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Subgroups of free idempotent generated semigroups need not be free

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ABSTRACT

We study the maximal subgroups of free idempotent generated semigroups on a biordered set by topological methods. These subgroups are realized as the fundamental groups of a number of 2-complexes naturally associated to the biorder structure of the set of idempotents. We use this to construct the first example of a free idempotent generated semigroup containing a non-free subgroup.

1. Introduction

Let S be a semigroup with set $E(S)$ of idempotents, and let $\langle E(S)\rangle$ denote the subsemigroup of S generated by $E(S)$. We say that S is an *idempotent generated* semigroup if $S = \langle E(S)\rangle$. Idempotent generated semigroups have received considerable attention in the literature. For example, an early result of J.A. Erdös [8] proves that the idempotent generated part of the semigroup of $n \times n$ matrices over a field consists of the identity matrix and all singular matrices. J.M. Howie [15] proved a similar result for the full transformation monoid on a finite set and also showed that every semigroup may be embedded in an idempotent generated semigroup. This result has been extended in many different ways, and many authors have studied the structure of idempotent generated semigroups. Recently, Putcha [24] gave necessary and sufficient conditions for a reductive linear algebraic monoid to have

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the property that every non-unit is a product of idempotents, significantly generalizing the results of J.A. Erdös mentioned above.

In 1979 K.S.S. Nambooripad [18] published an influential paper about the structure of (von Neumann) regular semigroups. Nambooripad observed that the set $E(S)$ of idempotents of a semigroup carries a certain structure (the structure of a "biordered set," or a "regular biordered set" in the case of regular semigroups) and he provided an axiomatic characterization of (regular) biordered sets in his paper. If E is a regular biordered set, then there is a free object, which we will denote by $RIG(E)$, in the category of regular idempotent generated semigroups with biordered set E. Nambooripad showed how to study $RIG(E)$ via an associated groupoid $\mathcal{N}(E)$. There is also a free object, which we will denote by $IG(E)$, in the category of idempotent generated semigroups with biordered set E for an arbitrary (not necessarily regular) biordered set E.

In the present paper we provide a topological approach to Nambooripad's theory by associating a 2-complex $K(E)$ to each regular biordered set E. The fundamental groupoid of the 2-complex $K(E)$ is Nambooripad's groupoid $\mathcal{N}(E)$. Our concern in this paper is in analyzing the structure of the maximal subgroups of $IG(E)$ and $RIG(E)$ when E is a regular biordered set. It has been conjectured that these subgroups are free [17], and indeed there are several papers in the literature (see for example, [17,19,20]) that prove that the maximal subgroups are free for certain classes of biordered sets. The main result of this paper is to use these topological tools to give the first example of non-free maximal subgroups in free idempotent generated semigroups over a biordered set. We give an example of a regular biordered set E associated to a certain combinatorial configuration such that $RIG(E)$ has a maximal subgroup isomorphic to the free abelian group of rank 2.

2. Preliminaries on biordered sets and regular semigroups

One obtains significant information about a semigroup by studying its ideal structure. Recall that if S is a semigroup and $a, b \in S$ then the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{J}$ and $\mathcal{D}$ are defined by $a\mathcal{R}b$ if and only if $aS^1 = bS^1$, $a\mathcal{L}b$ if and only if $S^1a = S^1b$, $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ and $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, so that $\mathcal{D}$ is the join of $\mathcal{R}$ and $\mathcal{L}$ in the lattice of equivalence relations on S. The corresponding equivalence classes of an element $a \in S$ are denoted by R_a, L_a, H_a, J_a and D_a respectively. Recall also that there are quasi-orders defined on S by $a \leqslant_{\mathcal{R}} b$ if $aS^1 \subseteq bS^1$, and $a \leqslant_{\mathcal{L}} b$ if $S^1a \subseteq S^1b$. As usual, these induce partial orders on the set of $\mathcal{R}$-classes and $\mathcal{L}$-classes respectively. The restrictions of these quasi-orders to $E(S)$ will be denoted by ω^r and ω^l respectively in this paper, in accord with the notation in Nambooripad's paper [18]. It is easy to see that if e and f are idempotents of S then $e\omega^r f$ (i.e. $eS \subseteq fS$) if and only if $e = fe$, that $e\omega^l f$ if and only if $e = ef$, that $e\mathcal{R}f$ if and only if $e = fe$ and $f = ef$, and that $e\mathcal{L}f$ if and only if $e = ef$ and $f = fe$.

Let e be an idempotent of a semigroup S. The set eSe is a submonoid in S and is the largest submonoid (with respect to inclusion) whose identity element is e. The group of units G_e of eSe, that is the group of elements of eSe that have two sided inverses with respect to e, is the largest subgroup of S (with respect to inclusion) whose identity is e and is called the maximal subgroup of S at e.

Recall also that if e and f are idempotents of S then the natural partial order on $E(S)$ is defined by $e\omega f$ if and only if $ef = fe = e$. Thus $\omega = \omega^r \cap \omega^l$. An element $a \in S$ is called *regular* if $a \in aSa$: in that case there is at least one *inverse* of a, i.e. an element b such that $a = aba$ and $b = bab$. Note that regular semigroups have in general many idempotents: if a and b are inverses of each other, then ab and ba are both idempotents (in general distinct). Standard examples of regular semigroups are the semigroup of all transformations on a set (with respect to composition of functions) and the semigroup of all $n \times n$ matrices over a field (with respect to matrix multiplication).

We recall the basic properties of the very important class of completely 0-simple semigroup. A semigroup S (with 0) is (0)-simple if ($S^2 \neq 0$ and) its only ideal is S (S and 0). A (0)-simple semigroup S is completely (0)-semigroup if S contains an idempotent and every idempotent is (0)-minimal in the natural partial order of idempotents defined above. It is a fundamental fact that every finite (0)-simple semigroup is completely (0)-simple.

Let S be a completely 0-simple semigroup. The Rees theorem [2,16] states that S is isomorphic to a regular Rees matrix semigroup over a group G, $M^0(A, G, B, C)$ and conversely that every such semigroup is completely (0)-simple. Here $A(B)$ is an index set for the $\mathcal{R}(\mathcal{L})$-classes of the non-zero

$\mathcal{J}$-class of S and $C: B \times A \to G^0$ is a function called the structure matrix. C has the property that for each $a \in A$ there is a $b \in B$ such that $C(b,a) \neq 0$ and for each $b \in B$ there is an $a \in A$ such that $C(b,a) \neq 0$. We always assume that A and B are disjoint. The underlying set of $M^0(A,G,B,C)$ is $A \times G \times B \cup \{0\}$ and the product is given by $(a,g,b)(a',g',b') = (a, gC(b,a')g', b')$ if $C(b,a') \neq 0$ and 0 otherwise.

We refer the reader to the books of Clifford and Preston [2] or Lallement [16] for standard ideas and notation about semigroup theory.

An *E-path* in a semigroup S is a sequence of idempotents $(e_1, e_2, \ldots, e_n)$ of S such that $e_i(\mathcal{R} \cup \mathcal{L})e_{i+1}$ for all $i = 1, \ldots, n-1$. This is just a path in the graph $(E, \mathcal{R} \cup \mathcal{L})$: the set of vertices of this graph is the set E of idempotents of S and there is an edge denoted (e,f) from e to f for $e, f \in E$ if $e\mathcal{R}f$ or $e\mathcal{L}f$. One can introduce an equivalence relation on the set of E-paths by adding or removing "inessential" vertices: a vertex (idempotent) e_i of a path $(e_1, e_2, \ldots, e_n)$ is called *inessential* if $e_{i-1}\mathcal{R}e_i\mathcal{R}e_{i+1}$ or $e_{i-1}\mathcal{L}e_i\mathcal{L}e_{i+1}$. Following Nambooripad [18], we define an *E-chain* to be the equivalence class of an E-path relative to this equivalence relation. It can be proved [18] that each E-chain has a unique canonical representative of the form $(e_1, e_2, \ldots, e_n)$ where every vertex is essential. We will often abuse notation slightly by identifying an E-chain with its canonical representative.

The set $\mathcal{G}(E)$ of E-chains forms a *groupoid* with set E of objects (identities) and with an E-chain $(e_1, e_2, \ldots, e_n)$ viewed as a morphism from e_1 to e_n. The product C_1C_2 of two E-chains $C_1 = (e_1, e_2, \ldots, e_n)$ and $C_2 = (f_1, f_2, \ldots, f_m)$ is defined and equal to the canonical representative of $(e_1, \ldots, e_n, f_1, \ldots, f_m)$ if and only if $e_n = f_1$: the inverse of $(e_1, e_2, \ldots, e_n)$ is $(e_n, \ldots, e_2, e_1)$. We refer the reader to [18] for more detail.

For future reference we give a universal characterization of $\mathcal{G}(E)$ in the category of small groupoids. Every equivalence relation R on a set X can be considered to be a groupoid with objects X and arrows the ordered pairs of R. There are obvious notions of free products and free products with amalgamations in the category of small groupoids. See [13] for details. Clearly the objects of any groupoid form a subgroupoid whose morphisms are the identities. We will identify the objects of a groupoid as this subgroupoid and call it the trivial subgroupoid. The proof of the following theorem appears in [18].

Theorem 2.1. *Let S be a semigroup with non-empty set of idempotents E. Then $\mathcal{G}(E)$ is isomorphic to the free product with amalgamation $\mathcal{L} *_E \mathcal{R}$ in the category of small groupoids.*

As mentioned above, we are considering E to be the trivial subgroupoid of $\mathcal{G}(E)$.

It is easy to see from the characterizations of $\mathcal{R}$ and $\mathcal{L}$ above that if $(f_1, f_2, \ldots, f_m)$ is the canonical representative equivalent to an E-path $(e_1, e_2, \ldots, e_n)$, then $e_1e_2\ldots e_n = f_1f_2\ldots f_m$ in S, since $efg = eg$ if $e\mathcal{R}f\mathcal{R}g$ or $e\mathcal{L}f\mathcal{L}g$. Standard results of Miller and Clifford [2] imply that $e_1\mathcal{R}e_1e_2\ldots e_n\mathcal{L}e_n$.

In 1972, D.G. Fitzgerald [9] proved the following basic result about the idempotent generated subsemigroup of any semigroup.

Theorem 2.2. *Let S be any semigroup with non-empty set $E = E(S)$ of idempotents and let x be a regular element of $\langle E(S)\rangle$. Then x can be expressed as a product of idempotents $x = e_1e_2\ldots e_n$ in an E-path $(e_1, e_2, \ldots, e_n)$ of S, and hence as a product of idempotents in an E-chain. If S is regular, then so is $\langle E(S)\rangle$.*

In 1979, Nambooripad introduced the notion of a *biordered set* as an abstract characterization of the set of idempotents E of a semigroup S with respect to certain basic products that are forced to be idempotents. We give the details that will be needed in this paper.

Recall that if $e, f \in E = E(S)$ for some semigroup S then $e\omega^r f$ if and only if $fe = e$, and $e\omega^l f$ if and only if $ef = e$. In the former case, ef is an idempotent that is $\mathcal{R}$-related to e and $ef\omega f$ in the natural order on E: similarly, in the latter case, fe is an idempotent that is $\mathcal{L}$-related to e and $fe\omega f$. Thus in each case both products ef and fe are defined within E, i.e. such products of idempotents must always be idempotent. Products of these types are referred to as *basic products*. The partial algebra E with multiplication restricted to basic products is called the *biordered set* of S.

Nambooripad [18] characterized the partial algebra of idempotents of a (regular) semigroup with respect to these basic products axiomatically. We refer the reader to Nambooripad's article [18] for the details. The axioms are complicated but do arise naturally in mathematics. For example, Putcha proved that pairs of opposite parabolic subgroups of a finite group of Lie type have the natural structure of a biordered set [23]. We will need one more concept, the sandwich set $S(e,f)$ of two idempotents e, f of S.

If e, f are (not necessarily distinct) idempotents of a semigroup S, then $S(e,f) = \{h \in E \mid ehf = ef,\ fhe = h\}$ is called the sandwich set of e and f (in that order). It is straightforward to prove that if $h \in S(e,f)$, then h is an inverse of ef. In particular, $S(e,f)$ is non-empty for any e, f if S is a regular semigroup. Nambooripad also gave an order theoretic definition of the sandwich set, but we will not need that in this paper.

As mentioned above, Nambooripad gave a definition of a biordered set as a partial algebra satisfying a collection of axioms. We do not need the details of these axioms because of the following theorems. He called a biordered set *regular* if the (axiomatically defined) sandwich set of any pair of idempotents is non-empty.

Theorem 2.3. *(See Nambooripad [18].) The set E of idempotents of a regular semigroup is a regular biordered set relative to the basic products in E. Conversely, every regular (axiomatically defined) biordered set arises as the biordered set of idempotents of some regular semigroup.*

This was extended to non-regular semigroups and non-regular biordered sets by Easdown [6]. We will give a more precise statement of Easdown's result in the next section.

3. Free idempotent generated semigroups on biordered sets

If E is a biordered set we denote by $IG(E)$ the semigroup with presentation

$IG(E) = \langle E\colon\ e^2 = e$ for all $e \in E$ and $e.f = ef$ if ef is a basic product in $E\rangle$.

If E is a regular biordered set, then we define

$RIG(E) = \langle E\colon\ e^2 = e$ for all $e \in E$ and $e.f = ef$ if ef is a basic product in E and $ef = ehf$ for all $e, f \in E$ and $h \in S(e,f)\rangle$

The semigroup $IG(E)$ is called the *free idempotent generated semigroup on E* and the semigroup $RIG(E)$ is called the *free regular idempotent generated semigroup on E*. This terminology is justified by the following results of Easdown [6], Nambooripad [18] and Pastijn [21].

Theorem 3.1. *(See [6].) The biordered set of idempotents of $IG(E)$ is E. In particular, every biordered set is the biordered set of some semigroup. If S is any idempotent generated semigroup with biordered set of idempotents isomorphic to E then the natural map $E \to S$ extends uniquely to a homomorphism $IG(E) \to S$.*

Theorem 3.2. *(See [18,21].) If E is a regular biordered set then $RIG(E)$ is a regular semigroup with biordered set of idempotents E. If S is any regular idempotent generated semigroup with biordered set biorder isomorphic to E, then the natural map $E \to S$ extends uniquely to a homomorphism $RIG(E) \to S$.*

There is an obvious natural morphism $\phi : IG(E) \to RIG(E)$ if E is a regular biordered set. However, we remark that this is not an isomorphism, and the semigroups $IG(E)$ and $RIG(E)$ can be very different when E is a regular biordered set. Also, the regular elements of $IG(E)$ do not form a subsemigroup in general, even if E is a regular biordered set.

The following simple examples illustrate these facts.

Example 1. Let E be the (non-regular) biordered set consisting of two idempotents e and f with trivial quasi-orders ω^r and ω^l. Clearly the rules $e^2 \to e$, $f^2 \to f$ constitute a terminating confluent rewrite system for the semigroup $IG(E)$. Canonical forms for words in $IG(E)$ are of the form $efef\ldots e$ or $efef\ldots f$ or $fefe\ldots f$ or $fefe\ldots e$. Clearly $IG(E)$ is an infinite semigroup with exactly two idempotents (e and f).

Example 2. Let F be the biordered set E above with a zero 0 adjoined. Thus F is a three-element semilattice, freely generated as a semilattice by e and f. It is easy to see that $RIG(F) = F$ since $ef = e0f = fe = f0e = 0$ from the presentation for $RIG(F)$ and since $0 \in S(e, f)$. But $IG(F)$ is $IG(E)^0$, where $IG(E)$ is the semigroup in Example 1. Thus $IG(F)$ is infinite, but $RIG(F)$ is finite.

We will give more information about the relationship between $IG(E)$ and $RIG(E)$, for E a regular biordered set, at the end of this section. In particular, we will show that the regular elements of $IG(E)$ are in one–one correspondence with the elements of $RIG(E)$ (even though the regular elements of $IG(E)$ do not necessarily form a subsemigroup of $IG(E)$).

Nambooripad studied the free regular idempotent generated semigroup $RIG(E)$ on a regular biordered set via his general theory of "inductive groupoids" in [18]. If S is a regular semigroup, then Nambooripad introduced an associated groupoid $\mathcal{N}(S)$ (that we refer to as the *Nambooripad groupoid of S*) as follows. The set of objects of $\mathcal{N}(S)$ is the set $E = E(S)$ of idempotents of S. The morphisms of $\mathcal{N}(S)$ are of the form (x, x') where x' is an inverse of x: (x, x') is viewed as a morphism from xx' to $x'x$ and the composition of morphisms is defined by $(x, x')(y, y') = (xy, y'x')$ if $x'x = yy'$ (and undefined otherwise). With respect to this product, $\mathcal{N}(S)$ becomes a groupoid, which in fact is endowed with much additional structure, making it an *inductive groupoid* in the sense of Nambooripad [18]. An inductive groupoid is an ordered groupoid whose identities (objects) admit the structure of a regular biordered set E, and which admits a way of evaluating products of idempotents in an E-chain as elements of the groupoid. There is an equivalence between the category of regular semigroups and the category of inductive groupoids. We refer the reader to Nambooripad's paper [18] for much more detail. In particular, it follows easily from Nambooripad's results that the maximal subgroup of S containing the idempotent e is isomorphic to the local group of $\mathcal{N}(S)$ based at the object (identity) e (i.e. the group of all morphisms from e to e in $\mathcal{N}(S)$).

In his paper [18], Nambooripad also showed how to construct the inductive groupoid $\mathcal{N}(RIG(E))$ associated with the free regular idempotent generated semigroup on a regular biordered set E directly from the groupoid of E-chains of E. We review this construction here.

Let E be a regular biordered set. An *E-square* is an E-path (e, f, g, h, e) with $e\mathcal{R}f\mathcal{L}g\mathcal{R}h\mathcal{L}e$ or (e, h, g, f, e) with $e\mathcal{L}h\mathcal{R}g\mathcal{L}f\mathcal{R}e$. We draw the square as: $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$. An E-square is degenerate if it is of one of the following three types:

$$\begin{bmatrix} e & e \\ e & e \end{bmatrix}, \qquad \begin{bmatrix} e & f \\ e & f \end{bmatrix}, \qquad \begin{bmatrix} e & e \\ f & f \end{bmatrix}.$$

Unless mentioned otherwise, all E-squares will be non-degenerate.

An idempotent $t = t^2 \in E$ *left to right singularizes* the E-square $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$ if $te = e, th = h, et = f$ and $ht = g$ where all of these products are defined in the biordered set E. Right to left, top to bottom and bottom to top singularization is defined similarly and we call the E-square *singular* if it has a singularizing idempotent of one of these types. Note that since $te = e \in E$ if and only if $e\omega^r t$, all of these products can also be defined in terms of the order structure as well.

The importance of singular E-squares is given by the next lemma.

Lemma 3.3. *Let $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$ be a singular E-square in a semigroup S. Then the product of the elements in the E-cycle (e, f, g, h, e) satisfies $efghe = e$.*

Proof. Let $t = t^2$ left to right singularize the E-square $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$. Then in any idempotent generated semigroup with biordered set E, $efghe = fh$ follows from the basic $\mathcal{R}$ and $\mathcal{L}$ relations of E. Furthermore, $fh = eth = eh = e$ which follows from the definition of left to right singularization. The other cases of singularization are proved similarly. □

In order to build the inductive groupoid of $RIG(E)$, we must therefore identify any singular E-cycle of $\mathcal{G}(E)$ from an idempotent e to itself with e. This is because any inductive groupoid whose identities form a biordered set E is an image of $\mathcal{G}(E)$ by Nambooripad's theory [18]. This leads to the following definition. For two E-chains $C = (e_1, e_2, \ldots, e_n)$ and $C' = (f_1, f_2, \ldots, f_m)$ define $C \to C'$ if there are E-chains C_1 and C_2 and a singular E-square γ such that $C = C_1C_2$ and $C' = C_1\gamma C_2$ and let $\sim$ denote the equivalence relation on $\mathcal{G}(E)$ induced by $\to$. The next theorem follows from [18], Theorems 6.9, 6.10 and ensures that the quotient groupoid $\mathcal{G}(E)/\sim$ defined above has an inductive structure and is isomorphic to the inductive groupoid of $RIG(E)$.

Theorem 3.4. *(See Nambooripad [18].) If E is a regular biordered set, then $\mathcal{N}(RIG(E)) \cong \mathcal{G}(E)/\sim$.*

It is convenient to provide a topological interpretation of this theorem of Nambooripad. We remind the reader that just as groups are presented by a set of generators and a set of words over the generating set as relators (giving the group as a quotient of the free group on the generating set), groupoids are presented by a graph and a set of cycles in the graph as relators (giving the groupoid as a quotient of the free groupoid on the graph). See [13] for more details.

It follows from Theorems 2.1 and 3.4 that we have the following presentation for $\mathcal{N}(RIG(E)) \cong \mathcal{G}(E)/\sim$.

Generators: The graph with vertices E and edges the relation $\mathcal{R} \cup \mathcal{L}$.

Relators: There are two types of relators:

(1) $((e, f), (f, g), (g, e)) = 1_e$ if $e\mathcal{R}f\mathcal{R}g$ or $e\mathcal{L}f\mathcal{L}g$;
(2) $((e, f), (f, g), (g, h), (h, e)) = 1_e$ if $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$ is a singular E-square.

We will always assume that there are no trivial relators in the list above. This means that for relators of type (1) all three elements e, f, g are distinct and for relators of type (2), all four elements e, f, g, h are distinct.

If E is a regular biordered set we associate a 2-complex $K(E)$ which is the analogue of the presentation complex of a group presentation. The 1-skeleton of $K(E)$ is the graph $(E, \mathcal{R} \cup \mathcal{L})$ described above. Since $\mathcal{R}$ and $\mathcal{L}$ are symmetric relations we consider the underlying graph to be undirected in the usual way. The 2-cells of $K(E)$ are of the following types:

(1) if $e\mathcal{R}f\mathcal{R}g$ or $e\mathcal{L}f\mathcal{L}g$ for $e, f, g \in E$ then there is a 2-cell with boundary edges (e, f), (f, g), (g, e).
(2) all singular E-squares bound 2-cells.

We note that our 2-complexes are combinatorial objects and we follow the notation of [26,28].

We denote the fundamental groupoid of a 2-complex K by $\pi_1(K)$: the fundamental group of K based at v will be denoted by $\pi_1(K, v)$. The following corollary is an immediate consequence of Nambooripad's work and the definition of the fundamental groupoid of a 2-complex (see, for example, [13]).

Corollary 3.5. *If E is a regular biordered set, then $\pi_1(K(E)) \cong \mathcal{G}(E)/\sim$ and hence $\pi_1(K(E)) \cong \mathcal{N}(RIG(E))$.*

It follows that the maximal subgroup of $RIG(E)$ containing the idempotent e is isomorphic to the fundamental group of $K(E)$ based at e. The next theorem shows that there is a one-to-one correspondence between regular elements of $IG(E)$ and $RIG(E)$ if E is a regular biordered set and that for every $e \in E$, the maximal subgroup at e in $IG(E)$ is isomorphic to the maximal subgroup at e in $RIG(E)$.

Theorem 3.6. *Let E be a regular biordered set. Then the natural map $\phi : IG(E) \to RIG(E)$ is a bijection when restricted to the regular elements of $IG(E)$. That is, for each element $r \in RIG(E)$ there exists a unique regular element $s \in IG(E)$ such that $\phi(s) = r$. In particular, the maximal subgroups of $IG(E)$ and $RIG(E)$ are isomorphic.*

Proof. It follows from Fitzgerald's theorem, Theorem 2.2 that every element of $RIG(E)$ is the product of the elements in an E-chain. But it follows from the Clifford–Miller theorem [2] that the product of an element in an E-chain is a regular element in any idempotent generated semigroup with biordered set E. It follows immediately that ϕ restricts to a surjective map from the regular elements of $IG(E)$ to $RIG(E)$.

If u and v are regular elements of $IG(E)$, then there are E-chains $(e_1, e_2, \dots, e_n)$ and $(f_1, f_2, \dots, f_m)$ such that $u = e_1 e_2 \dots e_n$ and $v = f_1 f_2 \dots f_m$ in $IG(E)$. Suppose that $\phi(u) = \phi(v)$. Clearly, on applying the morphism ϕ, $e_1 e_2 \dots e_n = f_1 f_2 \dots f_m$ in $RIG(E)$. We mentioned previously that it follows from the Clifford–Miller theorem [2] that $e_1 \mathcal{R} f_1$ and $e_n \mathcal{L} f_m$. Thus without loss of generality, we may assume that $e_1 = f_1$ since $e_1 f_1 f_2 \dots f_m = f_1 f_2 \dots f_m$ in $IG(E)$, and similarly we may assume that $e_n = f_m$. Applying [18], Lemma 4.11 and Theorem 3.4, it follows that $(e_1, e_2, \dots, e_n) \sim (f_1, f_2, \dots, f_m)$. Thus it is possible to pass from $(e_1, e_2, \dots, e_n)$ to $(f_1, f_2, \dots, f_m)$ by a sequence of operations of two types:

(a) inserting or deleting paths of length 3 corresponding to $\mathcal{R}$ or $\mathcal{L}$ related idempotents; and
(b) inserting or deleting E-cycles corresponding to singular E-squares.

Note that if (e, f, g, h, e) is a singular E-square then $efghe = e$ in any semigroup S with biordered set E by Lemma 3.3. It follows easily that if $(g_1, g_2, \dots, g_p)$ is obtained from $(e_1, e_2, \dots, e_n)$ by one application of an operation of type (a) or (b) above, then $e_1 e_2 \dots e_n = g_1 g_2 \dots g_p$ in any semigroup with biordered set E, and in particular this is true in $IG(E)$. It follows by induction on the number of steps of types (a) and (b) needed to pass from $(e_1, e_2, \dots, e_n)$ to $(f_1, f_2, \dots, f_m)$ that $u = e_1 e_2 \dots e_n = f_1 f_2 \dots f_m = v$ in $IG(E)$, so ϕ is one-to-one on regular elements, as desired.

To prove the final statement of the theorem, note that elements of the maximal subgroup of $IG(E)$ or $RIG(E)$ containing e come from E-chains that start and end at e, since $e_1 \mathcal{R} e_1 e_2 \dots e_n \mathcal{L} e_n$ for any E-chain $(e_1, e_2, \dots, e_n)$. This shows that the map ϕ is surjective on maximal subgroups: the first part of the theorem shows that it is injective on maximal subgroups. $\square$

4. Connections between the Nambooripad complex and the Graham–Houghton complex

In this section we use the Bass–Serre theoretic methods of [11] to study the local groups of $\mathcal{G}(E)$ and $\mathcal{N}(E)$. The local group of a groupoid G at the object v is the group of self morphisms $G(v, v)$. For $\mathcal{G}(E)$ we give a rapid topological proof of a result of Nambooripad and Pastijn [19] who showed that the local groups of $\mathcal{G}(E)$ are free groups. By applying [11] we are led directly to the graphs considered by Graham and Houghton [10,14] for studying completely 0-simple semigroups. We put a structure of a complex on top of the Graham–Houghton graphs in order to have tools to study the vertex subgroups of $\mathcal{N}(E)$, which by Theorems 3.4 and 3.6 are the maximal subgroups of $IG(E)$ and $RIG(E)$ when E is a regular biordered set.

Throughout this section, E will denote a regular biordered set. By Theorem 3.2 E is isomorphic to the biordered set of idempotents of $RIG(E)$ and we will use this identification throughout the section as well. Thus, we will refer to the elements of E as idempotents and talk about their Green classes within $RIG(E)$. We have seen in Theorem 2.1 that $\mathcal{G}(E)$ decomposes as the free product with amalgamation $\mathcal{G}(E) = \mathcal{L} *_E \mathcal{R}$, where by abuse of notation, E denotes the trivial subgroupoid. Since $\mathcal{L}$ and $\mathcal{R}$ also have the same objects as each other and as E, we can use the methods of [11] to study the maximal subgroup of $\mathcal{G}(E)$, since this paper was concerned with amalgams of groupoids in which the intersection of the two factors contains all the identity elements.

For every such amalgam of groupoids $G = A *_U B$, [11] associates a graph of groups in the sense of Bass–Serre theory [27] whose connected components are in one-to-one correspondence with the connected components of G and such that the fundamental group of a connected component is isomorphic to the local group of the corresponding component of G.

First note that there is a one-to-one correspondence between the $\mathcal{L}(\mathcal{R})$-classes of E and the $\mathcal{L}(\mathcal{R})$-classes of $RIG(E)$. This is because every $\mathcal{L}(\mathcal{R})$-class of $RIG(E)$ has an idempotent and the $\mathcal{L}(\mathcal{R})$ relation restricted to idempotents can be defined by basic products. We abuse notation by identifying an $\mathcal{L}(\mathcal{R})$ class of E with the $\mathcal{L}(\mathcal{R})$ class of $RIG(E)$ containing it.

We now describe explicitly the graph of groups associated to $\mathcal{G}(E)$. For more details, see [11]. The graph of groups G of $\mathcal{G}(E)$ consists of the following data: The set of vertices is the disjoint union of the $\mathcal{L}$ and $\mathcal{R}$-classes of E and its positive edges are the elements of E. If $e \in E$, its initial edge is its

$\mathcal{L}$-class and its terminal edge is its $\mathcal{R}$-class. That is, there is a unique positive edge from an $\mathcal{L}$-class L to an $\mathcal{R}$-class R if and only if the $\mathcal{H}$-class $L \cap R$ of $RIG(E)$ contains an idempotent. Each vertex group of G is the trivial group. This is an exact translation for $\mathcal{G}(E)$ of the graph of groups defined for an arbitrary amalgam on page 46 of [11].

Since the vertex groups of G are trivial, we can consider G to be a graph in the usual sense. Therefore its fundamental group is a free group and we have the following theorem of Nambooripad and Pastijn [19].

Theorem 4.1. *Every local subgroup of $\mathcal{G}(E)$ is a free group.*

Proof. It follows from Theorem 3 of [11] that for each element $e \in E$ the local subgroup of $\mathcal{G}(E)$ at e is isomorphic to the fundamental group of G based at the $\mathcal{L}$-class of e. Since the latter group is free by the discussion above, the theorem is proved. □

In the case that a connected component of $\mathcal{G}(E)$ has a finite number of idempotents, the rank of the free group will be the Euler characteristic of the corresponding component of G, that is, the number of edges of the graph minus the number of vertices plus 1. Thus if the connected component of $e \in E$ of $\mathcal{G}(E)$ has m $\mathcal{R}$-classes, n $\mathcal{L}$-classes and k idempotents, then the free group $\mathcal{G}(E)(e, e)$ has rank $k - (m + n) + 1$.

All the calculations of maximal subgroups of $RIG(E)$ or $IG(E)$ that have appeared in the literature [17,19,21] have been restricted to cases of biordered sets that have no non-degenerate singular squares. In this case it follows from Theorem 3.4 that $\mathcal{G}(E)$ is isomorphic to $\mathcal{N}(E)$. Since the local groups of $\mathcal{N}(E)$ are isomorphic to the maximal subgroups of $RIG(E)$ we have the following result of Nambooripad and Pastijn [19].

Theorem 4.2. *If E is a biordered set that has no non-degenerate singular squares, then every subgroup of $RIG(E)$ is free.*

Nambooripad and Pastijn's proof of Theorem 4.2 uses combinatorial word arguments. A topological proof of Theorem 4.2 in the special case that the (not necessarily regular) biordered set has no non-trivial biorder ideals was given by McElwee [17]. The graph that McElwee uses is the same as ours in this case, but without reference to the general work of [11] or the connection with the Graham–Houghton graph [10,14] that we discuss below. There are a number of interesting classes of regular semigroups whose biordered sets have no non-degenerate singular squares including locally inverse semigroups. See [19] for more examples.

Connected components of the graph G associated to $\mathcal{G}(E)$ defined above have arisen in the literature in connection with the theory of finite 0-simple semigroups and in particular with the theory of idempotent generated subsemigroups of finite 0-simple semigroups. Finite idempotent generated 0-simple semigroups have the property that all non-zero idempotents are connected by an E-chain. This follows from the Clifford–Miller theorem [2]. Thus the graph G corresponding to the biordered set of a finite 0-simple semigroup has a trivial component consisting of 0 and one other connected component. The graph defined independently by Graham and Houghton [10,14] associated to a finite 0-simple semigroup is exactly the graph that arises from Bass–Serre theory associated to $\mathcal{G}(E)$ that we have defined above. Graham and Houghton did not note the connection to Bass–Serre theory. A number of papers have given connections between completely 0-simple semigroups, the theory of graphs and algebraic topology [10,14,22]. The monograph [25] gives an updated version of these connections.

We now add 2-cells to G, the graph associated to $\mathcal{G}(E)$, one for each singular square $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$. Given this square and recalling that the positive edges of G are directed from the $\mathcal{L}$-class of an idempotent to its $\mathcal{R}$-class we sew a 2-cell onto G with boundary $ef^{-1}gh^{-1}$. We call this 2-complex the Graham–Houghton complex of E and denote it by $GH(E)$.

We note two important properties of $GH(E)$. Its 1-skeleton is naturally bipartite as each edge runs between an $\mathcal{L}$-class and an $\mathcal{R}$-class. Furthermore $GH(E)$ is a square complex in that each of its cells is a square bounded by a 4-cycle.

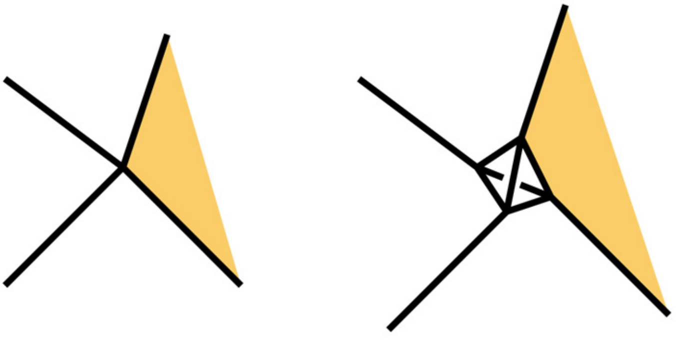

Fig. 1.

We now prove that the fundamental group of the connected component of $GH(E)$ containing the vertex $\mathcal{L}_e$ of an idempotent $e \in E$ is isomorphic to the fundamental group of the Nambooripad complex $K(E)$ containing the vertex e. We will then be able to use $GH(E)$ to compute the maximal subgroups of $RIG(E)$.

As we have seen above, the Nambooripad complex $K(E)$ has vertices E, the idempotents of S, edges (e, f) whenever $e\mathcal{R}f$ or $e\mathcal{L}f$, and two types of two cells: one triangular 2-cell $(e, f)(f, g)(g, e)$ for each unordered triple (e, f, g) of distinct elements satisfying $e\mathcal{R}f\mathcal{R}g$ or $e\mathcal{L}f\mathcal{L}g$, and one square 2-cell $(e, f)(f, g)(g, h)(h, e)$ for each non-degenerate singular E-square $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$.

The Graham–Houghton complex $GH(E)$ has one vertex for each $\mathcal{R}$ or $\mathcal{L}$-class of E, an edge labeled by $e \in E$ between $\mathcal{R}_a$ and $\mathcal{L}_b$ if $e \in \mathcal{R}_a \cap \mathcal{L}_b$ (giving a bipartite graph), and square 2-cells attached along (e, f, g, h) when $\left[\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right]$ is a non-degenerate singular E-square.

We now describe a sequence of transformations of complexes which starts with $GH(E)$ and ends with $K(E)$. Each step, we shall see, does not change the isomorphism class of the fundamental groups of the complex. This will imply that $GH(E)$ and $K(E)$ have isomorphic fundamental groups. The basic idea is that the vertices of $K(E)$ are the edges of $GH(E)$, and the vertices of $GH(E)$ are, in some sense, the edges of $K(E)$. The process basically "blows up" the vertices of $GH(E)$ to introduce the edges of $K(E)$, and then crushes the original edges of $GH(E)$ to points to create the vertices of $K(E)$. The blow-up process introduces the triangular 2-cells needed for $K(E)$, and the crushing process turns the square 2-cells of $GH(E)$ into the square 2-cells of $K(E)$. All of the topological facts used below may be found, for example, in [12,28]. More precisely, in the theorem below, we prove that $K(E)$ is the 2-skeleton of a complex that is homotopy equivalent to $GH(E)$ and in particular, they have isomorphic fundamental groups at each vertex. A result similar to the next theorem appears in Proposition 9.6 and Proposition 9.7 [29].

Theorem 4.3. *$\pi_1(K(E), e)$ is isomorphic to $\pi_1(GH(E), \mathcal{L}_e)$ for each $e \in E$.*

Proof. The first step is to blow up each vertex R or L of $GH(E)$ to an n-simplex, where n is the valence of the vertex. Fig. 1 shows the essential details. The basic idea is that the vertex R or L becomes the n-simplex, each edge of $GH(E)$ incident to R or L becomes an edge incident to a distinct vertex of the n-simplex, and any square 2-cell incident to the vertex receives an added edge of the n-simplex in its boundary, joining the two vertices which its original pair of edges are now incident to. Carrying out this process for all of the original vertices results in a complex which we will call Q_1. Note that Q_1 is homotopy equivalent to $GH(E)$, since $GH(E)$ may be obtained from Q_1 by crushing each n-simplex σ^n to a point (literally, taking the quotient complex Q_1/σ^n). Since each n-simplex is a contractible subcomplex of Q_1, the quotient map $Q_1 \to Q_1/\sigma^n$ is a homotopy equivalence [12, Proposition 0.17]; the result then follows by induction, since Q_1 with every one of the introduced simplices crushed to points is isomorphic to $GH(E)$. The original square 2-cells of $GH(E)$ have now become octagons in Q_1.

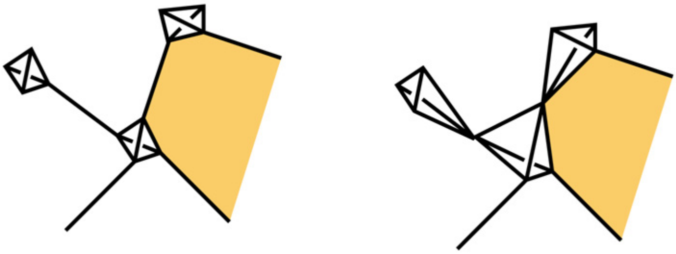

Fig. 2.

The complex Q_1 has a pair of vertices for each original edge of $GH(E)$, that is, for each element $e \in E$. One of the vertices lies in the 2-skeleton of the n-simplex corresponding to the $\mathcal{L}$-class of e, and the other in the corresponding $\mathcal{R}$-class. Our second step is to crush each of these original edges from $GH(E)$ to points, resulting in a complex which we will call Q_2; see Fig. 2. Each such edge forms a contractible subcomplex of Q_1, since its vertices are distinct—the 1-skeleton of $GH(E)$ is a bipartite graph, so the vertices of each edge lie on distinct n-simplices—so quotienting out by each edge is again a homotopy equivalence. Q_2 is therefore homotopy equivalent to Q_1. The vertices of Q_2 are now in one-to-one correspondence with E, since there is one vertex for each edge in $GH(E)$. The edges of Q_2 are precisely the edges in the n-simplices, so there is an edge from e to f precisely when e and f lie in the same $\mathcal{L}$- or $\mathcal{R}$-class, which are precisely the edges of the Nambooripad complex. Under the quotient map the octagonal 2-cells of Q_1 have become square 2-cells, whose boundaries are edge paths through the vertices e, f, g, h given by the edges in the boundaries of the square 2-cells of $GH(E)$. That is, they are precisely the singular E-squares of the Nambooripad complex.

Finally, the Nambooripad complex $K(E)$ is isomorphic to the 2-skeleton $Q_2^{(2)} \subseteq Q_2$ of Q_2. That is, $Q_2^{(2)}$ consists of the 1-skeleton, which is the 1-skeleton of $K(E)$, together with the singular squares and all of the 2-faces of the n-simplices, which are precisely the triangular 2-cells of $K(E)$ for e, f, g three distinct elements in the same $\mathcal{L}$- or $\mathcal{R}$-class. Having the same vertices, edges, and 2-cells, the two 2-complexes are therefore isomorphic.

Since the fundamental groupoid of the 2-skeleton of a complex is isomorphic to the fundamental groupoid of the complex, we have

$$\pi_1\big(K(E)\big) \cong \pi_1\big(Q_2^{(2)}\big) \cong \pi_1(Q_2) \cong \pi_1(Q_1) \cong \pi_1\big(GH(E)\big),$$

as desired. $\square$

5. An example of a free idempotent generated semigroup with non-free subgroups

In this section we present an example of a finite regular biordered set E such that $Z \times Z$, the free abelian group of rank 2, is isomorphic to a maximal subgroup of $RIG(E)$. This is the first example of a subgroup of a free idempotent generated semigroup that is not a free group.

Before presenting the example, we give more details on the connection between bipartite graphs and completely 0-simple semigroups. This will help us explain how we present our example.

Let $S = M^0(A, 1, B, C)$ be a combinatorial completely 0-simple semigroup. That is, the maximal subgroup is the trivial group 1. Thus we can represent elements as pairs $(a, b) \in A \times B$ with product $(a, b)(a', b') = (a, b')$ if $C(b, a') \neq 0$ and 0 otherwise. As in the general case of the Graham–Houghton graph that we described in the previous section, we associate a bipartite graph $\Gamma(S)$ to S. The vertices of $\Gamma(S)$ are $A \cup B$ (where as usual, we assume $A \cap B$ is empty). There is an edge between $b \in B$ and $a \in A$ if and only if $C(b, a) = 1$. Clearly $\Gamma(S)$ is a bipartite graph with no isolated vertices.

Conversely, let Γ be a bipartite graph with vertices the disjoint union of two sets A and B and no isolated vertices. We then have the incidence matrix $C = C(\Gamma): B \times A \to \{0, 1\}$ with $C(b, a) = 1$ if and only if $\{b, a\}$ is an edge of Γ. As usual we write C as a $\{0, 1\}$ matrix with rows labeled by

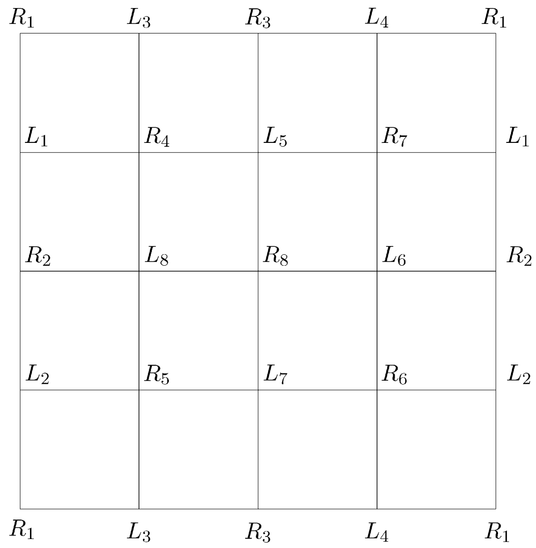

Fig. 3. The graph Γ.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Fig. 4. The transpose of the incidence matrix of the graph Γ.

elements of B and columns labeled by elements of A. Define $S(\Gamma)$ to be the Rees matrix semigroup $S(\Gamma) = M^0(A, 1, B, C(\Gamma))$. Then it follows from the fact that Γ has no isolated vertices that $S(\Gamma)$ is a combinatorial 0-simple semigroup. Clearly, these assignments give a one-to-one correspondence between combinatorial 0-simple semigroups and directed bipartite graphs with no isolated vertices. Isomorphisms of graphs are easily seen to correspond to isomorphisms of the corresponding semigroup and vice versa.

We now explain the idea of our example. We will define a bipartite graph Γ that embeds on the surface of a torus. The graph will represent the one skeleton of a square complex. We will then define a finite regular semigroup S that has Γ as the bipartite graph corresponding to a completely 0-simple semigroup that is an ideal of S and such that if we add the singular squares of the biordered set $E(S)$ as 2-cells to Γ (in the language of the previous section, we build the Graham–Houghton complex), we obtain a complex that has the fundamental group of the torus, that is, $Z \times Z$ as maximal subgroup.

We begin by drawing the graph Γ in Fig. 3.

We call the colors of the bipartition R and L to remind the reader of the Green relations $\mathcal{R}$ and $\mathcal{L}$ (but if the reader insists, she/he can think of them as Red and bLue). Thus there are 16 vertices in the graph and 32 edges. Fig. 3 is drawn in a way that the graph is really drawn on the torus obtained by identifying the top of the graph with the bottom and the left side with the right side.

Before continuing we define the incidence matrix of Γ. For our purposes, it is more convenient to write the transpose of the incidence matrix. Thus the matrix in Fig. 4 has rows labeled by $R_1, \ldots, R_8$ and columns labeled by $L_1, \ldots, L_8$. In particular, the matrix written this way defines the biordered set of the 0-simple semigroup $S(\Gamma)$ corresponding to Γ. That is, idempotents correspond to the $\mathcal{H}$-classes with entries 1, the $\mathcal{R}$ relation corresponds to being idempotents in the same row and the $\mathcal{L}$ relation corresponds to being idempotents in the same column.

Now consider the 2-complex one obtains by sewing on 2-cells corresponding to the 16 visual one-by-one squares that we see in the diagram of Γ. Notice that after identifying the graph on the surface of a torus, there are 24 4-cycles in the graph. There are the 16 4-cycles bounding 2-cells in our complex (such as R_1, L_3, R_4, L_1) that we see in Fig. 3: there are also the 8 4-cycles (such as R_1, L_3, R_3, L_4) that are obtained when we fold Γ into a torus, but these 4-cells do not bound cells in our complex. Clearly the fundamental group of this complex is $Z \times Z$. We have simply drawn subsquares on the usual representation of the torus as a square with opposite sides identified. By killing off these corresponding 16 4-cycles we have a space homeomorphic to the torus and thus its fundamental group is $Z \times Z$.

Furthermore, each of the 16 visual one-by-one squares in the diagram of the graph Γ corresponds to an E-square in the biordered set of the 0-simple semigroup $S(\Gamma)$ corresponding to Γ. Thus if we can find a regular semigroup S that has the biordered set corresponding to $S(\Gamma)$ as a connected component and also has exactly the 16 visible squares as the singular squares in this component, it follows from the results of the previous section that the maximal subgroup of the connected component corresponding to Γ in $RIG(E(S))$ is $Z \times Z$. We proceed to construct such a regular semigroup.

Let $X = \{L_1, \ldots, L_8\}$. The semigroup S will be defined as a subsemigroup of the monoid of partial functions acting on the right of X. Let C be the transpose of the matrix in Fig. 4. Thus C is the structure matrix of the 0-simple semigroup $S(\Gamma)$. To each element $s = (R_i, L_j) \in S(\Gamma)$ we associate the partial constant function $f_s : X \to X$ defined by $L_x f_s = L_j$ if $C(L_x, R_i) = 1$ and undefined otherwise. In the language of semigroup theory, f_s is the image of s under the right Schutzenberger representation of $S(\Gamma)$ [1,25].

The semigroup generated by $\{f_s \mid s \in S(\Gamma)\}$ is isomorphic to $S(\Gamma)$. This can be verified by direct computation by showing that for all $s, t \in S(\Gamma)$, $f_s f_t = f_{st}$, (where st is the product of s and t in $S(\Gamma)$) and that the assignment $s \mapsto f_s$ is one-to-one. This follows directly from the definition of f_s above. Alternatively, one can verify this by noting as we did above that the assignment of s to f_s is the right Schutzenberger representation. The structure matrix of $S(\Gamma)$, that is, the transpose of the matrix in Fig. 4, has no repeated rows and columns and this implies that both the right and left Schutzenberger representations are faithful [1,25].

Now we define two more functions e, k by the following tables:

$$e = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 \\ L_1 & L_6 & L_3 & L_7 & L_3 & L_6 & L_7 & L_1 \end{bmatrix},$$

$$k = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 \\ L_4 & L_2 & L_2 & L_4 & L_5 & L_5 & L_8 & L_8 \end{bmatrix}.$$

Let S be the semigroup generated by $\{e, k, f_s \mid s \in S(\Gamma)\}$. We claim that S is the semigroup that has the properties we desire. Notice that e and k are idempotents and that $S(\Gamma)$ is generated by its idempotents (this is known to be equivalent to the graph Γ being connected [10,18]), so in fact, S is an idempotent generated semigroup.

The subsemigroup T generated by $\{e, k\}$ has by direct computation 8 elements $\{e, k, (ek), (ke), (eke), (kek), h = (ek)^2, f = (ke)^2\}$. This semigroup consists of functions all of rank 4 and is a completely simple semigroup whose idempotents are e, f, k, h. We claim that $TS(\Gamma) \cup S(\Gamma)T \subseteq S(\Gamma)$. To see this we first note that for $(R_i, L_j) \in S(\Gamma)$, we have $(R_i, L_j)t = (R_i, L_j t)$ for $t \in \{e, k\}$. Therefore $S(\Gamma)T \subseteq S(\Gamma)$ follows by induction on the length of a product of elements in $\{e, k\}$.

We now list how e and k act on the left of $S(\Gamma)$. In the charts below, we note, for $t \in \{e, k\}$ and $(R_i, L_j) \in S(\Gamma)$, that $t(R_i, L_j) = (tR_i, L_j)$ for the left action $R_i \mapsto tR_i$ listed here. Again, all of this can be verified by direct computation:

$$e\colon \quad \begin{bmatrix} R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 \\ R_4 & R_2 & R_3 & R_4 & R_3 & R_6 & R_2 & R_6 \end{bmatrix},$$

$$k\colon \quad \begin{bmatrix} R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 \\ R_1 & R_5 & R_7 & R_8 & R_5 & R_1 & R_7 & R_8 \end{bmatrix}.$$

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
R_1	▼	▲	▼	▲				
R_2	●	■				●		■
R_3			●	■	■		●	
R_4	●		●		■			■
R_5		▲	▼				▼	▲
R_6		■		■		●	●	
R_7	▼			▲	▲	▼		
R_8					▲	▼	▼	▲

Fig. 5. The idempotent order on $E(S)$.

Idempotent	Symbol
h	□
e	•
k	△
f	▽

Fig. 6.

For readers who know the terminology, we have listed the images of T in the left Schutzenberger representation on $S(\Gamma)$ [1,25]. Our claim that $TS(\Gamma) \cup S(\Gamma)T \subseteq S(\Gamma)$ follows from these charts by induction on the length of a product from T. It follows that S is the disjoint union of T and $S(\Gamma)$. Thus S is a regular semigroup with 3 $\mathcal{J}$-classes one of them being T and the other 2 coming from $S(\Gamma)$ (its unique non-zero $\mathcal{J}$-class and 0). $S(\Gamma)$ is the unique 0-minimal ideal of S. The order of S is 73 and the order of $E(S)$ is 37.

We now look at the biorder structure on $E(S)$. We summarize the usual idempotent order relation in Fig. 5.

We explain the symbols in this diagram. Each symbol represents an idempotent in T according to Fig. 6.

An entry of a symbol in a box in Fig. 5 denotes a relation in the usual idempotent order. For example, the idempotent (R_1, L_1) of $S(\Gamma)$ is below f in the idempotent order. For example it follows from the diagram that $(R_2, L_1) <_{\mathcal{L}} f$ but that (R_2, L_1) is not below f in the idempotent order. The other relations in the regular biordered set $E(S)$ can be computed directly in S. For example, $f(R_2, L_1) = (R_7, L_1), k(R_2, L_2) = (R_5, L_2)$, etc.

The partial order on $E(S)$ has many pleasant properties. For example, each of the idempotents in T is above exactly 8 idempotents in $S(\Gamma)$ and every idempotent in $S(\Gamma)$ is below exactly one idempotent in T. The 8 idempotents in $S(\Gamma)$ below a given idempotent in T form an E-cycle. Thus the idempotents in $S(\Gamma)$ decompose into the disjoint union of 4 E-cycles of length 8. Below we give a more geometric definition of the semigroup S which will help explain some of these properties.

Finally, in Fig. 7, we give the precise information on which idempotents in T singularize squares in $E(S(\Gamma))$. Again, all of this can be verified by direct computation.

The explanation of Fig. 7 is as follows. An entry in a square of the symbol of an idempotent from T indicates that idempotent singularizes the corresponding 2×2 rectangular set in $E(S)$. For

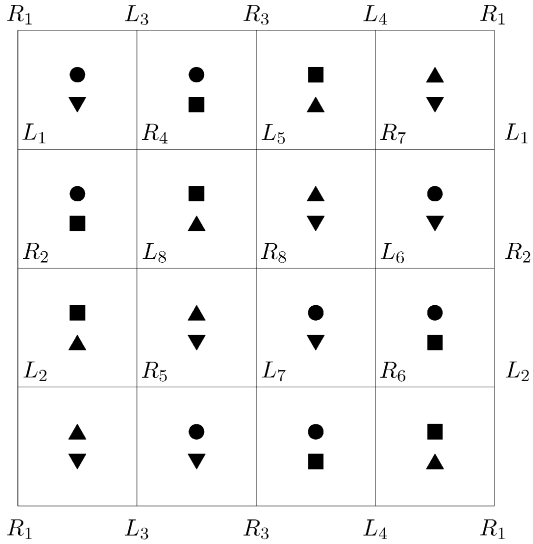

Fig. 7. Singularization of E-squares.

example, the square, $\left[\begin{smallmatrix}(R_1,L_1) & (R_1,L_3)\\ (R_4,L_1) & (R_4,L_3)\end{smallmatrix}\right]$, which is the square represented in the top left portion of Fig. 7 is singularized (bottom to top) by f and (top to bottom) by e. The diligent reader can verify all that we claim by direct computation in $E(S)$. In particular, exactly the 16 squares that we desire to be singularized in $S(\Gamma)$ are the ones singularized in S and therefore the free (regular) idempotent semigroup on the biordered set $E(S)$ has $Z \times Z$ as a maximal subgroup for the connected component corresponding to Γ as explained at the beginning of this section. This completes our first description of S. We now give a more geometric description of the semigroup S.

5.1. *Incidence structures and affine geometry over the field of order 2*

In this subsection we show that the semigroup S discussed above arises from a combinatorial structure related to affine 3-space over F_2, the field of order 2. We first recall some connections between incidence structures in the sense of combinatorics and finite 0-simple semigroups.

Up to now, we have used the tight connection between bipartite graphs and 0-simple semigroups over the trivial group to build our example. As is well known, $\{0,1\}$-matrices arise naturally to code information about other combinatorial structures besides bipartite graphs.

An *incidence system* is a pair $D = (V, \mathcal{B})$ where V is a (usually finite) set of points and $\mathcal{B}$ is a list of subsets of V called *blocks*. We allow for the possibility that a block, that is a certain subset of V, can appear more than once in the list $\mathcal{B}$. The *incidence matrix* of D is the $|\mathcal{B}| \times |V|$ matrix I_D (we will use the elements of $\mathcal{B}$ and V to name rows and columns) such that $I_D(b, v) = 1$ if $v \in b$ and 0 otherwise, where $b \in \mathcal{B}$ and $v \in V$. Sometimes, the transpose of this matrix is called the incidence matrix, but it is more convenient for our purposes to define things this way.

The semigroup $S(D)$ associated with D is the Rees matrix semigroup $M^0(\mathcal{B}, 1, V, C)$ where C is the transpose of I_D. It is straightforward to see that $S(D)$ is 0-simple if and only if the empty set is not a block and every point belongs to some block. We make these assumptions throughout. Conversely, it is easy to see that the transpose of the structure matrix of a combinatorial completely 0-simple semigroup is an incidence system with these two properties.

For example, if we consider the matrix in Fig. 4 as an incidence system, the points are $\{L_1, \dots, L_8\}$. The blocks are $R_1 = \{L_1, L_2, L_3, L_4\}$, $R_2 = \{L_1, L_2, L_6, L_8\}$, etc.

Now we show that this incidence system can be coordinatized as a certain affine configuration over the field of order 2 and that the semigroup S can be faithfully represented by affine partial functions that are "continuous" with respect to this structure in the sense of [3–5].

L_1	(0,0,0)
L_2	(1,0,0)
L_3	(0,1,0)
L_4	(1,1,0)
L_5	(0,1,1)
L_6	(1,0,1)
L_7	(1,1,1)
L_8	(0,0,1)

Fig. 8. The points of the structure.

Row	Block	Subset of V
R_1	$\{1,2,3,4\}$	$\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}$
R_2	$\{1,2,6,8\}$	$\{(0,0,0),(1,0,0),(1,0,1),(0,0,1)\}$
R_3	$\{3,4,5,7\}$	$\{(0,1,0),(1,1,0),(0,1,1),(1,1,1)\}$
R_4	$\{1,3,5,8\}$	$\{(0,0,0),(0,1,0),(0,1,1),(0,0,1)\}$
R_5	$\{2,3,7,8\}$	$\{(1,0,0),(0,1,0),(1,1,1),(0,0,1)\}$
R_6	$\{2,4,6,7\}$	$\{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\}$
R_7	$\{1,4,5,6\}$	$\{(0,0,0),(1,1,0),(0,1,1),(1,0,1)\}$
R_8	$\{5,6,7,8\}$	$\{(0,1,1),(1,0,1),(1,1,1),(0,0,1)\}$

Fig. 9. The blocks of the structure.

Let F_2 be the field of order 2 and let $V = F_2^3$ be 3-space over F_2. Consider the set of planes through the origin (i.e. 2 dimensional subspaces of V) that do not contain the vector $(1,1,1)$. An elementary counting argument shows that there are 4 such planes. We let $\mathcal{B}$ be the set of these 4 planes plus their 4 translates by the vector $(1,1,1)$. Therefore, $\mathcal{B}$ has 8 elements. We claim that by suitably ordering the points in V and the planes in $\mathcal{B}$, the incidence matrix of $(V,\mathcal{B})$ is the matrix in Fig. 4. We do this by making the assignment of vectors to the points $L_1,\dots,L_8$ according to Fig. 8.

With this identification of the L_i as vectors in V, we have the following way to identify the blocks of our structure. For simplicity of presentation, we write i in place of L_i in Fig. 9.

We can see from the preceding table that R_1, R_2, R_4, R_7 are precisely the 4 planes through the origin in V that do not contain the vector $(1,1,1)$ and that $R_3 = R_2 + (1,1,1)$, $R_5 = R_7 + (1,1,1)$, $R_6 = R_4 + (1,1,1)$, $R_8 = R_1 + (1,1,1)$ are their translates.

Now we show that the semigroup S defined in the previous subsection also has a natural interpretation with respect to this geometric structure. Let V be a vector space over an arbitrary field. An affine partial function on V is a partial function $f_{A,w}: V \to V$ of the form $vf = vA + w$, where $A: V \to V$ is a partial linear transformation, that is a linear transformation whose domain is an affine subspace of V and range an affine subspace of V and $w \in V$. The collection of all affine partial functions is a monoid $\mathit{Aff}(V)$. If we identify $f_{A,w}$ with the pair (A,w), then multiplication in $\mathit{Aff}(V)$ takes the form $(A,w)(A',w') = (AA', wA' + w')$ so that $\mathit{Aff}(V)$ is a semidirect product of the monoid of partial linear transformations on V with the additive group on V.

We claim that the idempotents e and k defined in the previous section in defining our semigroup S act as affine functions on F_2^3 using our translation of our structure in this section. Indeed, let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

considered as a matrix over F_2. Then it is easily checked that for $1 \leqslant i \leqslant 8$, $ie = j$ if and only if $v_i A = v_j$ where v_i is the vector corresponding to L_i in the table above and that if

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and $w = (1,1,0)$, then for $1 \leqslant i \leqslant 8$, $ik = j$ if and only if $v_iB + w = v_j$. Thus the completely simple subsemigroup T of our semigroup S is faithfully represented by affine functions over our geometric structure.

Furthermore, each element of T has the following property with respect to this structure: the inverse image of each plane in the structure is also in the structure. For example, $R_1e^{-1} = R_4$, $R_2e^{-1} = R_2$, $R_3e^{-1} = R_3$, $R_4e^{-1} = R_4$, $R_5e^{-1} = R_3$, $R_6e^{-1} = R_6$, $R_7e^{-1} = R_2$, $R_8e^{-1} = R_6$.

Each element (R_i, L_j) is also represented as an affine partial function, namely the partial function whose domain is R_i and sends all points in its domain to L_j. We can represent this as an affine partial function by taking A to be the 0 linear transformation restricted to R_i and w to be L_j. Clearly, the inverse image of a block R under this function is either R_i if $L_j \in R$ and the empty set otherwise.

Notice also, that for every element of S the closure of blocks under inverse image encodes left multiplication of e in the biordered set $E(S)$. For example, $e(R_1, L_1) = (R_1e^{-1}, L_1) = (R_4, L_1)$, $(R_1, L_1)(R_3, L_1) = 0$, etc.

Thus, there is an analogue of the action of the partial functions on our structure to continuous functions on a topological space. If we consider the blocks of our structure to be "open," then our functions preserve open sets under inverse image. The notion of continuous partial functions on combinatorial structures and its relationship to the semigroup theoretic notion of translational hull [2] has been explored in [3–5]. We see here that there is a close connection between building biordered sets with a specific connected component and the continuous partial functions on the corresponding 0-simple semigroup. We will explore this connection in future work.

6. Summary and future directions

We have shown how to represent the maximal subgroups of the free (regular) idempotent generated semigroup on a regular biordered set by a 2-complex derived from Nambooripad's [18] work. By applying the Bass–Serre techniques of [11], we are directly lead to the graph defined by Graham and Houghton for finite 0-simple semigroups [10,14]. We put the structure of a 2-complex on this graph and use that to construct an example of a finite regular biordered set that has a maximal subgroup that is isomorphic to the free abelian group of rank 2. This is the first example of a non-free group that appears in a free idempotent generated semigroup.

The biordered set arises from a certain combinatorial structure defined on a 3 dimensional vector space over the field of order 2. This suggests looking for further examples by either varying the field and looking at analogous structures over 3 dimensional spaces or by looking at higher dimensional analogues of the structure we have defined.

In related work we have proved, using completely different techniques, that if F is any field, and $E_3(F)$ is the biordered set of the monoid of 3×3 matrices over F, then the free idempotent generated semigroup over $E_3(F)$ has a maximal subgroup isomorphic to the multiplicative subgroup of F. In particular, finite cyclic groups of order $p^n - 1$, p a prime number appear as maximal subgroups of free idempotent generated semigroups [7].

This last example motivates an intended application of this work. We would like to apply Nambooripad's powerful theory of inductive groupoids [18] to study reductive linear algebraic monoids [23]. This very important class of regular monoids and their finite analogues have been intensively studied over the last 25 years. A basic example is the monoid of all matrices over a field.

The above discussion begs the question of describing the class of groups that are maximal subgroup of $IG(E)$ or $RIG(E)$ for a biordered set E. This seems to be a very difficult question at this time.

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References

[1] M. Arbib, K. Krohn, J. Rhodes, Algebraic Theory of Machines, Languages and Semigroups, Academic Press, New York, 1968.

[2] A.H. Clifford, G.P. Preston, The Algebraic Theory of Semigroups, Math. Surveys Monogr., vol. 7, Amer. Math. Soc., Providence, 1961, vol. I, Amer. Math. Soc., Providence, 1967, vol. II.
[3] J. Dinitz, S.W. Margolis, Continuous maps on block designs, Ars Combin. 14 (1982) 21–47.
[4] J. Dinitz, S.W. Margolis, Continuous maps in finite projective spaces, Congr. Numer. 35 (1982) 239–247.
[5] J. Dinitz, S.W. Margolis, Translational hulls and block designs, Semigroup Forum 27 (1983) 243–261.
[6] D. Easdown, Biordered sets come from semigroups, J. Algebra 96 (1985) 581–591.
[7] D. Easdown, M. Sapir, M. Volkov, Periodic elements of the free idempotent generated semigroup on a biordered set, Internat. J. Algebra Comput., in press.
[8] J.A. Erdös, On products of idempotent generated matrices, Glasg. Math. J. 8 (1967) 118–122.
[9] D.G. Fitz-Gerald, On inverses of products of idempotents in regular semigroups, J. Aust. Math. Soc. 15 (1) (1972) 335–337.
[10] R.L. Graham, On finite 0-simple semigroups and graph theory, Math. Syst. Theory 2 (4) (1968).
[11] S. Haataja, S.W. Margolis, J. Meakin, Bass–Serre theory for groupoids and the structure of full regular semigroup amalgams, J. Algebra 183 (1996) 38–54.
[12] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, England, 2002.
[13] P.J. Higgins, Categories and Groupoids, Van Nostrand Reinhold, New York, 1971.
[14] C.H. Houghton, Completely 0-simple semigroups and their associated graphs and groups, Semigroup Forum 14 (1977) 41–67.
[15] J. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, J. London Math. Soc. 41 (1966) 707–716.
[16] G. Lallement, Semigroups and Combinatorial Applications, Wiley, New York, 1979.
[17] B. McElwee, Subgroups of the free semigroup on a biordered set in which principal ideals are singletons, Comm. Algebra 30 (11) (2002) 5513–5519.
[18] K.S.S. Nambooripad, Structure of regular semigroups I, Mem. Amer. Math. Soc. 224 (1979).
[19] K.S.S. Nambooripad, F. Pastijn, Subgroups of free idempotent generated regular semigroups, Semigroup Forum 21 (1980) 1–7.
[20] F. Pastijn, Idempotent generated completely 0-simple semigroups, Semigroup Forum 15 (1977) 41–50.
[21] F. Pastijn, The biorder on the partial groupoid of idempotents of a semigroup, J. Algebra 65 (1980) 147–187.
[22] A. Pollatchek, Relationships between combinatorics and O-simple semigroup, Semigroup Forum 12 (1) (1978) 78–82.
[23] M. Putcha, Linear Algebraic Monoids, London Math. Soc., 1988.
[24] M. Putcha, Products of idempotents in algebraic monoids, J. Aust. Math. Soc. 80 (2006) 193–203.
[25] J. Rhodes, B. Steinberg, The Q-Theory of Finite Semigroups, Springer-Verlag, New York, 2009.
[26] J.J. Rotman, An Introduction to the Theory of Groups, fourth ed., Springer-Verlag, New York, 1994.
[27] J.P. Serre, Trees, Springer-Verlag, New York, 1980.
[28] E.H. Spanier, Algebraic Topology, Springer-Verlag, New York, 1981.
[29] B. Steinberg, A topological approach to inverse and regular semigroups, Pacific J. Math. 208 (2003) 367–396.