\mathfrak{m}}(M))$ is that it is one of the ingredients of a spectral sequence. In the following let M denote a finitely generated graded R-module. Choose $\underline{l} = l_1, \ldots, l_s$, $s \geq \dim_R M$, a generic linear system of elements with respect to M. Then consider the following complexes $K^\bullet$, the Čech complex, $K_\bullet(\underline{l}; M)$, the Koszul complex of M with respect to $\underline{l}$, and $C^\bullet := K^\bullet \otimes_R K_\bullet(\underline{l}; M)$. Then there is the following spectral sequence:

$$H^i_{\mathfrak{m}}(H_j(\underline{l}; M)) \Rightarrow H_{j-i}(C^\bullet).$$

Because of the choice of $\underline{l}$ it turns out that $H_j(\underline{l}; M) \simeq H^{s-j}(\underline{l}; M)(-s)$ are R-modules of finite length for all $j \in \mathbb{Z}$, see Lemma 3.2. Because of the basic properties of local cohomology it yields that

$$H^i_{\mathfrak{m}}(H_j(\underline{l}; M)) = \begin{cases} H_j(\underline{l}; M) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

Therefore the spectral sequence degenerates partially to the isomorphisms $H_j(C^\bullet) \simeq H_j(\underline{l}; M)$ for all $j \in \mathbb{Z}$. The second spectral sequence for the corresponding double complex is $H_j(\underline{l}; H^i_{\mathfrak{m}}(M)) \Rightarrow H_{j-i}(C^\bullet)$. Putting this together it proves the following

Lemma 3.5. *Let M be a finitely generated graded R-module. Let $\underline{l} = l_1, \ldots, l_s$, $s \geq \dim_R M$ be a generic linear system with respect to M. Then there is a spectral sequence*

$$E_2^{-j,i} = H_j(\underline{l}; H^i_{\mathfrak{m}}(M)) \quad \Rightarrow \quad E^{-j+i} = H_{j-i}(\underline{l}; M),$$

where all the derived homomorphisms are homogeneous of degree zero.

In the more special situation of a strongly generic linear system with respect to M not only $H_{j-i}(\underline{l}; M)$ but also $H_j(\underline{l}; H^i_{\mathfrak{m}}(M))$ are modules of finite length for all $i, j \in \mathbb{Z}$, see Theorem 3.4. Therefore there is an estimate for the length of $H_{j-i}(\underline{l}; M)$.

Corollary 3.6. *Suppose that $\underline{l} = l_1, \ldots, l_s$, $s \geq \dim_R M$, denotes a strongly generic linear system of elements with respect to M. Then*

$$L_A(H_n(\underline{l}; M)) \leq \sum_{i=0}^{\min\{s-n,d\}} L_A(H_{n+i}(\underline{l}; H^i_{\mathfrak{m}}(M))),$$

for all $n \in \mathbb{Z}$, where $d = \dim_R M$.

Proof. First note that $H^i_{\mathfrak{m}}(M) = 0$ for all $i > d$ resp. $H_j(\underline{l}; M) = 0$ for all $j > s$. Then the estimate follows by the same line of reasoning as in the proof of lemma 2.1. □

The spectral sequence in Lemma 3.5 has several more applications in Sections 4 and 5. Here we want to add just two simple consequences. They are helpful also in different situations.

Corollary 3.7. *Let M be a finitely generated graded R-module. Suppose that $\underline{l} = l_1,\dots,l_s$, $s \geq \dim_R M$ denotes a generic linear sytem with respect to M. Then*

(a) $H_{s-t}(\underline{l};M) \simeq H_s(\underline{l};H^t_{\mathfrak{m}}(M))$, *where* $t = \operatorname{depth}_R M$, *and*

(b) $H_{i-d}(\underline{l};M) \simeq H_i(\underline{l};H^d_{\mathfrak{m}}(M))$, *for all* $i \in \mathbb{Z}$, *provided M is a d-dimensional Cohen–Macaulay module.*

Proof. In order to prove (a) consider the spectral sequence in Lemma 3.5. Take the terms $E_2^{-j,i}$ with $j - i = s - t$. Then

$$E_2^{-j,i} = \begin{cases} 0 & \text{for } j > s \text{ or } i < t, \\ H_s(\underline{l};H^t_{\mathfrak{m}}(M)) & \text{for } j = s \text{ and } i = t. \end{cases}$$

But this means that the spectral sequence degenerates partially to the desired isomorphism.

The claim in (b) follows by a similar argument since $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq d$ in the case of a Cohen–Macaulay module M. □

For an extension of the results of this section to the situation of a finitely generated module over a local ring, see [14].

4. Bounds on the number of generators

For homogeneous ideals $I \subset R = K[x_1,\dots,x_r], r \geq 2$, such that I is a perfect ideal of codimension two it is known that

$$\mu(I) \leq a(I) + 1,$$

where $a(I) = \min\{n \in Z \mid I_n \neq 0\}$, the initial degree of I. Note that $a(I)$ is equal to the minimal degree of a non-zero form contained in I. This estimate is a generalization of a corresponding bound given by Dubreil in the case of $r = 3$, see [3]. For the proof see e.g. [2] resp. [12]. An approach related to Hilbert functions is developed in [2], while [12] contains a proof based on the Hilbert–Burch Theorem.

In the following put $\underline{x} = x_1, x_2$, $\underline{y} = x_3,\dots,x_r$, $r \geq 3$, where $x_1,\dots,x_r$ denotes a set of generators of $\mathfrak{m}$. In a certain sense the following result is a generalization of Dubreil's Theorem.

Theorem 4.1. *Let $I \subset R$ denote a homogeneous ideal of codimension at least two. Then*

$$\mu(I) \leq a(I) + 1 + \mu(H_1(\underline{y};R/I)),$$

where $\underline{y}$ is chosen generically with respect to R/I.

Proof. First put $S = R/\underline{y}R$ and $J = IS$. Then we obtain the bound

$$\mu(I) \le \mu(J) + \dim_K H_0(\underline{x}; H_1(\underline{y}; R/I)),$$

as follows by Corollary 2.2. By the generic choice of $\underline{y}$ it is known that $a(I) = a(J)$. Now J is a perfect ideal of codimension two in S. Therefore $\mu(J) \le a(J) + 1$. Finally the dimension of the vector space $H_0(\underline{x}; H_1(\underline{y}; R/I))$ coincides with the number of generators of $H_1(\underline{y}; R/I)$. □

In fact Theorem 4.1 is a generalization of Migliore's result, see [8, Corollary 3.3], in the case of the defining ideal $\mathcal{J}_C$ of a curve $C \subset \mathbb{P}^3_K$. Here we extend his result to an arbitrary projective scheme.

Corollary 4.2. *Let $I \subset R$ denote a homogeneous ideal with* $\operatorname{codim} I \ge 2$. *Put* $t = \operatorname{depth} R/I$. *Then*

$$\mu(I) \le a(I) + 1 + \mu(H_{t+1}(\underline{y}; H^t_{\mathfrak{m}}(R/I))),$$

where $\underline{y}$ is chosen strongly generic with respect to R/I.

Proof. By Corollary 3.7 there is the following isomorphism

$$H_1(\underline{y}; R/I) \simeq H_{t+1}(\underline{y}; H^t_{\mathfrak{m}}(R/I)).$$

Therefore the claim is a consequence of Theorem 4.1. □

In the situation of I the saturated defining ideal of curve $C \subset \mathbb{P}^3_K$ it follows that $t = 1$. Therefore $H_2(l_1, l_2; H^1_{\mathfrak{m}}(R/I))$ is just the submodule of $H^1_*(\mathcal{J}_C)$ annihilated by l_1, l_2, see [8, Corollary 3.3].

Besides of its vanishing it is known that Koszul homology is difficult to handle. So for the rest of this section there are several approaches in order to estimate the term $\mu(H_1(\underline{y}; R/I))$ in Theorem 4.1.

Corollary 4.3. *Let $I \subset R$ denote a homogeneous ideal of codimension at least two with* $d = \dim R/I$. *Then*

$$\mu(I) \le a(I) + 1 + \sum_{i=0}^{d} L_R(H_{i+1}(\underline{y}; H^i_{\mathfrak{m}}(R/I))).$$

Moreover, suppose that $H^i_{\mathfrak{m}}(R/I)$ are graded R-modules of finite length for $i = 0, 1, \dots, d-1$. *Then*

$$\mu(I) \le a(I) + 1 + \sum_{i=0}^{d-1} \binom{r-2}{i+1} L_R(H^i_{\mathfrak{m}}(R/I)) + L_R(H_{d+1}(\underline{y}; H^d_{\mathfrak{m}}(R/I))).$$

Here $\underline{y}$ is chosen strongly generic with respect to R/I.

Proof. Under the additional assumption that $\underline{y}$ is a strongly generic system of linear forms with respect to R/I it follows that $H_{i+1}(\underline{y}; H^i_{\mathfrak{m}}(R/I))$ are graded R-modules of finite length, see Theorem 3.4. Then the spectral sequence in Lemma 3.5 provides the following estimate:

$$L_R(H_1(\underline{y}; R/I)) \le \sum_{i=0}^{d} L_R(H_{i+1}(\underline{y}; H^i_{\mathfrak{m}}(R/I))).$$

By virtue of Theorem 4.1 this proves the first part of the claim.

Under the additional assumption of the finite length of $H^i_{\mathfrak{m}}(R/I)$ for $i = 0, 1, \dots, d-1$, it is easy to see that

$$L_R(H_{i+1}(\underline{y}; H^i_{\mathfrak{m}}(R/I))) \le \binom{r-2}{i+1} L_R(H^i_{\mathfrak{m}}(R/I)), \quad i = 0, \dots, d-1.$$

To this end consider the definition of the Koszul homology. Therefore the second bound follows. □

Note that Corollary 4.3 was shown by Martin and Migliore, [7, Theorem 2.5], under the additional assumption that $\operatorname{Proj} R/I$ is equidimensional and a Cohen–Macaulay scheme. Of particular interest is the case of $\operatorname{codim} I = 2$. In this situation the term $H_{d+1}(\underline{y}; H^d_{\mathfrak{m}}(R/I))$ does not occur since $d+1 = r-1 > r-2$, the number of elements of $\underline{y}$.

In the following let $\sigma(N)$ denote the socle dimension of N. That means $\sigma(N) = \dim_K \operatorname{Hom}_R(R/\mathfrak{m}, N)$ for an arbitrary R-module N.

Corollary 4.4. *Let $I \subset R$ denote a perfect homogeneous ideal of codimension at least three. Then*

$$\mu(I) \le a(I) + 1 + \sigma(H^{d+1}(\underline{y}; K_{R/I})),$$

where $\underline{y}$ is chosen strongly generic with respect to R/I. Here $K_{R/I}$ denotes the canonical module of R/I.

Proof. Because R/I is a Cohen-Macaulay ring we have to estimate $\mu(H_{d+1}(\underline{y}; H^d_{\mathfrak{m}}(R/I)))$, $d = \dim R/I$, see Corollary 4.2. But now

$$H_{d+1}(\underline{y}; H^d_{\mathfrak{m}}(R/I)) \simeq (H^{d+1}(\underline{y}; K_{R/I}))^{\vee}.$$

Therefore the dimension of $R/\mathfrak{m} \otimes_R H_{d+1}(\underline{y}; H^d_{\mathfrak{m}}(R/I))$ is equal to the socle dimension of $H^{d+1}(\underline{y}; K_{R/I})$ as easily seen. □

Of a particular interest is the case of a Gorenstein ideal of codimension three. In this situation it follows:

Corollary 4.5. *Let $I \subset R, \underline{y}$ be as in Corollary 4.4. Suppose that R/I is a Gorenstein ring and* $\operatorname{codim} I = 3$. *Then* $\mu(I) \le 2a(I) + 1$.

Proof. Because R/I is a Gorenstein ring it is known that $R/I \simeq K_{R/I}$. Put $\underline{z} = x_3, \ldots, x_{r-1}$, $y = x_r$. Now define $S = R/\underline{z}R$, $J = IS$. Then $H^{d+1}(\underline{y}; K_{R/I}) \simeq S/(J, yS)$. But now the socle dimension of $S/(J, yS)$ is equal to the type of $S/(J, yS)$, or what is the same, to the minimal number of generators of L minus one, $\mu(L) - 1$, where $T = S/yS$ and $L = JT$. Recall that L is a perfect ideal of codimenssion two in T. But then $\mu(L) \le a(L) + 1$ by Dubreil's Theorem. Finally $a(I) = a(L)$ since $\underline{y}$ is chosen generically. Therefore by Corollary 4.4 the claim is shown to be true. □

Note that this result follows also by the Buchsbaum–Eisenbud structure theorem for Gorenstein ideals of codimension three. For the details see [12]. A further result including the degree is the following:

Corollary 4.6. *Let $I \subset R$, $\underline{y}$ be as in Corollary 4.4. Suppose* $\operatorname{codim} I = 3$. *Then*

$$\mu(I) \le a(I) + 1 + e(R/I),$$

where $e(R/I)$ denotes the multiplicity of R/I.

Proof. First note that by Corollary 4.4 it is obviously true that

$$\sigma(H^{d+1}(\underline{y}; K_{R/I})) \le L_R(H^{d+1}(\underline{y}; K_{R/I})) \le L_R(K_{R/I}/\underline{z}K_{R/I}).$$

Here let $\underline{y}$ be generated by $y_1, \ldots, y_{d+1}$ and $\underline{z} = y_1, \ldots, y_d$. Then $\underline{z}$ forms a system of parameters for $K_{R/I}$ and R/I as well. Furthermore $L_R(K_{R/I}/\underline{y}K_{R/I}) \le L_R(K_{R/I}/\underline{z}K_{R/I})$. Because R/I is a Cohen–Macaulay ring, $K_{R/I}$ is a Cohen–Macaulay module and therefore

$$L_R(K_{R/I}/\underline{z}K_{R/I}) = e(\underline{z}; K_{R/I}) = e(\underline{z}; R/I).$$

Because of the generic choice of the linear elements in $\underline{z}$ this completes the proof. □

The bound in Corollary 4.6 is rather rough. It would be of some interest to find a common generalization of Corollaries 4.5 and 4.6.

5. Koszul homology and syzygies

As before let $R = K[x_1, \ldots, x_r]$ denote the polynomial ring in r variables. For a graded R-module M define

$$a(M) = \min\{n \in \mathbb{Z} \mid M_n \neq 0\} \quad \text{and} \quad e(M) = \max\{n \in \mathbb{Z} \mid M_n \neq 0\}.$$

It is well known that $e(H^i_{\mathfrak{m}}(M)) < \infty$ for all $i \in \mathbb{Z}$.

Definition 5.1. The Castelnuovo–Mumford regularity $\operatorname{reg} M$ of M is defined by

$$\operatorname{reg} M = \max\{e(H^i_{\mathfrak{m}}(M)) + i \mid i \in \mathbb{Z}\}.$$

Note that $e(0) = -\infty$.

It is a well known fact that

$$\operatorname{reg} M = \max\{e(\operatorname{Tor}_i^R(K,M)) - i \mid 0 \le i \le r\}.$$

So $\operatorname{reg} M$ yields a bound on the maximal degree in a minimal generating set of the syzygy modules of M. It reflects the structure of the minimal free resolution $F_\bullet$ of M over R, where

$$F_\bullet : 0 \to F_s \to \cdots \to F_i \to \cdots \to F_0 \to M \to 0,$$

with $F_i \simeq \bigoplus_{j\in\mathbb{Z}} R^{i_j}(-j)$ and $i_j = \dim_K \operatorname{Tor}_i^R(K,M)_j$. Suppose that M is a Cohen–Macaulay module. Then $\operatorname{reg} M = e(\operatorname{Tor}_c^R(K,M)) - c$, where $c = r - \dim_R M$ denotes the codimension of M. This follows easily since $\operatorname{Hom}_R(F_\bullet, R(-r))$ gives a minimal free resolution of $K_M = \operatorname{Ext}_R^c(M,R(-r))$, the canonical module of M.

On the other hand it was observed by Rao, see [11, Eq. (2.5)], that in the case of I the defining ideal of a curve $C \subset \mathbb{P}_K^3$ the Hartshorne–Rao module $M(C) \simeq H_{\mathfrak{m}}^1(R/I)$ gives certain information on the tail of the minimal free resolution of R/I.

In the following we shall generalize both of these observations. Firstly we describe $\operatorname{reg} M$ in terms of the Tors in a certain range. Secondly we shall clarify how the minimal free resolutions of $H_{\mathfrak{m}}^i(M)$, the 'modules of deficiency', determine the minimal free resolution of M. Both considerations turn out by a careful study of the spectral sequence given in Lemma 3.5.

Theorem 5.2. *Let M denote a finitely generated graded R-module. Let $s \in \mathbb{N}$, then the following two integers coincide*

(a) $\max\{e(H_{\mathfrak{m}}^i(M)) + i \mid 0 \le i \le s\}$ *and*

(b) $\max\{e(\operatorname{Tor}_j^R(K,M)) - j \mid r - s \le j \le r\}$.

In particular for $s = \dim_R M$ it follows that

$$\operatorname{reg} M = \max\{e(\operatorname{Tor}_j^R(K,M)) - j \mid c \le j \le r\},$$

where $c = r - \dim_R M$ denotes the codimension of M.

Before proving Theorem 5.2 we separate two partial results as lemmas. They concern results in this direction which seem to be of some independent interest.

Lemma 5.3. *Suppose that $H_s(\mathfrak{m};M)_{s+t} \neq 0$ for a certain $t \in \mathbb{Z}$ and $r - i \le s \le r$. Then there exists an $j \in \mathbb{Z}$ such that $0 \le j \le i$ and $H_{\mathfrak{m}}^j(M)_{t-j} \neq 0$.*

Proof. Assume the contrary, i.e. $H_{\mathfrak{m}}^j(M)_{t-j} = 0$ for all $0 \le j \le i$. Then consider the spectral sequence

$$[E_2^{-s-j,j}]_{t+s} = H_{s+j}(\mathfrak{m}; H_{\mathfrak{m}}^j(M))_{t+s} \;\Rightarrow\; [E^{-s}]_{t+s} = H_s(\mathfrak{m};M)_{t+s}$$

as defined in Corollary 4.5. Recall that all the homomorphisms are homogeneous of degree zero. Now the corresponding E_2-term is a subquotient of

$$\left[\bigoplus H^j_{\mathfrak{m}}(M)^{\binom{r}{s+j}}(-s-j)\right]_{t+s}.$$

Let $j \le i$. Then this vectorspace is zero by the assumption about the local cohomology. Let $j > i$. Then $s+j > s+i \ge r$ and $\binom{r}{s+j} = 0$. Therefore the corresponding E_2-term $[E_2^{-s-j,j}]_{t+s}$ is zero for all $j \in \mathbb{Z}$. But then also all the subsequent stages are zero, i.e. $[E_\infty^{-s-j,j}]_{t+s} = 0$ for all $j \in \mathbb{Z}$. Therefore $[E^{-s}]_{t+s} = H_s(\mathfrak{m}; M)_{t+s} = 0$, contradicting the assumption. □

The second partial result shows that a certain non-vanishing of $H^i_{\mathfrak{m}}(M))$ yields the existence of a minimal generator of a higher syzygy module.

Lemma 5.4. *Suppose that there are integers s, b such that the following conditions are satisfied*:

(a) $H^i_{\mathfrak{m}}(M)_{b+1-i} = 0$ *for all* $i < s$ *and*

(b) $H_r(\mathfrak{m}; H^s_{\mathfrak{m}}(M))_{b+r-s} \neq 0$.

Then it follows that $H_{r-s}(\mathfrak{m}; M)_{b+r-s} \neq 0$.

Note that the condition (b) in Lemma 5.4 means that $H^s_{\mathfrak{m}}(M)$ possesses a socle generator in degree $b-s$. Recall that r denotes the number of generators of $\mathfrak{m}$.

Proof. As above we consider the spectral sequence

$$E_2^{-r,s} = H_r(\mathfrak{m}; H^s_{\mathfrak{m}}(M)) \quad \Rightarrow \quad E^{-r+s} = H_{r-s}(\mathfrak{m}; M)$$

in degree $b+r-s$. The subsequent stages of $[E_2^{-r,s}]_{b+r-s}$ are derived by the cohomology of the following sequence:

$$[E_n^{-r-n,s+n-1}]_{b+r-s} \to [E_n^{-r,s}]_{b+r-s} \to [E_n^{-r+n,s-n+1}]_{b+r-s}$$

for $n \ge 2$. But now $[E_n^{-r-n,s+n-1}]_{b+r-s}$ resp. $[E_n^{-r+n,s-n+1}]_{b+r-s}$ are subquotients of

$$H_{r+n}(\mathfrak{m}; H^{s+n-1}_{\mathfrak{m}}(M))_{b+r-s} = 0 \quad \text{resp.} \quad H_{r-n}(\mathfrak{m}; H^{s-n+1}_{\mathfrak{m}}(M))_{b+r-s} = 0.$$

For the second module recall that it is a subquotient of

$$\left[\bigoplus H^{s-n+1}_{\mathfrak{m}}(M)^{\binom{r}{r-n}}(-r+n)\right]_{b+r-s} = 0, \quad n \ge 2.$$

Therefore $[E_2^{-r,s}]_{b+r-s} = [E_\infty^{-r,s}]_{b+r-s} \neq 0$ and

$$[E^{-r+s}]_{b+r-s} \simeq H_{r-s}(\mathfrak{m}; M)_{b+r-s} \neq 0$$

as follows by the filtration with the corresponding E_∞-terms. □

Proof of Theorem 5.2. First of all let us introduce two abbreviations. Put $a := \max\{e(\mathrm{Tor}_j^R(K,M)) - j \mid r - s \le j \le r\}$. Then by Lemma 5.3 it follows that $a \le b$, where $b := \max\{e(H_{\mathfrak{m}}^i(M)) + i \mid 0 \le i \le s\}$. On the other hand choose j an integer $0 \le j \le s$ such that $b = e(H_{\mathfrak{m}}^j(M)) + j$. Then $H_{\mathfrak{m}}^j(M)_{b-j} \neq 0$, $H_{\mathfrak{m}}^j(M)_{c-j} = 0$ for all $c > b$, and $H_{\mathfrak{m}}^i(M)_{b+1-i} = 0$ for all $i < j$. Recall that this means that $H_{\mathfrak{m}}^j(M)$ has a socle generator in degree $b-s$. Therefore Lemma 5.4 applies and $\mathrm{Tor}_{r-j}^R(K,M)_{b+r-j} \neq 0$. In other words, $b \le a$, as required. □

An easy byproduct of our investigations is the above mentioned fact that

$$\operatorname{reg} M = e(\mathrm{Tor}_c^R(K,M)) - c,\ c = r - \dim M,$$

provided M is a Cohen–Macaulay module.

Theorem 5.5. *Let M be a finitely generated graded R-module with $d = \dim_R M$. Suppose there is an integer $j \in \mathbb{Z}$ such that for all $q \in \mathbb{Z}$ either*

(a) $H_{\mathfrak{m}}^q(M)_{j-q} = 0$ *or*

(b) $H_{\mathfrak{m}}^p(M)_{j+1-q} = 0$ *for all* $p < q$ *and* $H_{\mathfrak{m}}^p(M)_{j-1-q} = 0$ *for all* $p > q$.

Then for $s \in \mathbb{Z}$ it follows that

(1) $\mathrm{Tor}_s^R(K,M)_{s+j} \simeq \bigoplus_{i=0}^{r-s} \mathrm{Tor}_{s+i}^R(K,H_{\mathfrak{m}}^i(M))_{s+j}$ *provided* $s > c$, *and*

(2) $\mathrm{Tor}_s^R(K,M)_{s+j} \simeq \bigoplus_{i=0}^{d-1} \mathrm{Tor}_{s+i}^R(K,H_{\mathfrak{m}}^i(M))_{s+j} \oplus \mathrm{Tor}_{c-s}^R(K,K_M)_{r-s-j}^{\vee}$, *provided* $s \le c$,

where $K_M = \mathrm{Ext}_R^c(M,R(-r))$, $c = \operatorname{codim} M$, *denotes the canonical module of M.*

Proof. As above consider the spectral sequence

$$E_2^{-s-i,i} = H_{s+i}(\mathfrak{m};H_{\mathfrak{m}}^i(M)) \quad \Rightarrow \quad E^{-s} = H_s(\mathfrak{m};M)$$

in degree $s+j$, see Lemma 3.5. Firstly we claim that $[E_2^{-s-i,i}]_{s+j} \simeq [E_\infty^{-s-i,i}]_{s+j}$ for all $s \in \mathbb{Z}$. Because $[E_2^{-s-i,i}]_{s+j}$ is a subquotient of

$$\left[\bigoplus H_{\mathfrak{m}}^i(M)^{\binom{r}{s+i}}(-s-i)\right]_{s+j}.$$

The claim is true provided $H_{\mathfrak{m}}^i(M)_{j-i} = 0$. Suppose that $H_{\mathfrak{m}}^i(M)_{j-i} \neq 0$. In order to prove the claim in this case too note that $[E_{n+1}^{-s-i,i}]_{s+j}$ is the cohomology at

$$[E_n^{-s-i-n,i+n-1}]_{s+j} \to [E_n^{-s-i,i}]_{s+j} \to [E_2^{-s-i+n,i-n+1}]_{s+j}.$$

Then the module at the left resp. the right is a subquotient of

$$H_{s+i+n}(\mathfrak{m};H_{\mathfrak{m}}^{i+n-1}(M))_{s+j} \quad \text{resp.} \quad H_{s+i-n}(\mathfrak{m};H_{\mathfrak{m}}^{i-n+1}(M))_{s+j}.$$

Therefore both of them vanish. But this means that the E_2-term coincides with the corresponding E_∞-term. So the target of the spectral sequence $H_s(\mathfrak{m};M)_{s+j}$ admits a finite filtration whose quotients are $H_{s+i}(\mathfrak{m};H_{\mathfrak{m}}^i(M))_{s+j}$. Because all of these modules are finite dimensional vectorspaces it follows that

$$H_s(\mathfrak{m};M)_{s+j} \simeq \bigoplus_{i=0}^{r-s} H_{s+i}(\mathfrak{m};H^i_{\mathfrak{m}}(M))_{s+j}$$

for all $s \in \mathbb{Z}$.

In the case of $s > c$ it is known that $r - s < d$. Hence the first part of the claim is shown to be true. In the remaining case $s \leq c$ the summation is taken from $i = 0,\ldots,d$. Therefore we have to interpret the summand $H_{s+d}(\mathfrak{m};H^d_{\mathfrak{m}}(M))_{s+j}$. By the Local Duality Theorem $H^d_{\mathfrak{m}}(M) \simeq (K_M)^\vee$. Therefore there are the following isomorphisms:

$$H_{s+d}(\mathfrak{m};(K_M)^\vee)_{s+j} \simeq (H^{s+d}(\mathfrak{m};K_M)^\vee)_{s+j} \simeq H_{r-d-s}(\mathfrak{m};K_M)^\vee_{r-s-j},$$

which proves the second part of the claim. □

As an application of Theorem 5.5 we derive Green's duality theorem [5, Section 2], see also [10, Theorem 1.2] for a similar approach of the original statement.

Corollary 5.6. *Suppose there exists an integer $j \in \mathbb{Z}$ such that*

$$H^q_{\mathfrak{m}}(M)_{j-q} = H^q_{\mathfrak{m}}(M)_{j+1-q} = 0$$

for all $q < \dim_R M$. Then

$$\mathrm{Tor}^R_s(K,M)_{s+j} \simeq \mathrm{Tor}^R_{c-s}(K,K_M)^\vee_{r-s-j},$$

for all $s \in \mathbb{Z}$, where $c = \operatorname{codim} M$.

Proof. It follows that the assumptions of Theorem 5.5 are satisfied for j because of $H^p_{\mathfrak{m}}(M)_{j-1-p} = 0$ for all $p > \dim M$. Therefore the isomorphism is a consequence of (1) and (2) in Theorem 5.5. To this end recall that

$$\mathrm{Tor}^R_{s+i}(K,H^i_{\mathfrak{m}}(M))_{s+j} \simeq H_{s+i}(\mathfrak{m};H^i_{\mathfrak{m}}(M))_{s+j} = 0,$$

as follows by the vanishing of $H^i_{\mathfrak{m}}(M)_{s+j}$ for all $j \in \mathbb{Z}$. □

Green's duality theorem in Corollary 5.6 relates the Betti numbers of M to those of K_M. Because of the strong vanishing assumptions in Corollary 5.6 very often it does not give strong information about Betti numbers. Often it says just the vanishing which follows also by different arguments, e.g. the regularity of M.

Theorem 5.5 is more subtle. In a certain sense it is an extension of Rao's argument, see [11, Eq. (2.5)]. We shall illustrate its usefulness by the following example.

Example 5.7. Let $C \subset \mathbb{P}^n_K$ denote a reduced integral non-degenerate curve over an algebraically closed field K. Suppose that C is non-singular and of genus $g(C) = 0$. Let $A = R/I$ denote its coordinate ring, i.e. $R = K[x_0,\ldots,x_n]$ and I its homogeneous defining ideal. Then

$$\mathrm{Tor}^R_s(K,R/I)_{s+j} \simeq \mathrm{Tor}^R_{s+1}(K,H^1_{\mathfrak{m}}(R/I))_{s+j}$$

for all $s \geq 1$ and all $j \geq 3$. To this end recall that A is a two-dimensional domain. Moreover it is well-known that $H^q_{\mathfrak{m}}(R/I) = 0$ for all $q \leq 0$ and $q > 2$. Furthermore it is easy to see that $H^1_{\mathfrak{m}}(R/I)_{j-1} = 0$ for all $j \leq 1$. Moreover $H^2_{\mathfrak{m}}(R/I)_{j-1-2} = 0$ for all $j \geq 3$ as follows because of $g(C) = 0$. That is for $j \geq 3$ one might apply Theorem 5.5. In order to conclude we have to show that $\operatorname{Tor}^R_{c-s}(K, K_{R/I})_{r-s-j} = 0$ for $j \geq 3$. To this end note that

$$H_{c-s}(\mathfrak{m}; K_{R/I})^{\vee}_{r-s-j} \simeq H_{s+2}(\mathfrak{m}; H^2_{\mathfrak{m}}(R/I))_{s+j}$$

as is shown in the proof of Theorem 5.5. But this vanishes for $j \geq 2$ as is easily seen.

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