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On generators of bounded ratios of minors for totally positive matrices

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Abstract

We provide a method for factoring all bounded ratios of the form

$$\det A(I_1|I_1')\det A(I_2|I_2')/\det A(J_1|J_1')\det A(J_2|J_2'),$$

where A is a totally positive matrix, into a product of more elementary ratios each of which is bounded by 1, thus giving a new proof of Skandera's result. The approach we use generalizes the one employed by Fallat et al. in their work on principal minors. We also obtain a new necessary condition for a ratio to be bounded for the case of non-principal minors.

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1. Introduction

An $n \times n$ matrix A is called totally positive if every minor of A is positive. If $I, I' \subseteq \{1, 2, \ldots, n\}$ with $|I| = |I'|$, we denote the minor of A with row set I and column set I' as $(I|I')(A) := \det A(I|I')$. If $S = ((I_1|I_1'), \ldots, (I_p|I_p'))$ is a sequence of p row and column sets, we define a function $S(A) = \det A(I_1|I_1') \cdot \det A(I_2|I_2') \cdots \det A(I_p|I_p')$. Please note that $S(A) > 0$ for any choice of S and for all totally positive matrices A. Similarly, if $T = ((J_1|J_1'), \ldots, (J_q|J_q'))$ is

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another sequence of q row and column sets, we say that $S \leqslant T$ (with respect to the class of totally positive matrices) if $S(A) \leqslant T(A)$ for all totally positive matrices A. Note that if we take the convention that $(\emptyset|\emptyset)(A) = 1$, we are free to assume that S and T are both sequences of the same size (i.e., $p = q$) by appending an appropriate number of $(\emptyset|\emptyset)$ to the shorter sequence.

It is also reasonable to ask when the ratio $S(A)/T(A)$ is bounded by some $k > 0$ for all totally positive matrices A. If this is true, we say that the ratio S/T is *bounded by* k. It is clear that $S \leqslant T$ if and only if S/T is bounded by 1. It has been conjectured that if S/T is bounded (by any number), then it is necessarily bounded by 1 (e.g., see [1]).

Recently, the problem of classifying all such ratios and inequalities has been a subject of much interest. Fallat et al. [2] were able to classify a large class of ratios of products of principal minors. In particular, they gave necessary and sufficient conditions for a ratio of products of two minors to be bounded over totally positive matrices. This result was later generalized to the case of non-principal minors by Skandera [3]. In this paper we generalize a necessary condition in [2] to the case of non-principal minors, and our main result is an explicit factorization of ratios of the form

$$\frac{\det A(I_1|I_1')\det A(I_2|I_2')}{\det A(J_1|J_1')\det A(J_2|J_2')}$$

into products of elementary ratios. This in particular implies the result of Skandera describing bounded ratios of this form. It has been conjectured by Gekhtman that all bounded ratios are products of these elementary ratios [2].

1.1. Planar networks and totally positive matrices

The relationship between totally positive matrices and directed acyclic weighted planar networks is well studied. It was first discussed by Karlin and McGregor in 1959 [4]. For a more modern presentation, refer to the paper by Fomin and Zelevinsky [5]. In an attempt to keep the manuscript mostly self-contained, we will present some relevant results from these papers.

A typical directed acyclic weighted planar network is shown in Fig. 1. Note that because the graph is acyclic, we can stretch the network in an appropriate fashion so that the direction of each edge is oriented from left-to-right. Furthermore, the network is assumed to have n labeled sources (on the left) and n labeled sinks (on the right). Both sources and sinks are labeled bottom to top. Additionally, to each edge of the network we associate a positive weight. In Fig. 1, these weights are shown as l_i, d_j, or u_k. Unmarked weights are assumed to be 1.

Let π be any path running left-to-right from source i to sink j. We define the weight of this path to be the product of the weights along each edge of the path and denote this as $w(\pi)$.

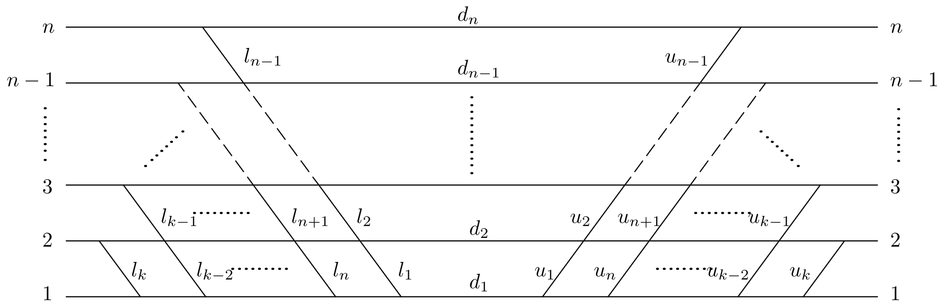

Fig. 1. General planar network.

To each such diagram, we can associate a totally positive matrix A with entries a_{ij} given by

$$a_{ij} = \sum_{\pi: i \to j} w(\pi), \tag{1}$$

where the summation is over all paths π that begin at source i and end at sink j. Formula (1) establishes a bijection between totally positive matrices and planar networks of the kind depicted in Fig. 1. This fact is equivalent to Anne Whitney's Reduction Theorem [6].

Let us define a path family π as a set of non-intersecting paths running from left-to-right starting at the sources in I and terminating at the sinks in I'. The weight of such a path family $w(\pi)$ is defined to be the product of the weights of each path in the path family. As shown in [7] the minor with row set I and column set I' is

$$\det A(I|I') = \sum_{\pi: I \to I'} w(\pi),$$

where the summation is over all such possible path families from I to I'.

Given the row set I and column set I', we have found it helpful to follow Skandera [3] in defining the set I'' which encapsulates both I and I'

$$I'' = I \cup \{2n + 1 - i | i \in I'^c\} \tag{2}$$

where $I'^c = \{1, 2, \ldots, n\} \setminus I'$. While this I'' may seem cryptic, it has a natural interpretation if one considers an embedding of totally positive matrices into the totally positive part of the Grassmannian $Gr(n, 2n)$.

1.2. Grassmannians

In this section we will discuss the real Grassmannian and refer the reader to Section 5.4 of [8] for more information. Recall that the real Grassmannian $Gr(n, 2n)$ is the set of n-dimensional subspaces of $\mathbb{R}^{2n}$, i.e.

$$Gr(n, 2n) = \{\text{Real } 2n \times n \text{ matrices of rank } n\}/GL(n, \mathbb{R}),$$

where we have factored out the action of right multiplication by an invertible $n \times n$ matrix.

It is clear that an element $\Lambda \in Gr(n, 2n)$ does not have a unique matrix representation, but rather a collection of matrix representatives which are unique up to right multiplication by an invertible $n \times n$ matrix.

If A is a $2n \times n$ matrix representative of Λ, we can define the *Plücker coordinates of Λ with respect to A* (or more briefly the *Plücker coordinates of A*) to be the vector of all $n \times n$ minors of the matrix A, i.e. an element of real $\binom{2n}{n}$-space.

We say an element $\Lambda \in Gr(n, 2n)$ is totally positive if there exists a matrix representative A of Λ such that every Plücker coordinate of A is positive. The totally positive part of the Grassmannian $Gr(n, 2n)$ is then defined to be

$$TPGr(n, 2n) = \{\Lambda \in Gr(n, 2n) : \Lambda \text{ is totally positive}\}.$$

If $\Lambda \in TPGr(n, 2n)$ we say that the *standard matrix representative* of Λ is the $2n \times n$ matrix representative $\overline{A}$ with lower $n \times n$ submatrix equal to

$$\begin{bmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ \pm 1 & & & \end{bmatrix}.$$

Note that such a matrix can always be chosen because the lower $n \times n$ block of any matrix representative of Λ is always of full rank.

Proposition 1. *There is a natural bijection*:

$$\{\textit{Totally positive } n \times n \textit{ matrices}\} \leftrightarrow TPGr(n, 2n).$$

Proof. Let $\Lambda \in TPGr(n, 2n)$, and let $\overline{A}$ be its standard matrix representative. We shall denote the upper $n \times n$ submatrix of $\overline{A}$ as A. Then the relation

$$\det A(I|I') = \det \overline{A}(I''|(1, 2, \ldots, n)),$$

where I'' is defined as in Eq. (2), and the positivity of all Plücker coordinates of $\overline{A}$ imply that A is a totally positive matrix.

This same relation allows us to pass from an $n \times n$ totally positive matrix A to an element $\Lambda \in TPGr(n, 2n)$ by choosing Λ to be the unique element with standard matrix representative having A as the upper $n \times n$ submatrix. □

With this bijection clearly established we will maintain the convention of using A to represent a totally positive $n \times n$ matrix and $\overline{A}$ as its corresponding standard matrix representative in $TPGr(n, 2n)$.

For additional notational convenience, and to distinguish between minors and Plücker coordinates, we will designate index sets representing Plücker coordinates using Greek letters and drop the bar notation where its meaning is unambiguous. That is, for an index set $\alpha_j \subset \{1, 2, \ldots, 2n\}$ of size n, we define

$$[\alpha_j](A) := \det \overline{A}(\alpha_j|(1, 2, \ldots, n)). \tag{3}$$

Unless stated otherwise, all index sets α_j in the remainder of the paper will be assumed to be cardinality n subsets of $\{1, \ldots, 2n\}$.

If we have a sequence of index sets $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$, we can define the function $\alpha(A)$ as a product of Plücker coordinates

$$\alpha(A) = \prod_{i=1}^{p} [\alpha_i](A),$$

where A is an $n \times n$ totally positive matrix.

If we similarly let $\beta = (\beta_1, \ldots, \beta_q)$ be another sequence of index sets, we write $\alpha \leqslant \beta$ (with respect to totally positive matrices) if $\alpha(A) \leqslant \beta(A)$ for all $n \times n$ totally positive matrices A. We say that α/β is bounded by k (with respect to totally positive matrices) if $\alpha(A)/\beta(A) \leqslant k$ for all totally positive matrices A. Note that $\alpha \leqslant \beta$ is equivalent to saying α/β is bounded by 1.

By construction we have $[(n + 1, n + 2, \ldots, 2n)](A) = 1$ for all totally positive matrices A, and thus in general we will assume that α and β each contain the same number of index sets.

Lastly, when we say

$$\frac{[\alpha_1] \cdots [\alpha_p]}{[\beta_1] \cdots [\beta_p]}$$

is bounded (resp. bounded by k), we mean that the ratio α/β is bounded (resp. bounded by k) where $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_p)$.

2. Operations which preserve bounded ratios

Before proving the main theorem in the work of Fallat et al. (see [2]), they developed several operators that preserved bounded ratios of minors, namely what they called the Complement, Reversal, Shift, Insertion, and Deletion operators. Of these operators, we will provide generalizations of the shift and reversal operators. The insertion and deletion operators were not generalized because they have little applicability to our situation in which the cardinality of each index set must remain fixed. The complement operator was not studied.

Definition 2 (*Cyclic shift*). For an index set α_j, define a *cyclic shift* of the elements of α_j as

$$\sigma(\alpha_j) = \{i + 1 \bmod 2n | i \in \alpha_j\}$$

which maps $i \in \alpha_j$ to $i + 1$ and $2n$ back to 1.

For a sequence $\alpha = (\alpha_1, \ldots, \alpha_p)$ of index sets, define $\sigma(\alpha) = (\sigma(\alpha_1), \ldots, \sigma(\alpha_p))$.

Lemma 3. *Let A be a $n \times n$ totally positive matrix. Then there exists a totally positive $n \times n$ matrix B and a positive constant c_A such that*

$$[\sigma(\alpha_j)](B) = c_A[\alpha_j](A)$$

for all index sets α_j, where σ is the cyclic shift operator as defined in Definition 2.

In particular, if $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_p)$ then α/β is bounded if and only if $\sigma(\alpha)/\sigma(\beta)$ is bounded.

Proof. Let $\overline{A}$ be the standard matrix representation of the embedding of A into $TPGr(n, 2n)$ as discussed in Section 1.2. Enumerate the rows of $\overline{A}$ as $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_{2n}$.

Form the element $\Lambda \in TPGr(n, 2n)$ which is represented by the matrix C having rows

$$C = [(-1)^{n-1}\overline{A}_{2n}; \overline{A}_1; \overline{A}_2; \overline{A}_3; \ldots; \overline{A}_{2n-1}].$$

Let $\overline{B}$ be the standard matrix representation of Λ, i.e. $\overline{B} = CX$ for some $X \in GL(n, \mathbb{R})$ where $\det X > 0$.

Then for any index set α_j, we have

$$\begin{aligned}
[\alpha_j](A) &= \det \overline{A}(\alpha_j|(1, 2, \ldots, n)) \\
&= \det C(\sigma(\alpha_j)|(1, 2, \ldots, n)) \\
&= \det \overline{B}(\sigma(\alpha_j)|(1, 2, \ldots, n)) \cdot \det X^{-1} \\
&= [\sigma(\alpha_j)](B) \cdot \det X^{-1},
\end{aligned}$$

where B is the totally positive matrix corresponding to Λ. □

Analogous to the cyclic shift operator is the reversal operator which is described by Fallat et al. but which behaves differently in this situation of non-principal minors (see [2, §3]).

Definition 4 (*Reversal*). For an index set α_j, define the *reversal* of the elements of α_j as

$$\rho(\alpha_j) = \{(2n + 1) - i | i \in \alpha_j\}.$$

For a sequence $\alpha = (\alpha_1, \ldots, \alpha_p)$ of index sets, define $\rho(\alpha) = (\rho(\alpha_1), \ldots, \rho(\alpha_p))$.

Lemma 5. *Let A be a $n \times n$ totally positive matrix. Then there exists a totally positive $n \times n$ matrix B and a positive constant c_A such that*

$$[\rho(\alpha_j)](B) = c_A[\alpha_j](A)$$

for all index sets α_j where ρ is the reversal operator as defined in Definition 4.

In particular, if $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_p)$ then α/β is bounded if and only if $\rho(\alpha)/\rho(\beta)$ is bounded.

We leave the details of the proof to the reader.

3. Necessary conditions for bounded ratios

For $i \in \{1, 2, \ldots, 2n\}$, let $f_\alpha(i)$ be the number of index sets in α that contain i. We now give a generalization of a simple, necessary, but not sufficient condition for a ratio to be bounded originally described by Fallat et al. (see [2,3]).

Definition 6 (*ST0 condition*). Let α and β be two sequences of index sets. If $f_\alpha(i) = f_\beta(i)$ for all i, we say the ratio α/β satisfies the ST0 (set-theoretic) condition.

Proposition 7. *Let $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_p)$ be two sequences of index sets, with each set containing the same number of elements. If α/β is bounded for all totally positive matrices, then the ratio satisfies the* ST0 *condition.*

Proof. Suppose that α/β does not satisfy the ST0 condition. By Lemma 3, we may assume without loss of generality that $f_\alpha(1) \neq f_\beta(1)$.

Let C be any totally positive matrix, for example the matrix arising from Fig. 1 when all weights are chosen to be 1. Let $\overline{C}$ be the standard matrix representation of the embedding of C into $TPGr(n, 2n)$, and enumerate the rows of $\overline{C}$ as $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_{2n}$.

Construct a new element in $TPGr(n, 2n)$ which has matrix representative $\overline{A}$ whose rows are

$$\overline{A} = [t\overline{C}_1; \overline{C}_2; \overline{C}_3; \ldots; \overline{C}_{2n}],$$

where t is chosen to be a positive indeterminate. (We can think of $\overline{A}$ as the embedding of $A = \operatorname{diag}(t, 1, 1, \ldots, 1) \times C$ into $TPGr(n, 2n)$).

Let α_j be any index set. Then either:

- $1 \in \alpha_j$ and $[\alpha_j](A) = c_i t$ for some positive constant c_i; or
- $1 \notin \alpha_j$ and $[\alpha_j](A) = c_i$ for some positive constant c_i.

Thus $\alpha(A)$ is a monomial in t of degree $f_\alpha(1)$ and $\beta(A)$ is a monomial in t of degree $f_\beta(1)$. Because we have assumed that $f_\alpha(1) \neq f_\beta(1)$, $\alpha(A)/\beta(A)$ must increase without bound as either $t \to 0$ or $t \to \infty$ so the ratio is not bounded. $\square$

In order to present another necessary condition for a ratio α/β to be bounded, we first need to discuss the concept of majorization. The following two definitions and one proposition can be found in [9].

Definition 8 (*Majorization*). Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two non-increasing sequences of non-negative integers. Then x majorizes y (written $x \succeq y$) if for each $k = 1, 2, \ldots, n$

$$\sum_{i=1}^{k} x_i \geqslant \sum_{i=1}^{k} y_i$$

with equality if $k = n$.

Definition 9 (*Conjugate sequence*). The conjugate sequence to $x = (x_1, \ldots, x_n)$ is given by $x^* = (x_1^*, \ldots, x_{x_1}^*)$ where

$$x_j^* = |\{i : x_i \geqslant j\}|.$$

Proposition 10. *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two non-increasing sequences of non-negative integers. Then $x \succeq y$ if and only if $y^* \succeq x^*$.*

The following notion of an interval is relied upon in the work of Fallat, Skandera, and others (see [2,3]). Note that we define an interval slightly differently as contiguous points on a labeled $2n$-gon rather than contiguous points on a line segment with $2n$ vertices, but Lemma 3 shows us that such a distinction is irrelevant in most cases.

Definition 11 (*Interval*). A subset $L \subseteq \{1, 2, \ldots, 2n\}$ is called an interval if either L or $L^c = \{1, 2, \ldots, 2n\} \setminus L$ has the form $\{i, i + 1, i + 2, \ldots, i + m\}$.

Unless mentioned otherwise, all intervals L will be assumed to be subsets of $\{1, \ldots, 2n\}$ of the specified form.

The following definition comes directly from the work of Fallat et al. but is applied to the case of non-principal minors (see [2, §2]).

Definition 12 (*Condition* (*M*)). Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ be two sequences of index sets. For any subset L of $\{1, \ldots, 2n\}$, define $m(\alpha, L)$ to be the non-increasing rearrangement of the sequence $(|\alpha_1 \cap L|, \ldots, |\alpha_p \cap L|)$. We say that a ratio α/β satisfies condition (M) if

$$m(\alpha, L) \succeq m(\beta, L)$$

for every interval L.

Remark 13. Condition (M) implies the ST0 condition by choosing the interval $L = \{j\}$ for $j = 1, \ldots, 2n$.

Before we show that condition (M) is necessary for a ratio α/β to be bounded, we give some lemmas which will aid in the proof.

Lemma 14. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ be two sequences of index sets and L be any interval. If $m(\alpha, L) \succeq m(\beta, L)$ then $m(\alpha, L^c) \succeq m(\beta, L^c)$.*

Proof. Denote the components of $m(\alpha, L)$, etc. by

$$m(\alpha, L) = (m_1, \ldots, m_p),$$
$$m(\beta, L) = (m'_1, \ldots, m'_p),$$
$$m(\alpha, L^c) = (n_p, \ldots, n_1),$$

$$m(\beta, L^c) = (n'_p, \ldots, n'_1).$$

Since $|\alpha_j \cap L| + |\alpha_j \cap L^c| = n$, we have $m_i + n_i = m'_i + n'_i = n$ for all i. Let $M = \sum_{i=1}^{p} m_i = \sum_{i=1}^{p} m'_i$. Then for any index $k \leqslant p$ we have

$$\begin{aligned} n_p + n_{p-1} + \cdots + n_{p-(k-1)} &= (n - m_p) + \cdots + (n - m_{p-(k-1)}) \\ &= kn - M + m_1 + m_2 + \cdots + m_{p-k} \\ &\geqslant kn - M + m'_1 + m'_2 + \cdots + m'_{p-k} \\ &\geqslant n'_p + n'_{p-1} + \cdots + n'_{p-(k-1)} \end{aligned}$$

and hence $m(\alpha, L^c) \succeq m(\beta, L^c)$. □

Thus a ratio α/β satisfies condition (M) if and only if $m(\alpha, L) \succeq m(\beta, L)$ for all intervals L with $|L| \leqslant n$.

The following theorem is a direct analog of a theorem of Fallat et al. and is proved in a similar fashion (see [2, Theorem 2.4]).

Theorem 15. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ be two sequences of index sets. If the ratio α/β is bounded for all totally positive matrices, then it satisfies condition* (M).

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ be two sequences of index sets such that α/β is bounded. By Lemmas 3 and 14, it is sufficient to show that $m(\alpha, L) \succeq m(\beta, L)$ for all intervals $L = \{1, \ldots, s\}$ with $s \leqslant n$.

Fix $s \leqslant n$ and let $L = \{1, \ldots, s\}$. We then construct totally positive matrices $A_1, A_2, \ldots, A_s$ as follows:

Consider the planar network shown in Fig. 2. Define the matrix A_i to be the matrix corresponding to this planar network (see Section 1.1) with weights $a_1 = a_2 = \cdots = a_i = t$ where t is a positive indeterminate and remaining weights $a_{i+1} = \cdots = a_s = 1$.

If α_i is any index set, then $[\alpha_i](A_k)$ is a polynomial in t and has a well-defined degree. In fact, recalling that $[\alpha_i](A_k)$ is the sum of weights of path families with sources at $\alpha_i \cap \{1, \ldots, n\}$ and sinks at $\{n+1, \ldots, 2n\} \setminus \alpha_i$ (see Section 1.1), we have $\deg[\alpha_i](A_k) = \min(k, |\alpha_i \cap L|)$ and hence

$$\deg \alpha(A_k) = \sum_{i=1}^{p} \deg[\alpha_i](A_k) = \sum_{i=1}^{p} \min(k, |\alpha_i \cap L|).$$

For α/β to be bounded as $t \to \infty$, we must have that $\deg \alpha(A_k) \leqslant \deg \beta(A_k)$, i.e.

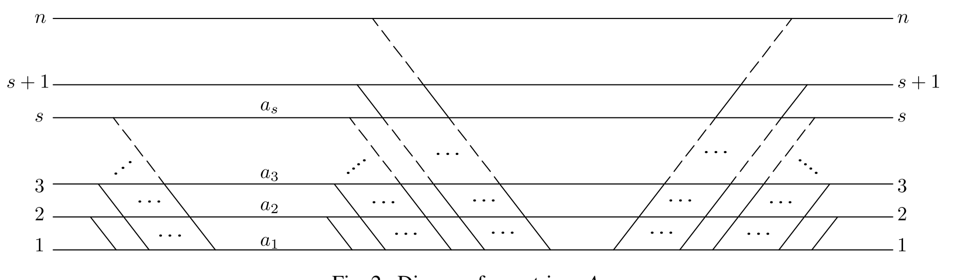

Fig. 2. Diagram for matrices A_i.

$$\sum_{i=1}^{p} \min(k, |\alpha_i \cap L)|) \leqslant \sum_{i=1}^{p} \min(k, |\beta_i \cap L|)$$

for each $1 \leqslant k \leqslant s$.

Note that if $m^*(\alpha, L) = (m_1^*(\alpha, L), m_2^*(\alpha, L), \ldots)$ is the conjugate sequence to $m(\alpha, L)$ we recognize the left side of the inequality as

$$\sum_{i=1}^{p} \min(k, |\alpha_i \cap L)|) = \sum_{j=1}^{k} |\{i : |\alpha_i \cap L| \geqslant j\}| = \sum_{j=1}^{k} m_j^*(\alpha, L).$$

and similarly for the summation with β.

Since this inequality holds for all $1 \leqslant k \leqslant s$ with equality when $k = s$ by the ST0 condition (see Proposition 7), we have that $m^*(\alpha, L) \preceq m^*(\beta, L)$ and hence $m(\alpha, L) \succeq m(\beta, L)$ as desired. $\square$

4. Basic and elementary bounded ratios

In this section we define two special classes of ratios of the form

$$\frac{[\alpha_1][\alpha_2]}{[\beta_1][\beta_2]},$$

where the $\alpha_1, \alpha_2, \beta_1$, and β_2 are index sets. In particular, ratios belonging to either of these classes will be bounded by 1.

Let α_i be an index set. If $\alpha_i = \gamma_1 \cup \cdots \cup \gamma_n$ then the notation $[\gamma_1, \ldots, \gamma_n]$ should be interpreted as $[\alpha_i]$. Furthermore, if γ_i consists of only a single element j, then we may simply write j instead of γ_i.

Definition 16. An elementary ratio is a ratio of the form

$$\frac{[i, j', \Delta][i', j, \Delta]}{[i, j, \Delta][i', j', \Delta]}$$

satisfying

(1) $|\Delta| = n - 2$;
(2) $i < i' < j < j'$ when considering each element as the mod $2n$ representative in $\{i, i + 1, \ldots, i + 2n - 1\}$; and
(3) i, i', j, and j' are not elements of Δ.

Proposition 17. *A ratio R of the form*

$$\frac{[i, j', \Delta][i', j, \Delta]}{[i, j, \Delta][i', j', \Delta]}$$

with $|\Delta| = n - 2$, $\Delta \cap \{i, i', j, j'\} = \emptyset$ and i, i', j, j' pairwise distinct is elementary if and only if it satisfies condition (M).

Proof. Suppose the ratio R is elementary, and let L be any interval. Set $\alpha_1 = \{i, j'\} \cup \Delta$, $\alpha_2 = \{i', j\} \cup \Delta$, $\beta_1 = \{i, j\} \cup \Delta$, and $\beta_2 = \{i', j'\} \cup \Delta$, so that $R = [\alpha_1][\alpha_2]/[\beta_1][\beta_2]$.

Because R satisfies the ST0 condition by construction, it suffices to verify that

$$\max(|\alpha_1 \cap L|, |\alpha_2 \cap L|) \geqslant \max(\beta_1 \cap L|, |\beta_2 \cap L|)$$

or equivalently

$$\max(|\{i, j'\} \cap L|, |\{i', j\} \cap L|) \geqslant \max(\{i, j\} \cap L|, |\{i', j'\} \cap L|)$$

noting that $|\Delta \cap L|$ appears in every term and thus may be omitted. We verify this last inequality by considering the possible values for the right hand side.

If the right hand side is 0, the inequality is trivially satisfied.

If the right hand side is 1, the interval L contains at least one of i, i', j, or j' and thus the left hand side is at least 1.

If the right hand side is 2, the interval L contains either i and j, or i' and j'. Assume for the moment that L contains both i and j. Then because R is an elementary ratio it must be that the interval L also contains either i' or j' (or possibly both), and hence the left hand side is 2. Similar reasoning holds if L had instead contained i' and j'.

Conversely, suppose that the ratio R satisfies condition (M). Consider the two intervals $L = \{i, i+1, \ldots, j\}$ and $L' = \{j, j+1, \ldots, i\}$, working with the elements modulo $2n$ as required. Because R satisfies condition (M), it must be that

$$\max(|\{i, j'\} \cap L|, |\{i', j\} \cap L|) \geqslant \max(\{i, j\} \cap L|, |\{i', j'\} \cap L|) = 2$$

and hence either i' or j' lies in L. Additionally, upon consideration of the complementary interval L', we see that

$$\max(|\{i, j'\} \cap L|, |\{i', j\} \cap L|) \geqslant \max(\{i, j\} \cap L|, |\{i', j'\} \cap L|) = 2$$

and hence either i' or j' lies in L'. Thus working modulo $2n$ and considering representatives in $\{i, i+1, \ldots, i+2n-1\}$, we have either $i < i' < j < j'$ or $i < j' < j < i'$. In the first case the ratio is elementary. In the latter case, a simple renaming $i \leftrightarrow j'$ and $i' \leftrightarrow j$ preserves the ratio and makes it elementary. $\square$

Remark 18. All elementary ratios are necessarily bounded by 1. Indeed, the short Plücker relation

$$[i, i', \Delta][j, j', \Delta] + [i, j', \Delta][i', j, \Delta] = [i, j, \Delta][i', j', \Delta] \tag{4}$$

together with the positivity of all Plücker coordinates over $TPGr(n, 2n)$ imply

$$[i, j', \Delta][i', j, \Delta] < [i, j, \Delta][i', j', \Delta]$$

as desired.

Computationally the elementary ratios are inefficient due to the large number of them. The solution to this problem is to consider instead a small subset of the elementary ratios, which we will call the *basic* ratios. We will show that every elementary ratio can be written as a product of positive powers of basic ratios. We will use this fact in the next section.

Definition 19. A basic ratio is one of the form

$$\frac{[i, j+1, \Delta][i+1, j, \Delta]}{[i, j, \Delta][i+1, j+1, \Delta]},$$

where $i, j \in \{1, \ldots, 2n\}$ and $\Delta \subset \{1, \ldots, 2n\}$ such that $|\Delta| = n-2$ and $i, i+1, j, j+1$ and Δ are all distinct. Here indices $i+1$ and $j+1$ are understood mod $2n$.

Clearly, a basic ratio is an elementary ratio with $i' = i + 1$, and $j' = j + 1$.

We define the complexity of a particular elementary ratio

$$R = \frac{[i, j', \Delta][i', j, \Delta]}{[i, j, \Delta][i', j', \Delta]}$$

as

$$\mu(R) = |\Delta \cap (\{i, i+1, \ldots, i'\} \cup \{j, j+1, \ldots, j'\})|.$$

To prove that every elementary ratio can be written as a product of basic ratios, we first consider the following special case.

Lemma 20. *An elementary ratio R with complexity $\mu(R) = 0$ can be written as a product of basic ratios.*

Proof. We define

$$\delta(R) = |\{i, i+1, \ldots, i'\} \cup \{j, j+1, \ldots, j'\}|.$$

Recall that R is a basic ratio if $i' = i + 1$ and $j' = j + 1$, or in other words $\delta(R) = 4$. We proceed by induction.

Assume that when $\mu(R) = 0$ and $\delta(R) = 4, 5, \ldots, k - 1$ we have a factorization of ratio R into a product of basic ratios.

Now consider a given elementary ratio R with $\mu(R) = 0$ and $\delta(R) = k > 4$. It cannot be the case that both $i' = i + 1$ and $j' = j + 1$ as $\delta(R) > 4$. Without loss of generality, assume that $i + 1 \neq i'$ (otherwise exchange the labels of i and i' with j and j' respectively).

Now the elementary ratio R factors as

$$\frac{[i, j', \Delta][i', j, \Delta]}{[i, j, \Delta][i', j', \Delta]} = \left(\frac{[i, j', \Delta][i+1, j, \Delta]}{[i, j, \Delta][i+1, j', \Delta]}\right)\left(\frac{[i+1, j', \Delta][i', j, \Delta]}{[i+1, j, \Delta][i', j', \Delta]}\right),$$

where each factor F on the right hand size has $\mu(F) = 0$ and $\delta(F) < k$. By induction, each factor on the right hand side can be expressed as a product of basic ratios. Hence R can we written as a product of basic ratios. □

Theorem 21. *Every elementary ratio can be written as a product of basic ratios.*

Proof. We shall proceed by induction on $\mu(R)$. By the previous lemma, when $\mu(R) = 0$ we have that R can be expressed as a product of basic ratios. Assume that for any ratio R with $\mu(R) = 0, 1, \ldots, k - 1$ we can express R as a product of basic ratios.

Now consider an elementary ratio R with $\mu(R) = k > 0$. It cannot be the case that both $\Delta \cap \{i, i+1, \ldots, i'\} = \emptyset$ and $\Delta \cap \{j, j+1, \ldots, j'\} = \emptyset$, so assume without loss of generality that $\Delta \cap \{i, i+1, \ldots, i'\} \neq \emptyset$ (if not, exchange the labels of i and i' with j and j' respectively).

Let $p \in \Delta \cap \{i, i+1, \ldots, i'\}$ be the element nearest to i and let $\Delta' = \Delta \setminus \{p\}$.

The ratio R then factors as

$$\frac{[i, j', \Delta][i', j, \Delta]}{[i, j, \Delta][i', j', \Delta]} = \left(\frac{[i, j', p, \Delta'][i, i', j, \Delta']}{[i, j, p, \Delta'][i, i', j', \Delta']}\right)\left(\frac{[i, i', j', \Delta'][i', j, p, \Delta']}{[i, i', j, \Delta'][i', j', p, \Delta']}\right),$$

where each factor F on the right hand side has $\mu(F) < k$ and hence may be written as a product of basic ratios. □

5. A factorization of (some) bounded ratios

In this section we give an alternative proof of a necessary and sufficient condition for a ratio of the form

$$\frac{[\alpha_1][\alpha_2]}{[\beta_1][\beta_2]} \tag{5}$$

to be bounded in terms of the four index sets, α_1, α_2, β_1, and β_2. In addition, we will show that this ratio can be written as a product of elementary ratios.

For the remainder of the section we will assume that R is a ratio of the form $[\alpha_1][\alpha_2]/[\beta_1][\beta_2]$ which satisfies the ST0 condition and condition (M). Denote the set of all such ratios by $\mathscr{B}$. We define

$$\begin{aligned}
&\Delta = \alpha_1 \cap \alpha_2 = \beta_1 \cap \beta_2;\\
&\gamma_1 = (\alpha_1 \cap \beta_1) \setminus \Delta;\\
&\gamma_2 = (\alpha_1 \cap \beta_2) \setminus \Delta;\\
&\delta_1 = (\alpha_2 \cap \beta_2) \setminus \Delta;\\
&\delta_2 = (\alpha_2 \cap \beta_1) \setminus \Delta; \quad \text{and}\\
&\Omega = \gamma_1 \cup \gamma_2 \cup \delta_1 \cup \delta_2,
\end{aligned}$$

so that

$$R = \frac{[\alpha_1][\alpha_2]}{[\beta_1][\beta_2]} = \frac{[\gamma_1, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]}.$$

(Recall that notationally $[\gamma_1, \gamma_2, \Delta]$ means $[\gamma_1 \cup \gamma_2 \cup \Delta]$, and that we necessarily have: $|\gamma_1| = |\delta_1|$; $|\gamma_2| = |\delta_2|$; and $|\gamma_1| + |\gamma_2| + |\Delta| = n$.)

An important property of the ratio R is the number of indices which are *not* shared by all index sets comprising the ratio. We shall denote this quantity as

$$\nu(R) = n - |\Delta| = |\Omega|/2.$$

Before proceeding, we investigate what information $\nu(R)$ holds.

Definition 22 (*Trivial ratio*). We say a ratio $[\alpha_1][\alpha_2]/[\beta_1][\beta_2]$ is trivial if either

- $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$; or
- $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_1$.

Note that a ratio R satisfying the ST0 condition with $\nu(R) = 0$ or 1 is trivial.

Lemma 23. *Suppose $R \in \mathscr{B}$ and $\nu(R) = 2$. Then either*

- *R is trivial*; *or*
- *R is an elementary ratio and can be written as a product of basic ratios.*

Proof. If R is not trivial, R must be of the form

$$\frac{[i, j', \Delta][i', j, \Delta]}{[i, j, \Delta][i', j', \Delta]}$$

with $|\Delta| = n-2$ and i, i', j, j', and Δ pairwise distinct. Proposition 17 establishes that R is an elementary ratio, and hence R may be written as a product of basic ratios by Theorem 21. $\square$

We will eventually show that any ratio $R \in \mathscr{B}$ with $\nu(R) \geqslant 3$ can be factored as $R = R_1 R_2$ with $R_i \in \mathscr{B}$ and $\nu(R_i) < \nu(R)$ for $i = 1, 2$. In order to do this, we will rely heavily upon the following definition, simple remark, and technical lemma.

Definition 24 (*Interlacing*). Suppose $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_k)$ are two subsequences of $(1, 2, \ldots, 2n)$. Then we say the sequence (i_s) interlaces the sequence (j_t) if either:

(1) $j_1 \leqslant i_1 \leqslant j_2 \leqslant i_2 \leqslant \cdots \leqslant j_k \leqslant i_k$; or
(2) $i_1 \leqslant j_1 \leqslant i_2 \leqslant j_2 \leqslant \cdots \leqslant i_k \leqslant j_k$.

Remark 25. Suppose the ratio $R = [\alpha_1][\alpha_2]/[\beta_1][\beta_2]$ satisfies the ST0 condition, and suppose that $\beta_1 \setminus \Delta$ interlaces with $\beta_2 \setminus \Delta$. Then R automatically satisfies condition (M).

For notational convenience, let $g(\alpha_1, \alpha_2, L) = \max(|\alpha_1 \cap L|, |\alpha_2 \cap L|)$.

Lemma 26 (Technical lemma). *Suppose that $R \in \mathscr{B}$ and we have a factorization of R as*

$$\frac{[\gamma_1, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]} = \frac{[\gamma_1, \gamma_2, \Delta][\gamma_{11}, \delta_{12}, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\gamma_{11}, \delta_{12}, \gamma_2, \Delta]} \cdot \frac{[\gamma_{11}, \delta_{12}, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_{11}, \delta_{12}, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]}$$

for some non-empty sets $\gamma_{11}, \gamma_{12}, \delta_{11}$, and δ_{12} where $\gamma_1 = \gamma_{11} \cup \gamma_{12}$ and $\delta_1 = \delta_{11} \cup \delta_{12}$ such that $|\gamma_{11}| = |\delta_{11}|$ and $|\gamma_{12}| = |\delta_{12}|$.

Suppose as well that $\gamma_{11} \cup \delta_{12}$ interlaces with $\gamma_{12} \cup \delta_{11}$.

Then each of the factors

$$R_1 = \frac{[\gamma_1, \gamma_2, \Delta][\gamma_{11}, \delta_{12}, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\gamma_{11}, \delta_{12}, \gamma_2, \Delta]} \quad \textit{and} \quad R_2 = \frac{[\gamma_{11}, \delta_{12}, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_{11}, \delta_{12}, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]}$$

are elements of $\mathscr{B}$, and $\nu(R_i) < \nu(R)$ for $i = 1, 2$.

Proof. Observe that

$$|\gamma_1 \cap L| + |\delta_1 \cap L| = |(\gamma_{11} \cup \delta_{12}) \cap L| + |(\gamma_{12} \cup \delta_{11}) \cap L|$$

for all intervals L. This, along with the hypothesis that $\gamma_{11} \cup \delta_{12}$ interlaces with $\gamma_{12} \cup \delta_{11}$, immediately gives

$$g(\gamma_1, \delta_1, L) \geqslant g(\gamma_{11} \cup \delta_{12}, \gamma_{12} \cup \delta_{11}, L)$$

for all intervals L.

Fix an interval L. Then there are three possible cases:

(1) $|\alpha_1 \cap L| > |\alpha_2 \cap L|$;
(2) $|\alpha_1 \cap L| < |\alpha_2 \cap L|$; or
(3) $|\alpha_1 \cap L| = |\alpha_2 \cap L|$.

Suppose case (1) holds. Then since $g(\alpha_1, \alpha_2, L) \geqslant g(\beta_1, \beta_2, L)$ it follows that $|\gamma_1 \cap L| \geqslant |\delta_1 \cap L|$ and $|\gamma_2 \cap L| \geqslant |\delta_2 \cap L|$. However, if $g(\gamma_1, \delta_1, L) \geqslant g(\gamma_{11} \cup \delta_{12}, \gamma_{12} \cup \delta_{11}, L)$ and $|\gamma_1 \cap$

$L| \geqslant |\delta_1 \cap L|$, then applying similar reasoning reveals that $|\gamma_{11} \cap L| \geqslant |\delta_{11} \cap L|$ and $|\gamma_{12} \cap L| \geqslant |\delta_{12} \cap L|$. Thus the following four inequalities hold:

(i) $|\gamma_1 \cap L| \geqslant |\delta_1 \cap L|$;
(ii) $|\gamma_2 \cap L| \geqslant |\delta_2 \cap L|$;
(iii) $|\gamma_{11} \cap L| \geqslant |\delta_{11} \cap L|$; and
(iv) $|\gamma_{12} \cap L| \geqslant |\delta_{12} \cap L|$.

But these preceding inequalities (i)–(iv) imply that both of the ratios

$$\frac{[\gamma_1, \gamma_2, \Delta][\gamma_{11}, \delta_{12}, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\gamma_{11}, \delta_{12}, \gamma_2, \Delta]} \quad \text{and} \quad \frac{[\gamma_{11}, \delta_{12}, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_{11}, \delta_{12}, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]},$$

satisfy condition (M), for the fixed interval L. Similar analysis holds for cases (2) and (3) when $|\alpha_1 \cap L| \leqslant |\alpha_2 \cap L|$ and is omitted here.

Lastly, note that $\nu(R_1) = \nu(R) - |\gamma_{11}| < \nu(R)$ and $\nu(R_2) = \nu(R) - |\delta_{12}| < \nu(R)$. □

Lemma 27. *Let $R \in \mathscr{B}$, and suppose that either*

- *γ_1 and δ_1 do not interlace; or*
- *γ_2 and δ_2 do not interlace (or both).*

Then we may write $R = R_1R_2$ for some ratios $R_1, R_2 \in \mathscr{B}$ with $\nu(R_i) < \nu(R)$ for $i = 1, 2$.

Proof. Without loss of generality, assume that γ_1 and δ_1 do not interlace. (If instead γ_2 and δ_2 do not interlace, interchange the labeling of α_1 and α_2).

Label the elements of $\gamma_1 \cup \delta_1$ as $\{i_1, i_2, \ldots, i_{2m}\}$ with $i_1 < i_2 < \cdots < i_{2m}$, and define $\gamma_{11} = \gamma_1 \cap \{i_1, i_3, \ldots, i_{2m-1}\}$, $\gamma_{12} = \gamma_1 \cap \{i_2, i_4, \ldots, i_{2m}\}$, $\delta_{12} = \delta_1 \cap \{i_1, i_3, \ldots, i_{2m-1}\}$, and $\delta_{11} = \delta_1 \cap \{i_2, i_4, \ldots, i_{2m}\}$.

Because γ_1 does not interlace with δ_1, we have constructed γ_{11}, γ_{12}, δ_{11}, and δ_{12} to all be non-empty. In addition, $\gamma_{11} \cup \delta_{12}$ interlaces with $\gamma_{12} \cup \delta_{11}$, thus satisfying the requirements of Lemma 26. □

We now examine the situation when both γ_1 interlaces with δ_1 and γ_2 interlaces with δ_2.

Claim 28 (Agreeable labeling). *Let $R \in \mathscr{B}$ be a ratio satisfying condition* (M)*, and suppose that γ_1 interlaces with δ_1 and γ_2 interlaces with δ_2. Set $\Omega = \gamma_1 \cup \gamma_2 \cup \delta_1 \cup \delta_2 = \{i_1, i_2, \ldots, i_{2m}\}$ with $i_1 < i_2 < \cdots < i_{2m}$. Then, up to a possible relabeling of α_1 and α_2 or β_1 and β_2, we may assume that*

(1) $\beta_1 \setminus \Delta = \gamma_1 \cup \delta_2 = \{i_1, i_3, \ldots, i_{2m-1}\}$;
(2) $\beta_2 \setminus \Delta = \delta_1 \cup \gamma_2 = \{i_2, i_4, \ldots, i_{2m}\}$; *and*
(3) $i_1 \in \gamma_1$.

We will say that ratio with a labeling satisfying conditions (1)–(3) *is agreeably labeled.*

Proof. Suppose that either $\gamma_1 \cup \delta_2$ or $\delta_1 \cup \gamma_2$ contained a consecutive pair of elements $i_l, i_{l+1} \in \Omega$. It cannot be that this pair lies entirely in one of $\gamma_1, \gamma_2, \delta_1$, or δ_2, as this would violate the interlacing hypotheses. However if i_l and i_{l+1} lie in different sets, for example γ_1 and δ_2, consideration of condition (M) with the interval $L = \{i_l, \ldots, i_{l+1}\}$ again leads to a contradiction.

Thus $\gamma_1 \cup \delta_2$ and $\delta_1 \cup \gamma_2$ contain no consecutive pairs of elements of Ω, and hence we may assume (up to relabeling of β_1 and β_2) that both (1) and (2) hold. Lastly, we may swap the labeling of α_1 and α_2 if necessary to ensure that $i_1 \in \alpha_1$ and hence (3) holds. $\square$

Under this labeling, the element i_2 may be in either δ_1 or γ_2. We investigate each case separately.

Lemma 29. *Let $R \in \mathscr{B}$ with $\nu(R) \geqslant 3$, and suppose that both γ_1 interlaces with δ_1 and γ_2 interlaces with δ_2. Assume that R is agreeably labeled (see Claim 28), and set $\Omega = \gamma_1 \cup \gamma_2 \cup \delta_1 \cup \delta_2 = \{i_1, i_2, \ldots, i_{2m}\}$ with $i_1 < i_2 < \cdots < i_{2m}$. Furthermore, assume that $i_2 \in \delta_1$.*

Then we may write $R = R_1 R_2$ for some ratios $R_1, R_2 \in \mathscr{B}$ with $\nu(R_i) < \nu(R)$ for $i = 1, 2$.

Proof. Because R is agreeably labeled, we know that $\gamma_1 \cup \delta_2 = \{i_1, i_3, \ldots\}$ with $i_1 \in \gamma_1$. Define $k \geqslant 1$ to be the value so that $\{i_1, i_3, i_5, \ldots, i_{2k-1}\} \subseteq \gamma_1$ and $i_{2k+1} \in \delta_2$. Similarly define $l \geqslant 1$ to be the value so that $\{i_2, i_4, \ldots, i_{2l}\} \subseteq \delta_1$ and $i_{2l+2} \in \gamma_2$.

To summarize, we have set

$$\begin{aligned}
\gamma_1 &= \{i_1, i_3, i_5, \ldots, i_{2k-1}, *\};\\
\gamma_2 &= \{i_{2l+2}, *\};\\
\delta_1 &= \{i_2, i_4, \ldots, i_{2l}, *\}; \quad \text{and}\\
\delta_2 &= \{i_{2k+1}, *\},
\end{aligned}$$

where the use of $*$ is understood to represent the remaining elements and is not the same in each instance.

Because of the interlacing hypothesis, we must have either

(a) $l = k$; or
(b) $l = k - 1$.

For case (a), let $\gamma_{11} = \{i_1\}$, $\gamma_{12} = \gamma_1 \setminus \{i_1\} = \{i_3, i_5, \ldots, i_{2k-1}, *\}$, $\delta_{11} = \{i_2\}$ and $\delta_{12} = \delta_1 \setminus \{i_2\} = \{i_4, \ldots, i_{2l}, *\}$.

We claim $\gamma_{12} \cup \delta_2$ and $\delta_{12} \cup \gamma_2$ interlace, as

$$\begin{aligned}
\gamma_{12} \cup \delta_2 &= (\gamma_1 \cup \delta_2) \setminus \{i_1\} = \{i_3, i_5, \ldots, i_{2m-1}\} \quad \text{and}\\
\delta_{12} \cup \gamma_2 &= (\delta_1 \cup \gamma_2) \setminus \{i_2\} = \{i_4, i_6, \ldots, i_{2m}\}.
\end{aligned}$$

Similarly $\gamma_{11} \cup \delta_2$ and $\delta_{11} \cup \gamma_2$ interlace, as

$$\begin{aligned}
\gamma_{11} \cup \delta_2 &= \{i_1\} \cup \delta_2 = \{i_1, i_{2k+1}, *\} \quad \text{and}\\
\delta_{11} \cup \gamma_2 &= \{i_2\} \cup \gamma_2 = \{i_2, i_{2l+2}, *\},
\end{aligned}$$

noting $l = k$ and γ_2 interlaces with δ_2 by hypothesis.

Therefore, we may write

$$\frac{[\gamma_1, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]} = \frac{[\gamma_1, \gamma_2, \Delta][\gamma_{11}, \delta_{12}, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\gamma_{11}, \delta_{12}, \gamma_2, \Delta]} \cdot \frac{[\gamma_{11}, \delta_{12}, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_{11}, \delta_{12}, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]},$$

where both ratios on the right hand side satisfy condition (M) by Remark 25. Labeling the ratios on the right hand side as R_1 and R_2 respectively, we see that $\nu(R_1) = \nu(R) - |\gamma_{11}|$ and $\nu(R_2) = \nu(R) - |\gamma_{12}|$. Now $|\gamma_{11}| = 1$, so $\nu(R_1) < \nu(R)$.

If $|\gamma_{12}| \geqslant 1$ we are finished with this case. If instead $|\gamma_{12}| = 0$ we deduce that $\gamma_1 = \{i_1\}$, $\gamma_2 = \{i_3, i_5, \ldots, i_{2m-1}\}$, $\delta_1 = \{i_2\}$, and $\delta_2 = \{i_4, i_6, \ldots, i_{2m}\}$. We can then factor R as

$$\frac{[i_1, i_4, i_6, \ldots, \Delta][i_2, i_3, i_5, \ldots, \Delta]}{[i_1, i_3, i_5, \ldots, \Delta][i_2, i_4, i_6, \ldots, \Delta]} = \frac{[i_1, i_4, i_6, \ldots, \Delta][i_2, i_4, i_5, i_7, \ldots, \Delta]}{[i_2, i_4, i_6, \ldots, \Delta][i_1, i_4, i_5, i_7, \ldots, \Delta]} \cdot \frac{[i_1, i_4, i_5, i_7, \ldots, \Delta][i_2, i_3, i_5, \ldots, \Delta]}{[i_2, i_4, i_5, i_7, \ldots, \Delta][i_1, i_3, i_5, \ldots, \Delta]},$$

where the ellipses indicate the sequence continues with the same parity subscripts. Note that $\{i_1, i_5, i_7, \ldots\}$ interlaces with $\{i_2, i_6, i_8, \ldots\}$ and that $\{i_1, i_3\}$ interlaces with $\{i_2, i_4\}$, hence both ratios on the right hand side satisfy condition (M) by Remark 25. Labeling the ratios on the right hand side as R_1 and R_2 respectively, we see that $\nu(R_1) = \nu(R) - 1 < \nu(R)$ and $\nu(R_2) = 2 < \nu(R)$.

Now we return to case (b), where $l = k - 1$. Let $\gamma_{11} = \{i_3, i_5, \ldots, i_{2k-1}\}$, $\gamma_{12} = \gamma_1 \setminus \gamma_{11}$, $\delta_{11} = \{i_2, i_4, \ldots, i_{2l}\}$, and $\delta_{12} = \delta_1 \setminus \delta_{11}$.

We claim $\gamma_{12} \cup \delta_2$ and $\delta_{12} \cup \gamma_2$ interlace, as

$$\gamma_{12} \cup \delta_2 = (\gamma_1 \cup \delta_2) \setminus \{i_2, i_3, i_4, \ldots i_{2l+1}\} \quad \text{and}$$
$$\delta_{12} \cup \gamma_2 = (\delta_1 \cup \gamma_2) \setminus \{i_2, i_3, i_4, \ldots i_{2l+1}\},$$

noting $\gamma_1 \cup \delta_2$ interlaces with $\delta_1 \cup \gamma_2$ and we have removed a section of consecutive elements of Ω.

Similarly, we claim $\gamma_{11} \cup \delta_2$ and $\delta_{11} \cup \gamma_2$ interlace, as

$$\gamma_{11} \cup \delta_2 = \{i_3, i_5, \ldots, i_{2k-1}, i_{2k+1}, *\} \quad \text{and}$$
$$\delta_{11} \cup \gamma_2 = \{i_2, i_4, \ldots, i_{2l}, i_{2l+2}, *\},$$

noting $l = k - 1$ and γ_2 interlaces with δ_2 by hypothesis.

Therefore, we may write

$$\frac{[\gamma_1, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]} = \frac{[\gamma_1, \gamma_2, \Delta][\gamma_{11}, \delta_{12}, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\gamma_{11}, \delta_{12}, \gamma_2, \Delta]} \cdot \frac{[\gamma_{11}, \delta_{12}, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_{11}, \delta_{12}, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]},$$

where both ratios on the right hand side satisfy condition (M) by Remark 25. Labeling the ratios on the right hand side as R_1 and R_2 respectively, we see that $\nu(R_1) = \nu(R) - |\gamma_{11}|$ and $\nu(R_2) = \nu(R) - |\gamma_{12}|$.

Observe that $i_2 \in \delta_{11}$ so $|\delta_{11}| = |\gamma_{11}| \geqslant 1$ and hence $\nu(R_1) < \nu(R)$. Similarly, $i_1 \in \gamma_{12}$ so $|\gamma_{12}| \geqslant 1$ and hence $\nu(R_2) < \nu(R)$.

This concludes the proof, as we have successfully dealt with both cases (a) and (b). □

Lemma 30. *Let $R \in \mathscr{B}$ with $\nu(R) \geqslant 3$, and suppose that both γ_1 interlaces with δ_1 and γ_2 interlaces with δ_2. Assume that R is agreeably labeled (see Claim 28), and set $\Omega = \gamma_1 \cup \gamma_2 \cup \delta_1 \cup \delta_2 = \{i_1, i_2, \ldots, i_{2m}\}$ with $i_1 < i_2 < \cdots < i_{2m}$. Furthermore, assume that $i_2 \in \gamma_2$.*

Then we may write $R = R_1 R_2$ for some ratios $R_1, R_2 \in \mathscr{B}$ with $\nu(R_i) < \nu(R)$ for $i = 1, 2$.

Proof. First, note that $i_3 \in \delta_2$, since otherwise $\{i_1, i_3\} \subset \gamma_1$ and hence γ_1 and δ_1 would not interlace.

We consider several possibilities. Suppose first that $|\gamma_1| = 1$, i.e. $\gamma_1 = \{i_1\}$. This then forces $\delta_2 = \{i_3, i_5, \ldots, i_{2m-1}\}$, $\gamma_2 = \{i_2, i_4, \ldots, i_{2m-2}\}$, and $\delta_1 = \{i_{2m}\}$. We can then factor R as

$$\begin{aligned} R &= \frac{[i_1, i_2, i_4, \ldots, i_{2m-2}, \Delta][i_3, i_5, \ldots, i_{2m-1}, i_{2m}, \Delta]}{[i_2, i_4, \ldots, i_{2m-2}, i_{2m}, \Delta][i_1, i_3, i_5, \ldots, i_{2m-1}, \Delta]} \\ &= \frac{[i_3, i_5, \ldots, i_{2m-1}, i_{2m}, \Delta][i_1, i_3, i_4, i_6, \ldots, i_{2m-2}, \Delta]}{[i_1, i_3, i_5, \ldots, i_{2m-1}, \Delta][i_3, i_4, i_6, \ldots, i_{2m-2}, i_{2m}, \Delta]} \\ &\quad \cdot \frac{[i_3, i_4, i_6, \ldots, i_{2m-2}, i_{2m}, \Delta][i_1, i_2, i_4, \ldots, i_{2m-2}, \Delta]}{[i_1, i_3, i_4, i_6, \ldots, i_{2m-2}, \Delta][i_2, i_4, \ldots, i_{2m-2}, i_{2m}, \Delta]}, \end{aligned}$$

where the ellipses indicate the sequence continues with the same parity subscripts. Note that $\{i_1, i_5, i_7, \ldots\}$ interlaces with $\{i_4, i_6, i_8, \ldots\}$ and that $\{i_1, i_3\}$ interlaces with $\{i_2, i_4\}$, hence both ratios on the right hand side satisfy condition (M) by Remark 25. Labeling the ratios on the right hand side as R_1 and R_2 respectively, we see that $\nu(R_1) = \nu(R) - 1 < \nu(R)$ and $\nu(R_2) = 2 < \nu(R)$.

If instead $|\gamma_1| > 1$, we may define $k \geqslant 2$ to be the value so that $\{i_3, i_5, \ldots, i_{2k-1}\} \subseteq \delta_2$ and $\{i_1, i_{2k+1}\} \subseteq \gamma_1$. Similarly let $l \geqslant 1$ be the value so that $\{i_2, i_4, \ldots, i_{2l}\} \subseteq \gamma_2$ and $i_{2l+2} \in \delta_1$. Observe that because both γ_1 interlaces with δ_1 and γ_2 interlaces with δ_2, we necessarily have $l = k - 1$.

Let $\delta_{21} = \{i_3\}$, $\delta_{22} = \delta_2 \setminus \delta_{21}$, $\gamma_{21} = \{i_2\}$, $\gamma_{22} = \gamma_2 \setminus \gamma_{21}$.

We claim $\gamma_1 \cup \delta_{21}$ interlaces with $\delta_1 \cup \gamma_{21}$, as

$$\begin{aligned} &\gamma_1 \cup \delta_{21} = \{i_1, i_3, i_{2k+1}, *\} \quad \text{and} \\ &\delta_1 \cup \gamma_{21} = \{i_2, i_{2l+2}, *\}, \end{aligned}$$

noting $2l + 2 = 2k$ and γ_1 interlaces with δ_1 by hypothesis.

Similarly, we claim $\gamma_1 \cup \delta_{22}$ interlaces with $\delta_1 \cup \gamma_{22}$, as

$$\begin{aligned} &\gamma_1 \cup \delta_{22} = \{i_1, i_5, i_7, \ldots, i_{2k+1}, *\} \quad \text{and} \\ &\delta_1 \cup \gamma_{22} = \{i_4, i_6, \ldots, i_{2l+2}, *\}, \end{aligned}$$

noting $2l + 2 = 2k$ and γ_1 interlaces with δ_1 by hypothesis.

Therefore, we may write

$$\frac{[\gamma_1, \gamma_2, \Delta][\delta_1, \delta_2, \Delta]}{[\gamma_1, \delta_2, \Delta][\delta_1, \gamma_2, \Delta]} = \frac{[\gamma_1, \gamma_2, \Delta][\delta_1, \delta_{21}, \gamma_{22}, \Delta]}{[\delta_1, \gamma_2, \Delta][\gamma_1, \delta_{21}, \gamma_{22}, \Delta]} \cdot \frac{[\gamma_1, \delta_{21}, \gamma_{22}, \Delta][\delta_1, \delta_2, \Delta]}{[\delta_1, \delta_{21}, \gamma_{22}, \Delta][\gamma_1, \delta_2, \Delta]},$$

where both ratios on the right hand side satisfy condition (M) by Remark 25. Labeling the ratios on the right hand side as R_1 and R_2 respectively, we see that $\nu(R_1) = \nu(R) - |\gamma_{22}|$ and $\nu(R_2) = \nu(R) - |\delta_{21}|$. Now $|\delta_{21}| = 1$, so $\nu(R_2) < \nu(R)$.

If $|\gamma_{22}| \geqslant 1$ we are finished. If instead $|\gamma_{22}| = 0$, we deduce that $\gamma_1 = \{i_1, i_5, i_7, \ldots, i_{2m-1}\}$, $\delta_1 = \{i_4, i_6, \ldots, i_{2m}\}$, $\gamma_2 = \{i_2\}$, and $\delta_2 = \{i_3\}$. We can then factor R as

$$\begin{aligned} \frac{[i_1, i_2, i_5, i_7, \ldots, \Delta][i_3, i_4, i_6, \ldots, \Delta]}{[i_2, i_4, i_6, \ldots, \Delta][i_1, i_3, i_5, i_7, \ldots, \Delta]} &= \frac{[i_1, i_2, i_5, i_7, \ldots, \Delta][i_3, i_5, i_6, i_8, \ldots, \Delta]}{[i_1, i_3, i_5, i_7, \ldots, \Delta][i_2, i_5, i_6, i_8, \ldots, \Delta]} \\ &\quad \cdot \frac{[i_2, i_5, i_6, i_8, \ldots, \Delta][i_3, i_4, i_6, \ldots, \Delta]}{[i_3, i_5, i_6, i_8, \ldots, \Delta][i_2, i_4, i_6, \ldots, \Delta]}, \end{aligned}$$

where the ellipses indicate the sequence continues with the same parity subscripts. Note that $\{i_1, i_3, i_7, \ldots\}$ interlaces with $\{i_2, i_6, i_8, \ldots\}$ and that $\{i_3, i_5\}$ interlaces with $\{i_2, i_4\}$, hence both ratios on the right hand side satisfy condition (M) by Remark 25. Labeling the ratios on the right hand side as R_1 and R_2 respectively, we see that $\nu(R_1) = \nu(R) - 1 < \nu(R)$ and $\nu(R_2) = 2 < \nu(R)$. □

Theorem 31. *Let $R \in \mathscr{B}$ with $\nu(R) \geqslant 3$. Then we may write $R = R_1 R_2$ for some ratios $R_1, R_2 \in \mathscr{B}$ with $\nu(R_i) < \nu(R)$ for $i = 1, 2$.*

Proof. If either γ_1 and δ_1 do not interlace, or γ_2 and δ_2 do not interlace, or both we may appeal to Lemma 27.

If instead both γ_1 interlaces with δ_1 and γ_2 interlaces with δ_2, we may assume without loss of generality that R is agreeably labeled (see Claim 28). Observe that with this labeling $i_2 \in \delta_1 \cup \gamma_2$.

If $i_2 \in \delta_1$, we may appeal to Lemma 29.

If $i_2 \in \gamma_2$, we may appeal to Lemma 30. □

We now state our main result.

Theorem 32 (Main theorem). *Let R be a ratio of the form $\alpha/\beta = [\alpha_1][\alpha_2]/[\beta_1][\beta_2]$ where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are index sets in $\{1, \ldots, 2n\}$. The following are equivalent:*

(1) *R satisfies the ST0 condition and*

$$\max(|\alpha_1 \cap L|, |\alpha_2 \cap L|) \geqslant \max(|\beta_1 \cap L|, |\beta_2 \cap L|) \tag{6}$$

for every interval $L \subseteq \{1, \ldots, 2n\}$, i.e. α/β satisfies condition (M).

(2) *R can be written as a product of basic ratios.*

(3) *R is bounded by* 1.

(4) *R is bounded.*

Proof Note that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) are clear, so what remains to show is that (1) $\Rightarrow$ (2).

By Theorem 31, we may write any ratio R of the specified form satisfying condition (M) with $\nu(R) \geqslant 3$ as a product of ratios $R_1 R_2$ of the same form where each satisfies condition (M) and $\nu(R_i) < \nu(R)$ for $i = 1, 2$. By Lemma 23 and the remarks directly preceding it, any ratio R of the specified form satisfying condition (M) with $\nu(R) \leqslant 2$ is either trivial or can be written as a product of basic ratios.

A simple induction argument on the value of $\nu(R)$ completes the proof. □

Remark 33. Much of the proof in this section extends similar techniques used by Fallat et al. to the case of non-principal minors (see [2]). The equivalence of (1), (3) and (4) in Theorem 32 is a result of Skandera (see [3]).

Remark 34. While we have shown that condition (M) implies boundedness for this specific class of ratios, this condition is not sufficient in general. For example, the ratio

$$\frac{[1,2,3,8][2,3,4,5][4,6,7,8]}{[1,4,6,8][2,3,4,8][2,3,5,7]}$$

satisfies condition (M) but is unbounded over the class of totally positive matrices. For example, when applied to the totally positive matrix

$$\begin{pmatrix} 1 & 3t^{-1} & 3t^{-2} & t^{-1} \\ 2+t^{-1} & 1+6t^{-1}+3t^{-2} & 2t^{-1}+6t^{-2}+3t^{-3} & 1+2t^{-1}+t^{-2} \\ t+2 & t+4+6t^{-1} & 3+5t^{-1}+6t^{-2} & 2t+2+2t^{-1} \\ t & t+3 & t+2+3t^{-1} & t^2+t+2 \end{pmatrix},$$

where t is a positive indeterminate, the exhibited ratio increases without bound as $t \to \infty$.

Conjecture 35. *A ratio α/β, where α and β are each sequences of an arbitrary number of index sets is bounded if and only if it can be written as a product of basic ratios.*

This conjecture was briefly hinted at by Fallat et al. with regards to a possible way to save a similar conjecture with respect to bounded ratios of principal minors (see [2, §6]).

6. Computational methods and computational results

Given this collection of basic ratios, a natural question to consider is: *What is the set of ratios generated by products of positive powers of the basic ratios?* Every ratio in this space is both bounded and expressible as a product of basic ratios.

We consider a typical element of this space to be a ratio of products of index sets. Recall there are $N = \binom{2n}{n}$ such index sets. Then each ratio can be described by giving the power to which each index set appears in the ratio; terms appearing in the denominator have negative exponent. This allows us to identify each ratio with a vector in $\mathbb{R}^N$ where each entry represents the power to which that index set appears in the ratio. The product of two ratios then simply corresponds to the sum of their two associated vectors in $\mathbb{R}^N$.

We write $v_1, \ldots, v_M$ as the vectors that correspond to each of the

$$M = n(2n-3)\binom{2n-4}{n-2}$$

basic ratios. Such a list of generating vectors can be easily computed using `Mathematica`.

The set of all ratios that are products of positive powers of the basic ratios is a polyhedral cone

$$P = \left\{ \sum_{i=1}^{M} \lambda_i v_i \middle| \lambda_i \in \mathbb{R}_{\geqslant 0} \right\}.$$

P can also be described as the intersection of finitely many linear half-spaces, namely

$$P = \left\{ x \in \mathbb{R}^N | A \cdot x \leqslant 0 \right\}$$

for some matrix A that can be computed. The software program `cdd+` [10] is useful in the conversion between these two representations of convex polyhedral cones.

We illustrate the utility of this by proving a non-trivial theorem. This result was first obtained in [1].

Proposition 36. *Every bounded ratio of minors of a* 3×3 *totally positive matrix can be written as a product of positive powers of the basic ratios. Furthermore, every such bounded ratio is bounded by* 1.

This can be verified computationally by computing the half planes of the cone generated by the basic ratios and then constructing a matrix in terms of a parameter t that satisfies the inequality listed above.

As mentioned in [2], one method of determining whether or not a ratio is bounded is to work with a totally positive matrix corresponding to the diagram in Fig. 1. The entries in this matrix are then polynomials in the variables l_i, d_j, and u_k. It is well known that this matrix will be totally positive if each variable is chosen to be positive, and all totally positive matrices may be arrived at by this construction for appropriate choices of the variables.

In this view, a ratio of minors R is a rational function p/q in the same variables. Some information about the ratio p/q may be gleaned by examining the difference $q - p$ as a polynomial in the variables l_i, d_j, and u_k. We denote this polynomial by p_R. For example, if every coefficient in p_R is positive (we call this 'subtraction free') then p_R will be positive for *any* choice of positive variables l_i, d_j, and u_k. This would imply that the ratio p/q is necessarily bounded by 1 over the class of totally positive matrices.

This suggests the following conjecture, formulated in [2,3]

Conjecture 37. *A ratio of minors R is bounded if and only if p_R is subtraction free.*

In other words, if p_R is not subtraction free, we conjecture that it is possible to find a family of totally positive matrices on which the ratio increases without bound. (It has *always* been possible in every ratio that we have examined.) This is significant, because of the existence of polynomials which remain non-negative but are not subtraction free. (e.g. $x^2 + y^2 - 2xy + 1$).

Remark 38. Observe that Conjecture 37 follows from Conjecture 35 and the short Plücker relation (Eq. (4)). Indeed, the short Plücker relation guarantees that all basic ratios are subtraction free. This fact extends to arbitrary products of basic ratios, noting that if $R = A/B \cdot C/D$ where both $p_{A/B}$ and $p_{C/D}$ are subtraction free, then $p_R = BD - AC = D(B - A) + A(D - C)$ is also subtraction free.

Using `Mathematica`, we considered the set of ratios of the form

$$\frac{[\alpha_1][\alpha_2][\alpha_3]}{[\beta_1][\beta_2][\beta_3]} \tag{7}$$

over the class of 4×4 totally positive matrices.

Of the ratios satisfying the required ST0 and majorization conditions, we found that approximately 98% could be written as a product of basic ratios. Those that could not be written as a product of basic ratios were found to be *not* subtraction free and actually unbounded over the class of totally positive matrices. The results of these computer experiments can be summarized in the following proposition.

Proposition 39. *For ratios of the form in Eq.* (7), *the following are equivalent when working over* 4×4 *totally positive matrices*:

(1) *The ratio is bounded*;
(2) *The ratio is bounded by 1*;
(3) *The ratio can be factored into a product of basic ratios*; *and*
(4) *The ratio is subtraction free*.

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