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On Auslander's n-Gorenstein rings

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Abstract

According to Auslander, a Noetherian ring R is called n-Gorenstein for $n \geq 1$ if in a minimal injective resolution $0 \to {}_RR \to E_0 \to E_1 \to \cdots \to E_n \to, \ldots,$ the flat dimension of each E_i is at most i for $i = 0, 1, \ldots, n-1$. We prove that for an n-Gorenstein ring R of self-injective dimension n, the last term E_n in a minimal injective resolution of ${}_RR$ has essential socle.

We also prove that the 1-Gorenstein property is inherited by a maximal quotient ring, and as a related result, we characterize a Noetherian ring of dominant dimension at least 2.

0. Introduction

A Noetherian ring R is called an *n-Gorenstein* ring if, in a minimal injective resolution $0 \to {}_RR \to E_0 \to E_1 \to \cdots \to E_n \to, \ldots,$ flat dimension of E_i is at most i for each i $(0 \leq i \leq n-1)$, and further R is called an *Auslander* ring if R is n-Gorenstein for all $n \geq 1$. The important thing is that the notion of an n-Gorenstein ring is left–right symmetric [9, Auslander's Theorem 3.7]. Moreover, Auslander raised a conjecture, which says that an Auslander Artinian algebra has finite self-injective dimension. Related to this conjecture, a ring R is said to have *self-injective dimension n* if both of ${}_RR$ and R_R have injective dimension n and a Noetherian ring R is called a *Gorenstein* ring if R has finite self-injective dimension. It is shown in Zaks [25] that, for a Noetherian ring R, if the injective dimensions of ${}_RR$ and R_R are finite, then they coincide.

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In the first part of this paper, we will derive a module theoretical property from a homological property on minimal injective resolutions of Gorenstein rings and n-Gorenstein rings. Main results are concerned with the last term in a minimal injective resolution for a Gorenstein ring. We will show that, for an n-Gorenstein ring of self-injective dimension n, the last term has essential socle. In Fuller and Iwanaga [10], it is proved that the last term has nonzero socle in this situation. Moreover we will show that, for a Noetherian ring of a finite global dimension and a fully bounded Noetherian ring of finite self-injective dimension, the last terms have nonzero socles. In the second part, we will consider a maximal quotient ring and show that the 1-Gorenstein property is inherited by maximal quotient rings.

Throughout this paper, we fix a minimal injective resolution for ${}_RR$ as above and similarly denote a minimal injective resolution for R_R by $0 \to R_R \to E'_0 \to E'_1 \to \cdots \to E'_n \to \cdots$. For a module M, $\mathrm{pd}(M)$, $\mathrm{id}(M)$ and $\mathrm{fd}(M)$ denote the projective, injective and flat dimension of M, respectively.

1. Minimal injective resolution

As we mentioned in the introduction, the notion of n-Gorenstein rings is left–right symmetric and further Auslander characterizes them by the so-called *Auslander condition* [5]. As we use it in our argument, we write down the result [9].

Theorem (Auslander). *The following are equivalent for a Noetherian ring R and an integer* $n \geq 1$:

(1) $\mathrm{fd}(E_i) \leq i$ *for any* i $(0 \leq i < n)$;

(2) $\mathrm{fd}(E'_i) \leq i$ *for any* i $(0 \leq i < n)$;

(3) *For any finitely generated right R-module* X_R *and any* j $(1 \leq j \leq n)$, *we have* $\mathrm{Ext}^i_R(M,R) = 0$ *if* ${}_RM$ *is a submodule of* $\mathrm{Ext}^j_R(X,R)$ *and if* $i < j$;

(4) *The dual of* (3).

In the commutative case, Gorenstein rings have beautiful properties as described by Bass [2]. If R is a commutative Noetherian ring of self-injective dimension n, then Krull dimension of R is n and $\mathrm{fd}(E_i) = i$ for all i $(0 \leq i \leq n)$. Moreover an injective indecomposable module $E = \mathrm{E}(R/P)$with $P \in \mathrm{Spec}(R)$ embeds in E_i if and only if $\mathrm{ht}(P) = i$. Hence, in particular, the last term E_n has essential socle. However, in the noncommutative case, there is a Noetherian ring of self-injective dimension n with $\mathrm{fd}(E_i) = n$ for all i. Thus it is reasonable to study the property of the last term E_n for a Noetherian ring of self-injective dimension n. We will actually discuss the following question.

Question. *Let R be a Noetherian ring of self-injective dimension n. Is the socle,* $\mathrm{Soc}(E_n)$, *of* E_n *nonzero and moreover is* $\mathrm{Soc}(E_n)$ *essential in* E_n?

The first result shows that an n-Gorenstein ring of self-injective dimension n is an Auslander ring and direct summands in the last term in a minimal injective resolution are homogeneous with respect to projective and flat dimension.

Proposition 1. *Let R be a Noetherian ring of self-injective dimension n. Then we have the following.*

(1) $\mathrm{Ext}_R^n(X, R) \neq 0$ *for any nonzero submodule X of E_n, and so* $\mathrm{pd}(X) = n$ *or* ∞.

(2) *For any finitely generated nonzero submodule U of E_n, there exist a simple right R-module S_R and finitely generated submodules V_1, V_2 of $E(S)$ satisfying*

$$\mathrm{Tor}_n^R(E(S), U),\ \mathrm{Tor}_n^R(E(S), E(U)) \neq 0$$

and

$$\mathrm{Tor}_n^R(V_1, U),\ \mathrm{Tor}_n^R(V_2, E(U)) \neq 0 .$$

Thus $\mathrm{fd}(E(S)) = \mathrm{fd}(E(U)) = n$.

(3) $\mathrm{pd}(E) = \mathrm{fd}(E) = n$ *for any nonzero direct summand E in E_n.*

Proof. (1) In the exact sequence

$$0 \to K_{n-1} \to E_{n-1} \to E_n \to 0 \quad \text{with } \mathrm{E}(K_{n-1}) = E_{n-1},$$

$X \subseteq E_n \cong E_{n-1}/K_{n-1}$ implies

$$0 \neq \mathrm{Ext}_R^1(X, K_{n-1}) \cong \mathrm{Ext}_R^n(X, R).$$

Hence $\mathrm{pd}(X) \geq n$, that is $\mathrm{pd}(X) = n$ or ∞ by [13, Theorem 2].

Proofs of (2) and (3). Taking $X = E$ as a nonzero direct summand of E_n in (1), we have $\mathrm{pd}(E) \geq n$ and so $\mathrm{pd}(E) = n$ by [13, Theorem 2].

For the remaining part of the proof, without loss of generality, E may be assumed indecomposable and then of the form $E = \mathrm{E}(U)$ for a finitely generated submodule U of E.

For any finitely generated submodule U of E_n, we have from (1)

$$0 \neq \mathrm{Ext}_R^1(U, K_{n-1}) \cong \mathrm{Ext}_R^n(U, R).$$

Now let

$$W_R = \bigoplus_{\lambda \in \Lambda} \mathrm{E}(S_\lambda)$$

be the direct sum of injective hulls of all non-isomorphic simple right R-modules S_λ ($\lambda \in \Lambda$). Then W_R is an injective cogenerator and hence we see

$$0 \neq \mathrm{Hom}_R(\mathrm{Ext}_R^n(U, R)_R, W_R) \cong \mathrm{Tor}_n^R(W, U) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{Tor}_n^R(\mathrm{E}(S_\lambda), U)$$

by [6, Ch. VI, Proposition 5.3]. Thus there is a simple right R-module S_R such that $\mathrm{Tor}_n^R(\mathrm{E}(S), U) \neq 0$ and then $\mathrm{fd}(\mathrm{E}(S)) = n$ by [13, Proposition 1]. Since $\mathrm{Tor}_n^R(E(S), -)$ is left exact, we get an embedding

$$0 \neq \mathrm{Tor}_n^R(\mathrm{E}(S), U) \hookrightarrow \mathrm{Tor}_n^R(\mathrm{E}(S), \mathrm{E}(U))$$

and so $\mathrm{fd}(\mathrm{E}(U)) = n$ again by [13, Proposition 1]. Next write

$$\mathrm{E}(S) = \varinjlim V_i,$$

where V_i are finitely generated submodules of $\mathrm{E}(S)$. Then, since Tor-functor commutes with direct limits, there exists a finitely generated submodule V_1 of $\mathrm{E}(S)$ with $\mathrm{Tor}_n^R(V_1, U) \neq 0$. Similarly, we have $\mathrm{Tor}_n^R(V_2, \mathrm{E}(U)) \neq 0$ for some finitely generated submodule V_2 of $\mathrm{E}(S)$. □

In view of the proof of Proposition 1(1), we have the following theorem, which is more general than our previous result in [14].

Theorem 2. *Let R be a ring with $\mathrm{id}({}_RR) = n \geq 1$. Then E_0 and E_n have no isomorphic direct summands.*

Proof. Let $E \neq 0$ be an indecomposable direct summand in E_0 and assume E is isomorphic to a direct summand in E_n. Then $E = \mathrm{E}(U)$ for some finitely generated submodule U of E and in this case, $V = U \cap R \neq 0$. In the exact sequence

$$\mathrm{Ext}_R^n(R, R) \to \mathrm{Ext}_R^n(V, R) \to \mathrm{Ext}_R^{n+1}(R/V, R)$$

which is induced from $0 \to V \to R \to R/V \to 0$ (exact), we see $\mathrm{Ext}_R^n(R, R) = 0 = \mathrm{Ext}_R^{n+1}(R/V, R)$ from $n \geq 1$ and $\mathrm{id}({}_RR) = n$. Hence we have $\mathrm{Ext}_R^n(V, R) = 0$. On the other hand, $0 \neq V \subseteq E \hookrightarrow E_n$ implies $\mathrm{Ext}_R^n(V, R) \neq 0$ by the same argument as the proof of Proposition 1(1), and this is a contradiction. □

To prove a main result, we need two facts. The first one indicates that, for an n-Gorenstein ring R of self-injective dimension n, the last term in a minimal injective resolution of ${}_RR$ involves a property different from the other terms.

Proposition 3. *Let R be a Noetherian ring. Then, R is n-Gorenstein if and only if, for any finitely generated right (resp. left) R-module X and any $j \leq n$, we have $\mathrm{Hom}_R(\mathrm{M}, E_i) = 0$ (resp. $\mathrm{Hom}_R(\mathrm{M}, E_i') = 0$) provided that M is a submodule of $\mathrm{Ext}_R^j(X, R)$ and $i < j$.*

Proof. Assume R is n-Gorenstein. Let X be a finitely generated right R-module and $j \leq n$. Then $\mathrm{fd}(E_i) \leq i$ for any $i < n$ implies

$$0 = \mathrm{Tor}_j^R(X, E_i) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^j(X, R), E_i) \quad \text{if } i < j.$$

Next, for any submodule M of $\mathrm{Ext}_R^j(X,R)$, we have an exact sequence

$$0 = \mathrm{Hom}_R(\mathrm{Ext}_R^j(X,R), E_i) \to \mathrm{Hom}_R(M, E_i) \to 0$$

since E_i is injective, and consequently we obtain $\mathrm{Hom}_R(M, E_i) = 0$.

To prove the sufficiency, let X be any finitely generated right R-module and $i < n$. Then we have

$$0 = \mathrm{Hom}_R(\mathrm{Ext}_R^{i+1}(X,R), E_i) \cong \mathrm{Tor}_{i+1}^R(X, E_i).$$

Hence we see $\mathrm{fd}(E_i) \le i$ for any $i < n$. $\square$

The next proposition is crucial for the proof of our main result.

Proposition 4. *Let R be an n-Gorenstein ring of self-injective dimension n. Then $\mathrm{Ext}_R^n(M,R)$ is an Artinian right R-module for any finitely generated left R-module M.*

Proof. First we claim that, for submodules $X \subseteq Y$ of $\mathrm{Ext}_R^n(M,R)$, we get

$$\mathrm{Hom}_R(Y/X, E'_j) = 0 = \mathrm{Ext}_R^j(Y/X, R) \quad \text{for any } j < n.$$

In fact, from $0 \to X \to Y \to Y/X \to 0$ (exact), we have an exact sequence

$$0 \to \mathrm{Hom}_R(Y/X, E'_j) \to \mathrm{Hom}_R(Y, E'_j) = 0 \quad \text{(by Proposition 3)}$$

and so the first equality follows. For the second equality, again $0 \to X \to Y \to Y/X \to 0$ (exact) induces exact sequences

$$0 \to \mathrm{Hom}_R(Y/X, R) \to \mathrm{Hom}_R(Y, R),$$

$$\mathrm{Ext}_R^{j-1}(X,R) \to \mathrm{Ext}_R^j(Y/X, R) \to \mathrm{Ext}_R^j(Y,R) \quad \text{for } j \ge 1.$$

Now by the Auslander condition, $\mathrm{Hom}_R(Y,R) = 0$ and $\mathrm{Ext}_R^{j-1}(X,R) = 0 = \mathrm{Ext}_R^j(Y,R)$ if $1 \le j < n$. Hence the second equality follows as well.

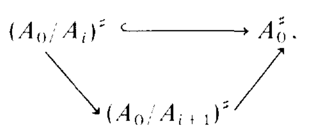
From now on, we denote $\mathrm{Ext}_R^n(N,R)$ by $N^\sharp$ for an R-module N. If $0 \to X \to Y \to Z \to 0$ is an exact sequence with $Y \subseteq \mathrm{Ext}_R^n(M,R)$, $0 \to Z^\sharp \to Y^\sharp \to X^\sharp \to 0$ is also exact by the n-Gorenstein property. Let

$$A_0 = \mathrm{Ext}_R^n(M,R) \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

be a descending chain of submodules of $\mathrm{Ext}_R^n(M,R)_R$. Then from the exact sequence

$$0 \to (A_0/A_i)^\sharp \to A_0^\sharp \to A_i^\sharp \to 0 \quad (\text{for any } i \ge 1)$$

and the commutative diagram



we can regard naturally each $(A_0/A_i)^\sharp$ as a submodule of $A_0^\sharp$ and we have the following ascending chain of submodules of $A_0^\sharp$:

$$(A_0/A_1)^\sharp \subseteq (A_0/A_2)^\sharp \subseteq \cdots \subseteq A_0^\sharp. \qquad (*)$$

Since $A_0 = \mathrm{Ext}_R^n(M,R)_R$ is finitely generated, so is $A_0^\sharp$ and thus it is Noetherian as a left R-module. Consequently the ascending chain $(*)$ terminates at some step m (say) and then from the exact sequence $0 \to A_m/A_{m+1} \to A_0/A_{m+1} \to A_0/A_m \to 0$, we have

$$(A_0/A_m)^\sharp \xrightarrow{\sim} (A_0/A_{m+1})^\sharp \longrightarrow (A_m/A_{m+1})^\sharp \longrightarrow 0 \text{ (exact)}.$$

Hence we get $(A_m/A_{m+1})^\sharp = 0$, that is, $\mathrm{Ext}_R^n(A_m/A_{m+1}, R) = 0$. On the one hand, the exact sequence

$$0 \to A_{m+1} \to A_m \to A_m/A_{m+1} \to 0 \quad \text{with } A_m, A_{m+1} \subseteq \mathrm{Ext}_R^n(M,R)$$

and $\mathrm{id}\,(R_R) = n$ imply

$$\mathrm{Ext}_R^j(A_m/A_{m+1}, R) = 0 \quad \text{for } \forall j \neq n.$$

Therefore $A_m/A_{m+1} = 0$ follows by Colby and Fuller [7, Theorem 2] and as a result, $A_0 = \mathrm{Ext}_R^n(M,R)$ is Artinian as a right R-module. □

As an application of Proposition 4, we have

Corollary 5. *Let R be an n-Gorenstein ring of self-injective dimension n and X a finitely generated right R-module. Then $\mathrm{Ext}_R^n(X,R)$ embeds in $E_n^{(t)}$ for some $t > 0$. Moreover, if M is a nonzero submodule of $\mathrm{Ext}_R^n(X,R)$, then we have $\mathrm{Ext}_R^n(M,R) \neq 0$.*

Proof. By [12, Theorem 2], $Y = \mathrm{Ext}_R^n(X,R)$ is embedded in a direct product of copies of $E_0 \oplus \cdots \oplus E_n$. Since $\mathrm{Hom}_R(Y, E_i) = 0$ for any $i < n$ from Proposition 3, Y embeds in a direct product of copies of E_n. Now Y is a Noetherian and Artinian R-module by Proposition 4, i.e. M has a composition series of finite length. Thus the socle, $\mathrm{Soc}(Y)$, of Y is essential in Y and finitely generated. Therefore $\mathrm{E}(\mathrm{Soc}(Y)) = \mathrm{E}(Y)$ is embedded in a direct sum of finitely many copies of E_n.

Let $M \subseteq \mathrm{Ext}_R^n(X,R) \hookrightarrow E_n^{(t)}$ and $M \neq 0$. Consider the exact sequence

$$0 \to K_{n-1} \to E_{n-1} \to E_n \to 0 \quad \text{with } \mathrm{E}(K_{n-1}) = E_{n-1},$$

then the exact sequence

$$0 \to K_{n-1}^{(t)} \to E_{n-1}^{(t)} \to E_n^{(t)} \to 0,$$

also satisfies $\mathrm{E}(K_{n-1}^{(t)}) = E_{n-1}^{(t)}$. Hence there exists a nonsplit exact sequence

$$0 \to K_{n-1}^{(t)} \to N \to M \to 0,$$

for some submodule N of $E_{n-1}^{(t)}$. Consequently we have

$$0 \neq \mathrm{Ext}_R^1(M, K_{n-1}^{(t)}) \cong \mathrm{Ext}_R^1(M, K_{n-1})^{(t)} \cong \mathrm{Ext}_R^n(M, R)^{(t)}. \quad \square$$

Now we can prove our first main result.

Theorem 6. *Let R be an n-Gorenstein ring of self-injective dimension n and $0 \to R \to E_0 \to \cdots \to E_n \to 0$ a minimal injective resolution of ${}_R R$. Then $\mathrm{Soc}(E_n)$ is essential in E_n.*

Proof. Any injective indecomposable module E is of the form $E = \mathrm{E}(U)$ for some finitely generated submodule U of E. Let E be a nonzero direct summand in E_n. Then by Proposition 1(2), there exists a finitely generated right R-module V with $\mathrm{Tor}_n^R(V, \mathrm{E}(U)) \neq 0$ and hence we have

$$0 \neq \mathrm{Tor}_n^R(V, \mathrm{E}(U)) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^n(V, R), \mathrm{E}(U)).$$

As a consequence, $E = \mathrm{E}(U)$ has a nonzero Artinian submodule by Proposition 4 and so a simple submodule S. Therefore we obtain $E = \mathrm{E}(S)$ since E is injective indecomposable, and thus E_n has essential socle. $\square$

As a byproduct of Theorem 6, we have

Corollary 7. *Let R be an n-Gorenstein ring of self-injective dimension n and $0 \to R \to E_0 \to \cdots \to E_n \to 0$ a minimal injective resolution of ${}_R R$. Then the following statements hold.*

(1) *E_n is a direct sum of injective indecomposables of the form $E(S)$ with S a simple left module.*

(2) *If E is injective indecomposable of flat dimension n, then $E \cong E(S)$ for some simple module S of projective dimension n or ∞.*

(3) *If $n \geq 1$, $E_0 \oplus \cdots \oplus E_{n-1}$ and E_n have no isomorphic direct summands in common.*

Proof. (1) immediately follows from Theorem 6.

(2) Hoshino (oral communication) proves that if E is injective indecomposable of flat dimension n, E appears as a direct summand of E_n. Thus $E \cong \mathrm{E}(S)$ for some simple left module S. Here, $\mathrm{pd}(S) = n$ or ∞ by Proposition 1(1).

(3) follows from Proposition 1(3). $\square$

Besides the case of Theorem 6, we have another two types of Noetherian rings of finite self-injective dimension such that the last terms in minimal injective resolutions have nonzero socles. Recall that a ring is *bounded* if any essential onesided ideal contains a nonzero twosided ideal and that a ring R is *fully bounded* if every prime factor ring R/P by a prime ideal P is bounded.

Proposition 8. *Let R be one of the following rings:*
(1) *a Noetherian ring of global dimension n,*
(2) *a fully bounded Noetherian ring of self-injective dimension n.*
Then the last term E_n in a minimal injective resolution of ${}_RR$ has nonzero socle.

Proof. (1) In this case, we know $\mathrm{id}({}_RR) = \mathrm{id}({}_RR) = \mathrm{gl.dim}\, R = n$. Thus, for any cyclic nonzero submodule ${}_RC$ of E_n, there exists an injective right R-module E' with $\mathrm{Tor}_n^R(E', C) \neq 0$ by Proposition 1(2). Now consider a class of left ideals

$$\mathscr{F} = \{I \mid \mathrm{Tor}_n^R(E', R/I) \neq 0\}.$$

Then $\mathscr{F}$ is nonempty and contains a maximal element L (say). Letting $U = R/L$, we may assume

$$\mathrm{Tor}_n^R(E', R) \neq 0,$$

$$\mathrm{Tor}_n^R(E', U/V) = 0 \quad \text{for any nonzero submodule } V \subseteq U.$$

Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be a family of all nonzero submodules of U and $S = \bigcap_{\lambda \in \Lambda} V_\lambda$. Then $S = 0$ or S is simple. Assume now $S = 0$. Since $\mathrm{Tor}_n^R(E', -)$ is left exact from $\mathrm{gl.dim}\, R = n$, the canonical embedding $U \hookrightarrow \prod_{\lambda \in \Lambda} U/V_\lambda$ induces

$$\mathrm{Tor}_n^R(E', U) \hookrightarrow \mathrm{Tor}_n^R\left(E', \prod_{\lambda \in \Lambda} U/V_\lambda\right).$$

Therefore we get

$$\mathrm{Tor}_n^R\left(E', \prod_{\lambda \in \Lambda} U/V_\lambda\right) \neq 0.$$

Write $E' = \varinjlim L_i$ (each L_i is finitely generated submodule of E'). Since R is Noetherian and each L_i is finitely generated, we have the following isomorphisms by Lenzing [17, Satz 2]:

$$\mathrm{Tor}_n^R\left(E', \prod_{\lambda \in \Lambda} U/V_\lambda\right) \cong \varinjlim \mathrm{Tor}_n^R\left(L_i, \prod_{\lambda \in \Lambda} U/V_\lambda\right)$$
$$\cong \varinjlim \prod_{\lambda \in \Lambda} \mathrm{Tor}_n^R(L_i, U/V_\lambda).$$

On the one hand, $\mathrm{Tor}_n^R(-, U/V_\lambda)$ is left exact for every λ from $\mathrm{gl.dim}\, R = n$ and so we have, for any i and λ,

$$\mathrm{Tor}_n^R(L_i, U/V_\lambda) \hookrightarrow \mathrm{Tor}_n^R(E', U/V_\lambda) = 0.$$

Consequently we have $\mathrm{Tor}_n^R(E', \prod_{\lambda \in \Lambda} U/V_\lambda) = 0$, as a contradiction.

Therefore S is simple and the exact sequence $0 \to S \to U \to U/S \to 0$ induces an exact sequence

$$0 \to \mathrm{Tor}_n^R(E', S) \to \mathrm{Tor}_n^R(E', U) \to \mathrm{Tor}_n^R(E', U/S)$$

with $\mathrm{Tor}_n^R(E', U) \neq 0$ and $\mathrm{Tor}_n^R(E', U/S) = 0$. It turns out that

$$0 \neq \mathrm{Tor}_n^R(E', S) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^n(S, R), E')$$

and so we get $\mathrm{Ext}_R^n(S, R) \neq 0$, i.e. $\mathrm{Soc}(E_n) \neq 0$.

(2) Let U'_R be any finitely generated nonzero submodule of E'_n. Then, by Proposition 1(2), there exist a simple left R-module ${}_RS$ and a finitely generated submodule U of $\mathrm{E}(S)$ satisfying $\mathrm{Tor}_n^R(\mathrm{E}(U'), U) \neq 0$. Here, since R is fully bounded Noetherian, U is Artinian as a left R-module by Jategaonkar [15, Corollary 3.6]. Thus

$$0 \neq \mathrm{Tor}_n^R(\mathrm{E}(U'), U) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^n(U, R), \mathrm{E}(U'))$$

implies

$$0 \neq \mathrm{Ext}_R^n(U, R) \cong \mathrm{Ext}_R^1(U, K_{n-1}),$$

where $0 \to K_{n-1} \to E_{n-1} \to E_n \to 0$ (exact) and $\mathrm{E}(K_{n-1}) = E_{n-1}$. Now we have an exact sequence

$$\mathrm{Hom}_R(U, E_n) \to \mathrm{Ext}_R^1(U, K_{n-1}) \to \mathrm{Ext}_R^1(U, E_{n-1}) = 0$$

and hence we see $\mathrm{Hom}_R(U, E_n) \neq 0$. It turns out that E_n has an Artinian submodule and in particular, we have $\mathrm{Soc}(E_n) \neq 0$. □

Example. Among noncommutative examples of n-Gorenstein rings, we can see by applying Theorem 6 that the last term in a minimal injective resolution of a ring has essential socle in the following cases:

(1) A Weyl algebra

$$\mathscr{A}_n(K) = K\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]$$

over an algebraically closed field K of characteristic zero is an Auslander ring of global dimension n [23].

(2) Let G be a finite subgroup of $GL_n(\mathbb{C})$ without pseudo-reflection. Then the natural G-action on $\mathscr{A}_n(\mathbb{C})$ gives rise to the G-invariant sub-ring, which is an Auslander ring of finite self-injective dimension but of infinite global dimension [18].

(3) A ring of differential operators

$$K\left[\!\left[x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]\!\right]$$

of a formal power series ring over a field K of characteristic zero is an Auslander ring of global dimension $2n$ [3].

(4) Malliavin [19] also discusses the last term for an enveloping algebra of a solvable Lie algebra (cf. Theorems 3.4 and 3.6 in [19]).

Concerning the homological aspects on Gorenstein rings with Auslander condition, we refer Björk's survey [4].

In the latter part of this section, we consider an Artinian n-Gorenstein ring R and study the second term E_1 of a minimal injective resolution of ${}_RR$. In this case, the first term E_0 is projective. (Such a ring has been called a QF-3 ring.) Thus any indecomposable summand of E_0 is isomorphic to a left ideal of R (cf. [8]). We are interested in E_1 and characterize a minimal projective resolution of E_1.

Let R be an Artinian n-Gorenstein ring with $n \geq 2$ and

$$0 \to \mathrm{P}^1(E_1) \to \mathrm{P}^0(E_1) \to E_1 \to 0$$

a minimal projective resolution for E_1. It is well known that a projective cover of an injective module is injective and so $\mathrm{P}^0(E_1)$ is projective–injective. However all projective–injective indecomposables do not necessarily appear as a direct summand of $\mathrm{P}^0(E_1)$. Direct summands of $\mathrm{P}^0(E_1)$ and $\mathrm{P}^1(E_1)$ can be characterized as follows.

Proposition 9. *Let R be an Artinian n-Gorenstein ring with $n \geq 2$, $0 \to {}_RR \to E_0 \to E_1 \to \cdots$ a minimal injective resolution and*

$$0 \to P^1(E_1) \to P^0(E_1) \to E_1 \to 0 \tag{$*$}$$

a minimal projective resolution for E_1. Then we have the following.

(1) *A projective indecomposable left module P is a direct summand of $P^1(E_1)$ if and only if* $\mathrm{id}(P) = 1$.

In particular, E_1 is projective if and only if there exists no projective module of injective dimension one.

(2) *A projective–injective indecomposable left module Q is a direct summand of $P^0(E_1)$ if and only if either of the following holds:*

(i) $\mathrm{Soc}(Q) \subseteq \mathrm{Soc}(P)$ *for some projective indecomposable P of injective dimension one,*

(ii) $\mathrm{Soc}(Q) \subseteq \mathrm{Soc}(E(P)/P)$ *for some projective indecomposable P.*

(3) *For any projective module P, there exists a projective module Q of injective dimension one such that $E^1(P) \cong E^1(Q)$ in* $\underline{\mathrm{Mod}}(R)$, *the projectively stable category. Here $E^1(P)$ stands for an injective hull of $E(P)/P$.*

Proof. (1) First of all, from $\mathrm{pd}(E_1) \leq 1$, E_1 is projective if and only if there is no projective module of injective dimension one. So we may assume $\mathrm{pd}(E_1) = 1$.

Since $(*)$ is also an injective resolution of $\mathrm{P}^1(E_1)$, we see $\mathrm{id}(\mathrm{P}^1(E_1)) \leq 1$ and hence any indecomposable direct summand of $\mathrm{P}^1(E_1)$ has injective dimension at most one. However $\mathrm{P}^1(E_1)$ is small in $\mathrm{P}^0(E_1)$ and so no direct summand of $\mathrm{P}^1(E_1)$ is injective, that is, every direct summand of $\mathrm{P}^1(E_1)$ has injective dimension one.

Conversely, let P be projective indecomposable of injective dimension one. Then $\mathrm{E}(P)/P$ is injective and $\mathrm{E}(P)/P$ is isomorphic to a direct summand of E_1. Now the exact sequence $0 \to P \to \mathrm{E}(P) \to \mathrm{E}(P)/P \to 0$ is a minimal projective resolution of $\mathrm{E}(P)/P$ by Tachikawa [24, (8.1) Lemma]. Hence $\mathrm{E}(P)$ isomorphic to a direct summand of $\mathrm{P}^0(E_1)$ and so P appears as a direct summand of $\mathrm{P}^1(E_1)$.

(2) Let $E_1 = E \oplus G$ such that G is projective–injective and $\mathrm{pd}(E) = 1$, and assume there is no projective direct summand in E. Then $\mathrm{P}^0(E_1) = \mathrm{P}^0(E) \oplus G$ and we have

two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{P}^1(E_1) & \longrightarrow & \mathrm{P}^0(E_1) & \longrightarrow & E_1 & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathrm{P}^1(E) & \underset{g}{\longrightarrow} & \mathrm{P}^0(E) \oplus G & \underset{(f,1_G)}{\longrightarrow} & E \oplus G & \longrightarrow & 0, \end{array}$$

where

$$0 \to \mathrm{P}^1(E) \xrightarrow{g} \mathrm{P}^0(E) \xrightarrow{f} E \longrightarrow 0$$

is a minimal projective resolution of E. Thus we can write as

$$\mathrm{P}^1(E_1) = \bigoplus_{i \in A} P_i$$

with $\mathrm{id}(P_i) = 1$ by (1). Then we have

$$\mathrm{P}^0(E) = \mathrm{E}(\mathrm{P}^1(E)) = \mathrm{E}(\mathrm{P}^1(E_1)) \cong \bigoplus_{i \in A} \mathrm{E}(P_i),$$

that is,

$$\mathrm{P}^0(E_1) \cong \bigoplus_{i \in A} \mathrm{E}(P_i) \oplus G.$$

Only if: Let Q be an indecomposable summand of $\mathrm{P}^0(E_1)$.

(i) If Q is a direct summand of $\bigoplus_{i \in A} \mathrm{E}(P_i)$, then we have $\mathrm{Soc}(Q) \subseteq \mathrm{Soc}(P_i)$ for some P_i.

(ii) If Q is a direct summand of G, then Q is a projective–injective direct summand of E_1. Now E_0/R is a direct sum of modules of the form $\mathrm{E}(P)/P$ with P projective indecomposables and further it is an essential submodule of E_1. Thus $\mathrm{Soc}(Q)$ is monomorphic to $\mathrm{Soc}(\mathrm{E}(P)/P)$ for some projective indecomposable P.

If. In case of (i), Q is a direct summand of $\mathrm{E}(P)$. Here $\mathrm{E}(P)$ is a direct summand of $\mathrm{P}^0(E_1)$, since $0 \to P \to \mathrm{E}(P) \to \mathrm{E}(P)/P \to 0$ is a minimal projective resolution of $\mathrm{E}(P)/P$ and $\mathrm{E}(P)/P$ is a direct summand of E_1. In case of (ii), Q is isomorphic to a direct summand of $\mathrm{E}(\mathrm{E}(P)/P)$, which is a direct summand of E_1. Thus Q is a direct summand of $\mathrm{P}^0(\mathrm{E}_1)$.

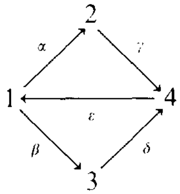
(3) Since $D = \mathrm{E}^1(P)$ is a direct summand of E_1, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{P}^1(E_1) & \longrightarrow & \mathrm{P}^0(E_1) & \longrightarrow & E_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{P}^1(D) & \longrightarrow & \mathrm{P}^0(D) & \longrightarrow & D & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{P}^1(D) & \longrightarrow & \mathrm{E}^0(P^1(D)) & \longrightarrow & \mathrm{E}^1(\mathrm{P}^1(D)) & \longrightarrow & 0 \end{array}$$

where the upper two rows are minimal projective resolutions and the bottom row is a minimal injective resolution. Now since R is especially a QF-3 ring, $\mathrm{P}^0(D)$ is projective–injective and hence $\mathrm{E}^0(\mathrm{P}^1(D))$ is a direct summand of $\mathrm{P}^0(D)$. Further $\mathrm{P}^1(D)$ has injective dimension one by (1). Let $\mathrm{P}^0(D) = \mathrm{E}^0(\mathrm{P}^1(D)) \oplus G$. Then G is projective–injective and we have $\mathrm{E}^1(P) \cong E^1(\mathrm{P}^1(D)) \cong G$. □

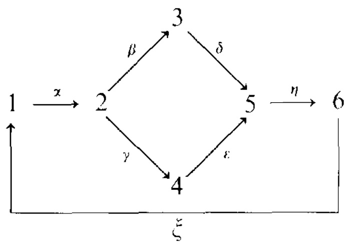
Examples for Proposition 9. We give examples for understanding Proposition 9.

(1) Let R be an algebra over a field given by the quiver



with relations $\gamma\alpha = \delta\beta$, $\varepsilon\gamma = \varepsilon\delta = \alpha\varepsilon = \beta\varepsilon = 0$. Then R is a 1-Gorenstein of self-injective dimension one. Corresponding to the vertices $1, 2, 3$ and 4, e_i are primitive idempotents and S_i are simple left R-modules for $i = 1, 2, 3, 4$. Re_1 and Re_4 are projective–injective, and $\mathrm{Soc}(Re_1) \cong S_4$ and $\mathrm{Soc}(Re_4) \cong S_1$. Projective indecomposables of injective dimension one are Re_2 and Re_3. S_4 embeds in Re_2 but S_1 is neither the case (i) nor (ii) in Proposition 9(2). Hence $\mathrm{P}^0(E_1)$ is a direct sum of copies of Re_1, and $\mathrm{P}^1(E_1)$ is a direct sum of copies of Re_2 and Re_3.

(2) Let R be an algebra over a field given by the quiver

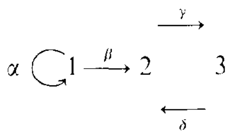


with relations $\delta\beta = \varepsilon\gamma$, $\eta\delta = \eta\varepsilon = \alpha\xi\eta = \gamma\alpha\xi = 0$. Then R is an Auslander ring of global dimension 5. The primitive idempotents e_i and simple left modules S_i are the same as in (1). Re_1, Re_5 and Re_6 are projective–injective and Re_3 is the only one projective indecomposable of injective dimension one. In the isomorphisms

$$\mathrm{Soc}(\mathrm{Re}_1) \cong S_5, \qquad \mathrm{Soc}(\mathrm{Re}_5) \cong S_1 \quad \text{and} \quad \mathrm{Soc}(\mathrm{Re}_6) \cong S_3,$$

we see $S_5 \hookrightarrow Re_3$, $S_1 \hookrightarrow \mathrm{E}(Re_2)/Re_2$ and $S_3 \hookrightarrow \mathrm{E}(Re_4)/Re_4$. Therefore all projective–injective indecomposables appear in $\mathrm{P}^0(E_1)$ as direct summands.

(3) Let R be an algebra over a field given by the quiver



with relations $\alpha^2 = \gamma\beta\alpha = \gamma\delta = 0$ and $\beta\alpha = \delta\gamma\beta$. Then R is 2-Gorenstein but not 3-Gorenstein, and R has infinite self-injective dimension. Let S_i be a simple left module corresponding to a vertex i ($i = 1, 2, 3$), $P_i = \mathrm{P}^0(S_i)$ a projective cover and $I_i = \mathrm{E}^0(S_i)$ an injective hull of S_i, respectively. Then we have

$P_1 \cong I_2$ are projective–injective

$\mathrm{id}(P_2) = 1, \qquad \mathrm{id}(P_3) = \infty,$

$\mathrm{pd}(I_1) = 1, \qquad \mathrm{pd}(I_3) = \infty.$

The minimal injective resolution of ${}_RR$ is

$$0 \to {}_RR \to I_2^{(3)} \to I_1 \oplus (I_2 \oplus I_1) \to I_1 \oplus I_3 \to \cdots,$$

and $I_1 = \mathrm{E}^1(P_2)$ and $I_2 \oplus I_1 = \mathrm{E}^1(P_3)$. Hence $\mathrm{E}^1(P_3) \cong \mathrm{E}^1(P_2)$ in the projectively stable category $\underline{\mathrm{Mod}}(R)$.

2. Maximal quotient ring of a 1-Gorenstein ring

In [22], a ring R is called left QF-3 if there is a (unique) minimal faithful left R-module, and it is shown that any intermediate ring between a left QF-3 ring and its maximal quotient ring is also left QF-3. In [10], it is shown that, for a serial 2-Gorenstein ring R, intermediate rings in the maximal quotient ring of R are also 2-Gorenstein. If a ring is Artinian, the notions of QF-3 rings and 1-Gorenstein rings coincide. Thus it is reasonable to consider the similar problem for (non-Artinian) 1-Gorenstein rings. We should recall that the left and right maximal quotient rings of a 1-Gorenstein ring are coincident [20, Proposition 2].

Proposition 10. *Let R be a 1-Gorenstein ring. Then the maximal quotient ring Q is semiprimary and $E({}_QQ) = E(Q_Q)$ is projective.*

Proof. First recall $\mathrm{E}({}_RR) = \mathrm{E}({}_QQ)$. We will show that any finitely generated submodule of $\mathrm{E}({}_QQ)$ is torsionless. Let ${}_QX = Qx_1 + \cdots + Qx_n$ be a finitely generated submodule of $\mathrm{E}({}_QQ)$ and consider an R-submodule ${}_RY = Rx_1 + \cdots + Rx_n$. Then, since $\mathrm{E}({}_RR)$ is flat, there exists an R-monomorphism $f\colon Y \to R^{(t)} \subseteq Q^{(t)}$ for some $t \geq 0$ by [16, Theorem 1]. On the other hand, each Qx_i/Rx_i is torsion (under Lambek torsion theory) and so is

$$\bigoplus_{i=1}^{n} (Qx_i/Rx_i).$$

Now there is an epimorphism

$$\bigoplus_{i=1}^{n} (Qx_i/Rx_i) \longrightarrow \bigoplus_{i=1}^{n} (Qx_i + Y)/Y \longrightarrow X/Y$$

and hence ${}_RX/Y$ is torsion. Therefore f may be extended to an R-homomorphism $g: X \to Q^{(t)}$ and then g is actually a Q-monomorphism.

In order to show that $\mathrm{E}({}_QQ)$ is flat as a Q-module, we use Lazard's result [6]. That is, we prove that, for any finitely generated left Q-module M and any Q-homomorphism u of M to $\mathrm{E}({}_QQ)$, u factors through a free Q-module. Let u factor as $u = jp$ with $p: M \to \mathrm{Im}(u)$ and $j: \mathrm{Im}(u) \hookrightarrow \mathrm{E}({}_QQ)$. As we saw above, $\mathrm{Im}(u)$ is torsionless and so embeds in a finitely generated free Q-module F, since Q is semiprimary by [21, Theorem 2]. Then this embedding is extended to a Q-homomorphism of F to $\mathrm{E}({}_QQ)$. Consequently u factors through a finitely generated free module F. □

Proposition 10 cannot be generalized to general n-Gorenstein rings for $n \geq 2$ as is mentioned in [10].

A ring R is said to have *dominant dimension* $\geq n$ if in a minimal injective resolution of ${}_RR$, all E_i for $0 \leq i \leq n-1$ are flat (Definition by Hoshino [11]). A Noetherian ring with dominant dimension $\geq n$ is, of course, n-Gorenstein. On the other hand, Ringel and Tachikawa [22, (2.1) Theorem] characterized a ring with dominant dimension ≥ 2 as an endomorphism ring of a generator–cogenerator. Now we have another characterization for such rings.

Proposition 11. *The following are equivalent for a Noetherian ring R:*

(1) *R has dominant dimension* ≥ 2;

(2) *R is* 1-*Gorenstein and is its own maximal quotient ring;*

(3) *R is an Artinian* 2-*Gorenstein ring and there is no injective left module of projective dimension one.*

Proof. First of all, we recall the fact that R is its own maximal left quotient ring if and only if E_0/R is embedded in a direct product of copies of E_0.

(1) ⇒ (3): It follows by [14, Proposition 7] that R is Artinian. Assume that there exists an injective indecomposable left module E of projective dimension one. Then the torsion submodule $t(E)$ is nonzero. For, if $t(E) = 0$, E embeds in a direct product of copies of E_0, which is projective. Thus E is projective, a contradiction. Let $0 \to P_1 \to P_0 \to E \to 0$ be a minimal projective resolution for E and take a pull back

diagram of two maps $t(E) \hookrightarrow E$ and $P_0 \to E$:

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^1 & \longrightarrow & L & \longrightarrow & t(E) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & E & \longrightarrow & 0.
\end{array}$$

Then the monomorphism $P_1 \to L$ is an essential extension. Hence, in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & L & \longrightarrow & t(E) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & P_1 & \longrightarrow & \mathrm{E}(P_1) & \longrightarrow & \mathrm{E}(P_1)/P_1 & \longrightarrow & 0,
\end{array}$$

α is monic and so is β. Thus we have the following embedding

$$t(E) \overset{\beta}{\hookrightarrow} \mathrm{E}(P_1)/P_1 \hookrightarrow \bigoplus_{I} (E_0/R) \hookrightarrow \bigoplus_{I} E_1 .$$

Therefore $\mathrm{E}(t(E)) = E$ since $t(E) \neq 0$, and E is embedded in $\bigoplus_{I} E_1$. However $\bigoplus_{I} E_1$ is projective and this is a contradiction.

(3) $\Rightarrow$ (2): By the assumption, E_1 is projective and hence R is its own maximal left quotient ring.

(2) $\Rightarrow$ (1) follows from the fact we mentioned in the beginning of the proof. □

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