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Imbeddings of the tensor product of graphs where the second factor is a complete graph

Ghidewon Abay Asmerom*

Virginia Commonwealth University, Richmond, VA 23284-2014, USA

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Abstract

In this paper we present genus result for the tensor product of graphs where the second factor is the complete graph K_m and m is a power of 2. We are going to view the second factor as a Cayley graph and we will use an imbedding technique that combines surgery and voltage graph theory. In this imbedding technique we start with a suitable imbedding of H on some surface and then modify the edges of H according to the nature of K_m to get a voltage graph H^*. The covering graph of this voltage graph H^* will be the desired tensor product of H with K_m. The genus of $H \otimes K_m$, where H is a graph with a quadrilateral imbedding and also H is either, bipartite, with bichromatic dual, or with straight-ahead imbedding is given

1. Introduction

Given two graphs H and G and the cartesian product of their vertex sets we can define several graph products. Among the common graph products defined this way are: the cartesian product, the lexicographic product or composition, the tensor product, and the strong tensor product. Graph products, in addition to their importance of constructing 'bigger' graphs out of 'small' ones, are useful in the recognition and decomposition of larger graphs. They are also important in the sense that we can get an insight about their structural and topological properties from the factor graphs. The topological parameter that we are interested in this paper is the (minimum) genus parameter and the product of our focus will be the tensor product.

The connection between graph products and genus imbeddings goes all the way back to Gerhard Ringel's [8] and independently Beineke and Harary's [3] proof of the genus of Q_n (the n-cube). This work can be cited as the first work on imbeddings and graph products. (The n-cube is the graph K_2 if $n = 1$, while for $n > 1$, Q_n is defined inductively as $Q_{n-1} \times K_2$.) This work was followed by White's generalization, among other things,

*E-mail:gasmerom@saturn.vcu.edu.

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to the cartesian product of even cycles [10]. Pisanski [7] later generalized White's work to find the genus of the cartesian product of regular bipartite graphs. Lately, there has been interest in the genus parameter of the tensor product as well: Zeleznik [13], Bouchet and Mohar [4], Abay-Asmerom [1], and Dakić and Pisanski [5].

The two commonly used imbedding techniques for finding the genus of graphs are surgery and voltage graph theory. If a graph is factorable by a product operation, a surgical technique is often used, and for graphs that are Cayley graphs of a group, voltage graph theory often proves to be an effective technique. However, if a graph is factorable by a product operation where the second factor is a Cayley graph, a technique that combines the above two techniques works well [1, 2]. This technique is called the *surgical-voltage* imbedding technique. It was first used by White [12] and later by Abay-Asmerom [1]. In [5], Dakić and Pisanski have generalized the method to that where the second factor is a regular graph. In this paper we will use the surgical-voltage method to find the genus parameter for the tensor product of graphs when one of the factors is K_m, the complete graph on m vertices where m is a power of two. In Section 2 we will define the tensor product, in Section 3 we will state some basic properties of the tensor product and results of topological graph theory that will be needed in the sequel, in Section 4 we will discuss the surgical-voltage imbedding technique and in Section 5 we will give genus results for four families of graphs.

2. Definitions and examples

If H and G are two given graphs, the tensor product, $H \otimes G$, involving H and G has a vertex set $V(H \otimes G) = V(H) \times V(G)$. $E(H \otimes G) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(H)$ and $v_1v_2 \in E(G)$. The tensor product is sometimes referred to as the *conjunction* [13] or the *Kronecker product* [8].

Let us note here that up to isomorphism $H \otimes G$ is commutative, but since we are taking G to be K_m, G will always come as the second factor. We next define a Cayley graph.

Given a finite group Γ, as a set Δ of its generators, the Cayley color graph $C_\Delta(\Gamma)$ has the vertex set Γ, with (g, g') a directed edge labeled with generator δ_i if and only if $g' = g\delta_i$. We assume the identity element of the group is not in Δ. Also if $\delta_i \in \Delta$, then unless δ_i is of order 2 in Γ, $\delta^{-1} \notin \Delta$. When δ_i is of order 2, the two directed edges $(g, g\delta_i)$ and $(g\delta_i, g)$ are represented by a single undirected edge $[g, g\delta_i]$ labeled δ_i. The graph obtained by deleting all labels (colors) and arrows (directions) from the arcs of $C_\Delta(\Gamma)$ is called the Cayley graph and we denote it by $G_\Delta(\Gamma)$. Thus, $V(G_\Delta(\Gamma)) = \Gamma$ and $E(G_\Delta(\Gamma)) = \{(g, g\delta) \mid g \in \Gamma, \delta \in \Delta^*\}$ where $\Delta^* = \Delta \cup \Delta^{-1}$, and $\Delta^{-1} = \{\delta^{-1} \mid \delta \in \Delta\}$.

3. Background results

When we say the order and size of a graph G, in this paper, they are meant to be the cardinality of the vertex set, $V(G)$, and edge set, $E(G)$, respectively, we will denote these as p and q in that order.

Lemma 1 (Weichsel [9]). *The tensor product, $H \otimes G$, of H and G is connected if, and only if both H and G are connected and at least one them has an odd cycle, that is, at least one of them is nonbipartite.*

From this we immediately see that for $m \geqslant 3$, $H \otimes K_m$ is connected if, and only if H is connected. In the case of $m = 2$, $H \otimes K_m$ is connected if, and only if H is connected and has an odd cycle.

Lemma 2. *The tensor product $H \otimes G$ is bipartite if, and only if at least one of H or G is bipartite.*

Proof. Suppose $H \otimes G$ is bipartite and H and G both have odd cycle(s). If we take one odd cycle from H of length say x and another odd cycle from G say of length y, then we has a cycle corresponding to these two cycles in $H \otimes G$. Its length is the lcm$[x, y]$, the least common multiple of x and y. But the lcm$[x, y]$ is odd, contrary to our assumption of $H \otimes G$ being biparite. Thus, at least one of H or G has to be bipartite.

Conversely, let us assume that at least one of H or G is bipartite. Without loss of generality, let us say that H is bipartite with partite sets H_1 and H_2. Because there are no edges between H_1 and H_2 in H, the vertex sets $H_1 \times V(G)$ and $H_2 \times V(G)$ form a partition of $H \otimes G$; hence, $H \otimes G$ is bipartite. $\square$

From this we see that $H \otimes K_m$, $m \geqslant 3$, is bipartite if, and only if H is bipartite. The following result, Lemma 3, can be deduced with simple calculations and observations from the Euler identity $p - q + r = 2 - 2\gamma$; this can be found in either [10] or [6]. Here r stands for the number of regions of the imbedding in the surface of genus γ.

Lemma 3. *If G is a connected bipartite graph with a quadrilateral imbedding, then*

$$\gamma(G) = (q/4) - (p/2) + 1.$$

Following is definition for voltage graph. A *voltage graph* is a triple (K, Γ, ϕ) where K is a pseudograph, Γ is a finite group and $\phi : K^* \to \Gamma$ satisfies $\phi(e^{-1}) = (\phi(e))^{-1}$ for all $e \in K^*$, where $K^* = \{(u, v) \mid uv \in E(K)\}$ and $e = (u, v)$, while $e^{-1} = (v, u)$. The *covering graph* $K \times_\phi \Gamma$ for (K, Γ, ϕ) has vertex set $V(K) \times \Gamma$ and $E(K \times_\phi \Gamma) = \{(u, g)(v, g\phi(e)) \mid e = uv \in E(K) \text{ and } g \in \Gamma\}$. Thus, every vertex v of K is covered by $|\Gamma|$ vertices (v, g), $g \in \Gamma$; and each edge $e = uv$ of K is covered by $|\Gamma|$ edges $(u, g)(v, g\phi(e))$, $g \in \Gamma$. Then, $K \times_\phi \Gamma$ is an $|\Gamma|$-fold covering graph of K, if we regard K as a topological space.

The following results is due to Alpert and Gross. Here we have adopted the version that appears in [10, Ch. 10].

Lemma 4. *If R is a k-gonal region for the imbedding of H^* in S_α (a surface of genus α) and if the net voltage on the boundary of R has order $|R|_\phi$ in the voltage group Γ, then the lift of R in S_γ has $(|\Gamma|/|R|_\phi)$ components, each a $k|R|_\phi$-gon of the imbedding of $H^* \times_\phi \Gamma$.*

4. Surgical-voltage imbedding technique

The *surgical-voltage* imbedding technique works as follows: We start with an appropriate imbedding of H on some surface S and a finite group Γ with generating set Δ. We then proceed to modify H by adding multiple edges to get H^*. This gives us a voltage graph (H^*, Γ, ϕ). The theory of voltage graphs is then used to lift this to $H^* \times_\phi \Gamma$, which is an $|\Gamma|$-fold covering graph of the voltage graph (H^*, Γ, ϕ). This in turn is the tensor product $H \otimes G_\Delta(\Gamma)$. We state this formally in the following lemma which is the direct result of this observation and the definition of the tensor product, its proof can be found in [1].

Lemma 5. *Let G be a Cayley graph $G(\Gamma)$ with generating set Δ. Then for any graph H, the tensor product $H \otimes G$ is the same as the covering graph for the voltage graph H^* imbedded into the same surface as H and obtained from H by replacing every edge $uv \in E(H)$ by $|\Delta^*|$ labeled directed edges, one for each element of Δ^*.*

5. Genus results

Theorem 1. *Let H be a bipartite (p, q)-graph with an orientable quadrilateral imbedding, and $m = 2^n$ for $n \geqslant 2$; then $\gamma(H \otimes K_m) = (m/4)\,[(m-1)q - 2p] + 1$.*

We will illustrate this theorem with the case of $H = C_4$ and K_4; we will take Γ to be Z_2^2. Let us modify C_4 as seen in Fig. 1.

Here, as we can see from the Fig. 1, each of the 2-sided regions has $|R|_\phi = 2$ and the two 4-gons have $|R|_\phi = 1$. The later are said to satisfy the *Kirchoff's Voltage Law.* (KVL). By Lemma 4 all these regions lift to 4-gons. We can also see in Fig. 1 that C_4^* has 12 edges and four vertices. This means $C_4 \otimes K_4$ will have 48 edges and 16 vertices. From Lemmas 2 and 1 we get that $C_4 \otimes K_4$ is bipartite and connected. As a result our imbedding is minimal and this implies that $\gamma(C_4 \otimes K_4) = \frac{48}{4} - \frac{16}{2} + 1 = 5$.

Proof of Theorem 1. Let us take $\Gamma = Z_2^n$ and modify every 4-gon of H as specified by Lemma 5. Due to the bipartite nature of H, our quadrilaterals will look like that in Fig. 1, except this time we have $2^n - 1$ edges, one for each nonzero element of Γ, instead of three. Since every edge label in this modification is of order two, the 2-sided

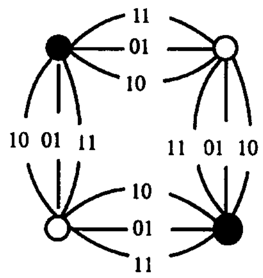


Fig. 1.

regions will lift to 4-gons. As they all satisfy the KVL, the same is true with the 4-gons. This means our modification guarantees a 4-gon imbedding for $H \otimes K_m$. But since H is bipartite Lemma 2 shows that $H \otimes K_m$ is bipartite too; it is also connected by Lemma 1. Finally, we can get the genus of $H \otimes K_m$ using Lemma provided we know the order and size of $H \otimes K_m$. But the order is p^{2^n} and the size is $(2^n - 1)q^{2^n}$ and, hence,

$$\begin{aligned}\gamma(H \otimes K_m) &= \frac{(2^n-1)q^{2^n}}{4} - \frac{p^{2^n}}{2} + 1 \\ &= 2^{n-2}[(2^n-1)q - 2p] + 1 \\ &= (m/4)\,[(m-1)q - 2p] + 1, \quad \text{because } m = 2^n. \quad \square\end{aligned}$$

Theorem 2. *Let $m = 2^n$, for $n \geqslant 2$, and P_3 the path on three vertices. Then*

$$\gamma(P_3 \otimes K_m) = (m/2)[m-4] + 1.$$

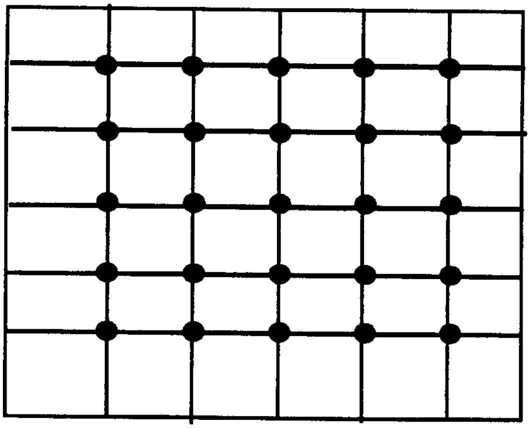
Proof. We let Γ be Z_2^n. The resulting modification of P_3 has $2^n - 1$ edges for each edge of P_3. This gives $2(2^n - 1)\,|\Gamma|$ edges in $P_3 \otimes K_m$. We also observe that $P_3 \otimes K_m$ is both connected and bipartite. Since every 2-sided region satisfies $|R|_\phi = 2$ all such regions lift to 4-gons, and the only 4-gon of P_3^* satisfies the KVL and thus, it also lifts to 4-gons. As a result, we have

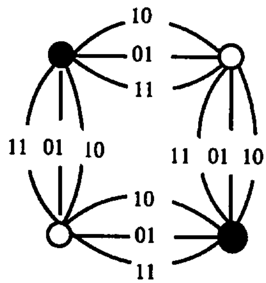
$$\gamma(P_3 \otimes K_m) = \frac{2^{n+1}(2^n-1)}{4} - \frac{3(2^n)}{2} + 1 = 2^{n-1}[2^n - 4] + 1,$$

but because $2^n = m$ we get $\gamma(P_3 \otimes K_m) = (m/2)\,[m-4] + 1$. $\square$

Let us note that in the proof of Theorem 2, the imbedding of P_3^* is planar, but the order of the parallel edges is more or less arbitrary. The only requirement we had was that the four edges of the only 4-gon have only two distinct voltages.

Theorem 3. *Let H be the cartesian product $C_r \times C_s$ of two cycles C_r and C_s; $r, s \geqslant 4$. If $m = 2^n$, $n \geqslant 1$, then $\gamma(H \otimes K_m) = (rsm/2)\,(m-2) + 1$.*





(a) (b)

Fig. 2.

Before proving this theorem, we illustrate the case $H = C_5 \times C_5$ and K_4. Let us take Γ to be Z_2^2. Fig. 2(a) shows a 4-gon imbedding of H on S_1. We can easily show that $C_5 \times C_5$ has girth four. And, consequently, $H \otimes K_4$ has girth four. If we modify every 4-gon of Fig. 2(a) as seen in Fig. 2(b), we see that our modification is consistent and $H^* \times_\phi \Gamma$ will have a 4-gon imbedding. Also from Lemma 1, $H^* \times_\phi \Gamma = H \otimes K_4$ is connected, because H is connected. Because of the fact that the girth of $H \otimes K_4$ is four, the embedding we got is minimal. The product $H \otimes K_4$ has $24rs$ edges and $4rs$ vertices and from Lemma 2 we see that the genus is $4rs + 1$.

Proof of Theorem 3. The proof of the theorem resembles that of our example except now we have $H \otimes K_m$ with $2^{n+1}rs(2^n - 1)$ edges and $2^n rs$ vertices. After noting that $H \otimes K_m$ is a connected graph of girth four, we conclude that

$$\gamma(H \otimes K_m) = \frac{2^{n+1}rs(2^n - 1)}{4} - \frac{2^n rs}{2} + 1$$

After substituting 2^n for m, we get the result $(rsm/2)\,[m - 2] + 1$. $\square$

Let H be a (p, q) graph with an orientable quadrilateral imbedding. The dual, H^d, of H is 4-regular. We define a binary relation $\sim$ on the edges of H^d in the following manner: $e_1 \sim e_2$ if in the imbedding of H edges e_1 and e_2 are 'parallel' sides of a quadrilateral. We can see that $\sim$ is an equivalence relation. We take the smallest equivalence relation induced by $\sim$ and consider its equivalence classes. We say as in [5] the graph has a *straight-ahead imbedding* if all equivalence classes under $\sim$ are cycles in H^d. Now, we can state Theorem 4 which is a generalization of Theorem 3.

Theorem 4. *Let H be a (p, q) graph with an orientable quadrilateral imbedding with a straight-ahead imbedding. If $m = 2^n$, $n \geqslant 1$, then $\gamma(H \otimes K_m) = (m/4)\,[q(m - 1) - 2p] + 1$.*

A graph G imbedded in some surface S_k is said to have a *bichromatic dual* for that particular imbedding if the geometric dual of G imbedded in S_k has a chromatic number equal to two.

Theorem 5. *Let H be a (p, q) graph with an orientable quadrilateral imbedding and a bichromatic dual. If $m = 2^n$, $n \geqslant 1$, then $\gamma(H \otimes K_m) = (m/4)\,[q(m-1) - 2p] + 1$.*

The results reported in this paper dealt only with the cases orientable imbeddings, work is going on to apply the same technique to nonorientable imbeddings of the tensor and other graph products.

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