Character Sheaves I

GEORGE LUSZTIG

Department of Mathematics, M.I.T.,
Cambridge, Massachusetts 02139


This paper is an attempt to construct a geometric theory of characters of a reductive algebraic group $G$ defined over an algebraically closed field. We are seeking a theory which is as close as possible to the theory of irreducible (complex) characters of the corresponding groups $G(F_q)$ over a finite field $F_q$ and yet it should have a meaning over algebraically closed fields.

The basic objects in the theory are certain irreducible ($l$-adic) perverse sheaves (in the sense of [1]) on $G$; they are the analogues of the irreducible ($l$-adic) representations of $G(F_q)$ and are called the character sheaves of $G$. The definition of character sheaves is suggested by the following result [3, Corollary 7.7]: any irreducible representation of $G(F_q)$ appears in at least one of the virtual representations $R^θ(w)$, defined by Deligne and Lusztig in [3, 1.9].

The virtual representations $R^θ(w)$ have a geometric analogue $K^w_{θ}$, (here $w$ is an element in the Weyl group and $L$ is a “tame” local system of rank 1 on the maximal torus of $G$). We shall define here $K^w_{θ}$ only in the case where $L$ is the constant local system $L_0 = Q_I$.

Let $Y_w$ be the variety of all pairs $(g, B)$, where $g$ is and element of $G$ and $B$ is a Borel subgroup of $G$ such that $B, gBg^{-1}$ are in relative position $w$; let $π_w: Y_w → G$ be the morphism defined by $π_w(g, B) = g$. We define $K^w_{θ}$ to be the direct image with compact support $(π_w)_!Q_I$. Then, $K^w_{θ}$ is an object in the derived category of constructible $l$-adic sheaves on $G$. (The definition of $K^w_{θ}$ is given in 2.4.) The character sheaves of $G$ are, by definition, those irreducible perverse sheaves which are constituents of a perverse cohomology sheaf $p^ιH^ι(K^w_{θ})$ for some $ι, w, L$.

We note the similarity of $K^w_{θ}$ and $R^ι(w)$: the virtual representation $R^ι(w)$ is defined as the alternating sum of the $G(F_q)$-modules $H^ι_c(X_w, Q_I)$, where $X_w$ is the variety of all Borel subgroups which are in relative position $w$ with their transform under the Frobenius map. (Thus, $Y_w$ is the analogue of $X_w$.)

Our objective in this paper and the ones following it is to classify the character sheaves of $G$ and to compute their cohomology sheaves.
The paper is organized as follows: Section 1 collects some of the basic results on perverse sheaves due to Beilinson–Bernstein–Deligne–Gabber [1]. Section 2 contains the definition of character sheaves. Apart from the definition in terms of $K_w$, we also give an equivalent definition in terms of some compactification $\pi_w$ of $Y_w \to G$. This compactification (which is analogous to the compactification [3, 9.10] of $X_w$) is essential to apply the deep results of [1, 2]. In Sections 3 and 4 we study the restriction and induction for character sheaves. (These are analogues of the familiar operations on representations of $G(F_q)$.) As a consequence of Theorem 4.4, the character sheaves of $G$ are a special case of the "admissible complexes of $G" defined in [4]; we hope to show elsewhere that these two classes of complexes on $G$ coincide. Section 5 contains some technical preliminaries to Section 6. The most difficult result of this paper is Theorem 6.9(a) which asserts that the restriction functor carries a character sheaf to a direct sum of character sheaves.

1. PERVERSE SHEAVES

1.1. The theory of perverse sheaves on algebraic varieties is due to Beilinson, Bernstein, Deligne, and Gabber. The basic reference is [1].

We shall review here some of the theory.

1.2. Let $k$ be an algebraically closed field. Unless otherwise specified, all algebraic varieties will be over $k$.

We denote by $\mathcal{D}X = \mathcal{D}_b^c(X, \mathcal{O}_l)$ the bounded derived category of $\mathcal{O}_l$ (constructible) sheaves on $X$ [1, 2.2.18]; here $l$ is a fixed prime number such that $l^{-1} \in k$ and $\mathcal{O}_l$ is an algebraic closure of the field of $l$-adic numbers.

Objects of $\mathcal{D}X$ are referred to as "complexes." For a complex $K \in \mathcal{D}X$, we denote by $\mathcal{H}^iK$ the $i$th cohomology sheaf of $K$ (a $\mathcal{O}_l$-sheaf on $X$); we denote by $DK \in \mathcal{D}X$ the Verdier dual of $K$.

1.3. Let $\mathcal{D}X^{<0}$ be the full subcategory of $\mathcal{D}X$ whose objects are those $K$ in $\mathcal{D}X$ such that, for any integer $i$, $\mathcal{H}^iK$ has support of dimension $\leq -i$. (In particular, we have $\mathcal{H}^iK = 0$ for $i > 0$.)

Let $\mathcal{D}X^{>0}$ be the full subcategory of $\mathcal{D}X$ whose objects are those $K$ in $\mathcal{D}X$ such that $DK \in \mathcal{D}X^{<0}$. Let $\mathcal{M}X$ be the full subcategory of $\mathcal{D}X$ whose objects are those $K$ in $\mathcal{D}X$ such that $K \in \mathcal{D}X^{<0} \cap \mathcal{D}X^{>0}$; the objects of $\mathcal{M}X$ are called perverse sheaves on $X$.

$\mathcal{M}X$ is an abelian category [1, 2.14, 1.3.6] in which all objects have finite length [1, 4.3.1].
1.4. The inclusion of $\mathcal{D}X^{<0}$ in $\mathcal{D}X$ has a right adjoint $p_{\tau_{<0}}$ and the inclusion of $\mathcal{D}X^{>0}$ in $\mathcal{D}X$ has a left adjoint $p_{\tau_{>0}}$, [1, 2.2.11, 1.3.3(i)]: we have natural morphisms $p_{\tau_{<0}}K \to K \to p_{\tau_{>0}}K (K \in \mathcal{D}X)$ and

$$\text{Hom}(A, p_{\tau_{<0}}B) = \text{Hom}(A, B) \quad \text{for all} \quad A \in \mathcal{D}X^{<0}, B \in \mathcal{D}X$$

and

$$\text{Hom}(p_{\tau_{>0}}A', B') = \text{Hom}(A', B') \quad \text{for all} \quad A' \in \mathcal{D}X, B' \in \mathcal{D}X^{>0}.$$

The functors $p_{\tau_{>0}}p_{\tau_{<0}}, p_{\tau_{<0}}p_{\tau_{>0}}, (\mathcal{D}X \to \mathcal{D}X)$, are canonically isomorphic [1, 1.3.5]. Hence, for any $K \in \mathcal{D}X$, the complex $p_{\tau_{>0}}p_{\tau_{<0}}K$ is a perverse sheaf; it is denoted $pH^0K$.

The functor $pH^0: \mathcal{D}X \to \mathcal{M}X$ is a cohomological functor [1, 1.3.6], i.e., for any distinguished triangle $(K, K', K'')$ in $\mathcal{D}X$ (notation of [1, 1.1.1]), the corresponding sequence $pH^0K \to pH^0K' \to pH^0K''$ is exact.

We define $pH^i: \mathcal{D}X \to \mathcal{M}X$ by $pH^iK = pH^0(K[i])$, where $[i]$ denotes "décalage," or shift. Then, it follows that for any distinguished triangle $(K, K', K'')$ in $\mathcal{D}X$ we have a long exact sequence of perverse sheaves

$$\ldots \to pH^iK \to pH^iK' \to pH^iK'' \to pH^{i+1}K \to \ldots.$$ 

Moreover, for any $K \in \mathcal{D}X$, we have $pH^iK = 0$ for all but a finite number of integers $i$.

1.5. The irreducible objects of $\mathcal{M}X$ can be described as follows [1, 4.3.1].

Let $V$ be a locally closed, smooth, irreducible subvariety of $X$, of dimension $d$ and let $\mathcal{L}$ be an irreducible $\overline{Q}_r$-local system on $V$. Then $\mathcal{L}[d]$ is an irreducible perverse sheaf on $V$ and there is a unique irreducible perverse sheaf $\mathcal{L}[d]$ on the closure $\overline{V}$, whose restriction to $V$ is $\mathcal{L}[d]$; we have $\mathcal{L}[d] = \text{IC}(\overline{V}, \mathcal{L})[d]$, where $\text{IC}(\overline{V}, \mathcal{L})$ is the intersection cohomology complex of Deligne–Goresky–MacPherson of $\overline{V}$ with coefficients in $\mathcal{L}$. The extension of $\mathcal{L}[d]$ to $X$ (by 0 outside $\overline{V}$) is an irreducible perverse sheaf on $X$, and all irreducible perverse sheaves on $X$ are obtained in this way.

1.6. Let $X$ be a smooth irreducible variety of dimension $d$, and let $D_1, D_2, \ldots, D_r$ be smooth divisors with normal crossings in $X$. Let $\mathcal{L}$ be a one-dimensional, $\overline{Q}_r$-local system on the open subset $X - (D_1 \cup \cdots \cup D_r)$, such that the corresponding representation of the fundamental group factors through a finite quotient of order invertible in $k$. The intersection complex $\text{IC}(X, \mathcal{L})$ can be represented in $\mathcal{D}X$ as a single constructible $\overline{Q}_r$-sheaf $\mathcal{P}$ (in degree 0). Let $I_0$ be the set of $i \in [1, r]$ such that the local monodromy of $\mathcal{L}$ around $D_i$ is nontrivial. Then $\mathcal{P}$ restricted to the open subset $X - \bigcup_{i \in I_0} D_i$ is a local system of rank 1 and $\mathcal{P}$ restricted to the closed subset $\bigcup_{i \in I_0} D_i$ is zero. (These statements can be reduced to the special case where dim $X = 1$.)
1.7. Let $f: X \to Y$ be a morphism between the algebraic varieties $X$, $Y$. Let $f^*: \mathcal{D}Y \to \mathcal{D}X$ be the inverse image functor and let $f_*: \mathcal{D}X \to \mathcal{D}Y$ be the direct image with compact support. They admit adjoint functors $f_*: \mathcal{D}X \to \mathcal{D}Y, f^!: \mathcal{D}Y \to \mathcal{D}X$; for any $A \in \mathcal{D}X, B \in \mathcal{D}Y$, we have:

\begin{align*}
(1.7.1) \quad & \text{Hom}(f^*B, A) = \text{Hom}(B, f_*A). \\
(1.7.2) \quad & \text{Hom}(f_*A, B) = \text{Hom}(A, f^!B). \\
(1.7.3) \quad & \text{If } f \text{ is proper, then } f_* = f^!. \\
(1.7.4) \quad & \text{If } f \text{ is smooth with connected fibres of dimension } d, \text{ then } f^! = f^*[2d](d), \text{ where } (d) \text{ denotes Tate twist; in this case, we set } f^! = f^*[d]. \\
(1.7.5) \quad & \text{Let}
\end{align*}

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow h & & \downarrow h' \\
Z & \xrightarrow{f'} & U
\end{array} 
\]

be a cartesian diagram with $f, f'$ smooth with connected fibres of dimension $d$. Then $h, h' = f'^*h': \mathcal{D}Y \to \mathcal{D}Z$.

1.8. Assume that $f: X \to Y$ is smooth, with connected fibres of dimension $d$.

Here are some properties of $f^!$ (see (1.7.4)):

\begin{align*}
(1.8.1) \quad & \text{If } K \in \mathcal{D}Y, \text{ then we have}
\quad \quad K \in \mathcal{D}Y^{<0} \Leftrightarrow f^!K \in \mathcal{D}X^{<0}, \\
\quad \quad K \in \mathcal{D}Y^{>0} \Leftrightarrow f^!K \in \mathcal{D}X^{>0}, \\
\quad \quad K \in \mathcal{M}Y \Leftrightarrow f^!K \in \mathcal{M}X, \\
\quad \quad \quad pH^i(f^!K) = f^!(pH^iK).
\end{align*}

\begin{align*}
(1.8.2) \quad & \text{If } K \in \mathcal{D}Y^{<0}, \; K' \in \mathcal{D}Y^{>0}, \text{ then } \text{Hom}_{\mathcal{D}Y}(K, K') = \\
\quad \quad \quad \text{Hom}_{\mathcal{D}X}(f^!K, f^!K'). \\
(1.8.3) \quad & \text{If } f: \mathcal{M}Y \to \mathcal{M}X \text{ is fully faithful.
}\end{align*}

\begin{align*}
(1.8.4) \quad & \text{If } K \in \mathcal{M}Y \text{ and } K' \in \mathcal{M}X \text{ is a subquotient of } f^!K \in \mathcal{M}X, \text{ then } \text{K' is isomorphic to } f^!K_1 \text{ for some } K_1 \in \mathcal{M}Y.
\end{align*}

(The proofs are in [1, 4.2.5, 4.2.6].)

1.9. Let $m: H \times Y \to Y$ be an action of a connected algebraic group $H$ on the variety $Y$. Let $\pi: H \times Y \to Y$ be the second projection. Both $m$ and $\pi$ are
smooth morphisms with fibres isomorphic to $H$. Hence, if $K \in \mathcal{M} Y$, then $\tilde{m} K$, $\tilde{p} K$ (see (1.7.4)) are perverse sheaves on $H \times Y$. We say that $K$ is $H$-equivariant if $\tilde{m} K$, $\tilde{p} K$ are isomorphic as perverse sheaves on $H \times Y$. (This is equivalent to the definition in [4, Sect. 0].)

(1.9.1) If $A \in \mathcal{M} Y$ is $H$-equivariant and $B \in \mathcal{M} Y$ is a subquotient of $A$, then $B$ is again $H$-equivariant.

(Apply (1.8.4) to $X = H \times Y$, $f = \pi$, $K = A$, $K' = \tilde{m} B$. We see that there exists $C \in \mathcal{M} Y$ such that $\tilde{m} B = \tilde{p} C$. Restricting this equality to $\{e\} \times Y \subset H \times Y$ we get $B = C$. Hence $\tilde{m} B = \tilde{p} B$.)

(1.9.2) Let $f : X \to Y$ be an $H$-equivariant morphism, with respect to actions of $H$ on $X$ and $Y$. If $K \in \mathcal{M} X$ is $H$-equivariant, then $p H f i^* K$ is $H$-equivariant for all $i$. If $K' \in \mathcal{M} Y$ is $H$-equivariant, then $p H f i^* K'$ is $H$-equivariant for all $i$. (The verification is left to the reader.)

(1.9.3) Assume that $f : X \to Y$ is as in (1.9.2), and that $H$ acts freely on $X$ and trivially on $Y$. Assume furthermore, that for each $y \in Y$, there is an open neighborhood $U \subset Y$, $(U \ni y)$, and an $H$-equivariant isomorphism $f^{-1}(U) \cong H \times U$ ($H$ acts on $H \times U$ by $h : (h', u) \mapsto (hh', u)$) such that $p r_2 \circ i = f : f^{-1} U \to U$. Then the following conditions for $K \in \mathcal{M} X$ are equivalent.

(a) $K$ is $H$-equivariant,

(b) $K$ is isomorphic to $f(K_i)$, for some $K_i \in \mathcal{M} Y$.

The implication (b) $\Rightarrow$ (a) is trivial, (see (1.9.2). Assume now that $K$ is $H$-equivariant. Let $d = \dim H$. According to [1, 4.2.6], (b) is equivalent to the statement that the canonical map $K \to \tilde{f}(p H^{-d} f^* K)$ is an isomorphism. For this, we may assume that $X = H \times Y$, $f = p r_2$, and $H$ acts on $X$ by left translation on the first factor. Let $m, \pi : H \times H \times Y \to H \times Y$ be defined by $m(h, h', y) = (hh', y)$, $\pi(h, h', y) = (h', y)$ and let $i : H \times Y \to H \times Y \times Y$ be defined by $i(h, y) = (h, e, y)$. By our assumption, $m K \approx \pi K$, hence $i m K \approx i^* \pi K$ or equivalently, $K \approx f^* j^* K$, where $j : Y \to H \times Y$ is defined by $j(y) = (e, y)$. Let $K_1 = j^* K[-d] \in \mathcal{D} Y$. Then $K = \tilde{f} K_1$. It remains to show that $K_1 \in \mathcal{M} Y$. This follows from (1.8.1), since we know that $\tilde{f} K_1 \in \mathcal{M} X$.

1.10. Let $X$ be an algebraic variety, let $X'$ be an open subset of $X$ and let $X''$ be the complement of $X'$ in $X$. Let $j' : X' \hookrightarrow X$, $j'' : X'' \hookrightarrow X$ be the natural inclusions. For any $K \in \mathcal{D} X$, there is a canonical distinguished triangle in $\mathcal{D} X$: $(j'_! i^* K, K, j''_! i''^* K)$. Hence, if $f : X \to Y$ is a morphism, then we have a canonical distinguished triangle $(f_! i^* K, f K, i''_! i''^* K)$ in $\mathcal{D} Y$.

1.11. Let $n \geq 1$ be an integer invertible in $k$. Let $\mu_n = \{x \in k^* \mid x^n = 1\}$. 

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Consider the principal fibration $\rho_n : k^\ast \to k^\ast (x \to x^n)$ with group $\mu_n$. The finite group $\mu_n$ acts naturally on the direct image local system $(\rho_n)_\ast \mathcal{O}_T$; we denote by $\mathcal{E}_{n,\psi}$ the summand of $(\rho_n)_\ast \mathcal{O}_T$ on which $\mu_n$ acts according to the character $\psi : \mu_n \to \mathcal{O}_T$. Then $\mathcal{E}_{n,\psi}$ is a $\mathcal{O}_T$-local system of rank 1 on $k^\ast$. The following result is well known:

(1.11.1) If $m \geq 1$ is an integer not divisible by $n$ and if $\psi$ is injective, then $H^i_c(k^\ast, \mathcal{E}_{n,\psi}^\otimes m) = 0$ for all $i$.

1.12. A complex $K \in \mathcal{O}X$ is said to be split if $K$ is isomorphic in $\mathcal{O}X$ to a direct sum $\bigoplus_i \mathcal{O}^iK[-i]$. If $K$ is split, then $K[j]$ is split for any $j$. If $K' \in \mathcal{O}X$ is a direct summand of $K \in \mathcal{O}X$ with $K$ split, then $K'$ is split.

A complex $K \in \mathcal{O}X$ is said to be semisimple if it is split and each $\mathcal{O}^iK$ is a semisimple object of $\mathcal{O}X$. If $K$ is semisimple and $K' \in \mathcal{O}X$ is a direct summand of $K$, then $K'$ is semisimple.

2. Definition of Character Sheaves.

2.1. Let $G$ be a connected reductive algebraic group over $k$. We fix a Borel subgroup $B \subset G$ with unipotent radical $U$ and a maximal torus $T \subset B$.

Let $R \subset \text{Hom}(T, k^\ast)$ be the set of roots and $R^- \subset \text{Hom}(k^\ast, T)$ the set of coroots; the canonical bijection $R \leftrightarrow R^-$ is denoted $\alpha \leftrightarrow \alpha^-$. Let $R^+$ be the set of positive roots determined by $B$ and let $R^- = R - R^+$. Let $W = N_G(T)/T$ be the Weyl group. An element $w \in W$ may be regarded as an automorphism $w : T \to T : w(t) = t^{w\alpha^-}(t \in T)$. Here $\mathcal{C} \in N_G(T)$ is a representative for $w$ in $N(T)$. Let $S$ be the set of simple reflections in $W$ (defined by $R^+$) and let $l : W \to \mathbb{N}$ be the corresponding length function.

2.2. Let $\mathcal{L}(T)$ be the set of isomorphism classes of $\mathcal{O}_T$-local systems of rank 1 on $T$ which are of the form $\lambda^\ast(\mathcal{E}_{n,\psi})$, (see 1.12), for some character $\lambda \in \text{Hom}(T, k^\ast)$, some integer $n \geq 1$ invertible in $k$, and some imbedding $\psi : \mu_n \hookrightarrow \mathcal{O}_T^\ast$; tensor product makes $\mathcal{L}(T)$ an abelian group.

We may (and shall) assume that $\psi$ is the restriction to $\mu_n$ of a fixed injective homomorphism $\psi : \{\text{group of roots of 1 in } k^\ast\} \hookrightarrow \mathcal{O}_T^\ast$, which is independent of $\lambda$ and $n$.

The choice of $\psi$ gives rise to a group isomorphisms $\lambda \otimes (1/n) \to \lambda^\ast(\mathcal{E}_{n,\psi})$:

(2.2.1) $\text{Hom}(T, k^\ast) \otimes (\mathbb{Q}^\prime/\mathbb{Z}) \simeq \mathcal{L}(T)$, where $\mathbb{Q}^\prime = \{m/n \in \mathbb{Q} \mid m \in \mathbb{Z}, n \in \mathbb{Z}, n \geq 1 \text{ invertible in } k\}$.

The Weyl group $W$ operates on $\mathcal{L}(T)$ by $w : \mathcal{L} \to (w^{-1})^\ast\mathcal{L}$, where $(w^{-1})^\ast$ denotes inverse image under $w^{-1} : T \to T$; it also operates on $\text{Hom}(T, k^\ast)$ by $w(\lambda)(t) = \lambda(w^{-1}(t))$, $t \in T$, $\lambda \in \text{Hom}(T, k^\ast)$. These actions are compatible with (2.2.1).
For $\mathcal{L} \in \mathcal{S}(T)$, we set

$$W'_{\mathcal{L}} = \{w \in W \mid (w^{-1})^* \mathcal{L} = \mathcal{L}\}.$$

(2.2.2) The following conditions on $w \in W$ and $\mathcal{L} = \lambda^*(\mathcal{S}_{\alpha,0}) \in \mathcal{S}(T)$ are equivalent:

(a) The local system $\mathcal{L}$ is $T$-equivariant for the action of $T$ on $T$
    given by $t_0 \cdot t = w^{-1}(t_0) t t_0^{-1}$

(b) There exists a character $\lambda \in \text{Hom}(T, k^*)$ such that $w(\lambda) = \lambda w_1$

(c) $w \in W'_{\mathcal{L}}$.

2.3. For $\mathcal{L} \in \mathcal{S}(T)$ we define

$$R_{\mathcal{L}} = \{a \in R \mid r_a \in W'_{\mathcal{L}}\} = \{a \in R \mid \langle a^*, \lambda \rangle \equiv 0 \pmod{n}\}$$

where $r_a$ is the reflection in $W$ corresponding to $a$, and $\langle , , \rangle$ is the natural pairing \text{Hom}(k^*, T) \times \text{Hom}(T, k^*) \to \mathbb{Z}$. We define

$$W_{\mathcal{L}} = \text{subgroup of } W \text{ generated by the } r_a, a \in R_{\mathcal{L}}.$$

Then $R_{\mathcal{L}}$ is a root system with Weyl group $W_{\mathcal{L}}$. The set $R^+_{\mathcal{L}} = R_{\mathcal{L}} \cap R^+$ is a set of positive roots for $R_{\mathcal{L}}$; let $S_{\mathcal{L}}$ be the corresponding set of simple reflections for $W_{\mathcal{L}}$. (The set $S_{\mathcal{L}}$ is not in general contained in the set $S$.)

2.4. Let $\mathcal{B}$ be the variety of all Borel subgroups of $G$. For each $w \in W$, we consider the subvariety $O(w)$ of $\mathcal{B} \times \mathcal{B}$ defined by $O(w) = \{(B', B'') \in \mathcal{B} \times \mathcal{B} \mid \exists g \in G: gB'B^{-1} = B, \ gB''g^{-1} = \hat{w}B\hat{w}^{-1}\}$. We define a morphism

$$\pi_w : Y_w \to G$$

as follows:

$$Y_w = \{(g, B') \in G \times \mathcal{B} \mid (B', gB'B^{-1}) \in O(w)\}, \quad \pi_w (g, B') = g.$$

Let $pr_w : BW \to T$ be the map defined by $pr_w(uu') = t$ $(u, u' \in U, t \in T)$. Let $\hat{Y}_w = \{(g, hU) \in G \times (G/U) \mid h^{-1}gh \in BW\}$. The map $\hat{Y}_w \to T$ given by $(g, hU) \to pr_w(h^{-1}gh)$ is $T$-equivariant with respect to the action $t_0 : (g, hU) \to (g, h t_0^{-1}(U))$ (of $T$ on $\hat{Y}_w$) and $t_0 : t \to (t^{-1} t_0 t)(t_0^{-1})$ (of $T$ on $T$).

Hence, if $\mathcal{L} \in \mathcal{S}(T)$ and $w \in W'_{\mathcal{L}}$, then the inverse image $\hat{\mathcal{L}}$ of $\mathcal{L}$ under $\hat{Y}_w \to T$ is $T$-equivariant. The map $Y_w \to Y_w ((g, hU) \to (g, hBh^{-1}))$ is a principal fibration with group $T$, (the action of $T$ on $\hat{Y}_w$ has been described
above). It follows that there is a unique $\bar{Q}_l$-local system of rank 1, $\mathcal{P}$ on $Y_w$ whose inverse image under $\hat{Y}_w \to Y_w$ is $\mathcal{P}$. It is easy to see that the isomorphism class of $\mathcal{P}$ is independent of the choice of representative $\hat{w}$.

We shall set for $w \in W_{\mathcal{P}}$:

$$K_w^\mathcal{P} = (\pi_w)_! \mathcal{P} \in \mathcal{O} G.$$

2.5. More generally, let $w = (w_1, w_2, \ldots, w_r)$ be a sequence in $W$ and let $w = w_1 w_2 \cdots w_r$.

We define a morphism

$$\pi_w : Y_w \to G$$

as follows:

$$Y_w = \{(g, B_0, B_1, \ldots, B_r) \in G \times B \times B \times \cdots \times B \mid (B_{i-1}, B_i) \in O(w_i) \ (1 \leq i \leq r), B_r = g B_0 g^{-1}\},$$

$$\pi_w(g, B_0, B_1, \ldots, B_r) = g.$$

Let $\hat{Y}_w = (g, h_0 U, h_1 B_1, \ldots, h_r B)$: $h_i^{r-1} h_i \in Bw_i B \ (1 \leq i \leq r), h_r^{r-1} g h_0 \in B$. Define a map $\hat{Y}_w \to T$ by $(g, h_0 U, h_1 U, \ldots, h_r U) \to \hat{w}^{r-1} n_1 n_2 \cdots n_r$, where $n_i \in N_G(T)$ are defined by $h_i^{r-1} h_i \in U n_i U$ and $\tau \in T$ is defined by $h_r^{r-1} g h_0 \in \tau U$. This map is $T$-equivariant with respect to the action $t_0 : (g, h_0 U, h_1 B_1, \ldots, h_r B) \to (g, h_0 t_0^{-1} U, h_1 B_1, \ldots, h_r B)$ (of $T$ on $\hat{Y}_w$) and $t_0 : t \to (\hat{w}^{r-1} t_0 \hat{w}) t_0^{-1}$ (of $T$ on $T$). Hence, if $\mathcal{P} \in \mathcal{F}(T)$ and $w \in W_{\mathcal{P}}$, then the inverse image $\mathcal{P}$ of $\mathcal{P}$ under $\hat{Y}_w \to T$ is $T$-equivariant. The map $Y_w \to Y_w$ given by $(g, h_0 U, h_1 B_1, \ldots, h_r B) \to (g, h_0 B h_0^{-1}, h_1 B h_1^{-1}, \ldots, h_r B h_r^{-1})$ is a principal fibration with group $T$. It follows that there is a unique $Q_l$-local system of rank 1, $\mathcal{P}$ on $Y_w$ whose inverse image under $\hat{Y}_w \to Y_w$ is $\mathcal{P}$. We shall set

$$K_w^\mathcal{P} = (\pi_w)_! \mathcal{P} \in \mathcal{O} G.$$

(This is defined only when $w_1 w_2 \cdots w_r \in W_{\mathcal{P}}$.)

(2.5.1) When $w$ reduces to a single element $w$, the variety $Y_w$ may be identified with the variety $Y_w$ in 2.4: $(g, B_0, B_1) \in Y_w$ corresponds to $(g, B_0) \in Y_w$. This is compatible with the maps $\pi_w, \pi_{w'}$ and with the local systems $\mathcal{P}$ (if $w \in W_{\mathcal{P}}$). Hence $K_w^\mathcal{P} = K_w^{\mathcal{P}}$.

(2.5.2) In general, $Y_w$ is smooth and connected.

An equivalent statement is $\{(g, x_0, x_1, \ldots, x_r) \in G \times G \times \cdots \times G \mid x_{i-1} x_i \in B w_i B \ (1 \leq i \leq r), x_r^{-1} g x_0 \in B\}$ is smooth and connected. By the substitution $b = x_r^{-1} g x_0$, $x_{i-1} x_i = y_i \ (1 \leq i \leq r)$, we are reduced to showing that $\{(b, y_0, y_1, \ldots, y_r) \in B \times G \times \cdots \times G \mid y_i \in B w_i B \ (1 \leq i \leq r)\}$ is smooth and connected, and this is clear.
2.6. For any sequence \( s = (s_1, s_2, ..., s_r) \) in \( S \cup \{e\} \) (\( e = \) neutral element of \( W \)) we define a proper morphism 
\[
\tilde{\pi}_s \colon \tilde{Y}_s \to G
\]
as follows:
\[
\tilde{Y}_s = \{(g, B_0, B_1, ..., B_r) \in G \times \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B} \mid (B_{i-1}, B_i) \in O(s_i) \quad (1 \leq i \leq r), \quad B_r = gB_0 g^{-1}\}
\]
Here, \( \overline{O(s_i)} \) denotes the Zariski closure of \( O(s_i) \) in \( \mathcal{B} \times \mathcal{B} \). It is \( O(s_i) \cup O(e) \) if \( s_i \in S \), and it is \( O(e) \) if \( s_i = e \).

Let \( J_0 = \{j \in [1, r] \mid s_j \in S\} \). For such subset \( J \subset J_0 \), we consider the \( r \) element sequence \( s_j \) in \( S \cup \{e\} \) whose \( i \)th term is \( s_i \) if \( i \notin J \) and \( e \) if \( i \in J \). Then \( Y_{s_j} \) (see 2.6) may be identified with the locally closed subvariety of \( \tilde{Y}_s \) defined by the conditions \( B_{i-1} = B_i \) if \( i \in J \), \( (B_{i-1}, B_i) \in O(s_i) \) if \( i \in [1, r] \setminus J \). The sets \( Y_{s_j} \) (\( J \subset J_0 \)) form a partition of \( \tilde{Y}_s \). We have \( s_{j_0} = s \) and the corresponding piece \( Y_{s_{j_0}} = Y_s \) is open dense in \( \tilde{Y}_s \). For each \( j \in J_0 \), we write \( s_j \) instead of \( s_{j_0} \).

**Lemma 2.7.** \( \tilde{Y}_s \) is smooth, connected. The closures of \( Y_{s_j} \) (for various \( j \in J_0 \)) are smooth divisors on \( \tilde{Y}_s \) with normal crossings.

**Proof.** An equivalent statement is: the variety
\[
\{(g, x_0, x_1, ..., x_r) \in G \times G \times \cdots \times G \mid x_{i-1}^{-1}x_i \in \overline{B_{s_i}B} \quad (1 \leq i \leq r), \quad x_r^{-1}gx_0 \in B\}
\]
is smooth and connected and its subvarieties
\[
\{(g, x_0, x_1, ..., x_r) \in G \times G \times \cdots \times G \mid x_{i-1}^{-1}x_i \in \overline{B_{s_i}B} \quad (1 \leq i \leq r, i \neq j_0), x_{j_0-1}^{-1}x_{j_0} \in B, \quad x_r^{-1}gx_0 \in B\}
\]
\( (j_0 \in J_0) \), are smooth divisors with normal crossings. By the substitution \( b = x_r^{-1}gx_0, \quad x_{i-1}^{-1}x_i = y_i \quad (1 \leq i \leq r) \), we are reduced to the following statement: the variety
\[
\{(b, x_0, y_1, y_2, ..., y_r) \in B \times G \times \cdots \times G \mid y_i \in \overline{B_{s_i}B} \quad (1 \leq i \leq r)\}
\]
is smooth and connected and its subvarieties
\[
\{(b, x_0, y_1, y_2, ..., y_r) \in B \times G \times \cdots \times G \mid y_i \in \overline{B_{s_i}B} \quad (1 \leq i \leq r, i \neq j_0), \quad y_{j_0} \in B\}
\]
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(J_0 \in J_0) are smooth divisors with normal crossings. This is, in turn, a consequence of the following obvious statement: if s_i \in S, then Bs_iB = (Bs_iB) \cup B is smooth and connected and B is a smooth divisor on it.

2.8. Assume now that s is such that s_1s_2 \cdots s_r \in W_T. By 1.6 and 2.7, there is a well-defined (constructible) \Omega_T-sheaf on \mathcal{E} on \overline{Y}, such that \mathcal{E} = IC(\overline{Y}, \mathcal{F}) in D\overline{Y}; (here we regard \mathcal{E} as a \Omega_T-local system on Y, as in 2.5, and we identify Y with an open dense subset of \overline{Y}, as in 2.6).

We shall set

\overline{K}_s = (\pi_*)_! \mathcal{F} \in D\overline{Y}.

(Here \mathcal{F} is regarded as an object in D\overline{Y}, concentrated in degree 0.)

We can now state the following result.

PROPOSITION 2.9. Let \mathcal{L} \in \mathcal{S}(T) and let A be an irreducible perverse sheaf on G. The following conditions on A are equivalent:

(a) A is a constituent of \text{^pH}^i(K^s_\mathcal{F}) for some w \in W_\mathcal{F} and some i \in \mathbb{Z}.

(b) A is a constituent of \text{^pH}^i(K^s_\mathcal{F}) for some sequence w = (w_1, w_2, \ldots, w_r) in W such that w_1w_2 \cdots w_r \in W_\mathcal{F} and for some i \in \mathbb{Z}.

(c) A is a constituent of \text{^pH}^i(K^s_\mathcal{F}) for some sequence s = (s_1, s_2, \ldots, s_r) in S \cup \{e\} such that s_1s_2 \cdots s_r \in W_\mathcal{F} and for some i \in \mathbb{Z}.

(d) A is a constituent of \text{^pH}^i(K^s_\mathcal{F}) for some sequence s = (s_1, s_2, \ldots, s_r) in S \cup \{e\} such that s_1s_2 \cdots s_r \in W_\mathcal{F} and for some i \in \mathbb{Z}.

The proof will be given in 2.11–2.16.

2.10. DEFINITION. For \mathcal{L} \in \mathcal{S}(T), we denote by \hat{G}_\mathcal{F} the set of isomorphism classes of irreducible perverse sheaves A on G which satisfy the equivalent conditions 2.9(a)–(d) with respect to \mathcal{L}.

A character sheaf on G is an irreducible perverse sheaf on G, which is in \hat{G}_\mathcal{F} for some \mathcal{L} \in \mathcal{S}(T). The set of isomorphism classes of character sheaves on G is denoted by \hat{G}.

Note that the character sheaves of the torus T are the perverse sheaves \mathcal{L}[d] (\mathcal{L} \in \mathcal{S}(T)), where d = \dim T.

2.11. We now begin the proof of 2.9. The implication (a) \Rightarrow (b) follows from (2.5.1). The implication (c) \Rightarrow (b) is trivial. We now prove the implication (b) \Rightarrow (c). Let w = (w_1, \ldots, w_r) be a sequence in W, and let, for some i (1 \leq i \leq r), w'_i, w''_i be elements of W such that w_i = w'_iw''_i and l(w_i) = l(w'_i) + l(w''_i). Let \hat{w} = (w_1, \ldots, w_{i-1}, w'_i, w''_i, w_{i+1}, \ldots, w_r). The map
\[(g, B_0, B_1, \ldots, B_{i-1}, B_i, B_{i+1}, \ldots, B_{r+1}) \rightarrow (g, B_0, B_1, \ldots, B_{i-1}, B_i, B_{i+1}, \ldots, B_{r+1})\]
defines an isomorphism \(Y_w \approx Y_w\). It is compatible with the maps \(\pi_w, \pi_w\) and with the local systems \(\mathcal{L}\) defined on \(Y_w, Y_w\) in terms of \(\mathcal{L}\) as in 2.5 (assuming \(w_1 w_2 \cdots w_r \in W_\mathcal{L}\)). Hence

\[K_\mathcal{L} = K_\mathcal{L}.\]

Applying this repeatedly, we see that \(K_\mathcal{L}\) is equal to \(K_\mathcal{L}\) for some sequence \(s\) in \(S \cup \{e\}\). This proves the implication (b) \(\Rightarrow\) (c).

For the proof of the equivalence (c) \(\Leftrightarrow\) (d) we shall need the following result.

**Lemma 2.12.** Let \(s = (s_1, \ldots, s_r)\) be a sequence in \(S \cup \{e\}\) such that \(s_1 s_2 \cdots s_r \in W_\mathcal{L}\). Let \(I_s = \{j \in [1, r] \mid s_i \in S, s_{i-1}s_i \cdots s_r \in W_\mathcal{L}\}\).

(a) \(\mathcal{L}\) is a \(\mathbb{Q}_l\)-local system of rank 1 on the open subset \(\bigcup_{j \in I_s} Y_s\) of \(\overline{Y_s}\) (see 2.6) and is zero on its complement.

(b) If \(J \subset I_s\), the restriction of \(\mathcal{L}\) to \(Y_s\) is isomorphic to the local system \(\mathcal{L'}\) on \(Y_s\) (defined in 2.5 for \(s_s\) instead of \(s\)); note that, for \(J \subset I_s\), the product of the elements in the sequence \(s_j\) belongs to \(W_\mathcal{L}\), hence \(\mathcal{L}\) is defined on \(Y_s\).

**Proof.** We first prove (a). Let \(j \in [1, r]\) be such that \(s_j \in S\). Then \(Y_{s_j}\) is a smooth divisor in the smooth variety \(Y_s \cup Y_s\) (see 2.7). By a computation which takes place essentially in \(SL_2\), we see that the local monodromy of the local system \(\mathcal{L}\) (on \(Y_s\)) along the divisor \(Y_{s_j}\) is the same as the monodromy of the local system \(\mathcal{L}' \otimes \mathcal{L}_{\text{tor}}\) on \(k^*\) at 0, where \(m = (\beta_j, \lambda), \beta_j\) is the root corresponding to the reflection \(s_r s_{r-1} \cdots s_j s_{r-1} s_r\) and \(\mathcal{L}' = \lambda^* \mathcal{L}_{\text{tor}}\), as in 2.2. Hence, this local monodromy is trivial if and only if \((\beta_j, \lambda) \equiv 0 \pmod{n}\), i.e., if \(s_r s_{r-1} s_j \cdots s_{r-1} s_r \in W_\mathcal{L}\). Hence (a) follows from 1.6.

To prove (b), we may assume that \(J\) consists of a single element \(j \in I_s\). Then \(s_j\) has the same entries as \(s\) except for the \(j\)th entry which is \(e\) for \(s_j\) and \(s_j\) for \(s\).

Let \(\overline{G} \rightarrow G\) be a surjective homomorphism of algebraic groups whose kernel is a central torus in \(\overline{G}\) and such that \(\overline{G}\) is a reductive group with simply connected derived group. The varieties \(Y_s, \overline{Y}_s\) for \(\overline{G}\) are locally trivial fibrations over the corresponding varieties for \(G\) with connected smooth fibres (isomorphic to a torus). Hence if (b) is true for \(\overline{G}\), then it is also true for \(G\). Thus, we may assume that \(G\) has simply connected derived group.

Let \(\overline{w}_1, \overline{s}_j, \overline{w}_2\) be representatives in \(N_e(T)\) for \(s_1 s_2 \cdots s_{j-1}, s_j, s_{j+1} \cdots s_{r-1}s_r\) respectively, and let \(\overline{w} = \overline{w}_1 \overline{s}_j \overline{w}_2\). We shall assume (as we
may) that \( s_j \) is a product of three unipotent elements in \((B_{s_j}B) \cup B\).

Consider the smooth variety

\[
Z = \{(b, x_0, y_1, \ldots, y_r) \in B \times G \times \cdots \times G \mid y_i \in B_{s_j}B \}
\]

\[(1 \leq i \leq r, i \neq j), y_j \in (B_{s_j}B) \cup B \}
\]

and the smooth divisor \( D \subset Z \) defined by the equation \( y_j \in B \). Let \( f: Z - D \to k^*, f': D \to k^* \) be the maps given by

\[
f(b, x_0, y_1, \ldots, y_r) = \lambda(\hat{w}_2^{-1}s_j^{-1}\hat{w}_1^{-1}n_1n_2 \cdots n_r\tau),
\]

\[
f'(b, x_0, y_1, \ldots, y_r) = \lambda(\hat{w}_2^{-1}\hat{w}_1^{-1}n_1n_2 \cdots n_r\tau),
\]

where \( n_i \in N_G(T) \) are defined by \( y_i \in Un_iU \) and \( \tau \in T \) is defined by \( b \in \tau U \).

As in the proof of 2.7, we are reduced to proving the following statement

\[(2.12.1) \text{ If } \langle \beta_j, \lambda \rangle \equiv 0 \pmod{n}, \text{ then there exists a local system on } Z \text{ whose restriction to } Z - D \text{ is } f^*\mathcal{F}_{n,\phi}, \text{ and whose restriction to } D \text{ is } f'^*\mathcal{F}_{n,\phi}.\]

We can write \( \langle \beta_j, \lambda \rangle = nn_1 \), where \( n_1 \) is an integer. Since \( G \) is simply connected derived group, there exists \( \lambda' \in \text{Hom}(T, k^*) \) such that \( \langle \beta_j, \lambda' \rangle = n_1 \). Then \( \langle \beta_j, \lambda'^{-n}\lambda \rangle = 0 \). Replacing \( \lambda \) by \( (\lambda')^{-n}\lambda \) does not change \( f^*\mathcal{F}_{n,\phi}, f'^*\mathcal{F}_{n,\phi} \). Hence, we may assume in (2.12.1) that \( \langle \beta_j, \lambda \rangle = 0 \).

In this case, there is a unique homomorphism of algebraic groups \( \gamma: (B_{s_j}B) \cup B \to k^* \) such that

\[
\gamma(t) = \lambda(\hat{w}_2^{-1}t\hat{w}_2) \quad \text{for all } t \in T.
\]

Since \( s_j \) is a product of unipotent elements in \((B_{s_j}B) \cup B\), we must have \( \gamma(s_j) = 1 \). We define a morphism \( \bar{f}: Z \to k^* \) by

\[
\bar{f}(b, x_0, y_1, \ldots, y_r) = \gamma(s_j^{-1}\hat{w}_1^{-1}(n_1n_2 \cdots n_{j-1}) y_j(n_{j+1} \cdots n_r)\hat{w}_2^{-1}),
\]

where \( n_i \in N_G(T) \) are defined by \( y_i \in Un_iU \) (\( i \neq j \)), and \( \tau \in T \) is defined by \( b \in \tau U \).

We show that \( f = \bar{f} \mid Z - D, f' = \bar{f} \mid D \). If \( y_j \in B_{s_j}B \), we write \( y_j \in Un_jU, n \in N(T) \) and we have

\[
\bar{f}(b, x_0, y_1, \ldots, y_r) = \gamma(s_j^{-1}\hat{w}_1^{-1}(n_1n_2 \cdots n_{j-1}) n_j(n_{j+1} \cdots n_r)\hat{w}_2^{-1})
\]

\[
= \lambda(\hat{w}_2^{-1}s_j^{-1}\hat{w}_1^{-1}n_1n_2 \cdots n_r)
\]

\[
= f(b, x_0, y_1, \ldots, y_r).
\]

If \( y_j \in B \), we write \( y_j \in n_jU, n_j \in T \), and we have

\[
\bar{f}(b, x_0, y_1, \ldots, y_r) = \gamma(\hat{w}_1^{-1}(n_1n_2 \cdots n_{j-1}) n_j(n_{j+1} \cdots n_r)\hat{w}_2^{-1})
\]

\[
= \lambda(\hat{w}_2^{-1}\hat{w}_1^{-1}n_1n_2 \cdots n_r\tau)
\]

\[
= f'(b, x_0, y_1, \ldots, y_r).
\]
It follows that the local system \( f^*\mathcal{G}_{\mathcal{R},\omega} \) on \( Z \) has the property required in (2.12.1). The lemma is proved.

2.13. Let \( s, \mathcal{C} \) be as in 2.9(d). Consider the sequence of closed subsets \( Z^{(i)} \subset Y_s \), defined by

\[
Z^{(i)} = \bigcup_{J \in J_0, |J| > i} Y_s \quad (i \in \mathbb{Z})
\]

(see 2.6). We have \( Z^{(i+1)} \subset Z^{(i)} \). If \( \phi^{(i)}: Z^{(i)} \subset Y_s, \psi^{(i)}: Z^{(i)} - Z^{(i+1)} \subset Y_s \) are the inclusion maps, we have (by 1.10) a natural distinguished triangle in \( G \):

\[
((\bar{\pi}_s), \psi_1^{(i)}(\psi^{(i)})*\mathcal{Z}_s, (\bar{\pi}_s), \phi_1^{(i)}(\phi^{(i)})*\mathcal{Z}_s, (\bar{\pi}_s), \phi_1^{(i+1)}(\phi^{(i+1)})*\mathcal{Z}_s).
\]

It gives rise to a long exact sequence in \( \mathcal{H}_G \) (for each \( i \)):

\[
\begin{align*}
\cdots & \rightarrow pH^{i-1}((\bar{\pi}_s), \phi_1^{(i+1)}(\phi^{(i+1)})*\mathcal{Z}_s) \rightarrow \bigoplus_{J \in J_0, |J| = i} pH^i(K_{\mathcal{C}_s}) \\
& \rightarrow pH^i((\bar{\pi}_s), \phi_1^{(i)}(\phi^{(i)})*\mathcal{Z}_s) \rightarrow pH^i((\bar{\pi}_s), \phi_1^{(i+1)}(\phi^{(i+1)})*\mathcal{Z}_s) \\
& \rightarrow \bigoplus_{J \in J_0, |J| = i} pH^{i+1}(K_{\mathcal{C}_s}) \rightarrow \cdots.
\end{align*}
\]

Here we have used the isomorphism

\[
((\bar{\pi}_s), \psi_1^{(i)}(\psi^{(i)})*\mathcal{Z}_s) = \bigoplus_{J \in J_0, |J| = i} K_{\mathcal{C}_s}
\]

which follows from Lemma 2.12. Note that

\[
(2.13.2) \quad pH^i((\bar{\pi}_s), \phi_1^{(i)}(\phi^{(i)})*\mathcal{Z}_s) = \begin{cases} \quad pH^i(K_s) & \text{for } i \leq 0, \\ 0 & \text{for } i > |I_s|. \end{cases}
\]

2.14. We now prove the equivalence of (c) and (d) in 2.9. Let \( A \) be as in 2.9. For a sequence \( s = (s_1, \ldots, s_r) \) in \( S \cup \{e\} \), we denote by \( m(s) \) the number of \( i \in [1, r] \) such that \( s_i \in S \). If \( m(s) = 0 \), then \( Y_s = \bar{Y}_s \) and \( K_{\mathcal{C}_s} = K_{\mathcal{Z}_s} \) (if defined) hence \( A \) is a constituent of \( pH^i(K_{\mathcal{C}_s}) \), if and only if it is a constituent of \( pH^i(K_{\mathcal{Z}_s}) \). It is enough to prove the following statement.

\[
(2.14.1) \quad \text{Assume that } s \text{ satisfies } m(s) = m \geq 1, s_1, s_2, \ldots, s_r \in W_{\mathcal{C}_s}, \text{ and that for any sequence } s' \text{ in } S \cup \{e\}, \text{ with product in } W_{\mathcal{C}_s}, \text{ and with } m(s') < m, \text{ and any integer } j, A \text{ is not a constituent of } pH^j(K_{\mathcal{C}_s}). \text{ Then, for any } j, A \text{ is a constituent of } pH^j(K_{\mathcal{C}_s}) \text{ if and only if it is a constituent of } pH^j(K_{\mathcal{Z}_s}).
\]
Using our hypothesis and (2.13.1) for \( i > 0 \), we see that for any \( i > 0 \), and any \( j \), we have:

\[
A \text{ is a constituent of } p^h((\pi_i), \phi^{(i)}(\phi^{(i)})) \text{ if and only if } A \text{ is a constituent of } p^h((\pi_i^0), \phi^{(0)}(\phi^{(0)}))
\]

Applying this repeatedly for \( i = 1, 2, 3, \ldots \) and using (2.13.2), we see that for any \( j \), \( A \) is not a constituent of \( p^h((\pi_i), \phi^{(i)}(\phi^{(i)})) \) which, by (2.13.2) is the same as \( p^h((\pi_i), \phi^{(i)}(\phi^{(i)})) \). Thus, (2.14.1) and hence the equivalence (c) \( \iff \) (d) in 2.9, are verified.

2.15. Let \( s = (s_1, s_2, \ldots, s_r) \) be a sequence in \( S \cup \{e\} \) such that \( s_1s_2 \cdots s_r \in W'_{\alpha} \). Assume that, for some \( h \) (\( 2 < h < r \)), we have \( shhl = sk \in S \). We have a partition \( Y_s = Y'_s \cup Y''_s \) where \( Y'_s \) (resp. \( Y''_s \)) is the subvariety of \( Y_s \) defined by \( (B_{h-2}, B_h) \in \Omega(s_h) \) (resp. by \( B_{h-2} = B_h \)). Then \( Y'_s \) is open in \( Y_s \) and \( Y''_s \) is closed in \( Y_s \), so that, if we denote \( \pi'_s \) (resp. \( \pi''_s \)) the restriction of \( \pi_s \) to \( Y'_s \) (resp. \( Y''_s \)) we have a natural distinguished triangle

\[
((\pi'_s), \mathcal{D}, K_s^\alpha, (\pi''_s), \mathcal{D})
\]

Here, we denote the restriction of \( \mathcal{D} \) from \( Y_s \) to \( Y'_s \) or \( Y''_s \) again by \( \mathcal{D} \). It follows that we have a natural long exact sequence in \( \mathcal{M} G \):

\[
\cdots \to p^h((\pi'_s), \mathcal{D}) \to p^h((\pi''_s), \mathcal{D}) \to p^h((\pi''_s), \mathcal{D})
\]

(2.15.1) Let \( s' \) be the sequence \( (s_1, s_2, \ldots, s_{h-1}, s_{h+1}, \ldots, s_r) \) and let \( s'' \) be the sequence \( (s_1, s_2, \ldots, s_{h-1}, s_{h+1}, \ldots, s_r) \). Then \( (g, B_0, B_1, \ldots, B_r) \to (g, B_0, B_1, \ldots, B_{h-1}, B_h, B_{h+1}, \ldots, B_r) \) makes \( Y'_s \) into a locally trivial \( k^* \)-bundle over \( Y_s \), and \( (g, B_0, B_1, \ldots, B_r) \to (g, B_0, B_1, \ldots, B_{h-2}, B_{h+1}, \ldots, B_r) \) makes \( Y''_s \) into a locally trivial affine line bundle over \( Y_s \).

The local system \( \mathcal{D} \) on \( Y''_s \) is just the inverse image of the local system \( \mathcal{D} \) on \( Y_s \) (obtained by the construction in 2.5 applied to \( s'' \), whose product is in \( W'_{\alpha} \)). The local system \( \mathcal{D} \) on \( Y'_s \) is the inverse image of the local system \( \mathcal{D} \) on \( Y''_s \), if \( h \in I_s \) (by the argument in the proof of (a) in 2.12); if \( h \in I_s \), the direct image with compact support of \( \mathcal{D} \) under \( Y_s \to Y''_s \) is zero, (using 1.11.1). It follows that

\[
(\pi''_s), \mathcal{D} = K_s^\alpha[-2](-1),
\]

and, if \( h \in I_s \), we have \( (\pi'_s), \mathcal{D} = 0 \). If \( h \in I_s \), we have a natural distinguished triangle (1.10) in \( \mathcal{D} G \):

\[
((\pi'_s), \mathcal{D}, K_s^\alpha[-2](-1), K_s^\alpha).
\]
Hence, we have long exact sequences in $\mathcal{M}G$:

\begin{equation}
\cdots \rightarrow pH^i((\pi'_s)^\sharp) \rightarrow pH^i(K)' \rightarrow pH^{i-1}(K)'(-1) \rightarrow \cdots
\end{equation}

\begin{equation}
\cdots \rightarrow pH^{i+1}((\pi'_w)^\sharp) \rightarrow \cdots
\end{equation}

if $h \in I_s$, and isomorphisms

\begin{equation}
pH^i(K)' \simeq pH^{i-2}(K)'(-1), \quad \text{if } h \notin I_s.
\end{equation}

2.16. We now prove the implication (c) $\Rightarrow$ (a) in 2.9.

Assume that $A$ is a constituent of $pH^i(K)'$ for some sequence $s = (s_1, s_2, ..., s_r) \subseteq S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W_p$ and some $i$. We may assume that $r$ is minimum possible (for $A$), which implies that all $s_j$ are in $S$. We want to prove that $A$ is a constituent of $pH^i(K)'$ for some $w \in W_p$ and some $j$.

If $l(s_1, s_2, ..., s_r) = r$, then $K_s' = K_w'$, where $w = s_1 s_2 \cdots s_r$ (see 2.11, (2.5.1)) and the desired conclusion follows. Hence we may assume that $l(s_1, s_2, ..., s_r) < r$. We shall show that this contradicts the minimality of $r$. We can find $h$ ($2 < h < r$) such that $s_h \cdots s_{r-1} s_r$ is a reduced expression and $s_{h-1} s_h \cdots s_r$ is not a reduced expression. We can find $s_h', s_{h-1}', s_r'$ in $S$ such that $s_h' \cdots s_{r-1}' s_r' = s_h \cdots s_{r-1} s_r = y$ and $s_h' = s_{h-1}'$. Let $s = (s_1, s_2, ..., s_{h-1}, s_h', s_{h+2}, ..., s_r')$ and $\tau = (s_1, s_2, ..., s_{h-1}, y)$. From 2.11, we see that $K_s' = K_s''$, $K_s'' = K_s'''$. Hence $K_s' = K_s'''$. Hence we may assume that $s_{h-1} = s_h$, so that the discussion in 2.15 is applicable.

If $h \notin I_s$, then (2.15.4) shows that $A$ is a constituent of $pH^{i-2}(K)'$; the sequence $s''$ in $S$ has length $r - 2$. This contradicts the minimality of $r$.

Assume now that $h \in I_s$. By the minimality of $r$, $A$ is not a constituent of $pH^i(K)'$ (see 2.15) for any $i$. From (2.15.3) it then follows that $A$ is not a constituent of $pH^i((\pi'_w)^\sharp)$ for any $j$. This, together with (2.15.2) shows that $A$ is a constituent of $pH^{i-2}(K)'$. This again contradicts the minimality of $r$.

Thus, the implication (c) $\Rightarrow$ (a) in 2.9 is proved. This completes the proof of 2.9.

Proposition 2.17. Let $s = (s_1, s_2, ..., s_r)$ be a sequence in $S \cup \{e\}$ such that $s_1 s_2 \cdots s_r \in W_p$, $\mathcal{L} \in \mathcal{S}(T)$. Let $m$ be the number of indices $i \in [1, r]$ such that $s_i \in S$, and let $m' = m + \dim G$.

(a) $\overline{K}' \in \mathcal{D}G$ is semisimple (see 1.12).

(b) $pH^i(\overline{K}'')$ is isomorphic to $pH^{2m'-j}(\overline{K}'')$ (in $\mathcal{M}G$) for any $j$. 

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Proof. (a) is a special case of the "decomposition theorem" [1, 6.2.5], and (b) is a special case of the "relative hard Lefschetz theorem" [1, 6.2.10] applied to the projective morphism $\tilde{\pi}_s : \tilde{Y}_s \to G$ and to the perverse sheaf $\mathcal{L}'(m)$ on $\tilde{Y}_s$.

PROPOSITION 2.18. (a) If $K \in \dot{G}$, then $K$ is $G$-equivariant for the conjugation action of $G$ on $G$.

(b) More precisely, if $K \in \dot{G}$, $\mathcal{L} = \lambda^*\mathcal{E}_{a,y}$ (see 2.2), and $\mathcal{L}_G^0$ is the connected centre of $G$, then $K$ is $G \times \mathcal{L}_G^0$-equivariant for the action $(g_0, z) : g \to z^ng_0g_0^{-1}$ of $G \times \mathcal{L}_G^0$ on $G$.

Proof. Define an action of $G \times \mathcal{L}_G^0$:

(i) On $T$ by $(g_0, z) : t \mapsto z^nt$.

(ii) On $\dot{Y}$ by $(g_0, z) : (g, hU) \mapsto (z^ng_0g_0^{-1}, g_0hU)$ (see 2.4).

(iii) On $Y_w$ by $(g_0, z) : (g, B') \mapsto (z^ng_0g_0^{-1}, g_0B')$.

If $\mathcal{L}'$ is as in (b), then $\mathcal{L}'$ is $T$-equivariant for the action of $T$ on itself given by $t_0 : t \mapsto t_0t$. Hence, it is $\mathcal{L}_G^0$-equivariant since $\mathcal{L}_G^0$ is a subgroup of $T$ and $G$ acts trivially on $T$. With the notation in 2.4, $\dot{Y}_w \to T$ is $G \times \mathcal{L}_G^0$-equivariant hence the local system $\mathcal{L}'$ on $\dot{Y}_w$ is $G \times \mathcal{L}_G^0$-equivariant. Since $\dot{Y}_w \to Y_w$ is $G \times \mathcal{L}_G^0$-equivariant, the local system $\mathcal{L}'$ on $Y_w$ is $G \times \mathcal{L}_G^0$-equivariant. Since $\mathcal{L}'$ is defined in terms of $g'$ in the same way as $\mathcal{L}'$ is defined in terms of $g$, it follows that $\mathcal{L}_G^0(\mathcal{L}'(K_s'))$ is $G \times \mathcal{L}_G^0$-equivariant for all $i$ hence, by (1.9.1) any subquotient of $\mathcal{L}_G^0(K_s')$ (in $\mathcal{M}G$) is $G \times \mathcal{L}_G^0$-equivariant). Thus (b) is proved; (a) is a special case of (b).

2.19. Consider a sequence $s = (s_1, s_2, \ldots, s_r)$ in $S \cup \{e\}$ such that $s_1s_2 \cdots s_r \in W'_{p'}$. Let $s' = (s_2, s_3, \ldots, s_r, s_1)$; we have $s_2s_3 \cdots s_r s_1 \in W'_{p'}$, where $\mathcal{L}' = s_1^*\mathcal{L}$. We have a natural isomorphism $Y_s \to Y_s'$ (over $G$) defined by $(g, B_0, B_1, \ldots, B_r) \to (g, B_1, B_2, \ldots, B_r, gB_0g^{-1})$. One can verify that this isomorphism carries $Y_s \to Y_s'$ on $Y_s$. $\mathcal{L}'$ is defined in terms of $\mathcal{L}'$ in the same way as $\mathcal{L}'$ is defined in terms of $\mathcal{L}$). It follows that $K_s = K_s'$. Applying this property $r$ times, we obtain the following result.

(2.19.1) Let $s = (s_1, s_2, \ldots, s_r)$, $s' = (s'_1, s'_2, \ldots, s'_r)$ be two sequences in $S \cup \{e\}$ such that $s_1s_2 \cdots s_r, s'_1s'_2 \cdots s'_r \in W'_{p'}$. Let $ss'$ be the sequence $(s_1, s_2, \ldots, s_r, s'_1, s'_2, \ldots, s'_r)$ and let $s's$ be the sequence $(s'_1, s'_2, \ldots, s'_r, s_1, s_2, \ldots, s_r)$. Then $K_{ss'} = K_{s's}$.

3. Restriction

3.1. We now fix a parabolic $P$ of $G$ such that $P \supset B$ and we denote by $U_P$ its unipotent radical and by $L$ the Levi subgroup of $P$ containing $T$. We
denote by $\pi_p$ the canonical homomorphism of $P$ onto $L$. Let $B^* = B \cap L$; it is a Borel subgroup of $L$. We shall denote by $R^*, W^*, S^*, W^*_r, S^*_r, W^*_\varphi, S^*_\varphi, O^*(w)$ the objects defined by replacing $G$ by $L$ in the definition of $R, W, S, W_\varphi, S_\varphi, W_\varphi, O(w)$. (We regard $T$ also as a maximal torus of $L$.)

Let

$$W_\times = \{ y \in W | y \text{ has minimal length among elements in } W^*y \}.$$

The correspondence $y \to W^*y$ is a 1–1 correspondence $W_\times \to W^* \setminus W$. The set $W^* \setminus W$ is also in 1–1 correspondence with the set of $P$-orbits on $\mathscr{B}$: to the coset $W^*y$ ($y \in W$), corresponds the $P$-orbit of $yB^y^{-1}$; we denote this $P$-orbit by $v(y)$.

3.2. If $v$ is a $P$-orbit on $\mathscr{B}$ and $w \in W$, we define a new $P$-orbit $vw$ by: $vw = v(yw)$, where $v = v(y)$.

We may assume here that $y \in W_\times$. If $s \in S$, there are three possibilities for $vs$:

(a) $ys \in W_\times$ and $ys > y$; then $v \subset vs - vs$,

(b) $ys \in W_\times$ and $ys < y$; then $vs \subset \overline{v} - v$,

(c) $ys \notin W_\times$; then $ysy^{-1} \in S^* \text{ and } vs = v$.

3.3. Let $s = (s_1, s_2, \ldots, s_r)$ be a sequence in $S$ such that $s_1 s_2 \cdots s_r \in W^*_\varphi$ ($\varphi \in \mathscr{V}(T)$). Let $\tilde{Y}'$ be the closed subvariety of $\tilde{Y}_s$ defined by

$$\tilde{Y}' = \{ (g, B_0, B_1, \ldots, B_r) \in \tilde{Y}_s | g \in P \},$$

let $\tilde{\mathscr{P}}'$ be the restriction of $\tilde{\mathscr{P}}$ (see 2.8) from $\tilde{Y}_s$ to $\tilde{Y}'$, and let $\tilde{\pi}' : \tilde{Y}' \to L$ be the map defined by $\tilde{\pi}'(g, B_0, B_1, \ldots, B_r) = \pi_p(g)$.

Any sequence $v = (v_0, v_1, \ldots, v_r)$ of $P$-orbits on $\mathscr{B}$ defines a locally closed subvariety $\tilde{Y}_v$ of $\tilde{Y}_s$:

$$\tilde{Y}_v = \{ (g, B_0, B_1, \ldots, B_r) \in \tilde{Y}_s | g \in P, B_i \in v_i (0 \leq i \leq r) \}.$$

It is clear that $\tilde{Y}_v$ is empty unless $v$ satisfies

$$(3.3.1) \quad v_i = v_{i-1} \text{ or } v_{i-1}s_i \text{ for all } i, 1 \leq i \leq r, \text{ and } v_0 = v_r.$$

Let $\tilde{\mathscr{P}}_v'$ be the restriction of $\tilde{\mathscr{P}}'$ to $\tilde{Y}_v$, and let $\tilde{\pi}_v'$ be the restriction of $\tilde{\pi}'$ to $\tilde{Y}_v$. We associate with $v$ (satisfying (3.3.1)) the sequence $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_r)$ in $S \cup \{ e \}$ defined by

$$(3.3.2) \quad \tilde{s}_i = \begin{cases} s_i & \text{if } v_i = v_{i-1}s_i, \\ e & \text{if } v_i \neq v_{i-1}s_i. \end{cases}$$
We then have \( v_i = v_{i-1} \Delta_i \) (1 \( \leq i \leq r \)), hence \( v_0 \Delta_1 \Delta_2 \cdots \Delta_r = v_0 \).

(3.3.3) Let \( y_i \in W_\ast \) be defined by \( v_i = v(y_i) \) (0 \( \leq i \leq r \)). We define

\[
(3.3.4) \quad t_i = y_{i-1} \Delta_i y_i^{-1} = \begin{cases} y_{i-1} \Delta_i y_{i-1}^{-1}, & \text{if } v_{i-1} \Delta_i = v_{i-1}, \\ e, & \text{if } v_{i-1} \Delta_i \neq v_{i-1} \end{cases}
\]

(1 \( \leq i \leq r \)). Then \( t = (t_1, t_2, \ldots, t_r) \) is a sequence in \( S^* \cup \{e\} \).

3.4. The formula \( (g, B_0, B_1, \ldots, B_r) \rightarrow (\pi_p(g), \pi_p(B_0 \cap P), \pi_p(B_1 \cap P), \ldots, \pi_p(B_r \cap P)) \) defines a morphism \( \rho: V_\ast \rightarrow \tilde{Y}_t \ast \), where

\[
\tilde{Y}_t \ast = \{(l, B_0, B_1, \ldots, B_r) \in L \times \mathcal{B}^* \times \cdots \times \mathcal{B}^* | (B_{i-1}^*, B_i^*) \in O^*(t_i), B_r^* = lB_0^* l^{-1} \}.
\]

This morphism is a locally trivial fibration. Its fibre over any point \( (l, B_0^*, B_1^*, \ldots, B_r^*) \in \tilde{Y}_t \ast \) is isomorphic to the affine space of dimension

\[
(3.4.1) \quad d(v) = \dim U_p + \# \{i \in [1, r] | v_i s_i \subset \Delta_i - v_i\}.
\]

Indeed, the set of all \( B_0 \in v_0 \) such that \( \pi_p(B_0 \cap P) = B_0^* \) is an affine space of dimension \( l(y_0) \). If \( (B_0, B_1, \ldots, B_r) \) are already determined, the set of all \( B_i \in v_i \) such that \( (B_i, B_{i+1}) \in O^*(s_i) \) and \( \pi_p(B_i \cap P) = B_i^* \) is an affine line if \( v_i s_i \subset \Delta_i - v_i \) and is a point, otherwise. Finally, if \( B_0, B_1, \ldots, B_r \) are already determined, the set of all \( g \in \pi_p^{-1}(l) \) such that \( B_r = gB_0 g^{-1} \) is an affine space of dimension \( \dim U_p - l(y_0) \), (since \( y_0 = y_n \)). Hence our fibre is an affine space of dimension \( l(y_0) + \# \{i \in [1, r] | v_i s_i \subset \Delta_i - v_i\} + (\dim U_p - l(y_0)) = d(v) \). We now state

**Lemma 3.5.** Let notations be as in 3.3; we assume that \( v \) satisfies

(3.3.1). Let \( I_\ast \subset [1, r] \) be as in 2.12 and let \( J = J_\ast = \{i \in [1, r] | \Delta_i = e\} \):

(a) If \( J \not\subset I_\ast \), then \( \pi_\ast(\bar{\mathcal{F}}_t) = 0 \).

(b) If \( J \subset I_\ast \), then \( \pi_\ast(\bar{\mathcal{F}}_t) = \tilde{K}_t^\mathcal{G} [-2d(v)](-d(v)) \) where \( \mathcal{L}_1 = (y_0^{-1})^* \mathcal{L} \) and \( \tilde{K}_t^\mathcal{G} \in D L \) is defined as in 2.8, with respect to \( L \).

**Proof.** Let \( \pi_\ast \) be the canonical projection \( \bar{Y}_t \ast \rightarrow L \). We have \( (\pi_\ast)_\ast = (\bar{\pi}_\ast)_! \rho_\ast \) (\( \rho \) is as in 3.4). Hence it is enough to prove:

(a') If \( J \not\subset I_\ast \), then \( \bar{\mathcal{F}}_t = 0 \).

(b') If \( J \subset I_\ast \), then \( \rho_\ast(\bar{\mathcal{L}}_t) = \mathcal{L}_1 [-2d(v)](-d(v)) \), where \( \bar{\mathcal{D}}_t \) is the constructible sheaf on \( \bar{Y}_t \ast \) defined in the same way as \( \bar{\mathcal{D}} \) in 2.8, but replacing \( s, L, G, B \) by \( t, \mathcal{L}_t, L \).

(For \( \bar{\mathcal{L}}_t \) to be defined, we must know that \( t_1 t_2 \cdots t_r \in W_r^\ast \) or that \( y_0^{-1} t_1 t_2 \cdots t_r y_0 \in W_r^\ast \). We have \( y_0^{-1} t_1 t_2 \cdots t_r y_0 = (y_0^{-1} t_1 y_1)(y_1^{-1} t_2 y_2) \cdots (y_r^{-1} t_r y_r) = \Delta_1 \Delta_2 \cdots \Delta_r \) (since \( y_0 = y_r \)), and this is in \( W_r^\ast \), since \( J \subset I_\ast \).)
Since \( \rho \) is a locally trivial fibration with fibres \( \cong k^{d(v)} \) (see 3.4) we see that (b') is a consequence of

\[
(b'') \quad \mathcal{L}'_\mathbf{v} = \rho^*(\mathcal{L}_1), 
\]
if \( J \subset I_\mathbf{s} \).

First, assume that \( J \not\subset I_\mathbf{s} \) and let \( j \) be an index in \( J - I_\mathbf{s} \). If \( (g, B_0, B_1, \ldots, B_r) \in Y_{t_\mathbf{v}} \), we have \( B_{i-1} = B_j \) (since \( j \in J \)) and \( (B_{i-1}, B_i) \in O(J) \) for all \( i \in [1, r] - \{j\} \). Since \( j \notin I_\mathbf{s} \), from 2.12(a) it follows that the stalk of \( \mathcal{L}' \) (or \( \mathcal{L}' \)) at \( (g, B_0, B_1, \ldots, B_r) \) is 0, hence \( \mathcal{L}'_\mathbf{v} = 0 \), proving (a').

Next, we assume that \( J \subset I_\mathbf{s} \). Let \( H_0 = \{i \in [1, r] \mid t_i \neq e\} \). Then \( J \cap H_0 = \emptyset \). For any subset \( H \subset H_0 \) we have the locally closed subvariety of \( \mathcal{Y}_{t_\mathbf{v}} \)

\[
Y_{t_\mathbf{v}}^* = \{(l, B_0^*, B_1^*, \ldots, B_r^*) \in \mathcal{Y}_{t_\mathbf{v}}^* \mid (B_{i-1}^*, B_i^*) \in O^*(t_i) \}
\]

if \( i \notin H, B_{i-1}^* = B_i^* \) if \( i \in H \). These form a partition of \( \mathcal{Y}_{t_\mathbf{v}}^* \). Define \( Y_{t_\mathbf{v}}^{*, H} = \rho^{-1}Y_{t_\mathbf{v}}^* \), for all \( H \subset H_0 \). The subvarieties \( Y_{t_\mathbf{v}}^{*, H} \) form a partition of \( \mathcal{Y}_{t_\mathbf{v}}^* \) into locally closed pieces.

Let \( I_{t_\mathbf{v}} \) be the set of all \( j \in [1, r] \) such that \( t_j \neq e \) and \( t_{t_{r-1}} \cdots t_1, \ldots, t_{r-1} t_r \in W_{t_\mathbf{v}}^* \). Then \( I_{t_\mathbf{v}} = I_{t_\mathbf{v}} \cap H_0 \). Applying 2.12(a) to \( \mathcal{L}'_1 \), \( t_\mathbf{v} \), and \( L \), we see that \( \mathcal{L}'_1 \) is a local system of rank 1 on the open subset \( \bigcup_{H \subset H_0} Y_{t_\mathbf{v}}^{*, H} \) of \( \mathcal{Y}_{t_\mathbf{v}}^* \) and is zero on its complement.

It follows that \( \rho^*(\mathcal{L}'_1) \) is a local system of rank 1 on the open subset \( \bigcup_{H \subset H_0} Y_{t_\mathbf{v}}^{*, H} \) of \( \mathcal{Y}_{t_\mathbf{v}}^* \) and is zero on its complement in \( \mathcal{Y}_{t_\mathbf{v}}^* \).

With the notations in 2.6, we have \( Y_{t_\mathbf{v}}^{*, H} = Y_{s_{H \cup J}} \cap \mathcal{Y}_{t_\mathbf{v}}^* \). For a set \( H \subset H_0 \), the conditions \( H \subset I_{t_\mathbf{v}} \) and \( H \cup J \subset I_\mathbf{s} \) are equivalent. By 2.12(a), \( \mathcal{L}' \) (and hence \( \mathcal{L}' \)) is a local system of rank 1 on the open subset \( \bigcup_{H \subset H_0} Y_{t_\mathbf{v}}^{*, H} \) of \( \mathcal{Y}_{t_\mathbf{v}}^* \) and is zero on its complement in \( \mathcal{Y}_{t_\mathbf{v}}^* \). To prove (b'') it is then enough to show that the local systems on \( \bigcup_{H \subset H_0} Y_{t_\mathbf{v}}^{*, H} \) defined by \( \mathcal{L}' \) and \( \rho^*(\mathcal{L}'_1) \) are isomorphic. Since \( Y_{t_\mathbf{v}}^{*, \emptyset} \) is open dense in the smooth variety \( \bigcup_{H \subset H_0} Y_{t_\mathbf{v}}^{*, H} \) it is even enough to show that the local systems on \( Y_{t_\mathbf{v}}^{*, \emptyset} \) defined by \( \mathcal{L}' \) and \( \rho^*(\mathcal{L}'_1) \) are isomorphic.

The local system defined by \( \mathcal{L}' \) on \( Y_{t_\mathbf{v}}^{*, 0} \) is the restriction of the local system \( \mathcal{L}' \) from \( Y_{s_{\mathbf{v}}} \) to \( Y_{t_\mathbf{v}}^{*, \emptyset} \), \( \mathcal{L}' \) is constructed explicitly in 2.5).

The local system defined \( \rho^*(\mathcal{L}'_1) \) on \( Y_{t_\mathbf{v}}^{*, \emptyset} \) is the inverse image under \( \rho: Y_{t_\mathbf{v}}^{*, 0} \to Y_{t_\mathbf{v}}^{*, \emptyset} \) of the local system \( \mathcal{L}'_1 \), which is explicitly constructed as in 2.5 (for \( \mathbf{t}, \mathbf{L}' \), \( L \) instead of \( \mathbf{s}, \mathcal{L}, G \)). From these explicit constructions, we get immediately an explicit isomorphism between our two local system. Thus, (b'') follows, completing the proof of the lemma.

3.6. We consider the sequence of closed subsets of \( \mathcal{Y}_{t_\mathbf{v}}^* \) defined by

\[
Z_i = \bigcup_{c(v) \leq i} \mathcal{Y}_{t_\mathbf{v}}^*.
\]
where \( v \) runs over all sequences \( v = (v_0, v_1, \ldots, v_r) \) of \( P \)-orbits on \( \mathcal{B} \) satisfying (3.3.1) and \( c(v) \leq i \), where

\[
c(v) = \dim v_0 + \dim v_1 + \cdots + \dim v_r.
\]

If \( \beta_i \) is the inclusion \( Z'_i \hookrightarrow \overline{Y}' \), and \( \gamma_i \) is the inclusion \( Z'_i - Z'_{i-1} \hookrightarrow \overline{Y}' \), then we have a natural distinguished triangle (1.10) in \( \mathcal{O}L \):

\[
(\overline{\pi}'(\gamma_i), \overline{\gamma}'^* \overline{\mathcal{L}}', \overline{\pi}'(\beta_i), \beta_i^* \overline{\mathcal{L}}', \overline{\pi}'(\beta_{i-1}), \beta_{i-1}^* \overline{\mathcal{L}}').
\]

It gives rise to a long exact sequence in \( \mathcal{M}L \) (for each \( i \))

\[
\cdots \to pH^{i-1}(\overline{\pi}'(\beta_{i-1}), \beta_{i-1}^* \overline{\mathcal{L}}') \xrightarrow{\delta} \bigoplus_{c(v) = i} pH^i((\overline{\pi}'_v), \overline{\mathcal{L}}') \to pH^i(\overline{\pi}'(\beta_i), \beta_i^* \overline{\mathcal{L}}') \to pH^i(\overline{\pi}'(\beta_{i-1}), \beta_{i-1}^* \overline{\mathcal{L}}') \to \cdots
\]

Here we have used the isomorphism \( \overline{\pi}'(\gamma_i), \overline{\gamma}'^* \overline{\mathcal{L}}' = \bigoplus_{c(v) = i} ((\overline{\pi}'_v), \overline{\mathcal{L}}') \). Note that

\[
(3.6.2) \quad \overline{\pi}'(\beta_i), \beta_i^* \overline{\mathcal{L}}' = \begin{cases} \overline{\pi}'_v \overline{\mathcal{L}}' & \text{for large } i, \\ 0 & \text{for } i < 0. \end{cases}
\]

We now prove the following result.

**Lemma 3.7.** (a) For each integer \( i \), the maps \( \delta \) in (3.6.1) are zero.

(b) For each integer \( i \), the complex \( \overline{\pi}'(\beta_i), \beta_i^* \overline{\mathcal{L}}' \in \mathcal{O}L \) is semisimple (1.12); it is isomorphic in \( \mathcal{O}L \) to the direct sum \( \bigoplus_{c(v) < i} ((\overline{\pi}'_v), \overline{\mathcal{L}}') \).

(c) The complex \( \overline{\pi}' \overline{\mathcal{L}}' \in \mathcal{O}L \) is semisimple; it is isomorphic in \( \mathcal{O}L \) to the direct sum \( \bigoplus_{c(v) = i} ((\overline{\pi}'_v), \overline{\mathcal{L}}') \).

**Proof:** From (3.6.2) we see that (c) is a special case of (b), (for large \( i \)). Assuming that (a) and the first assertion of (b) are proved, we prove the second assertion of (b) as follows. Since both complexes in question are semisimple (see 3.5 and 2.17), it is enough to show that they have the same \( pH^j \) for all \( j \). Using (a) we see that (3.6.1) decomposes into short exact sequences of semisimple objects in \( \mathcal{M}(L) \). Hence

\[
pH^i(\overline{\pi}'(\beta_i), \beta_i^* \overline{\mathcal{L}}') \cong pH^i(\overline{\pi}'(\beta_{i-1}), \beta_{i-1}^* \overline{\mathcal{L}}') \oplus \bigoplus_{c(v) = i} pH^j((\overline{\pi}'_v), \overline{\mathcal{L}}'').
\]
This proves the desired equality for $^pH^j$ by induction on $i$. (The case $i < 0$ is trivial by (3.6.2).)

It remains to prove (a) and the first assertion of (b). By general principles [1, Sect. 6] it is enough to prove them in the case where the ground field $k$ is the algebraic closure of a finite field. In this case, we can realize (3.6.1), (3.6.2) in the category of mixed perverse sheaves over $G_0$ (a split $F_q$ form of $G$ with $B, T, P$ defined over $F_q$) for sufficiently large $F_q \subset k$, depending on $\mathcal{L}$. The isomorphisms in Lemma 3.5 can also be realized in that category (possibly with an even larger $F_q$). Now $K_i$ in that lemma is a pure complex of weight 0 (by Deligne's theorem [2, 6.2.6] applied to the proper map $\pi_i: \tilde{Y}_i \to G$ and to $\mathcal{L}$ which is pure of weight 0, as we can see either directly, or from Gabber's purity theorem [1, 5.3.4]), after applying to it $[-2d(v)](-d(v))$, it remains pure of weight 0, see [1, 6.1.4]. Hence, by 3.5, $(\pi'_i)(\mathcal{L}'_i')$ are pure complexes of weight 0; it follows that

$$\bigoplus (\pi'_i)^*(\mathcal{L}'_i')$$

in (3.6.1) are pure complexes of weight $j$.

We now show by induction on $i$ that $^pH^j(\pi'_i(\beta_i)_!\beta'_i^*\mathcal{L}'_i')$ is a pure complex of weight $j$ for any $i$. This is obvious for $i < 0$, by (3.6.2). If we assume that this is true for $i - 1$, the statement for $i$ follows from (3.6.1), using (3.7.1), the statement for $i - 1$ and the following fact: if $K_1 \to K_2 \to K_3$ is an exact sequence of mixed perverse sheaves with $K_1, K_3$ pure of weight $j$, then $K_2$ is also pure of weight $j$.

Now using [1, 5.4.4] it follows that $\pi'_i(\beta_i)_!\beta'_i^*\mathcal{L}'_i'$ is pure of weight 0. Using the "decomposition theorem" [1, 5.4.5, 5.3.8] it follows that $\pi'_i(\beta_i)_!\beta'_i^*\mathcal{L}'_i'$ is semisimple.

The vanishing of $\delta$ in (3.6.1) follows from the fact that $\delta$ is a morphism between two pure perverse sheaves of different weights. This completes the proof of the lemma.

3.8. We define a functor res: $\mathcal{D}G \to \mathcal{D}L$ by res $A = (\pi_p)_!i^*A(\alpha)$, where $i: P \subseteq G$ is the inclusion and $\alpha = \dim U_p$. It is clear that, with the notation in 3.3, we have

$$\text{res } K_i^\sigma = \pi_i^!\mathcal{L}'(\alpha) \in \mathcal{D}L.$$  

Hence 3.7(c) and 3.5 imply

$$\text{res } K_i^\sigma \in \mathcal{D}L$$

is semisimple; more precisely it is a direct sum of finitely many complexes of the form $A'[i]$, where $A' \in \mathcal{E}$ and $i$ is an integer.

We can now state

**Proposition 3.9.** If $A \in \hat{G}$, then \text{res } A \in \mathcal{D}L is semisimple; more
precisely, it is a direct sum of finitely many complexes of form \( A'[i] \), where \( A' \in \mathcal{L} \) and \( i \) is an integer.

**Proof.** We can find a sequence \( s = (s_1, s_2, \ldots, s_r) \) in \( S \) and \( \mathcal{L} \in \mathcal{S}(T) \) such that \( s_1s_2 \cdots s_r \in W_{\mathcal{L}} \) and such that \( A \) is a constituent of \( ^pH^i(\mathcal{K}_x^\mathcal{L}) \). From 2.17(a), it follows that \( A[-j] \) is a direct summand of \( \mathcal{K}_x^\mathcal{L} \). Since \( \text{res} \) transforms direct sums into direct sums, it follows that \( \text{res}(A)[-j] \) is a direct summand of \( \text{res}(\mathcal{K}_x^\mathcal{L}) \) which is semisimple by (3.8.2). By 1.12, \( \text{res} A[-j] \) must be also semisimple. Now \( ^pH^i(\text{res} A) \) is a direct summand of \( ^pH^{i+j}(\text{res} \mathcal{K}_x^\mathcal{L}) \) which, by (3.8.2) has all its irreducible subquotients in \( \mathcal{L} \); hence all irreducible subquotients of \( ^pH^i(\text{res} A) \) are in \( \mathcal{L} \). The proposition follows.

**Definition 3.10.** A character sheaf \( A \in \mathcal{G} \) is said to be **cuspidal** if for any parabolic subgroup \( P \subseteq G \) containing \( B \) (with Levi subgroup \( L \supseteq T \)), we have \( \text{res} A[-l] \in \mathcal{D} L \subseteq \mathcal{G} \) (with \( \text{res} \) defined with respect to \( P \)) or, equivalently, \( \dim \text{supp} \mathcal{A}(\text{res} A) < -l \) for all \( l \). The cuspidal character sheaves form a subset \( \mathcal{G}(o) \) of \( \mathcal{G} \).

3.11. For any \( g \in G \), we denote by \( g_s \) the semisimple part of \( g \) and we define \( H_G(g) \) to be the centralizer in \( G \) of the connected centre of \( Z^0(G(g_s)) \). Then \( H_G(g) \) is the smallest Levi subgroup of a parabolic subgroup containing \( Z^0(G(g_s)) \). We say that \( g \) (or its conjugacy class) is isolated if \( H_G(g) = G \). (When \( G \) is semisimple, it has only finitely many isolated classes.)

Following [4, 3.1], we now define a partition of \( G \) into finitely many locally closed smooth irreducible subvarieties stable by conjugation. The pieces in the partition are parametrized by pairs \( (L, \Sigma) \) up to \( G \)-conjugacy, where \( L \) is a subgroup of \( G \), which is the Levi subgroup of some parabolic subgroup of \( G \), and \( \Sigma \) is a subset of \( L \), which is the inverse image under \( L \to L/Z^0_L \) of a connected centre of \( L \) of an isolated conjugacy class of \( L/Z^0_L \). For such \( (L, \Sigma) \), we define

\[
\Sigma_{\text{reg}} = \{ g \in \Sigma | H_G(g) = L \} = \{ g \in \Sigma | Z^0_G(g_s) \subseteq L \}
\]

and \( Y_{(L, \Sigma)} = \bigcup_{x \in G} x(\Sigma_{\text{reg}})x^{-1} \).

The \( Y_{(L, \Sigma)} \) form the required partition of \( G \).

**Proposition 3.12.** Let \( A \in \mathcal{G}(o) \) be cuspidal, where \( \mathcal{L} = \lambda \ast \mathcal{G}_{n,o} \) is as in 2.2. Consider the action of \( G \times \mathcal{L}^0_G \) on \( G \) defined in 2.18(b). Then there is a unique \( G \times \mathcal{L}^0_G \)-orbit \( \Sigma_o \subset G \) and a unique irreducible, \( G \times \mathcal{L}^0_G \)-equivariant \( \mathbb{Q}_l \)-local system \( \mathcal{E} \) on \( \Sigma_o \) such that \( A = IC(\Sigma_o, \mathcal{E})[d] \), where \( d = \dim \Sigma_o \).
Moreover, the image of $\Sigma_0$ in $G/\mathcal{Z}_G^0$ is an isolated conjugacy class (see 3.11) of $G/\mathcal{Z}_G^0$. If $g \in \Sigma_0$ and $H$ is the centralizer of $g$ in $G$, then $H^0/\mathcal{Z}_G^0$ is a unipotent group.

Proof. Let $V$ be a locally closed smooth irreducible subvariety of $G$ which is dense in the support of $\mathcal{A}$ and is such that $A \mid V$ is isomorphic to $\mathcal{E}[d]$, where $\mathcal{E}$ is an irreducible $\overline{Q}_l$-local system on $V$ and $d = \dim V$. We shall assume (as we may be 2.18(b)) that $V$ is $G \times \mathcal{Z}_G^0$-stable and that $\mathcal{E}$ is $G \times \mathcal{Z}_G^0$-equivariant.

Since $V$ is irreducible, there is a unique piece $Y_{(u, \Sigma)}$ in the finite partition of $G$ described in 3.11 such that $V \cap Y_{(u, \Sigma)}$ is open dense in $Y_{(u, \Sigma)}$. Since $Y_{(u, \Sigma)}$ is $G \times \mathcal{Z}_G^0$-stable, we may assume (by replacing $V$ by $V \cap Y_{(u, \Sigma)}$) that $V \subset Y_{(u, \Sigma)}$. We may also assume that $L \supset T$ and is the Levi subgroup of a parabolic subgroup $P$ of $G$ containing $B$. Let $U_P, \pi_p$ be defined as in 3.1 and let $i$ be the inclusion $P \subset G$. Let $g \in \Sigma_{\text{reg}} \cap V \subset L$. The orbit of $g$ under the conjugation action of $U_P$ is closed (it is an orbit of a unipotent group acting on an affine variety) and is contained in $gU_P$ (since $\pi_p(g) = \pi_p(ugu^{-1})$ for $u \in U_P$). The isotropy group of $g$ in $U_P$ is contained in $U_P \cap Z_G(g) \subset U_P \cap Z_G^0(g)$; hence it is trivial since $g \in \Sigma_{\text{reg}}$. Hence the dimension of the $U_P$-orbit of $G$ is equal to $\dim U_P$; this orbit being closed in $gU_P$, it must be equal to $gU_P$. In particular, we have $gU_P \subset V$ (since $V$ is stable by conjugation). The restriction of $\mathcal{E}$ to $gU_P$ is a $U_P$-equivariant local system (for the conjugation action of $U_P$) on the $U_P$-orbit $gU_P$, with trivial isotropy group. It follows that $\mathcal{E}$ is a constant nonzero local system on $gU_P$, hence $H^2_{\text{ct}}(gU_P, \mathcal{E}) \neq 0$, ($\alpha = \dim U_P$). This means that the stalk of the cohomology sheaf $\mathcal{E}^{2\alpha-d}((\pi_p)_!*A)$ at $g$ is nonzero. Thus, we have shown that

$$\Sigma_{\text{reg}} \cap V \subset \sup \mathcal{Z}^{2\alpha-d}((\pi_p)_!*A).$$

Let $G_1 = \{ g \in N_G(L) \mid g\Sigma g^{-1} = \Sigma \} = \{ g \in N_G(L) \mid g\Sigma g^{-1} \cap \Sigma \neq \emptyset \}$. The group $G_1$ acts on $G \times (\Sigma_{\text{reg}} \cap V)$ by $g_1: (g, \sigma) \to (gg_1^{-1}, g_1\sigma g_1^{-1})$. The map $G \times (\Sigma_{\text{reg}} \cap V) \to V$ defined by $(g, \sigma) \to g\sigma g^{-1}$ is surjective (since $V \subset Y_{(u, \Sigma)}$) and its fibres are precisely the orbits of the $G_1$-action just described. It follows that

$$\dim(\Sigma_{\text{reg}} \cap V) + \dim G = \dim V + \dim G_1 = \dim V + \dim L = d + \dim G - 2\alpha;$$

hence $\dim(\Sigma_{\text{reg}} \cap V) = d - 2\alpha$. From (3.12.1) it now follows that $\dim \text{supp} \mathcal{Z}^{2\alpha-d}((\pi_p)_!*A) \geq d - 2\alpha$. Since $A$ is assumed to be cuspidal, it follows that $L = P = G$. In this case, $Y_{(u, \Sigma)} = \Sigma$ is a single $G \times \mathcal{Z}_G^0$-orbit on
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$G$, and therefore $V$ must also be a single $G \times Z_0^G$-orbit; the image of $V = \Sigma$ in $G/\mathcal{Z}_0^G$ is isolated. The last assertion follows from [4, 2.8], since $(\Sigma, \mathcal{E})$ is a cuspidal pair for $G$ in the sense of [4, 2.4]. (The condition [4, 2.4(a)] follows from $G \times Z_0^G$-equivariance; the condition [4, 2.4(b)] follows from the fact that $A$ is cuspidal.)

4. Induction

4.1. Let $P, L, U_p, \pi_p$ be as in 3.1. Consider the diagram

$$L \leftarrow \pi \rightarrow V_1 \rightarrow \pi' \rightarrow V_2 \rightarrow G,$$

where

$$V_1 = \{(g, h) \in G \times G \mid h^{-1}gh \in P\},$$

$$V_2 = \{(g, h) \in G \times (G/P) \mid h^{-1}gh \in P\},$$

$$\pi(g, h) = \pi_p(h^{-1}gh), \quad \pi'(g, h) = (g, hP), \quad \pi''(g, hP) = g.$$ 

Then $\pi, \pi'$ are smooth morphisms with connected fibres.

We associate with any perverse sheaf $K \in \mathcal{M}_L$ (which is $L$-equivariant for the conjugation action of $L$ on $L$) a complex $\text{ind} K \in \mathcal{D}_G$, as follows. The perverse sheaf $\tilde{\pi}_L K \in \mathcal{M}_L$ is $P$-equivariant for the action $p: (g, h) \rightarrow (g, hp^{-1})$ of $P$ on $V_1$ and the action $p: l \rightarrow \pi_p(l) l' \pi'_p(l')^{-1}$ of $P$ on $L$. Since $\pi'$ is a locally trivial principal $P$-bundle, there is (1.9.3) a well-defined perverse sheaf $K_1 \in \mathcal{M}_V$ such that $\tilde{\pi}_L K = \tilde{\pi}_L K_1$. We define $\text{ind} K = (\pi'')_* K_1$. We shall sometimes write $\text{ind}^G_P K$ instead of $\text{ind} K$.

In the case where $L = P = G$, we have $\text{ind}^G_P K = K$, as we see immediately from the definition of $G$-equivariance of $K$. From (1.9.2) it follows easily that

$$(4.1.1) \quad \sigma^i(\text{ind}^G_P K)$$

is a $G$-equivariant perverse sheaf on $G$ (for the conjugation action), for all $i$.

We shall now state a transitivity property of induction. Let $Q$ be a parabolic subgroup of $L$ containing $B^* = B \cap L$, let $M$ be the Levi subgroup of $Q$ containing $T$, and let $\pi_Q: Q \rightarrow M$ be the canonical projection.

Let $\tilde{Q} = QU_p$; it is parabolic subgroup of $G$, $B \subset \tilde{Q} \subset P$.

**Proposition 4.2.** Let $K \in \mathcal{M}(M)$ be $M$-equivariant (for the conjugation action). Assume that $\text{ind}^G_Q K$ is in $\mathcal{M}_L$. Then $\text{ind}^G_Q (\text{ind}^L_Q K) = \text{ind}^G_Q K$. 

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
M & \\
\downarrow^{d_0} & \downarrow^{d_1} & \downarrow^{d_2} \\
X_0 & \overset{h_1}{\longleftarrow} & X_1 \overset{h_2}{\longrightarrow} & X_2 \\
\downarrow^{f_0} & \downarrow^{f_1} & \downarrow^{f_2} \\
Y_0 & \overset{o_1}{\longleftarrow} & Y_1 \overset{o_2}{\longrightarrow} & Y_2 \\
\downarrow^{s_0} & \downarrow^{s_1} & \downarrow^{s_2} \\
L & \overset{\pi}{\longleftarrow} & V_1 \overset{\pi'}{\longrightarrow} & V_2 \\
& \downarrow^\pi & & \downarrow^\pi \\
& & G & \\
\end{array}
\]

where

\[
\begin{align*}
X_0 &= \{(x, y) \in L \times L \mid y^{-1}xy \in Q\}, \\
X_1 &= \{(y, z, v') \in P \times G \times G \mid v'^{-1}zv' \in \tilde{Q}\}, \\
X_2 &= \{(z, v') \in G \times G \mid v'^{-1}zv' \in \tilde{Q}\}, \\
Y_0 &= \{(x, yQ) \in L \times L/Q \mid y^{-1}xy \in \tilde{Q}\}, \\
Y_1 &= \{(y, z, v') \in P \times G \times G \mid v'^{-1}zv' \in \tilde{Q}\} \text{ mod. action of } \tilde{Q}: \\
q &: (y, z, v') \rightarrow (yq^{-1}, z, v'q^{-1}), \\
Y_2 &= \{(z, v'\tilde{Q}) \in G \times G/\tilde{Q} \mid v'^{-1}zv' \subset \tilde{Q}\}, \\
V_1, V_2, \pi, \pi', \pi'' & \text{ are as in 4.1,} \\
e_1(y, z, v') &= (yv'^{-1}zv'y^{-1}, yQ), \\
e_2(y, z, v') &= (z, v'\tilde{Q}), \\
h_1(y, z, v') &= (yv'^{-1}zv'y, y), \\
h_2(y, z, v') &= (z, v'), \\
d_0(x, y) &= \pi_0(y^{-1}xy), \\
d_1(y, z, v') &= \pi_p(v'^{-1}zv') \\
d_2(z, v') &= \pi_p(v'^{-1}zv')
\end{align*}
\]
\[ f_0(x, y) = (x, yQ), \]
\[ f_1(y, z, v') = \tilde{Q}\text{-orbit of } (y, z, v'), \]
\[ f_2(z, v') = (z, v'\tilde{Q}), \]
\[ g_0(x, yQ) = x \]
\[ g_1(y, z, v') = (z, v'y^{-1}), \]
\[ g_2(z, v'\tilde{Q}) = (z, v'P). \]

The two lowest squares (\( e_i, g_0, g_1, \pi \)) and (\( e_2, g_2, g_1, \pi' \)) are cartesian and the maps \( e_i, f_i, \pi, \pi' \) are smooth with connected fibres. It follows that

\[
\begin{align*}
(4.2.1) & \quad (g_1)_i \bar{\pi}_0 = \bar{\pi}(g_0)_i, \\
(4.2.2) & \quad (g_1)_i \bar{\pi}_1 = \bar{\pi}(g_2)_i, \\
(4.2.3) & \quad \tilde{f}_i: \mathcal{M}Y_i \to \mathcal{M}X_i \text{ is fully faithful } (i = 0, 1, 2).
\end{align*}
\]

Since \( K \) is \( M \)-equivariant, \( \tilde{d}_0 K \in \mathcal{M}X_0 \) is in the image of \( \tilde{f}_0: \mathcal{M}Y_0 \to \mathcal{M}X_0 \). We shall write \( (\tilde{f}_0)^{-1}\tilde{d}_0 K \) for the object in \( \mathcal{M}Y_0 \) which maps under \( \tilde{f}_0 \) to \( \tilde{d}_0 K \). Again, since \( K \) is \( M \)-equivariant, \( \tilde{d}_2 K \in \mathcal{M}X_2 \) is in the image of \( \tilde{f}_2: \mathcal{M}Y_2 \to \mathcal{M}X_2 \); we write \( (\tilde{f}_2)^{-1}\tilde{d}_2 K \) for the object in \( \mathcal{M}Y_2 \) which maps under \( \tilde{f}_2 \) to \( \tilde{d}_2 K \).

Let \( K' = (g_2)_i(\tilde{f}_2)^{-1}\tilde{d}_2 K \in \mathcal{M}V_2 \). It is enough to prove the following three statements:

(a) \( K' \in \mathcal{M}V_2 \),
(b) \( \bar{\pi}(g_0)_i(\tilde{f}_0)^{-1}\tilde{d}_0 K = \bar{\pi}'K' \),
(c) \( \pi_i K' = \pi_i(g_2)_i(\tilde{f}_2)^{-1}\tilde{d}_2 K \).

Property (c) is obvious from the definition of \( K' \). We now prove (b).

From \( d_i = d_0 h_i = d_2 h_2 \), we see that
\[ \bar{h}_2 \tilde{d}_2 K = \bar{h}_1 \tilde{d}_0 K \in \mathcal{M}X_1. \]

This can also be written as
\[ \bar{h}_2 \tilde{f}_2(f_2)^{-1}\tilde{d}_2 K = \bar{h}_1 \tilde{f}_0(f_0)^{-1}\tilde{d}_0 K \in \mathcal{M}X_1. \]

Now, using \( f_1 h_2 = e_2 f_1, f_0 h_1 = e_1 f_1 \), we have
\[ \tilde{f}_1 \bar{e}_2(f_2)^{-1}\tilde{d}_2 K = \tilde{f}_1 \bar{e}_1(f_0)^{-1}\tilde{d}_0 K \in \mathcal{M}X_1, \]

Using (4.2.3), we can suppress \( \tilde{f}_1 \):
\[ \bar{e}_2(f_2)^{-1}\tilde{d}_2 K = \bar{e}_1(f_0)^{-1}\tilde{d}_0 K \in \mathcal{M}Y_1. \]
We now apply \((g_1)_t\) to both sides of this equality and use (4.2.1), (4.2.2); we get
\[
\tilde{\pi}'(g_2)(f_2)^{-1}\tilde{a}_2 K = \tilde{\pi}(g_0)(f_0)^{-1}\tilde{a}_0 K,
\]
hence (b) is proved.

By assumption, \((g_0)_t f_0 \tilde{a}_0 K \in \mathcal{M}L\), hence the left-hand side of (b) is in \(\mathcal{M}L\). By (b), we have \(\tilde{\pi}'K' \in \mathcal{M}V_1\) and from (1.8.1) it follows that \(K' \in \mathcal{M}V_2\). This completes the proof.

4.3. Let \(\Sigma\) be a subset of \(L\) which is the inverse image under \(L \to L/\mathcal{Z}_L^0\) of an isolated conjugacy class of \(L/\mathcal{Z}_L^0\) and let \(\mathcal{E}\) be a \(\mathbb{Q}_\ell\)-local system on \(\Sigma\) which is equivariant for the action of \(L \times \mathcal{Z}_L^0\) on \(\Sigma\) defined by \((l, z) : \sigma \to z^n l z^{-1}\); \(n \geq 1\) is a fixed integer invertible in \(k\). Then \(\text{IC}(\Sigma, \mathcal{E})\)[\(d\)], \((d = \dim \Sigma)\) is an \(L\)-equivariant perverse sheaf on \(L\). The following result is proved in [4, 4.5]:

\[(4.3.1) \text{ ind}^G_{\Sigma}(\text{IC}(\Sigma, \mathcal{E})\)[\(d\)]) \text{ is a perverse sheaf on } G; \text{ it is a direct sum of irreducible perverse sheaves with support } \overline{Y}_{(u, z)}, \text{ (see 3.11)}.\]

Now, using 3.12, we deduce:

\[(4.3.2) \text{ If } A_1 \in \hat{L} \text{ is cuspidal, then } \text{ ind}^G_{\Sigma} A_1 \in \mathcal{M}G \text{ and is semisimple}.\]

We can now state

**Theorem 4.4.** (a) For any \(A \in \hat{G}\), there exists \(L \subset P\) as in 3.1 and \(A_1 \in \hat{L}^{(0)}\) such that \(A\) is a direct summand of \(\text{ ind}^G_{\Sigma} A_1\).

(b) If \(L \subset P\) is as in 3.1, and \(A_1 \in \hat{L}\), then \(\text{ ind}^G_{\Sigma} A_1 \in \mathcal{M}G\).

(c) If \(L \subset P\) is as in 3.1, and \(A \in \hat{G}\), then \(\text{ res } A \in \mathcal{M} L^{<0}\).

(d) If \(L \subset P\) is as in 3.1, \(A \in \hat{G}\) and \(A_1 \in \hat{L}\), then
\[
\text{Hom}_{\mathcal{M}}(\text{ res } A, A_1) \cong \text{Hom}_{\mathcal{M}}(A, \text{ ind } A_1).
\]

When \(G\) is a maximal torus, the theorem is obvious. Assume now that \(G\) is not a torus and that the theorem is already proved for \(G\) replaced by \(L\) for any \(L \subset P\) as in 3.1, with \(P \neq G\). We shall prove the theorem for \(G\) itself, in 4.5–4.6, using this inductive assumption.

4.5. We first show that

\[(4.5.1) \text{ 4.4(b) holds for } G.\]

Indeed it is enough to check 4.4(b) in the case where \(P \neq G\). Then (a) applies to \(L\), hence there exists \(M \subset Q\) in \(L\) as in 4.2 and \(A_2 \in \mathcal{M}^{(0)}\) such that \(A_1\) is a direct summand of \(\text{ ind}^G_{\Sigma}(A_2)\) (which is in \(\mathcal{M}L\) by (4.3.2)). By
4.2. we have $\text{ind}_G^G(\text{ind}_G^G(A_2)) = \text{ind}_G^G(\bar{\mathcal{O}}) \subset \mathcal{M}$. Again using (4.3.2), we have $\text{ind}_G^G(A_2) \in \mathcal{M}$ and $\text{ind}_G^G(\bar{\mathcal{O}}) \subset \mathcal{M}$. Since $A_1$ is a direct summand of $\text{ind}_G^G(A_2)$ and $\text{ind}_G^G$ takes direct sums to direct sums, it follows that $\text{ind}_G^G(A_1)$ is a direct summand (in $\mathcal{D}$) of an object in $\mathcal{M}$. This clearly forces $\text{ind}_G^G(A_1)$ to be in $\mathcal{M}$, as required.

4.6. Consider the commutative diagram

\[
\begin{array}{ccc}
V_2 & \overset{f_1}{\longrightarrow} & G \\
\downarrow{f_2} & & \downarrow{\iota} \\
D & \overset{\phi}{\longrightarrow} & G \times P \\
\beta \downarrow & & \theta \downarrow \theta' \downarrow \\
D' & \overset{\gamma}{\longrightarrow} & L \\
\end{array}
\]

where

\[
D = \{(x, l) \in G \times L \} \quad \text{modulo the $P$-action}
\]

\[
p: (x, l) \sim (xp^{-1}, \pi_p(p)xl\pi_p(p)^{-1}).
\]

\[
D' = \{(x, l) \in G \times L \},
\]

\[
f_1(g, xP) = g,
\]

\[
f_2(g, xP) = (x, \pi_p(x^{-1}gx)),
\]

\[
\beta: \text{obvious map},
\]

\[
\gamma(x, l) = l,
\]

\[
\rho(x, p) = (xp^{-1}, xP),
\]

\[
\phi(x, p) = (x, \pi_p(p)),
\]

\[
\zeta(x, p) = xpx^{-1},
\]

\[
\zeta'(x, p) = p,
\]

\[
\theta'(x, p) = p,
\]

\[
\theta(x, p) = \pi_p(p).
\]

Let $A_1 \in \mathcal{L}$. Then $\bar{\mathcal{Y}}A_1$ is $P$-equivariant for a $P$-action on $D'$ which makes $\beta$ a locally trivial principal $P$-bundle. By (1.9.3), we have $\bar{\mathcal{Y}}A_1 = \bar{\beta}A_1'$, $(A_1' \in \mathcal{M}D)$. We have

\[(4.6.1) \quad \text{ind} A_1 = (f_1)_* \bar{f}_2 A_1'.\]
(Indeed, it is enough to show that $\bar{\beta}_2 A'_1 = \bar{\theta} A_1$. But $\bar{\beta}_2 A'_1 = \bar{\theta} A'_1 = \bar{\gamma}_2 A'_1 = \bar{\theta} A_1$.)

Let $A \subset \hat{G}$. Define $\text{Res} A = (f_z)_! f_i^* A | \alpha | (\alpha) \subset \mathcal{D}, \ (\alpha = \dim U)$. We show that

\begin{equation}
(4.6.2) \quad \tilde{\gamma}(\text{Res} A) = \bar{\beta}(\text{Res} A).
\end{equation}

Indeed, we have

\begin{align*}
\tilde{\gamma}(\text{Res} A) | (-\dim G) (-\alpha) = \gamma^*(\pi_p) i^* A = \phi, \theta^' i^* A = \phi_i \zeta^* A, \\
\bar{\beta}(\text{Res} A) | (-\dim G) (-\alpha) = \beta^*(f_2)_! f_i^* A = \phi_i \rho f_i^* A = \phi_i \zeta^* A
\end{align*}

(we have used $\beta^*(f_2)_! = \phi_i \rho^*$, $\gamma^*(\pi_p)_! = \phi_i \theta^*$ which follow from (1.7.5)). But $\zeta, \zeta': G \times P \to G$ are compositions of $G \times P \hookrightarrow G \times G$ with the maps $G \times G \to G$ given, respectively, by $(g_1, g_2) \to g_1 g_2 g_1^{-1}$, $(g_1, g_2) \to g_2$. Hence, the $G$-equivariance of $A$ implies $\zeta^* A = \zeta'^* A$, so that $\phi_i \zeta^* A = \phi_i \zeta'^* A$ and (4.6.2) follows.

Next, we show that, for any integer $i$, we have

\begin{equation}
(4.6.3) \quad \text{Hom}_{\mathcal{D}}(\text{Res} A, A'| i) \cong \text{Hom}_{\mathcal{G}}(A, \text{ind} A'_i)
\end{equation}

Indeed, the left-hand side is

\begin{align*}
\text{Hom}( (f_z)_! f_i^* A | \alpha | (\alpha), A'_i | i) \\
= \text{Hom}( f_i^* A | \alpha | (\alpha), f_i^* A'_i | i) \quad \text{(by (1.7.2))} \\
= \text{Hom}( f_i^* A | \alpha | (\alpha), f_i^* A'_i | 2\alpha + i ) \quad \text{(by (1.7.4))} \\
= \text{Hom}( f_i^* A, f_i^* A'_i | \alpha + i ).
\end{align*}

The right-hand side in (4.6.3) is

\begin{align*}
\text{Hom}(A, (f_z)_! f_i^* A'_i | i) = \text{Hom}(A, (f_z)_! f_i^* A'_i | i) \quad \text{(by (1.7.3))} \\
= \text{Hom}( f_i^* A, f_i^* A'_i | i) \quad \text{(by (1.7.1))} \\
= \text{Hom}( f_i^* A, f_i^* A'_i | \alpha + i ) \quad \text{(by (1.7.4))};
\end{align*}

and (4.6.3) follows.

If $i < 0$, we have $\text{Hom}_{\mathcal{D}}(A, \text{ind} A'_i | i) = 0$, since $A, \text{ind} A_i \in \mathcal{M} G$, (see (4.5.1)). From (4.6.3), it now follows that

\begin{equation}
(4.6.4) \quad \text{Hom}_{\mathcal{D}}(\text{Res} A, A'_i | i) = 0 \text{ for } i < 0.
\end{equation}

According to 3.9, there exists a sequence $C_1, C_2, \ldots, C_t$ in $L$ and a sequence of integers $n_1, n_2, \ldots, n_t$ such that

\[
\text{res} A = \bigoplus_{j=1}^t C_j | n_j | \quad \text{in } \mathcal{D} L.
\]
Attach $C'_j \in \mathcal{D}$ to $C_j$ in the same way as $A'_j$ was attached to $A_1$. We have

\[ \beta(p^i \text{Res } A) = p^i(\beta \text{ Res } A) \]
\[ = p^i(\gamma \text{ res } A) \]
\[ = p^i \left( \bigoplus_{j=1}^t C_j[n_j] \right) \]
\[ = \bigoplus_{1 \leq j \leq t, n_j = -i} \beta C_j \]
\[ = \bigoplus_{1 \leq j \leq t, n_j = -i} \beta C'_j. \]

Since $\beta$ is fully faithful (1.8.3), we have

\[ (4.6.5) \quad p^i \text{ Res } A = \bigoplus_{1 \leq j \leq t, n_j = -i} C'_j. \]

Now we show that

\[ (4.6.6) \quad p^i \text{ Res } A = 0 \text{ for all } i > 0. \]

Assume that this is not so; let $i$ be the largest integer such that $p^i \text{ Res } A \neq 0$; then $i > 0$ and there exists a nonzero morphism $\text{Res } A \to p^i \text{ Res } A[-i]$. Now using (4.6.5), we see that there exists a nonzero morphism $\text{Res } A \to C'_j[-i]$ for some $j \in [1, t]$. Since $-i < 0$, this contradicts (4.6.4) with $A'_j$ replaced by $C'_j$. Thus, (4.6.6) is proved. We can also formulate it as stating that

\[ (4.6.7) \quad \text{Res } A \in \mathcal{D}^- \subseteq 0. \]

Applying (1.8.1) to $\beta$, we deduce that $\beta(\text{Res } A) \in \mathcal{D}^- \subseteq 0$. Using (4.6.2), we have then $\alpha(\text{res } A) \in \mathcal{D}^- \subseteq 0$. Applying (1.8.1) to $\gamma$, we deduce that $\text{res } A \in \mathcal{D}^\ell \subseteq 0$. Hence

\[ (4.6.8) \quad 4.4(\text{c}) \text{ holds for } G. \]

We have

\[ \text{Hom}(A, \text{ind } A_1) = \text{Hom}(\text{Res } A, A'_1) \quad \text{(by (4.6.3) with } i = 0) \]
\[ = \text{Hom}(\beta \text{ Res } A, \beta A'_1) \quad \text{(by (1.8.2) and (4.6.7))} \]
\[ = \text{Hom}(\gamma \text{ res } A, \gamma A_1) \quad \text{(by (4.6.2))} \]
\[ = \text{Hom}(\text{res } A, A_1) \quad \text{(by (1.8.2) and (4.6.8))}. \]

\[ (4.6.9) \quad \text{Hence } 4.4(\text{d}) \text{ holds for } G. \]
Finally, we show that 4.4(a) holds for any \( A \in \hat{G} \). If \( A \) is cuspidal, there is nothing to prove. Thus, we may assume that there exists \( L \subset \mathcal{P} \neq \mathcal{G} \) as in 3.1 such that \( \text{res} \, A[-1] \notin \mathcal{D} \mathcal{L} \), or equivalently, such that \( p^{H} \) \( \text{res} \, A \neq 0 \) for some \( i > 0 \). By (4.6.8) we have \( p^{H} \) \( \text{res} \, A = 0 \) for \( i > 0 \). It follows that \( p^{H} \) \( \text{res} \, A \neq 0 \) and that there exists a nonzero morphism \( \text{res} \, A \to p^{H} \) \( \text{res} \, A \) (in \( \mathcal{D} \mathcal{L} \)). Since \( p^{H} \) \( \text{res} \, A \) is a direct sum of objects in \( \hat{L} \) (see 3.9), it follows that there exists \( A_{1} \in \hat{L} \) and a nonzero morphism \( \text{res} \, A \to A_{1} \) (in \( \mathcal{D} \mathcal{L} \)). Using (4.6.9), it follows that there exists a nonzero morphism \( A \to \text{ind}_{\hat{Q}}^{\hat{G}} A_{1} \), in \( \mathcal{D} \mathcal{G} \) (or \( \mathcal{M} \mathcal{G} \)). This must be injective, since \( A \) is irreducible. By our inductive assumption, \( A_{1} \) is a direct summand in \( \text{ind}_{\hat{Q}}^{\hat{G}}(A_{2}) \) for some \( M \subset Q \) as in 4.2 and some \( A_{2} \in \hat{M} \). By transitivity of induction (4.2), \( \text{ind}_{\hat{Q}}^{\hat{G}} A_{1} \) is a direct summand in \( \text{ind}_{\hat{Q}}^{\hat{G}} A_{2} \), where \( \hat{Q} = QU_{P} \). Hence \( A \) is isomorphic to a subobject of \( \text{ind}_{\hat{Q}}^{\hat{G}}(A_{2}) \). By (4.3.2) \( \text{ind}_{\hat{Q}}^{\hat{G}}(A_{2}) \) is a semisimple object of \( \mathcal{M} \mathcal{G} \), hence \( A \) is a direct summand of it. Thus, 4.4(a) holds for \( \mathcal{G} \). This completes the proof of Theorem 4.4.

4.7. Let \( L \subset \mathcal{P} \), \( U_{P} \), \( W^{*}, W^{\#}, \mathcal{F}^{*} \) be as in 3.1, \( (\mathcal{F} \subset \mathcal{F}(T)) \). Let \( s = (s_{1}, s_{2}, ..., s_{r}) \) be a sequence in \( S^{*} \cup \{ e \} \). We can consider \( s \) also as a sequence in \( S \cup \{ e \} \). Let \( s_{L} : \mathcal{Y}_{s,L} \to L, \mathcal{Y}_{s,L}, \mathcal{K}_{s,L} \) be defined in terms of \( s, \mathcal{L}^{*}, L \) in the same way as \( \mathcal{K}_{s}, \mathcal{Y}_{s} \to G, \mathcal{Y}_{s}, \mathcal{K}_{s} \) is defined in 2.8 in terms of \( s, \mathcal{L}, G \). We shall prove the following result.

**Proposition 4.8.** (a) For any \( i \), we have \( \text{ind}_{p}^{G}(p^{H} \mathcal{K}_{s,L}) = p^{H^{i+d_{0}-d_{L}}} \mathcal{K}_{s,L} \), where \( d_{0} = \dim G, d_{L} = \dim L \).

(b) For any \( A_{L} \in L_{s,L} \), \( \text{ind}_{p}^{G}(A_{L}) \) is semisimple in \( \mathcal{M} \mathcal{G} \), and its irreducible components are in \( \mathcal{G}_{s,L} \).

**Proof:** We first show that (a) implies (b). If \( A_{L} \in L_{s,L} \), we may assume that \( A_{L} \) is a direct summand of \( p^{H^{i}}(\mathcal{K}_{s,L}) \), (2.17(a)). From (a), it follows that \( \text{ind}_{p}^{G}(A_{L}) \) is a direct summand of \( p^{H^{i+d_{0}-d_{L}}} \mathcal{K}_{s,L} \) which is semisimple by 2.17(a); (b) follows.

We now prove (a). Consider the commutative diagram

\[
\begin{align*}
\mathcal{Y}_{s,L} & \xleftarrow{p_{1}} V_{1} \times_{\mathcal{F}} \mathcal{Y}_{s,L} \xrightarrow{\mathcal{Y}_{s,L}} \mathcal{Y}_{s} \\
L & \xleftarrow{\pi} V_{1} \xrightarrow{\mathcal{Y}_{s,L}} \mathcal{Y}_{s} \xrightarrow{\pi^{\prime}} V_{2} \xrightarrow{\pi^{\prime\prime}} G
\end{align*}
\]

whose bottom row is defined in 4.1; \( \zeta \) is defined by

\[\zeta(g, B_{0}, B_{1}, ..., B_{r}) = (g, x_{0}P)\]
where \( x_0 \in G \) is any element such that \( B_0 = x_0 B x_0^{-1} \). The map \( \hat{\pi}' \) is defined as follows: for \((g, h) \in V_1\) and \((l, B_0^*, B_1^*, ..., B_r^*) \in \tilde{Y}_{s, l}\) such that \( \pi_p(h^{-1} gh) = l \), we set
\[
\hat{\pi}'((g, h), (l, B_0^*, B_1^*, ..., B_r^*)) = (g, hB_0^* U_p h^{-1}, hB_1^* U_p h^{-1}, ..., hB_r^* U_p h^{-1}).
\]
Both squares in the diagram are cartesian and the maps \( \pi, \pi', pr_2, \hat{\pi}' \) are smooth with connected fibres.

Using (1.7.5) we see that
\[
(4.8.1) \quad \tilde{\pi} \circ \pi = (p r_l)^* \tilde{\pi}'
\]
Let \( K' \in \mathcal{D} V_2 \) be defined by \( K' = \zeta_i \mathcal{L}' \). By the decomposition theorem [1, 6.2.5], \( K' \) is semisimple.

From the definitions on \( \mathcal{L}_i, \mathcal{L}'_j \) we can check easily that \( pr_2^* \mathcal{L}_i = \hat{\pi}' \mathcal{L}' \). It follows that \( (pr_1)_* pr_r^* \mathcal{L}'_j = (pr_1)_* \hat{\pi}' \mathcal{L}'_j \); using (4.8.1) we obtain \( \pi' \delta_i \mathcal{L}_j = \pi' \mathcal{L}_j \), hence \( \pi' \mathcal{K}' \mathcal{L}_j = \pi' K' \). We have \( \pi = \pi(\frac{1}{2} d_G - \frac{1}{2} d_l) \), \( \pi' = \pi(\frac{1}{2} d_G + \frac{1}{2} d_l) \). It follows that \( \pi \mathcal{K}' \mathcal{L}_j = \pi' K'[d_G - d_l] \).

Applying (1.8.1) to \( \pi \) and \( \pi' \), we have
\[
\hat{\pi}((p H^i K' \mathcal{L}_j)[d_G - d_l]) = \hat{\pi}'((p H^i K'[d_G - d_l]) = \hat{\pi}'((p H^i K'[d_G - d_l]).
\]

By the definition of induction, we have
\[
\text{ind}_{\pi}^G((p H^i K' \mathcal{L}_j)[d_G - d_l]) = \text{ind}_{\pi'}^G((p H^i K'[d_G - d_l]).
\]
We have
\[
(4.8.2) \quad \bigoplus_i \text{ind}_{\pi}^G((p H^i \mathcal{K}' \mathcal{L}_j)[d_G - d_l]) = \bigoplus_i ((p H^i K'[d_G - d_l])[-1], \text{ in } \mathcal{D} G.
\]
Indeed, the left-hand side is
\[
\bigoplus_i \pi''((p H^i K'[d_G - d_l])[-1] = \pi'' \bigoplus_i ((p H^i K'[d_G - d_l])[-1]) = \pi''(K'[d_G - d_l]) = \mathcal{K}'[d_G - d_l],
\]
which is equal to the right-hand side of (4.8.2), since \( \mathcal{K}' \) is semisimple (2.17(a)). In (4.8.2), we have \( \text{ind}_{\pi}^G((p H^i \mathcal{K}' \mathcal{L}_j)) \in \mathcal{M} G \) for all \( i \); indeed, \( p H^i (\mathcal{K}' \mathcal{L}_j) \) is a direct sum of objects of form \( A_i \in \mathcal{L} \) and for each such \( A_i \),
we have \( \text{ind}_G^F(A_1) \in \mathcal{M} \) (see 4.4(b)). Taking \( ^pH^i \) for both sides of (4.8.2), we therefore find \( \text{ind}_G^F( ^pH^{i+j} K')_s \) and the proposition is proved.

5. Sequences in the Weyl Group

5.1. We fix \( \mathcal{S} \in \mathcal{S}(T) \). Besides the notations in 2.3, we shall use the following notation. Let \( \Omega_\mathcal{S} = \{ w \in W_\mathcal{S} | w(R_\mathcal{S}) = R_\mathcal{S}^+ \} \). Then \( W_\mathcal{S} \) is the semidirect product of \( \Omega_\mathcal{S} \) and \( W_\mathcal{S}^\perp \), with \( W_\mathcal{S}^\perp \) normal.

Let \( \mathcal{L} : W_\mathcal{S}^+ \to \mathbb{N} \) be the function defined by \( \mathcal{L}(w) = \# \{ \alpha \in R_\mathcal{S}^+ | w(\alpha) \in R^- \} \). Then \( \mathcal{L} \) extends the length function of the Coxeter group \( (W_\mathcal{S}, S_\mathcal{S}) \).

5.2. Let \( s = (s_1, s_2, \ldots, s_r) \) be a sequence of elements in \( S \cup \{ e \} \) such that \( s_1 s_2 \cdots s_r \in W_\mathcal{S} \). When \( s_i \neq e \), we shall write \( \alpha_i \) for the simple root in \( R \) corresponding to \( s_i \). Define

\[
I_s = \{ i \in [1, r] | s_i \neq e, s_r \cdots s_{i+1} s_is_{i+1} \cdots s_r \in W_\mathcal{S} \}.
\]

We have the following

**Lemma 5.3.** \( |I_s| \geq \mathcal{L}(s_1 s_2 \cdots s_r) \), with equality if \( l(s_1 s_2 \cdots s_r) = l(s_1) + \cdots + l(s_r) \).

**Proof.** Let \( X = \{ \alpha \in R_\mathcal{S}^+ | (s_1 s_2 \cdots s_r)(\alpha) \in R^- \}, \) \( X' = \{ \alpha \in R_\mathcal{S}^+ | \exists i \in [1, r], s_i \neq e, \alpha = s_is_{r-1} \cdots s_{i+1}(\alpha_i) \} \). It is clear that \( X \subseteq X' \). We have \( |X| = \mathcal{L}(s_1 s_2 \cdots s_r) \) hence \( \mathcal{L}(s_1 s_2 \cdots s_r) \leq |X'| \). Let \( \phi : I_s \to R_\mathcal{S}^+ \) be defined by \( \phi(i) = s_is_{r-1} \cdots s_{i+1}(\alpha_i) \); then \( X' = \phi(I_s) \cap R_\mathcal{S}^+ \). Hence \( |X'| \leq |\phi(I_s)| \leq |I_s| \) so that \( \mathcal{L}(s_1 s_2 \cdots s_r) \leq |I_s| \), as required. Assume now that \( l(s_1 s_2 \cdots s_r) = l(s_1) + \cdots + l(s_r) \). Then the roots \( s_r s_{r-1} \cdots s_{i+1}(\alpha_i) (1 \leq i \leq r, s_i \neq e) \) are distinct and positive. Hence, for \( i \in I_s \), the roots \( s_is_{r-1} \cdots s_{i+1}(\alpha_i) \) are distinct elements of \( X \), so that \( |I_s| \leq |X| \). It follows that \( |I_s| = |X| \).

**Lemma 5.4.** Let \( J \subseteq I_s \); we define \( s_J \) to be the sequence \( (s'_1, s'_2, \ldots, s'_r) \) with \( s'_i = s_i \) for \( i \in J \), \( s'_i = e \) for \( i \in J \). We have \( I_{s_J} = I_s - J \).

**Proof.** Let \( h \in I_s - J \). We have \( s_r s_{r-1} \cdots s_h \cdots s_{r-1}s_h \in W_\mathcal{S} \). Hence if \( a_1 > a_2 > \cdots > a_p \) are the indices in \( J \cap [h + 1, r] \), we have

\[
s_r s_{r-1} \cdots s_{a_p} \cdots s_h \cdots s_{r-1}s_{r} - (s_r s_{r-1} \cdots s_{a_p} \cdots s_{r-1}s_{r})(s_r s_{r-1} \cdots s_h \cdots s_{r-1}s_{r}) \times (s_r s_{r-1} \cdots s_{a_p} \cdots s_{r-1}s_{r}) \in W_\mathcal{S}
\]
This shows that $h \in I_s$. The same computation (in the opposite direction) shows that if $h' \in I_s$, then $h' \in I_s - J$.

5.5. We write the elements of $I_s$ in ascending order: $i_1 < i_2 < \cdots < i_a$. Define

$$
\sigma_a = s_r s_{r-1} \cdots s_{i_a} \cdots s_{r-1} s_r,
$$
$$
\sigma_{a-1} = s_r s_{r-1} \cdots \hat{s}_{i_{a-1}} \cdots \hat{s}_{i_1} \cdots s_{r-1} s_r,
$$
$$
\vdots
$$
$$
\sigma_1 = s_r s_{r-1} \cdots \hat{s}_{i_1} \cdots s_{r-1} s_r,
$$
$$
\omega = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots s_r.
$$

**Proposition 5.6.** (a) $\sigma_1, \sigma_2, \ldots, \sigma_a \in S_{\mathcal{Y}}$ (see 2.3) and $\omega \in \Omega_{\mathcal{Y}}$, (see 5.1).

(b) $s_1 s_2 \cdots s_r = w \sigma_1 \sigma_2 \cdots \sigma_a$.

(c) More generally, if $J$ is a subset of $I_s$, then

$$
\prod_{1 \leq i < r} s_i = \omega \prod_{1 \leq j \leq a} \sigma_j
$$

(in both products, the factors are written in ascending order of indices).

(d) If $l(s_1 s_2 \cdots s_r) = l(s_1) + \cdots + l(s_r)$, then $\sigma_1 \sigma_2 \cdots \sigma_a$ is a reduced expression in $W_{\mathcal{Y}}$.

**Proof.** We set $h = i_a \in I_s$. Let $s'$ be the sequence $(s_r, s_{r-1}, \ldots, s_h, \ldots, s_{r-1}, s_r)$. We show that $I_{s'}$ has a single element. We have $s_r s_{r-1} \cdots s_i \cdots s_{r-1} s_r \in W_{\mathcal{Y}}$. But $s_r s_{r-1} \cdots s_{h+1} \cdots s_{r-1} s_r, s_r s_{r-1} \cdots s_{h+2} \cdots s_{r-1} s_r \in W_{\mathcal{Y}}$, since $h$ is the largest index in $I_s$. Hence the middle term in $s'$ has an index in $I_s$, but all the terms following it have an index outside $I_s'$. We now show that the term in $s'$ immediately preceding the middle term has an index outside $I_s'$. If this is not so, we would have $s_r s_{r-1} \cdots$
s_{h+1} s_h s_{h+1} s_h s_{h+1} \ldots s_{r-1} s_r \in W_{\sigma'}. \] Multiplying on the left and right by

\[ s_r s_{r-1} \ldots s_h \cdot s_{r-1} s_r, \]

we find that \([s_r s_{r-1} \ldots s_{h+1} s_{r-1} s_r] \in W_{\sigma'},\]

a contradiction. Similarly, we see that all terms in \(s',\) preceding the middle
term have an index outside \(I_{\sigma} + \). Thus, \(I_{\sigma'}\) has a single element. By 5.3,
we have

\[ \tilde{\eta}(\sigma_a) = \tilde{\eta}(s_r s_{r-1} \ldots s_{h+1} s_{r-1} s_r) \leq 1. \]

Since \(\sigma_a\) has odd length in \(W\), it
must be \(\neq e\) hence \(\tilde{\eta}(\sigma_a) = 1\). We have \(\sigma_a \in W_{\sigma'}\), hence \(\sigma_a \in S_{\sigma'}\).

We now prove (a) and (b) by induction on \(a\). Assume first that \(a = 0\). By

5.3, we have \(\tilde{\eta}(s_1 s_2 \ldots s_r) = 0\), hence \(s_1 s_2 \ldots s_r \in \Omega_{\sigma'}\) and (a), (b) are clear.

Assume now that \(a \geq 1\) and that (a), (b) are proved for \(a - 1\). Consider the
sequence \(s_J\), where \(J = \{i_a\}\), (see 5.4). By 5.4, we have \(I_{s_J} = I_s - \{i_a\}\). The
induction hypothesis applies to \(s_J\). It follows that \(\sigma_{a-1}, \ldots, \sigma_i \in S_{\sigma'}\)
and \(\omega \in \Omega_{\sigma'}\). We have checked already that \(\sigma_a \in S_{\sigma'}\). Hence (a) for \(s\) follows.

The induction hypothesis shows that \(s_1 \ldots s_{i_a} \sigma_{a-1} \sigma_a \in S_{\omega}\). It
follows that

\[ s_1 s_2 \ldots s_r = (s_1 s_2 \ldots s_{i_a}) (s_{r-1} s_{r-2} \ldots s_r) \omega \sigma_1 \sigma_2 \ldots \sigma_{a-1} \sigma_a \]

hence (b) for \(s\) follows. The more general statement (c) follows from (b)
using 5.4. Statement (d) follows from 5.3.

5.7. Let \(s = (s_1, s_2, \ldots, s_r), s' = (s'_1, s'_2, \ldots, s'_r)\) be two sequences in \(S \cup \{e\}\)
such that \(s_1 s_2 \ldots s_r \in W_{\sigma'}, s'_1 s'_2 \ldots s'_r \in W_{\sigma'}\). Let \(\omega, \sigma_{i_1}, \sigma_{i_2} \ldots \sigma_{i_a}\) be the
elements attached to \(s\) in 5.5, let \(\omega', \sigma_{j_1}, \sigma_{j_2} \ldots \sigma_{j_a}\) be the elements defined in
the same way for \(s'\) instead of \(s\), and let \(\omega'', \sigma_h, \sigma_{h_1}, \sigma_{h_2}, \ldots, \sigma_{h_a}\) be the elements
defined in the same way for the sequence

\[ ss' = (s_1, s_2, \ldots, s_r, s'_1, s'_2, \ldots, s'_r). \]

Then, we have

\[ a'' = a + a', \quad \omega'' = \omega \omega', \quad \sigma_h = \omega^{-1} \sigma_{i_1} \omega', \ldots, \sigma_{h_a} = \omega^{-1} \sigma_{i_a} \omega', \]

\[ \sigma_{h_a+1} = \sigma_{j_1}, \ldots, \sigma_{h_a+u} = \sigma_{j_a}. \]

(This follows easily from the definitions.)

5.8. We let \(S^* \subset S\), \(W^* \subset W\), \(R^* \subset R\), be as in 3.1. The statements
(5.8.1), (5.8.2) below are well known.

(5.8.1) Any coset \(W^* y \subset W\) contains a unique element of minimal
length \(y_0\); it is characterized by the property \(y_0^{-1}(R^* \cap R^+) \subset R^+\).

(5.8.2) Any coset \(zW \subset W\) contains a unique element of minimal
length \(z_0\); it is characterized by the property \(z_0 (R_+ \cap R^+) \subset R^+\).

(5.8.3) Let \(w \in W, s \in S\) be such that \(w\) has minimal length in \(wW_{\sigma'}\).
Then either (a) \(sw\) has minimal length in \(swW_{\sigma'}\) or (b) \(w^{-1}sw \in W_{\sigma'}\).

Indeed, assume that (a) does not hold, so that there exist \(a \in R_{\sigma'}^+\) such that
\(sw(a) < 0\). By our assumption we have \(w(a) > 0\). Hence \(w(a)\) must be the
simple root $\alpha_i$ corresponding to $s$. Thus, $w^{-1}(\alpha_i) = \alpha \in R_{\mathcal{Q}}$, hence $w^{-1}s w \in W_{\mathcal{Q}}$, hence (b) holds. This proves (5.8.3).

(5.8.4) Let $w, w' \in W$ be such that $W^*W_{\mathcal{Q}} = W^*wW_{\mathcal{Q}}$. Assume that

(a) $w$ has minimal length in $wW_{\mathcal{Q}}$ and also in $W^*w$,

(b) $w'$ has minimal length in $w'W_{\mathcal{Q}}$ and also in $W^*w'$.

Then $w = w'$.

Indeed, our assumption implies that there exist $s_1, s_2, \ldots, s_t \in S^*$, such that $s_1s_2 \cdots s_tw \in W^*W_{\mathcal{Q}}$. Assume that there exists $i \in [2, t]$, such that $s_1 \cdots s_tw$ has minimal length in $s_1 \cdots s_tw W_{\mathcal{Q}}$ and $s_1s_2 \cdots s_tw$ does not have minimal length in $s_1s_2 \cdots s_tw W_{\mathcal{Q}}$. Then by (5.8.3), we have $s_1s_2 \cdots s_tw = s_1s_2 \cdots s_t \sigma$ for some $\sigma \in W_{\mathcal{Q}}$. Hence $s_1 \cdots s_t s_1 \cdots s_tw = s_1 \cdots s_t \sigma \in W^*W_{\mathcal{Q}} \sigma = w'W_{\mathcal{Q}}$. Iterating this, we are reduced to the case where for all $i \in [1, t]$, $s_1 \cdots s_i_w$ has minimal length in $s_1 \cdots s_i w W_{\mathcal{Q}}$. In particular, $s_1s_2 \cdots s_tw$ has minimal length in $s_1s_2 \cdots s_tw W_{\mathcal{Q}} = w'W_{\mathcal{Q}}$. Since $w'$ has also minimal length in $w'W_{\mathcal{Q}}$, we have $s_1s_2 \cdots s_tw = w'$, by (5.8.2). Thus, $W^*w = W^*w'$. Since $w, w'$ both have minimal length in $W^*w = W^*w'$, we have $w = w'$, by (5.8.1). This proves (5.8.4). We can now state:

**Proposition 5.9.** Any double coset $W^*yW_{\mathcal{Q}}$ contains a unique element $y_0$ of minimal length. It is characterized by the property: $y_0^{-1}(R^* \cap R^+) \subset R^+$ and $y_0(R_{\mathcal{Q}}^+) \subset R^+$.

**Proof.** The existence of an element $y_0$ of minimal length in $W^*yW_{\mathcal{Q}}$ is obvious. It is clear that $y_0$ must have minimal length in $y_0W_{\mathcal{Q}}$ and also in $W^*y_0$ the proposition follows from (5.8.4), (5.8.1), (5.8.2).

### 6. Hecke Algebras

**6.1.** We fix $\mathcal{Q} \in S(T)$. Let $\mathcal{A} = \mathbb{Z}[u^{1/2}, u^{-1/2}]$, where $u$ is an indeterminate.

Let $H'_{\mathcal{Q}}$ be the Hecke algebra (over $\mathcal{A}$) corresponding to $W'_{\mathcal{Q}}$; it is a free $\mathcal{A}$-module with basis $T_w$ ($w \in W'_{\mathcal{Q}}$). The multiplication is characterized by

$$T_w T_{w'} = T_{ww'}, \quad \text{if} \quad w, w' \in W'_{\mathcal{Q}} \text{ satisfy } \widetilde{t}(w) + \widetilde{t}(w') = \widetilde{t}(ww')$$

$$(T_\sigma + 1)(T_\sigma - u) = 0, \quad \text{if} \quad \sigma \in S_{\mathcal{Q}}.$$

(Recall that $\widetilde{t}$ is defined in 5.1.)
Let $\tilde{H}'_{\omega}$ be the free $\mathcal{A}$-module with basis $e_s$ indexed by the sequences $s = (s_1, s_2, ..., s_r) \in S \cup \{e\}$ ($r \geq 1$), such that $s_1, s_2, ..., s_r \in W'_{\omega}$.

Let $\Gamma_\mathcal{G}$ be the abelian group with generators $[A]$ (corresponding to the various isomorphism classes of objects $A$ in $\mathcal{M}(G)$) and relations $[A] + [A'] = [A \oplus A']$ for any two objects $A, A' \in \mathcal{M}(G)$.

Define an $\mathcal{A}$-linear map $\gamma: \tilde{H}'_{\omega} \to \Gamma_\mathcal{G} \otimes_\mathcal{A} \mathcal{A}$ by $\gamma(e_s) = \sum_{i \in I} (-1)^i [\nu \tilde{H}'(\tilde{K}_s^\omega)] \otimes u^{i/2}$.

Define an $\mathcal{A}$-linear map $\delta: \tilde{H}'_{\omega} \to H'_{\omega}$ by $\delta(e_s) = T_\omega (1 + T_{\sigma_1}) \cdots (1 + T_{\sigma_a}) u^{(m - a + \dim G)/2}$, where $\omega, \sigma_1, \sigma_2, ..., \sigma_a$ are the elements of $W'_{\omega}$ associated to $s = (s_1, s_2, ..., s_r)$ in 5.6, and $m = \# \{ i \in [1, r] \mid s_i \neq e \}$.

With these definitions, we can state

**Proposition 6.2.** (a) There is a unique $\mathcal{A}$-linear map $\varepsilon: H'_{\omega} \to \Gamma_\mathcal{G} \otimes_\mathcal{A} \mathcal{A}$ such that the diagram

$$
\begin{array}{ccc}
\tilde{H}'_{\omega} & \xrightarrow{\delta} & H'_{\omega} \\
\downarrow{\gamma} & & \downarrow{\varepsilon} \\
\Gamma_\mathcal{G} \otimes \mathcal{A} & & \\
\end{array}
$$

is commutative.

(b) We have $\varepsilon(h, h_2) = \varepsilon(h_2, h_1)$ for all $h, h_2 \in H'_{\omega}$.

(c) Let $\cdot: H'_{\omega} \to H'_{\omega}$ be the ring involution defined by $T_w \to T_{w^{-1}}$, ($w \in W'_{\omega}$) and $u^{1/2} = u^{-1/2}$. Let $\cdot: \Gamma_\mathcal{G} \otimes_\mathcal{A} \mathcal{A}$ be the group involution defined by $[A] \otimes u^{1/2} \to [A] \otimes u^{-1/2}$. Then $\varepsilon(h) = \varepsilon(h)$ for all $h \in H'_{\omega}$.

First, note that $\delta$ is surjective. Indeed, given $\omega \in \Omega_{\omega}$, and a sequence $\sigma_1, \sigma_2, ..., \sigma_a$ in $S_{\omega}$, we consider reduced expressions $\omega = t_1, t_2, ..., t_p$, $\sigma_j = \tau_{j_1} \tau_{j_2} \cdots \tau_{j_{r_j}} \cdots \tau_{j_2} \tau_{j_1}$ ($1 \leq j \leq a$) in $S$, and let $s$ be the sequence

$$
(t_1, t_2, ..., t_p, \tau_{11}, \tau_{12}, ..., \tau_{1r_1}, ..., \tau_{11}, \tau_{21}, \tau_{22}, ..., \tau_{2r_2}, ..., \tau_{a1}, \tau_{a2}, ..., \tau_{a r_a}, ..., \tau_{a1})
$$

in $S$. It is easy to see that $\delta(s) = T_\omega (1 + T_{\sigma_1}) \cdots (1 + T_{\sigma_a})$; these elements clearly generate $H'_{\omega}$ as an $\mathcal{A}$-module, so that $\delta$ is surjective.

It follows that $\varepsilon$ is unique (if it exists). Assume that (a) is already proved. To prove (c), it is enough in view of surjectivity of $\delta$ to show that $\varepsilon(\omega e_s) = \varepsilon(e_s \omega)$ for all basis elements $e_s$ of $\tilde{H}'_{\omega}$. We have $\delta e_s = u^{-(m + \dim G)} \delta e_s$, since $1 + T_{\sigma_{1}} = u^{-1}(1 + T_{\sigma_1})$, $\tilde{T}_\omega = T_\omega$. Hence, we must check that $u^{-m} \varepsilon(\omega e_s) = \varepsilon(e_s \omega)$, $(m' = m + \dim G)$ or, equivalently, that $u^{-m'} \gamma(e_s) = \gamma(e_s)$. This is equivalent to the statement 2.17(b). Thus, (c) follows from (a).
It remains to prove (a) and (b). The statement (a) is a consequence of the following statement:

(6.2.1) Let \((e_1, e_2, \ldots, e_r)\), \((e_1', e_2', \ldots, e_r')\) be two sequences of basis elements of \(\tilde{H}'\) and let \((n_1, n_2, \ldots, n_r)\), \((n_1', n_2', \ldots, n_r')\) be two sequences of integers. Assume that

\[
\sum_{i=1}^r e_i \otimes u^{n_{i/2}} - \sum_{i=1}^{r'} e_i' \otimes u^{n_{i/2}}
\]
is in the kernel of \(\delta: \tilde{H}' \to H'\). Then for any integer \(j\), the perverse sheaves \(\oplus_{i=1}^r p\tilde{H}^{l-n_i}(K'^e_i)\) and \(\oplus_{i=1}^{r'} p\tilde{H}^{l-n_i}(K'^e_i)\) are isomorphic in \(\mathcal{C}G\).

By general principles [1, Sect. 6], the statement (6.2.1) for general \(k\) is a consequence of the statement (6.2.1) for \(k\) an algebraic closure of a finite field. The same applies to (b). Thus, it is enough to prove (6.2.1) and (b) in the special case where \(k\) is an algebraic closure of a finite field.

6.3. We now prove (6.2.1) under the assumption that \(k\) is an algebraic closure of a finite field. Since the two perverse sheaves in (6.2.1) are semisimple (2.17(a)), they are isomorphic if and only if they define the same element of the Grothendieck group \(\mathcal{C}G\) of the abelian category \(\mathcal{A}G\). Hence, if \(\rho: \Gamma \to \mathcal{C}G\) is the natural homomorphism, it is enough to prove that there exists an \(\mathcal{A}'\)-linear map \(\epsilon': \mathcal{A}' \to \mathcal{A}G\) such that

(6.3.3) \(\epsilon' \delta = (\rho \otimes 1) \gamma\).

We may regard \(K'^e, K'^e, K'_e\) as well as the complexes and morphisms appearing in 2.13–2.16 (for fixed \(\mathcal{L}\)) as being in the derived category of mixed complexes over \(G_0\) (a split \(F_q\)-form of \(G\)) with \(B, T\) defined over \(F_q\), for a sufficiently large \(F_q \subset k\). Then the \(pH^l\) of these complexes will have natural weight filtrations (see [1, 5.3.5]) whose subquotients (denoted \(pH^j\)) are pure perverse sheaves of weight \(j\). For any mixed complex \(K\) on \(G_0\), we define

\[
\chi_u(K) = \sum_{i,j} (-1)^i \langle pH^j(K) \rangle \otimes u^{j/2} \in \mathcal{A}G \otimes_{\mathcal{A}} \mathcal{C}.
\]

Here \(\langle pH^j(K) \rangle\) denotes the image of \(pH^j(K)\) in the Grothendieck group \(\mathcal{C}G\).

We define an \(\mathcal{A}'\)-linear map \(\epsilon': \mathcal{A}' \to \mathcal{A}G \otimes_{\mathcal{A}} \mathcal{C}\) by

(6.3.2) \(\epsilon'(T_{\sigma}) = \chi_u(K'^e) u^{{-\ell(\sigma) + T(\sigma) - \dim G)/2}}\).

Let \(s = (s_1, s_2, \ldots, s_r)\) be a sequence in \(S \cup \{e\}\) such that \(s_1, s_2, \ldots, s_r \in W'\), and let \(\omega, \sigma_1, \sigma_2, \ldots, \sigma_r\) be the elements of \(W'\) associated to \(s\) in 5.5. We shall prove by induction on \(m = \#(i \in [1, r] | s_i \neq e)\) that

(6.3.3) \(\chi_u(K'^e) = u^{(m-a+\dim G)/2} \epsilon'(T_\omega T_{\sigma_1} T_{\sigma_2} \cdots T_{\sigma_a})\).
We can assume that all $s_i$ are in $S$ by dropping the ones which are $e$. Then $m = r$.

When $m = 0$, we have $K_{s_{i}}^{a_{i}} = K_{w_{i}}^{a_{i}}$, $a = 0$, and (6.3.3) follows from (6.3.2).

Assume now that $m \geq 1$ and that (6.3.3) is already known for sequences $m$ replaced by $m' < m$.

Assume first that $l(s_{1}, s_{2}, \ldots, s_{r}) = r$ so that $K_{s_{i}}^{a_{i}} = K_{w_{i}}^{a_{i}}$, where $w = s_{1} s_{2} \cdots s_{r}$ (see 2.11, (2.5.1)). By 5.6, we have $\tilde{l}(\omega) = 0$, $\tilde{l}(\sigma) = \cdots = \tilde{l}(\sigma_{a}) = 1$, $\tilde{l}(\omega \sigma_{1} \sigma_{2} \cdots \sigma_{a}) = a$, hence $T_{\omega} T_{\sigma_{1}} T_{\sigma_{2}} \cdots T_{\sigma_{a}} = T_{\omega \sigma_{1} \sigma_{2} \cdots \sigma_{a}} = T_{s_{1} s_{2} \cdots s_{r}} = T_{w}$ so that

$$u^{(m-a+\dim G)/2} e'\left(T_{\omega} T_{\sigma_{1}} T_{\sigma_{2}} \cdots T_{\sigma_{a}}\right) = u^{(l(w)-l(w)+\dim G)/2} e'\left(T_{w}\right) = \chi_{a}(K_{w}^{a_{i}}) = \chi_{a}(K_{s_{i}}^{a_{i}})$$

as required.

Assume next that $l(s_{1}, s_{2}, \ldots, s_{r}) < r$. Then we can find $h$ $(2 \leq h \leq r)$ such that $s_{h} \cdots s_{r-1} s_{r}$ is a reduced expression and $s_{h-1} s_{h} \cdots s_{r}$ is not a reduced expression. We can find $s_{h}', \ldots, s_{h-1}', s_{r}'$ in $S$ such that $s_{h}' \cdots s_{r-1}' s_{r}' = s_{h} \cdots s_{r-1} s_{r}$ and $s_{h}' = s_{h-1}'$.

Let $\sigma = (s_{1}, s_{2}, \ldots, s_{h-1}, s_{h}', \ldots, s_{r-1}, s_{r}')$. As shown in 2.16, we have $K_{s_{i}}^{a_{i}} = K_{\sigma_{i}}^{a_{i}}$; hence $\chi_{a}(K_{s_{i}}^{a_{i}}) = \chi_{a}(K_{\sigma_{i}}^{a_{i}})$. The definition 5.5 of $\omega$, $\sigma_{1}, \ldots, \sigma_{a}$ attached to $s$ can be also applied to $\sigma$ instead of $s$, and it leads to the same sequence $\omega$, $\sigma_{1}, \ldots, \sigma_{a}$. Hence to prove (6.3.3) for $s$ it is enough to prove it for $\sigma$. Thus, we are reduced to the case where $s$ satisfies $s_{h-1} = s_{h}$. In this case, we shall use the notations in 2.15.

If $h \notin I_{s}$, then from (2.15.4) we have $\chi_{a}(K_{s_{i}}^{a_{i}}) = u \cdot \chi_{a}(K_{s_{i}}^{a_{i}})$. By the induction hypothesis, we have $\chi_{a}(K_{s_{i}}^{a_{i}}) = u^{(r-2-a+\dim G)/2} e'(T_{\omega} T_{\sigma_{1}} \cdots T_{\sigma_{a}})$ hence $\chi_{a}(K_{s_{i}}^{a_{i}}) = u^{(r-a+\dim G)/2} e'(T_{\omega} T_{\sigma_{1}} \cdots T_{\sigma_{a}})$.

If $h \in I_{s}$, then from (2.15.2) we have

$$\chi_{a}(K_{s_{i}}^{a_{i}}) = \chi_{a}((\pi_{s_{i}})^{a_{i}}) + u \chi_{a}(K_{s_{i}}^{a_{i}})$$

and from (2.15.3) we have

$$u \chi_{a}(K_{s_{i}}^{a_{i}}) = \chi_{a}((\pi_{s_{i}})^{a_{i}}) + \chi_{a}(K_{s_{i}}^{a_{i}}).$$

(Indeed, since weight filtrations are strictly compatible with morphisms [1, 5.3.5] the exact sequences (2.15.2), (2.15.3) remain exact when each $pH^{j}$ is replaced by $pH^{j}$ for fixed $j$.) It follows that

$$\chi_{a}(K_{s_{i}}^{a_{i}}) = u \chi_{a}(K_{s_{i}}^{a_{i}}) + (u-1) \chi_{a}(K_{s_{i}}^{a_{i}}).$$
The induction hypothesis is applicable to $K_s^{\varphi}$, $K_s^{\varphi'}$:

$$
\chi_u(K_s^{\varphi}) = u^{(r-1)-(a-1)+\dim G/2} e'(T_\omega T_{\sigma_1} \cdots T_{\sigma_{h-1}} \hat{T}_{\sigma_h} \cdots T_{\sigma_a}),
$$

$$
\chi_u(K_s^{\varphi'}) = u^{(r-2)-(a-2)+\dim G/2} e'(T_\omega T_{\sigma_1} \cdots \hat{T}_{\sigma_{h-1}} \cdots T_{\sigma_a}).
$$

Moreover, we have $\sigma_{h-1} = \sigma_h$ so that $uT_{\sigma_{h-1}} + (u-1)T_{\sigma_h} = T_{\sigma_{h-1}} T_{\sigma_h}$. Hence

$$
\chi_u(K_s^{\varphi'}) = u^{(r-a+\dim G)/2} e'(T_\omega T_{\sigma_1} \cdots T_{\sigma_a}),
$$

as required. Thus, (6.3.3) is proved.

We now prove that with the notation in (6.3.3), we have

$$
(6.3.4) \quad \chi_u(K_s^{\varphi'}) = u^{(m-a+\dim G)/2} e'(T_\omega (1 + T_{\sigma_1}) (1 + T_{\sigma_2}) \cdots (1 + T_{\sigma_a})).
$$

We shall use the notation in 2.13. From (2.13.1) (or rather, from the corresponding exact sequences obtained by considering the subquotients of fixed weight of the weight filtrations), we get

$$
\chi_u(\prod_{i \in I_s} \sigma_i^{(i)} \Phi_i^{(i)} \Phi_i^{(i)} \ast \varphi') = \chi_u(\prod_{i \in I_s} \sigma_i^{(i)} \Phi_i^{(i+1)} \ast \varphi') + \sum_{|J| = 1} \chi_u(K_s^{\varphi'}),
$$

for any $i$.

Summing these equalities over all $i$, $0 \leq i \leq |I_s|$ and taking into account (2.13.2), we find

$$
\chi_u(K_s^{\varphi'}) = \sum_{J \subseteq I_s} \chi_u(K_s^{\varphi'}).
$$

We now use (6.3.3) for each $s_J$ is the last sum, and 5.6(c); (6.3.4) follows.

The mixed complex $K_s^{\varphi'}$ is pure of weight 0 (see the proof of 3.7) hence

$$
\rho H^j(K_s^{\varphi'}) = \begin{cases} 
\rho H^j(K_s^{\varphi'}) & \text{if } j = i, \\
0 & \text{if } j \neq i.
\end{cases}
$$

It follows that $\chi_u(K_s^{\varphi'}) = (\rho \otimes 1) \gamma(e_s)$. On the other hand, the right-hand side of (6.3.4) is equal to $e'(\delta e_s)$. Hence (6.3.4) implies (6.3.1). This completes the proof of 6.2(a).

6.4. We shall now prove 6.2(b) assuming again (without loss of generality) that $k$ is an algebraic closure of a finite field. We again place ourselves in the setup of 6.3. It is enough to prove the following statement:

$$
(6.4.1) \quad \text{Let } s = (s_1, s_2, \ldots, s_r), \ s' = (s'_1, s'_2, \ldots, s'_r) \text{ be two sequences in } S \text{ as in 5.7 and let } (\omega, \sigma_1, \sigma_2, \ldots, \sigma_a), (\omega', \sigma'_1, \sigma'_2, \ldots, \sigma'_a) \text{ be the sequence in } W_\varphi.
$$
attached to them in 5.5. Then 
\[
\varepsilon'(\prod_{a} T_{a}) = \varepsilon'(\prod_{a} T_{a}').
\]
Let \( s' \), \( s' \) be defined as in (2.19.1). Using 5.7 and (6.3.3) we see that the equality (6.4.1) is equivalent to the equality

\[
\chi_{u}(K_{s'}) = \chi_{u}(K_{s}).
\]
But this follows from (2.19.1).

This completes the proof of Proposition 6.2.

6.5. Let us define for any \( K \in G \),

\[
\chi(K) = \sum_{i} (-1)^{i} \varepsilon_{i}(K) \in G.
\]

The proofs in 6.3 and 6.4 (specialized for \( u = 1 \)) give the following result:

Let \( \varepsilon' \colon \mathbb{Z}[W_{\mathcal{F}}] \to \mathcal{H}(G) \) be the homomorphism defined by \( \varepsilon'_{1}(w) = \chi(K_{s'}) \).

Then \( \varepsilon' \) is constant on conjugacy classes in \( W_{\mathcal{F}} \). With the notations in (6.3.3), we have

\[
\chi'(K) = \varepsilon'(s_{1}, s_{2}, \ldots, s_{r}) = \varepsilon'(\omega_{1} \sigma_{1} \cdots \sigma_{a}),
\]

\[
\chi(K_{s'}) = \varepsilon'(\omega(1 + \sigma_{1})(1 + \sigma_{2}) \cdots (1 + \sigma_{a})).
\]

6.6. We now return to the setting in 3.1. Let \( s = (s_{1}, s_{2}, \ldots, s_{r}) \) be a sequence in \( S \) such that \( s_{1}, s_{2}, \ldots, s_{r} \in W'_{\mathcal{F}} \) (\( \mathcal{F} \subset \mathcal{F}(T) \)). We apply the functor \( \mathcal{F} \to D \) to \( K_{s} \). We wish to describe \( H'_{L}(\mathcal{F}(T)) \) in terms of the function \( \varepsilon'_{L} : H'_{L} \to \mathcal{H}(L) \) (defined as \( \varepsilon \) in 6.2, for \( L \) instead of \( G \)); here \( H'_{L} \), \( H'_{L} \) is \( H'_{L} \) defined with respect to \( L \) instead of \( G \). We shall denote by \( \mathcal{F} \) (resp. \( \mathcal{F}_{L} \)) the \( \mathcal{F} \)-submodule of \( H'_{L} \) (resp. \( H'_{L} \)) spanned by the elements \( T_{w}, w \in W'_{\mathcal{F}} \), (resp. by the elements \( T_{w}, w \in W'_{\mathcal{F}} \)).

We shall denote by \( \mathcal{F} \) the set of elements \( y_{0} \) in \( W \) which have minimal length in their \( W^{*} - W'_{\mathcal{F}} \) double coset.

Let \( \omega, \sigma_{1}, \sigma_{2}, \ldots, \sigma_{a} \) be the sequence in \( W'_{\mathcal{F}} \) attached to \( s \) in 5.5. Thus, \( \omega \in \Omega_{\mathcal{F}}, \sigma_{i} \in S_{\mathcal{F}} \). If \( y_{0} \in \mathcal{F} \), then \( \omega' = y_{0} \omega y_{0}^{-1} \) is in \( \Omega_{\mathcal{F}}, \sigma_{i}' = y_{0} \sigma_{i} y_{0}^{-1} \) are in \( S_{\mathcal{F}} \), where \( \mathcal{F}' = (y^{-1})*\mathcal{F} \). If we assume that \( \omega' \in W^{*} \), then conjugation by \( \omega' \) is an automorphism of the Coxeter group \( (W'_{\mathcal{F}}), S_{\mathcal{F}} \) leaving stable its length function \( \tilde{n} \) and its parabolic subgroup \( W'_{\mathcal{F}} \). Hence it also leaves stable the set

\[
\mathcal{F} = \{ z \in W'_{\mathcal{F}} \mid z \text{ has minimal } \tilde{n} \text{-length in the coset } (W^{*} \cap W'_{\mathcal{F}})z \}.
\]

For any \( h \in H'_{\mathcal{F}}, \) and any \( z \in \mathcal{F} \), there are well-defined elements \( x_{z}z'(h) \in H'_{L} \) (\( z' \in \mathcal{F} \)), such that

\[
T_{w}^{-1}z_{w} = \sum_{z' \in \mathcal{F}} x_{z}z'(h) T_{z}, \text{ in } H'_{L}.
\]

(Indeed, \( H_{L} \) is free as a left \( H'_{L} \) module with basis \( z_{z}', z' \in \mathcal{F} \).)
With these notations, we set

\[(6.6.2) \gamma(y_0) = \epsilon_{\mathscr U, L}(T_{\omega}) \sum_{x_{\epsilon, L}} x_{\epsilon, L}((1 + T_{\omega})(1 + T_{\omega}^{-1}) \cdots (1 + T_{\sigma_d})) \in \mathcal{G}_L \otimes Z \mathcal{A}.
\]

We can now state:

**PROPOSITION 6.7.** The following identity holds in \(\mathcal{G}_L \otimes Z \mathcal{A}:

\[(6.7.1) \sum_{\epsilon} (-1)^{\epsilon} [\frac{H}{\mathcal{A}(\mathcal{G})}] u^{(j - m')/2} = \sum_{y_0} \gamma(y_0) u^{-a/2}.
\]

where \(y_0\) runs over all elements of \(\mathcal{F}\) such that \(y_0 \omega y_0^{-1} \in W^*\), and \(m' = r + \dim G\).

**Proof.** We shall give the proof in the case where \(\mathcal{L}\) is the constant sheaf \(\mathcal{Q}_l\). In this case we have \(W = W' = W'_L\); we denote \(H = H' = H'_L\), \(H_L = H'_L\), \(\mathcal{A}_L = \mathcal{A}'_L\). We have also \(\omega = e, a = r, \sigma_i = s_i (1 \leq i \leq r)\).

Using (3.7(c), 3.5, and (3.8.1) we see that the left-hand side of (6.7.1) is equal to

\[(6.7.2) \sum_{\epsilon} \sum_{\gamma} (-1)^{\epsilon} [\frac{P}{\mathcal{A}(\mathcal{G})}] u^{(j - m')/2},
\]

where \(\gamma\) runs over all sequences \(\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_j)\) of \(P\)-orbits on \(\mathcal{F}\) satisfying (3.3.1), \(d(\gamma)\) is defined by (3.4.1), \(t\) is the sequence in \(S^* \cup \{e\}\) defined in (3.3.4), and \(\tau\) is the sequence in \(S^*\) obtained from \(t\) by dropping all \(t_i\) which are equal to \(e\). (Thus, \(\tau\) is completely determined by \(\gamma\).) We denote by \(\mathcal{K}\), the complex in \(\mathcal{G}_L\) defined in 2.8 in terms of \(\tau, \mathcal{L} = \mathcal{Q}_l, L\).

Now using (6.2(a), we can rewrite (6.7.2) as

\[(6.7.3) \epsilon_L (\sum_{\gamma} (1 + T_{\tau_1})(1 + T_{\tau_2}) \cdots (1 + T_{\tau_j}) u^{d(\gamma) - \dim U_\gamma} u^{-r/2},
\]

where \(\gamma\) and \(\tau = (\tau_1, \tau_2, \ldots, \tau_j)\) are as in (6.7.2).

For any \(y \in W^*, s \in S\), we have

\[T_y (1 + T_s) = \begin{cases} T_y + T_{ys}, & \text{if } ys \in W^*, ys > y, \\
u(T_y + T_{ys}), & \text{if } ys \in W^*, ys \leq y, \\
(1 + T_{ysy^{-1}}) T_y, & \text{if } ys \in W^* (\text{so that } ysy^{-1} \in S^*).\end{cases}
\]

Applying this repeatedly, we see that for any \(y \in W^*,\) we have

\[(6.7.4) T_y (1 + T_{\tau_1})(1 + T_{\tau_2}) \cdots (1 + T_{\tau_j}) = \sum_{\gamma} a^{(\gamma)} (1 + T_{\tau_1})(1 + T_{\tau_2}) \cdots (1 + T_{\tau_j}) T_{\gamma};
\]

here, the sum is over all sequences \(\gamma = (y_{\gamma_0}, y_{\gamma_1}, \ldots, y_{\gamma_j})\) in \(W^*\) such that \(y = y_0, W^* y_i = W^* y_{i-1}\) or \(W^* y_{i-1} s_i (1 \leq i \leq r)\), \(\delta(y)\) is defined to be \#\{i \in [1, r] | y_i > y_i s_i, y_i s_i \in W^*\} and \(\tau = (\tau_1, \tau_2, \ldots, \tau_j)\) is the sequence in \(S^*\) consisting of those terms in \((y_{\gamma_0} s_i y_{\gamma_i}^{-1}, y_{\gamma_1} s_1 y_{\gamma_1}^{-1}, \ldots, y_{\gamma_r} s_r y_{\gamma_r}^{-1})\) which are in \(S^*.\)
Now (6.7.1) (in the case $\mathcal{L} = \mathcal{Q}_l$) follows directly from (6.7.3), (6.7.4), and the definition (6.6.2) of the function $\gamma$. The proof in the general case is similar, but the notation is more complicated; we shall omit it.

**Corollary 6.8.** For any $j$, we have $^pH^j(\text{res } \bar{K}_s^z) \cong ^pH^{2m'-j}(\text{res } \bar{K}_s^z)$ in $\mathcal{D}L$, where $m' = r + \dim G$.

**Proof.** Since $^pH^j(\text{res } \bar{K}_s^z)$ are semisimple objects of $\mathcal{M}L$ for all $j$, it is enough to show that $[^pH^j(\text{res } \bar{K}_s^z)] = [^pH^{2m'-j}(\text{res } \bar{K}_s^z)]$ (equality in $\Gamma_L$). By (6.7.1), it is then enough to show that for any $y_0$ in the sum (6.7.1), the expression $\gamma(y_0) u^{-a/2} \in \mathcal{F}_L \otimes \mathcal{A}$ is fixed by the involution $-^\ast$ of $\mathcal{F}_L \otimes \mathcal{A}$ defined in 6.2(c). Since $\varepsilon_{\mathcal{F}_L}$ commutes with the involutions $-^\ast$ (see 6.2(c)) and $(1 + T_{a_1}) \cdots (1 + T_{a_d}) u^{-a/2}$ is fixed by the bar involution, we see that it is enough to prove the following statement:

\begin{equation}
(6.8.1) \quad \varepsilon_{\mathcal{F}_L} = \sum_{z \in \mathcal{F}} x_{z,z}(h) = \varepsilon_{\mathcal{F}_L} (\sum_{z \in \mathcal{F}} x_{z}(\bar{h})) \quad \text{for all } h \in H_{s',L} \text{ (notation as in 6.6)}. \end{equation}

Applying the involution $-^\ast : H_{s',L} \to H_{s',L}$ to the identity (6.6.1), we get

\begin{equation}
(6.8.2) \quad \bar{h} = \sum_{z \in \mathcal{F}} \overline{x_{z,z}(h)} \bar{T}_z \quad (z \in \mathcal{F}). \end{equation}

Since $(T_z)_{-\omega}(\bar{h})$, $(z \in \mathcal{F})$, form two bases of $H_{s',s'}$, as a free left $H_{L,\mathcal{A}_{s'}}$ module, we have

\begin{equation}
(6.8.3) \quad \bar{T}_z = \sum_{z \in \mathcal{F}} r_{z,z} T_{z'}, T_{z} = \sum_{z \in \mathcal{F}} q_{z,z} \bar{T}_z \quad (z \in \mathcal{F}), \end{equation}

where $r_{z,z'} \in H_{L,\mathcal{A}_{s'}}$, $q_{z,z'} \in H_{L,\mathcal{A}_{s'}}$.

Introducing in (6.8.2) we get

\begin{align*}
\sum_{z \in \mathcal{F}} x_{z,z}(h) \bar{T}_z & = \sum_{z \in \mathcal{F}} r_{\omega^{-1}z,\omega z} T_{z'} \bar{h} \\
& = \sum_{z, z'' \in \mathcal{F}} r_{\omega^{-1}z,\omega z} x_{z,z''} \omega_{\omega^{-1}z} = \sum_{z \in \mathcal{F}} \overline{x_{z,z}(h)} \bar{T}_z. \\
& = \sum_{z, z'' \in \mathcal{F}} \overline{x_{z,z}(h)} \sum_{z, z'' \in \mathcal{F}} q_{z,z} \bar{T}_z.'
\end{align*}

From this, we deduce

\begin{equation}
\sum_{z \in \mathcal{F}} x_{z,z}(h) = \sum_{z, z'' \in \mathcal{F}} \overline{x_{z,z}(h)} \sum_{z, z'' \in \mathcal{F}} q_{z,z} \bar{T}_z.'
\end{equation}
Multiply both sides with $T_\omega$, and apply $\varepsilon_{\mathcal{L}',L}$:

$$
\varepsilon_{\mathcal{L}',L} \left( T_\omega \sum_{z \in \mathcal{L}} x_{z,L}(h) \right)
= \varepsilon_{\mathcal{L}',L} \left( T_\omega \sum_{z,z' \in \mathcal{L}, z' \in \mathcal{L}} \frac{r_{\omega^{-1} z \omega', z''} x_{\omega', z''}}{q_{z', z''}} \left( h \right) \right)
= \varepsilon_{\mathcal{L}',L} \left( T_\omega \sum_{z,z' \in \mathcal{L}, z' \in \mathcal{L}} \frac{q_{z', z''} T_\omega, r_{\omega^{-1} z \omega', z''} x_{\omega', z''}}{z''} \left( h \right) \right) \text{ by 6.2(b)}
= \varepsilon_{\mathcal{L}',L} \left( T_\omega \sum_{z,z' \in \mathcal{L}, z' \in \mathcal{L}} \delta_{\omega^{-1} z \omega', z''} x_{\omega', z''} \left( h \right) \right)
= \varepsilon_{\mathcal{L}',L} \left( T_\omega \sum_{z \in \mathcal{L}} x_{z,L}(h) \right).
$$

This proves (6.8.1) and hence the corollary.

We can now state

**Theorem 6.9.**

(a) If $L \subset P$ is as in 3.1, and $A \in \hat{G}$, then $\text{res } A \in \mathcal{M}L$; moreover, $\text{res } A$ is semisimple and its irreducible components are in $\mathcal{L}$.

(b) $A \in \hat{G}$ is cuspidal (see 3.10) if and only if for any $L \subset P$ as in 3.1 with $P \neq G$, we have $\text{res } A = 0$.

**Proof:** Assuming that (a) holds, the proof of (b) is immediate: if $A \in \hat{G}$ is cuspidal, then $^{\mathcal{H}}H^i(\text{res } A) = 0$ for all $i \geq 0$ and by (a), $^{\mathcal{H}}H^i(\text{res } A) = 0$ for all $i \neq 0$; hence $^{\mathcal{H}}H^i(\text{res } A) = 0$ for all $i$, so that $\text{res } A = 0$.

We now prove that in (a), we have $\text{res } A \in \mathcal{M}L$ for $A \in \hat{G}$. (The other statement in (a) follows from 3.9.) Let $s = (s_1, \ldots, s_r)$ be a sequence in $S$ such that $s_1 s_2 \cdots s_r \in W' \subset \mathcal{L}' \subset \mathcal{L}(T)$.

Let $K = K'_s \left[ m' \right]$, $K' = \text{res } K$, $K_i = ^{\mathcal{H}}H^i K$, $K'_i = \text{res } K_i$, ($m' = r + \dim G$). It is enough to prove that $K'_i \in \mathcal{M}L$ (since res $A$ may be assumed to be direct summand of $K'_i$).

Fix $A' \in \mathcal{L}$ and let $b_{ij}$ be the multiplicity of $A'$ in $^{\mathcal{H}}H^i(K'_j)$. Then $b_{ij} \geq 0$ and it is enough to prove that $b_{ij} = 0$ whenever $j \neq 0$. Let $b_j$ be the multiplicity of $A'$ in $^{\mathcal{H}}H^j(K')$. From 2.17(a), we have $^{\mathcal{H}}H^j(K') = \sum_i b_i \cdot [\text{res } K_i \left[ -i \right]] = \sum_i ^{\mathcal{H}}H^{i-j}(K'_i)$, hence $b_j = \sum_i b_{i,j-1}$. From 6.8, we have $b_j = b_{-j}$ for all $j$, hence

$$
0 = \sum_j jb_j = \sum_{i,j} jb_{i,j-1} = \sum_{i,j} (i + j) b_{i,j}.
$$
From 2.17(b), we have $K_i = K_{-i}$; it follows that $b_{ij} = b_{-i,j}$, so that $\sum_{i,j} ib_{i,j} = 0$. Introducing this into (6.9.1), we find $\sum_{i,j} jb_{ij} = 0$.

From 4.4(c) and 2.17(a) we see that $b_{ij} = 0$ for all $j > 0$. Therefore, we have $\sum_{i,j \leq 0} jb_{ij} = 0$. Since $jb_{ij} \leq 0$ for all terms in the previous sum, we must have $jb_{ij} = 0$ for all $i, j$. It follows that $b_{ij} = 0$ for $j \neq 0$ and the theorem is proved.

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