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On an Explicitly Soluble System of Nonlinear Differential Equations Related to Certain Toda Lattices

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In our paper [1] we introduced a system of nonlinear differential equations which in a certain sense was an analog of the Korteweg–de Vries equation. Our system was discovered by a probabilistic analogy that, in part at least, also explained why exponential unharmonicity is the natural one.

The present note shows how simply the semi-infinite and the finite cases fit into the inverse scattering scheme yielding at the same time alternative (and independently arrived at) derivations of some results recently obtained by J. Moser [2].

At this point we should like to acknowledge our debt to H. Flaschka [3] who first solved the doubly infinite Toda lattice by applying a discrete version of the inverse scattering problem. The strategy we use is essentially that of Flaschka although the details of execution are somewhat different.

If gives us particular pleasure to include this note in a volume dedicated to S. M. Ulam because it is a direct, though by far not the most illustrious, descendant of the classic Fermi, Pasta, Ulam paper.

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1

Consider the system of nonlinear differential equations

$$dR_1/dt = -e^{-R_2(t)}, \tag{1.1a}$$

$$dR_n/dt = e^{-R_{n-1}(t)} - e^{-R_{n+1}(t)}, \qquad n \geqslant 2, \tag{1.1b}$$

and let $Q(t)$ be the (semi-infinite) matrix

$$Q(t) = \begin{pmatrix} 0 & \frac{1}{2}e^{-\frac{1}{2}R_1(t)} & 0 & 0 & \cdots \\ \frac{1}{2}e^{-\frac{1}{2}R_1(t)} & 0 & \frac{1}{2}e^{-\frac{1}{2}R_2(t)} & 0 & \cdots \\ 0 & \frac{1}{2}e^{-\frac{1}{2}R_2(t)} & 0 & \frac{1}{2}e^{-\frac{1}{2}R_3(t)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix} \tag{1.2}$$

If $B(t)$ is the antisymmetric matrix whose elements are given by the formulas

$$B_{k,k+2} = \tfrac{1}{2}e^{-\frac{1}{2}(R_k(t)+R_{k+1}(t))}, \tag{1.3a}$$

$$B_{k,k-2} = -\tfrac{1}{2}e^{-\frac{1}{2}(R_{k-2}(t)+R_{k-1}(t))}, \quad \text{and} \tag{1.3b}$$

$$B_{k,l} = 0 \qquad \text{otherwise}, \tag{1.3c}$$

then we can check immediately that

$$dQ/dt = BQ - QB. \tag{1.4}$$

We can thus use the method of Lax [2] and define unitary matrices $V(t)$ by the equation

$$dV(t)/dt = BV, \qquad V(0) = I. \tag{1.5}$$

We then verify directly that

$$V^{-1}(t)\,Q(t)\,V(t) = Q(0), \tag{1.6}$$

and thus if $\psi_0(\lambda; n)$ $(n = 1, 2, ...)$ is such that

$$Q(0)\,\psi_0(\lambda; n) = \lambda\psi_0(\lambda; n), \tag{1.7}$$

then

$$\psi_t(\lambda; n) = \sum_{j=1}^{\infty} V_{nj}(t)\,\psi_0(\lambda; j) \tag{1.8}$$

clearly satisfy

$$Q(t)\,\psi_t(\lambda; n) = \lambda\psi_t(\lambda; n). \tag{1.9}$$

It now follows from (1.8) that

$$\frac{d}{dt}\psi_t(\lambda; n) = \sum_{j=1}^{\infty} B_{nj}\psi_t(\lambda; j) = B_{n,n+2}\psi_t(\lambda; n+2) + B_{n,n-2}\psi_t(\lambda; n-2). \tag{1.10}$$

For large n and $\lambda = \cos\theta$ we have

$$\psi_t(\lambda; n) \sim A(\theta; t)\, e^{in\theta} + B(\theta; t)\, e^{-in\theta}, \tag{1.11}$$

and since we shall be interested only in solutions for which

$$R_t(n) \to 0, \qquad n \to \infty, \tag{1.12}$$

we must have

$$B_{n,n+2} \sim \tfrac{1}{2}, \qquad B_{n-2,n} \sim -\tfrac{1}{2}, \qquad n \to \infty. \tag{1.13}$$

From (1.10), (1.11) and (1.13) it follows that

$$dA/dt = i(\sin 2\theta)A,$$
$$dB/dt = -i(\sin 2\theta)B$$

so that

$$A(\theta; t) = e^{it\sin 2\theta}\, A(\theta; 0),$$
$$B(\theta; t) = e^{-it\sin 2\theta}\, B(\theta; 0).$$

It thus follows that the phase shift $\delta_t(\theta)$ of $\psi_t(\cos\theta; n)$ is given by the formula

$$\delta_t(\theta) = \delta_0(\theta) + t\sin 2\theta. \tag{1.14}$$

Assuming (for the sake of simplicity only) that there are no bound states we can use formula (5.17) of [5] to determine the spectral function $\rho_t(\lambda)$ of $Q(t)$ in terms of the spectral function $\rho_0(\lambda)$ of $Q(0)$ and the result is

$$\rho_t(\lambda) = \begin{cases} 0, & \lambda < -1, \\ \dfrac{\int_{-1}^{\lambda} e^{4t\mu^2}\, d\rho_0(\mu)}{\int_{-1}^{1} e^{4t\mu^2}\, d\rho_0(\mu)}, & -1 \leqslant \lambda \leqslant 1, \\ 1, & \lambda > 1. \end{cases} \tag{1.15}$$

We determine orthogonal plynomials $\phi_t(\lambda; n)$ such that

$$\int_{-\infty}^{\infty} \phi_t(\lambda; m)\, \phi_t(\lambda; n)\, d\rho_t(\lambda) = \delta_{mn} \tag{1.16}$$

($\phi_t(\lambda; n)$ is of degree $n - 1$) and obtain the solution to our problem (1.1) in the formulas

$$\tfrac{1}{2}e^{-\frac{1}{4}R_k(t)} = \int_{-\infty}^{\infty} \lambda\phi_t(\lambda; k)\, \phi_t(\lambda; k+1)\, d\rho_t(\lambda). \tag{1.17}$$

It is easy to see that, even if there are bound states, formula (1.15) is still valid in the slightly modified form

$$\rho_t(\lambda) = \frac{\int_{-\infty}^{\lambda} e^{4t\mu^2}\, d\rho_0(\mu)}{\int_{-\infty}^{\infty} e^{4t\mu^2}\, d\rho_0(\mu)}\, . \tag{1.18}$$

2

Having found the solution of the system (1.1) by an application of a reasonably sophisticated method, we may note that a direct verification is extremely simple.

Denoting by $\mu_{2k}(t)$ the even moments of $\rho_t(\lambda)$, i.e.,

$$\mu_{2k}(t) = \int \lambda^{2k}\, d\rho_t(\lambda), \tag{2.1}$$

we find that the first few orthonormal polynomials $\phi_t(\lambda; m)$ are

$$\phi_t(1; \lambda) = 1,$$

$$\phi_t(2; \lambda) = \frac{\lambda}{(\mu^2(t))^{1/2}}\, ,$$

$$\phi_t(3; \lambda) = \frac{\lambda^2 - \mu_2}{(\mu_4 - \mu_2{}^2)^{1/2}}\, ,$$

$$\phi_t(4; \lambda) = \frac{(\mu_2)^{1/2}}{(\mu_2\mu_6 - \mu_4{}^2)^{1/2}} \left(\lambda^3 - \frac{\mu_4}{\mu_2}\lambda\right)$$

etc.

Thus, e.g.,

$$\tfrac{1}{2}e^{-\frac{1}{2}R_1(t)} = \int \lambda\phi_t(1;\lambda)\,\phi_t(2;\lambda)\,d\rho_t(\lambda) = (\mu_2(t))^{1/2}$$

$$\tfrac{1}{2}e^{-\frac{1}{2}R_2(t)} = \int \lambda\phi_t(2;\lambda)\,\phi_t(3;\lambda)\,d\rho_t(\lambda) = \frac{(\mu_4(t) - \mu_2^2(t))^{1/2}}{(\mu_2(t))^{1/2}},$$

$$\tfrac{1}{2}e^{-\frac{1}{2}R_3(t)} = \int \lambda\phi_t(3;\lambda)\,\phi_t(4;\lambda)\,d\rho_t(\lambda) = \frac{(\mu_2(t)\,\mu_6(t) - \mu_4^2(t))^{1/2}}{(\mu_2(t))^{1/2}\,(\mu_4(t) - \mu_2^2(t))^{1/2}},$$

and hence

$$e^{-R_1(t)} = 4\mu_2, \qquad e^{-R_3(t)} = 4\,\frac{\mu_2\mu_6 - \mu_4^2}{\mu_2(\mu_4 - \mu_2^2)}$$

$$R_2(t) = -\log 4 - \log(\mu_4 - \mu_2^2) + \log \mu_2 .$$

Now,

$$\frac{dR_2(t)}{dt} = -\frac{d\mu_4/dt - 2(d\mu_2/dt)}{\mu_4 - \mu_2^2} + \frac{d\mu_2/dt}{\mu_2}$$

and

$$d\mu_2/dt = 4(\mu_4 - \mu_2^2),$$

$$d\mu_4/dt = 4(\mu_6 - \mu_2\mu_4).$$

It is thus easy to see that

$$dR_2(t)/dt = e^{-R_1(t)} - e^{-R_3(t)}$$

and that to check (1.1) in general one only needs the easily verifiable formula

$$d\mu_{2k}/dt = 4(\mu_{2k+2} - \mu_{2k}\mu_2). \tag{2.2}$$

It now becomes clear that for *every distribution function* $\rho_0(\lambda)$ and a correspondingly defined $\rho_t(\lambda)$ (see formula (1.15)) the formulas (1.17) provide the solution of the system (1.1).

If, in particular, $\rho_0(\lambda)$ is purely discontinuous with jumps $\Delta_1, \Delta_2, ..., \Delta_{N-1}$ at $\lambda_1 < \lambda_2 < \cdots < \lambda_{N-1}$[1], $(\Delta_1 + \Delta_2 + \cdots + \Delta_{N-1} = 1)$ $\rho_t(\lambda)$ is also purely discontinuous with jumps at the same points (i.e., $\lambda_1, \lambda_2, ..., \lambda_{N-1}$) but with the jump at λ_i given by the expression

$$\Delta_i(t) = \frac{e^{4\lambda_i^2 t}\,\Delta_i}{\sum_{j=1}^{N-1} e^{4\lambda_j^2 t}\,\Delta_j}. \tag{2.3}$$

[1] It should be clear that the λ's come in positive–negative pairs so that $\lambda_1 = -\lambda_N$, $\lambda_2 = -\lambda_{N-1}$, etc.

It is clear that the eigenvalues λ_i are the roots of $\phi_t(\lambda; N)$, i.e.,

$$\phi_t(\lambda_i; N) = 0, \qquad i = 1, 2,..., N-1, \tag{2.4}$$

and hence

$$\begin{aligned} \tfrac{1}{2}e^{-\frac{1}{2}R_N(t)} &= \int \lambda\phi_t(\lambda; N-1)\,\phi_t(\lambda; N)\, d\rho_t(\lambda) \\ &= 0. \end{aligned} \tag{2.5}$$

In this way we arrive at the solution of the finite system

$$dR_1/dt = -e^{-R_2(t)}, \tag{2.6a}$$

$$dR_k/dt = e^{-R_{k-1}(t)} - e^{-R_{k+1}(t)} \qquad k = 2,..., N-1, \tag{2.6b}$$

$$dR_{N-1}/dt = e^{-R_{N-2}(t)}, \tag{2.6c}$$

which has been suggested by the application of the inverse scattering problem to the solution of the infinite system (1.1).

3

We shall now show how the solution of the system (2.6) yields also the solution to the finite Toda chain with two free ends, a problem that has been recently solved by Moser [2].

Let N be even ($N = 2n$) and set

$$r_k(t) = R_{2k}(t) + R_{2k+1}(t), \qquad k = 1, 2,..., n-1, \tag{3.1}$$

$$p_k = -(e^{-R_{2k}(t)} + e^{-R_{2k-1}(t)}) + \alpha, \qquad k = 1, 2,..., n-1, \tag{3.2a}$$

$$p_n = -e^{-R_{2n-1}(t)} + \alpha, \tag{3.2b}$$

where α is to be defined later on.

Let us finally set

$$r_k = q_{k+1} - q_k, \qquad k = 1, 2,..., n-1, \tag{3.3}$$

and verify at once that

$$dr_k/dt = (dq_{k+1}/dt) - (dq_k/dt) = p_{k+1} - p_k, \qquad k = 1, 2,..., n-1. \tag{3.4}$$

We shall also *require* that

$$dq_n/dt = p_n \tag{3.5}$$

which determines $q_n(t)$ once $q_n(0)$ is given and $p_n(t)$ determined.

Equations (3.4) and (3.5) imply at once that

$$dq_k/dt = p_k\,, \qquad k = 1, 2,..., n. \tag{3.6}$$

Going back to (3.2) and (2.6) we verify that

$$\begin{aligned} dp_1/dt &= -e^{-r_1} = -e^{-(q_2-q_1)}, \\ dp_k/dt &= e^{-r_{k-1}} - e^{-r_k} = e^{-(q_k-q_{k-1})} - e^{-(q_{k+1}-q_k)}, \qquad k = 2,..., n-1, \\ dp_n/dt &= e^{-r_{n-1}} = e^{-(q_n-q_{n-1})}, \end{aligned} \tag{3.7}$$

and hence that (3.6) and (3.7) are the Hamilton equations corresponding to the Hamiltonian

$$\mathscr{H} = \frac{1}{2}\sum_1^n p_k^2 + \sum_{k=1}^{n-1} e^{-(q_{k+1}-q_k)}. \tag{3.8}$$

It remains to show that given $q_k(0)$, $p_k(0)$, $k = 1, 2,..., n$, one can always determine $R_k(0)$, $k = 1, 2,..., 2n-1$. In other words, given $r_k(0) = q_{k+1}(0) - q_k(0)$, $k = 1, 2,..., n-1$ and $p_k(0)$, $k = 1, 2,..., n$ one can find *real* solutions $R_k(0)$ of the equations

$$r_k(0) = R_{2k}(0) + R_{2k+1}(0), \qquad k = 1, 2,..., n-1, \tag{3.9a}$$

$$p_k(0) = -(e^{-R_{2k}(0)} + e^{-R_{2k-1}(0)}) + \alpha, \qquad k = 1, 2,..., n-1, \tag{3.9b}$$

$$p_n(0) = -e^{-R_{2n-1}(0)} + \alpha. \tag{3.9c}$$

It is immediately clear that since

$$\alpha - p_n(0) = e^{-R_{2n-1}(0)} > 0$$

α must be chosen sufficiently large and what we shall show is that α can be chosen so large as to make the system (3.9) solvable.

Setting

$$\xi_k = e^{-R_{2k}(0)}, \qquad \eta_k = e^{-R_{2k+1}(0)}, \qquad k = 1, 2,..., n-1, \tag{3.10}$$

we rewrite the system (3.9) as follows

$$\alpha - p_n(0) = \eta_n ,$$

$$e^{-r_k(0)} = \xi_k \eta_{k+1} , \qquad k = 1, 2, \dots, n-1,$$

$$\alpha - p_k(0) = \xi_k + \eta_k , \quad k = 1, 2, \dots, n-1,$$

and we note that

$$\xi_{n-1} = \frac{e^{-r_{n-1}(0)}}{\alpha - p_n(0)} ,$$

$$\eta_{n-1} = \alpha - p_{n-1}(0) - \frac{e^{-r_{n-1}(0)}}{\alpha - p_n(0)} ,$$

$$\xi_{n-2} = \frac{e^{-r_{n-2}(0)}}{\alpha - p_{n-1}(0) - e^{-r_{n-1}(0)}/(\alpha - p_n(0))} ,$$

$$\eta_{n-2} = \alpha - p_{n-2}(0) - \frac{e^{-r_{n-2}(0)}}{\alpha - p_{n-1}(0) - e^{-r_{n-1}(0)}/p_n(0)}$$

etc.

It is clear that if α is chosen sufficiently large the ξ's and η's will be positive and hence their logarithms (which are the negatives of $R_k(0)$) real.

It is somewhat curious that while the p's and q's are uniquely determined the R's are not owing to the arbitrariness of α.

4

To appreciate a little better the rather neutral role of α, consider the case $N = 4$ which can be solved in a completely elementary way obtaining

$$R_1(t) = C + \log[D - \sqrt{E} \tanh(t \sqrt{E} + \gamma)], \tag{4.1a}$$

$$R_2(t) = \log[D - \sqrt{E} \tanh(t \sqrt{E} + \gamma)] - \log[E \operatorname{sech}^2(t \sqrt{E} + \gamma)], \tag{4.1b}$$

$$R_3(t) = -\log[D - \sqrt{E} \tanh(t \sqrt{E} + \gamma)], \tag{4.1c}$$

where $D > 0$, $E > 0$ and

$$D^2 = e^{-C} + E. \tag{4.2}$$

The constants E, C, γ are expressible in terms of $R_1(0)$, $R_2(0)$, $R_3(0)$ by the formulas

$$E = \tfrac{1}{4}(e^{-R_1(0)} + e^{-R_2(0)} - e^{-R_3(0)})^2 + e^{-R_2(0)}\, e^{-R_3(0)}, \tag{4.3a}$$

$$e^{-C} = e^{-R_1}\, e^{-R_3(0)}, \tag{4.3b}$$

$$\tanh\gamma = \frac{e^{-R_1(0)} + e^{-R_2(0)} - e^{-R_3(0)}}{[(e^{-R_1(0)} + e^{-R_2(0)} - e^{-R_3(0)})^2 + 4e^{-R_2(0)}\, e^{-R_3(0)}]^{1/2}}, \tag{4.3c}$$

whence by (4.2) it follows that

$$D = \tfrac{1}{2}(e^{-R_1(0)} + e^{-R_2(0)} + e^{-R_3(0)}). \tag{4.3d}$$

Recall now that

$$p_1(t) = \alpha - (e^{-R_1(0)} + e^{-R_2(0)}), \tag{4.4a}$$

$$p_2(t) = \alpha - e^{-R_3(t)}, \tag{4.4b}$$

$$r_1(t) = R_2(t) + R_3(t) = q_2(t) - q_1(t), \tag{4.4c}$$

so that

$$p_1(0) = \alpha - (e^{-R_1(0)} + e^{-R_2(0)}),$$

$$p_2(0) = \alpha - e^{-R_3(0)},$$

$$e^{-r_1(0)} = e^{-R_2(0)}\, e^{-R_3(0)},$$

and it follows from (4.3) that

$$E = \tfrac{1}{4}(p_1(0) - p_1(0))^2 + e^{-r_1(0)}, \tag{4.5a}$$

$$\tanh\gamma = \frac{p_2(0) - p_1(0)}{[(p_2(0) - p_1(0))^2 + 4e^{-r_1(0)}]^{1/2}}, \tag{4.5b}$$

$$e^{-C} = \alpha - \frac{p_1(0) + p_2(0)^2}{2} - \tfrac{1}{4}(p_1(0) - p_2(0)^2 - e^{-r_1(0)}, \tag{4.5c}$$

$$D = \alpha - \frac{p_1(0) + p_2(0)}{2}. \tag{4.5d}$$

Since D and e^{-C} are to be positive, α has to be chosen sufficiently large.

Once α has been so chosen, we note using (4.4a) and (4.4b) that

$$p_1(t) = \frac{p_1(0) + p_2(0)}{2} - \sqrt{E}\tanh(t\sqrt{E} + \gamma), \tag{4.6a}$$

$$p_2(t) = \frac{p_1(0) + p_2(0)}{2} + e\tanh(t\sqrt{E} + \gamma), \tag{4.6b}$$

and by (4.5a), (4.5b) E and γ do not depend on α.

References

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5. K. M. Case and M. Kac,[1] A discrete version of the inverse scattering problem, *J. Mathematical Phys.* **14** (1973), 594–603.

[1] We take this opportunity to correct a number of minor but annoying errors and misprints: 1, In formula (3.13) insert d/dx after c. 2, The right-hand side of (4.22) should be $(2/\pi)(1 - \lambda^2)^{1/2}\, d\lambda/|\,A(\cos^{-1}\lambda)|^2$ and correspondingly formula (4.21) and the one which precedes it should be corrected. 3, In the line just above formula (5.11) $z \to 0$ should be replaced by $z \to c$.