Existence and concentration of ground states of coupled nonlinear Schrödinger equations

Gong-Ming Wei \(^{a,b}\)

\(^{a}\) School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China
\(^{b}\) College of Science, University of Shanghai for Science and Technology, Shanghai 200093, PR China

Received 4 April 2006
Available online 28 November 2006
Submitted by V. Radulescu

Abstract

In this paper we consider the existence and concentration of ground states of coupled nonlinear Schrödinger equations with trap potentials. When the interaction between two states is repulsive, we prove the existence of ground states. Then concentration phenomenon of these ground states is studied as the small perturbed parameter (Planck constant) approaches zero. Roughly speaking, we prove that components of the ground states concentrate at the unique global minimum points of their potentials. Moreover, we prove the existence of ground states when the interaction is attractive.

© 2006 Elsevier Inc. All rights reserved.

Keywords: Concentration of ground states; Solution manifold; Critical point theory; Coupled nonlinear Schrödinger equations

1. Introduction

In this paper, we consider the existence and concentration of ground states of the following coupled nonlinear Schrödinger equations

\[
\begin{align*}
\hbar^2 \Delta u - V_1(x)u + \mu_1 u^3 + \beta uv^2 &= 0 \\
\hbar^2 \Delta v - V_2(x)v + \mu_2 v^3 + \beta u^2 v &= 0
\end{align*}
\]

in \(\mathbb{R}^N\)
as \( h \to 0^+ \), where \( N \leq 3 \), \( \mu_i \)'s are positive constants, \( \beta \in ( -\sqrt{\mu_1 \mu_2}, 0 ) \). When \( \beta > 0 \) is sufficiently large, we consider the existence of ground states of

\[
\begin{align*}
\Delta u - V_1(x)u + \mu_1 u^3 + \beta uv^2 &= 0 \quad \text{in } \mathbb{R}^N, \\
\Delta v - V_2(x)v + \mu_2 v^3 + \beta u^2 v &= 0
\end{align*}
\]  

(2)

But, for simplicity, in this paper we only prove the existence of ground states of

\[
\begin{align*}
\Delta u - V(x)u + u^3 + \beta uv^2 &= 0 \quad \text{in } \mathbb{R}^N, \\
\Delta v - V(x)v + v^3 + \beta u^2 v &= 0
\end{align*}
\]  

(3)

Conditions on potentials \( V_i(x)'s \) and \( V(x) \) will be given later. By ground state we mean a solution with the least energy among all strictly nontrivial solutions. By strictly nontrivial we mean that each component of the solution is nontrivial.

The above systems model many physical problems, especially in the Hartree–Fock theory for a double Bose–Einstein condensate. In fact, this system is satisfied by solitary waves of some time-dependent nonlinear Schrödinger equations. \( \mu_i > 0 \) means the interaction of the single \( i \)th state is attractive. The coupling constant \( \beta \) describes the interaction between the two hyperfine states. The interaction is attractive as \( \beta > 0 \), and the interaction is repulsive if \( \beta < 0 \).

From PDE point of view, the interaction term in these equations makes difficulties not only in proving existence but also in the analysis of asymptotic behaviors of ground states. Although these systems are variational very well, their associated energy functionals are indefinite and (PS) condition is not satisfied. These are two features of this type system.

The stationary Gierer–Meinhardt system

\[
\begin{align*}
\frac{d}{d} \Delta u - u + \frac{u^p}{v^q} &= 0 \quad \text{in } \Omega, \\
D \Delta v - v + \frac{u^r}{v^s} &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]  

(GM)

(0 < \( d \ll 1 \), \( D \gg 1 \)) and the partial differential equation in its shadow system

\[
\varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Omega
\]  

(SS)

(\( \varepsilon \ll 1 \)) have been extensively studied by many authors since the work of Lin, Ni and Takagi [16,20–22,27]. Readers can find a good review in [19] and many recent references in [31]. For techniques of approximate-solution manifold and Liapunov–Schmidt reduction for Gierer–Meinhardt type problems, one can find interesting developments in [2,15]. It is not our ambition to give a review of this fast developing field and what we want to emphasize here is that most of the interesting results on (SS) are based on understanding the following equation:

\[
\Delta w - w + \frac{w^p}{v^s} = 0 \quad \text{in } \mathbb{R}^N.
\]  

(Eq.1)

This is one of our motivations to first consider problems on the whole space. Compared with so many results on Gierer–Meinhardt type problems there are few results on singularly perturbed Schrödinger type systems. This is another motivation of this paper. Recently, Ramos and Yang [25] studied spiked–layered solutions for a singularly perturbed elliptic system (without interactions) on bounded domain, but they used an energy functional different from here.

For singularly perturbed Schrödinger equation with potential

\[
h^2 \Delta u - V(x)u + u^p = 0 \quad \text{in } \mathbb{R}^N, 
\]  

(Eq.2)
under a mathematical restriction on $V$, using minimax arguments combined with Ekeland’s variational principle, Rabinowitz [24] proved the existence of positive ground states. Then Wang [29] studied the behaviors of Rabinowitz’s ground states as $h \to 0$ and proved that they concentrate at a global minimum point of $V$. Wang and Zeng [30] gave a new viewpoint to study nonlinear Schrödinger equations both for existence and concentration of ground states, especially for equations with bounded potentials. Ambrosetti, Malchiodi and Secchi [4] gave multiplicity results of semiclassical solutions (solutions when $h \ll 1$) by studying stationary points of $V$. Badiale and D’Aprile [5] and Ambrosetti, Malchiodi and Ni [3] firstly proved the existence of concentrating sphere of radially symmetric solutions, especially [3] determines the limit radii as stationary points of an auxiliary potential function. One of the referees informed us that there has been a lot of work on peaks and multiple peaks for (Eq.2) and one can see, for example, work of Shusen Yan and of Cao and Noussair. One of the reviewers also informed us that problem (Eq.2) has been studied in a nonsmooth framework in F. Gazzola and V. Radulescu [14]. On singularly perturbed Neumann problems with potentials

$$
\varepsilon^2 \text{div} (J(x) \nabla u) - V(x) u + u^p = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
$$

(Eq.3)

using similar techniques as for (SS), Pomponio [23] studied the existence of single-peaked solutions and determined the concentrating points by potentials. For more general potentials for nonlinear Schrödinger equations, one can refer to [1,6,10]. For multiplicity results, one can refer to [8,9,11].

Recently, spike–layer solutions of singularly perturbed 2-coupled nonlinear Schrödinger equations

$$
\begin{align*}
&h^2 \Delta u - \lambda_1 u + \mu_1 u^3 + \beta uv^2 = 0 \\
&h^2 \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0
\end{align*}
$$

in $\Omega$ (4)

and ground states of

$$
\begin{align*}
&\Delta u - \lambda_1 u + \mu_1 u^3 + \beta uv^2 = 0 \\
&\Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0
\end{align*}
$$

in $\mathbb{R}^N$ (5)

are studied mathematically by Tai-Chia Lin and Juncheng Wei [17,18]. Problem (4) arises in the Hartree–Fock theory for a double condensate and (5) is from standing waves of time-dependent 2-coupled nonlinear Schrödinger equations in nonlinear optics. Similar ideas as for (SS), the study on spikes of (4) depends on understanding ground state of (5). One of the main tool for (5) has been the employ of Nehari’s solution manifold, which has been used by Conti, Terracini and Verzini [13] to study a class of competing species systems. Their main conclusions for (5) are (see [18]): (a) there exist ground states when $0 < \beta < \beta_0 < \sqrt{\mu_1 \mu_2}$ for some small constant $\beta_0$; (b) there do not exist ground states when $\beta < 0$.

When $\lambda_i$’s are replaced by potentials $V_i(x)$’s, existence and concentration of ground states are totally open. This is also a question asked by Lin and Wei in their paper [17]. Our goal is to study such problems. We organize this paper as follows.

Suppose the following condition (V) always holds in this paper. After giving some preliminaries in Section 2, we consider existence of ground states in Section 3 and study concentration of these ground states in Section 4. Precisely, in Section 3, we prove that: (i) when $\beta \in (-\sqrt{\mu_1 \mu_2}, 0)$, there exist ground states for problem (2); (ii) when $\beta > 1$, there exist ground states for problem (3). In Section 4, after scaling the ground states for (1), we study the behaviors of the scaled ground states as $h \to 0$. Along a sequence $h_k \to 0$, we prove that: (a) when $V_1$ and
V₂ have the same unique global minimum points, the ground states converge weakly to bound state of type (5) equations; (b) when V₁ and V₂ have different unique global minimum points, components of the ground states converge to ground states of type (Eq.1) equation. Case (b) implies the occurrence of concentration of ground states of the original equations (1).

The conditions on Vᵢ, Vᵢ’ s are as follows:

\[ V_i ∈ C^∞(\mathbb{R}^N), \quad V_i(x) ≥ a_i > 0, \quad \text{and} \]
\[ \forall M > 0, \quad \text{meas}\{x ∈ \mathbb{R}^N \mid V_i(x) < M\} < ∞. \]  

(V)

This type condition is a generalization of [24] and first appears in [8].

2. Preliminaries

The energy functional for (2) is

\[ I(u, v) := \int \left( \frac{1}{2} |\nabla u|^2 + \frac{V_1(x)}{2} u^2 - \frac{\mu_1}{4} u^4 + \frac{1}{2} |\nabla v|^2 + \frac{V_2(x)}{2} v^2 - \frac{\mu_2}{4} v^4 - \frac{\beta}{2} u^2 v^2 \right), \]

where \((u, v) ∈ E\)

\[ E = \{ (u, v) \mid u, v ∈ H^1(\mathbb{R}^N), \int V_1(x)u^2 < ∞, \int V_2(x)v^2 < ∞ \}. \]

Hence the existence of ground states of (2) is reduced to the existence of critical points of \(I\) on \(E\) with the least energy among all strictly nontrivial solutions of (2). This means that we will study the constrained minimization problem

\[ c = \inf_{(u, v) ∈ M} I(u, v) \]

where

\[ M = \left\{ (u, v) ∈ T : \frac{1}{2} |\nabla u|^2 + V_1(x)u^2 = \frac{\mu_1}{4} u^4 + \beta u^2 v^2 \right\} \]

and

\[ T = \{ (u, v) ∈ E \mid u ≠ 0 \text{ and } v ≠ 0 \}. \]

For convenience, set

\[ \|u\|^2_1 = \int |\nabla u|^2 + V_1(x)u^2, \quad \|v\|^2_2 = \int |\nabla v|^2 + V_2(x)v^2 < ∞, \]

\[ \|(u, v)\|^2 = \|u\|^2_1 + \|v\|^2_2. \]

Let

\[ B_i = \left\{ u ∈ H^1(\mathbb{R}^N) \mid \int V_i(x)u^2 < ∞ \right\}. \]

From [8], we have

**Lemma 2.1.** \(B_i\) is compactly imbedded in \(L^p(\mathbb{R}^N)\) for \(2 ≤ p < 2^*\).
Lemma 2.2. Suppose \(-\sqrt{\mu_1\mu_2} < \beta < 0\). Then:

1. There exists a constant \(\gamma > 0\) such that \(\|u\|_{L^4(\mathbb{R}^N)} \geq \gamma\), \(\|u\|_{H^1(\mathbb{R}^N)} \geq \gamma\), \(\|v\|_{L^4(\mathbb{R}^N)} \geq \gamma\) for any \((u, v) \in M\).

2. \(\forall (u, v) \in T\), there exists unique \(s = s(u, v) > 0, t = t(u, v) > 0\) such that \((\sqrt{su}, \sqrt{tv}) \in M\) and \(\phi(u, v) = f(s, t)\).

3. \(\phi(u, v)\) is weakly lower semi-continuous on \(T\).

4. \(c = d\).

Proof. (1) Since \((u, v) \in M\), \(\int|\nabla u|^2 + V_1(x)u^2 = \int \mu_1u^4 + \beta u^2v^2 \leq \int \mu_1u^4\). By imbedding theorem and \(L^p\) interpolation theory (or Caffarelli–Kohn–Nirenberg inequality [12]),

\[
|\nabla u|^2 + V_1(x)u^2 \leq \mu_1\|u\|_{L^4(\mathbb{R}^N)}^4.
\]

Therefore, \(\gamma_1 > 0, \gamma_2 > 0\) such that \(\|u\|_{L^4(\mathbb{R}^N)} \geq \gamma_1, \|u\|_{H^1(\mathbb{R}^N)} \geq \gamma_2\). Similar results hold for \(v\).

(2) From the definition of \(I\) and \(f\),

\[
f(s, t) = \int \left(\frac{s}{2}|\nabla u|^2 + \frac{s}{2}V_1(x)u^2 - \frac{s^2\mu_1}{4}u^4 + \frac{t}{2}|\nabla v|^2 + \frac{t}{2}V_2(x)v^2 - \frac{t^2\mu_2}{4}v^4 - \frac{s\beta}{2}u^2v^2\right).
\]

for \(s \geq 0, t \geq 0\). By direct calculus,

\[
f_s(s, t) = \int \frac{1}{2}|\nabla u|^2 + \frac{V_1(x)}{2}u^2 - \frac{s\mu_1}{2}u^4 - \frac{t\beta}{2}u^2v^2,
\]

\[
f_t(s, t) = \int \frac{1}{2}|\nabla v|^2 + \frac{V_2(x)}{2}v^2 - \frac{t\mu_2}{2}v^4 - \frac{s\beta}{2}u^2v^2,
\]

\[
f_{ss}(s, t) = -\frac{\mu_1}{2}\int u^4, \quad f_{st}(s, t) = -\frac{\beta}{2}\int u^2v^2, \quad f_{tt}(s, t) = -\frac{\mu_2}{2}\int v^4.
\]

Since \(-\sqrt{\mu_1\mu_2} < \beta < 0, f_{ss} < 0, f_{tt} < 0, f_{ss}f_{tt} - f_{st}^2 > 0\), the matrix \(\begin{pmatrix} f_{ss} & f_{st} \\ f_{st} & f_{tt} \end{pmatrix}\) is negative definite. Hence \(f(s, t)\) is convex in \(s \geq 0, t \geq 0\). Noting that \(f(s, t) > 0\) for \(0 < s, t < 1\) and \(f(s, t) \to -\infty\) as \(s \to \infty\) or \(t \to \infty\), there exists a unique maximum point \((s_0, t_0)\) of \(f\) such that \(s_0 > 0, t_0 > 0\). Indeed, if \(s_0 > 0, t_0 = 0\), then \(f_s(s_0, t_0) = 0, f_t(s_0, t_0) \leq 0\), i.e.

\[
\int |\nabla u|^2 + V_1(x)u^2 = \int s_0\mu_1u^4, \quad \int |\nabla v|^2 + V_2(x)v^2 \leq \beta \int s_0u^2v^2.
\]

This is a contradiction with \(\beta < 0\). Therefore, \(f_s(s_0, t_0) = 0, f_t(s_0, t_0) = 0\). This implies \((\sqrt{s_0}u, \sqrt{t_0}v) \in M\).
(3) Assume \((u_n, v_n) \rightharpoonup (u_0, v_0)\) in \(E\). From Lemma 2.1, \((u_n, v_n) \rightharpoonup (u_0, v_0)\) in \(L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)\) for \(2 \leq p < 2^*\). From Lemma 2.2(2), i.e. by the above result of this lemma,
\[
\phi(u_0, v_0) = \max_{s>0, t>0} I(\sqrt{s}u_0, \sqrt{t}v_0) \\
= I(\sqrt{s}(u_0, v_0)u_0, \sqrt{t}(u_0, v_0)v_0) \\
\leq \liminf_{n \to \infty} I(\sqrt{s}(u_0, v_0)u_n, \sqrt{t}(u_0, v_0)v_n) \\
\leq \liminf_{n \to \infty} \max_{s>0, t>0} I(\sqrt{s}u_n, \sqrt{t}v_n) \\
= \liminf_{n \to \infty} \phi(u_n, v_n).
\]

So \(\phi\) is weakly lower semi-continuous.

(4) For any \((u, v) \in M\),
\[
I(u, v) = \max_{s>0, t>0} I(\sqrt{s}u, \sqrt{t}v) = \phi(u, v) \geq \inf_{(u, v) \in T} \phi(u, v) = d.
\]

By the definition of \(c\), \(c \geq d\). On the other hand, since \((\sqrt{s}(u, v)u, \sqrt{t}(u, v)v) \in M\),
\[
d = \inf_{(u, v) \in T} \phi(u, v) = \inf_{(u, v) \in T} I(\sqrt{s}(u, v)u, \sqrt{t}(u, v)v) \geq c.
\]

Thus \(c = d\). \(\Box\)

The following two lemmas are well known (see e.g. [17,18] and [7,26]).

**Lemma 2.3.** There exists a unique positive ground state \(w\) of
\[
\begin{cases}
\Delta w - w + w^3 = 0, & x \in \mathbb{R}^N \ (N \leq 3), \\
\max_{x \in \mathbb{R}^N} w(x) = w(0), \\
w(x) \to 0 \text{ as } |x| \to \infty
\end{cases}
\]

which is radially symmetric about the origin and exponentially decay at infinity.

Set \(w_i(x) = \sqrt{\frac{\lambda_i}{\mu_i}} w(\sqrt{x_0}x), \) where \(\lambda_i > 0, \mu_i > 0\) for \(i = 1, 2\). Then \(w_i\) is a unique positive ground state of
\[
\Delta w_i - \lambda_i w + \mu_i w^3 = 0, \quad x \in \mathbb{R}^N
\]

and the corresponding least energies are
\[
I_i = \int \frac{1}{2} (|\nabla w_i|^2 + \lambda_i w_i^2) - \frac{\mu_i}{4} w_i^4 = \lambda_i^{-\frac{N}{2}} \mu_i^{-1} I_0,
\]

where
\[
I_0 = \int \frac{1}{2} (|\nabla w|^2 + w^2) - \frac{1}{4} w^4 = \frac{1}{4} \int |\nabla w|^2 + w^2.
\]

**Lemma 2.4.** \(H_1^1(\mathbb{R}^N)\) is compactly imbedded in \(L^p(\mathbb{R}^N)\) for \(2 \leq p < 2^*\) and \(N \geq 2\), where \(H_1^1(\mathbb{R}^N)\) denotes the set of functions in \(H^1(\mathbb{R}^N)\) which are radially symmetric.
3. Existence of ground state

**Theorem 3.1.** Suppose \(-\sqrt{\mu_1\mu_2} < \beta < 0\) and \(V_i, \ i = 1, 2\), satisfy condition (V). Then \(c\) is achieved and the minimizer is a ground state of (2).

**Proof.** Step 1. We claim that \(c\) is achieved, i.e. there exists \((u_c, v_c) \in M\) such that \(c = I(u_c, v_c)\). Let \((u_n, v_n)\) be a minimizing sequence for \(c\): \((u_n, v_n) \in M\) and \(I(u_n, v_n) \to c\) as \(n \to \infty\). By the definition of \(M\), we have

\[
\int |\nabla u_n|^2 + V_1(x)u_n^2 + |\nabla v_n|^2 + V_2(x)v_n^2 = \int \mu_1 u_n^4 + \mu_2 v_n^4 + 2\beta u_n^2 v_n^2
\]

and hence

\[
I(u_n, v_n) = \frac{1}{2} \int |\nabla u_n|^2 + \frac{V_1(x)}{2} u_n^2 - \mu_1 u_n^4 + \frac{1}{2} |\nabla v_n|^2 + \frac{V_2(x)}{2} v_n^2 - \mu_2 v_n^4 - \frac{\beta}{2} u_n^2 v_n^2
\]

\[
= \frac{1}{4} \int |\nabla u_n|^2 + V_1(x)u_n^2 + |\nabla v_n|^2 + V_2(x)v_n^2.
\]

This implies that \((u_n, v_n)\) is bounded in \(E\). From Lemma 2.1, there exists a subsequence, denoted also by \((u_n, v_n)\), such that \((u_n, v_n) \rightharpoonup (u_0, v_0)\) in \(E\) and \((u_n, v_n) \to (u_0, v_0)\) in \(L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)\) for \(2 \leq p < 2^*\).

By Lemma 2.2, \((u_0, v_0) \in T\) and there exist \(\theta_1 > 0, \theta_2 > 0\) such that \((\sqrt{\theta_1}u_0, \sqrt{\theta_2}v_0) \in M\). From the weakly lower semicontinuity of \(\phi\),

\[
d \leq \phi(u_0, v_0) \leq \liminf_{n \to \infty} \phi(u_n, v_n) = \liminf_{n \to \infty} I(u_n, v_n) = c.
\]

Thus \(\phi(u_0, v_0) = c\). By the definition of \(\phi\) and \(c\),

\[
c \leq \phi(\sqrt{\theta_1}u_0, \sqrt{\theta_2}v_0) = \max_{s > 0, t > 0} I(\sqrt{s}u_0, \sqrt{t}v_0) = \phi(u_0, v_0) = c.
\]

Let \(u_c = \sqrt{\theta_1}u_0, v_c = \sqrt{\theta_2}v_0\). Then \((u_c, v_c) \in M\) and it is a minimizer for \(c\).

**Step 2.** We claim that \((u_c, v_c)\) is a critical point of \(I\).

Define

\[
g(u, v) = \int |\nabla u|^2 + V_1(x)u^2 - \mu_1 u^4 - \beta u^2 v^2,
\]

\[
h(u, v) = \int |\nabla v|^2 + V_2(x)v^2 - \mu_2 v^4 - \beta u^2 v^2.
\]

Then

\[
\begin{align*}
g_u(u, v) &= -2\Delta u + 2V_1(x)u - 4\mu_1 u^3 - 2\beta uv^2, \\
g_v(u, v) &= -2\beta uv^2, \\
h_u(u, v) &= -2\beta uv^2, \\
h_v(u, v) &= -2\Delta v + V_2(x)v - 4\mu_2 v^3 - 2\beta u^2 v.
\end{align*}
\]

Since \((u_c, v_c)\) is a minimizer for \(c\) under the constrained condition \(g(u_c, v_c) = 0, h(u_c, v_c) = 0\), there exist two Lagrange multiplier \(\alpha_1, \alpha_2\) such that

\[
\nabla I + \alpha_1 \nabla g + \alpha_2 \nabla h = 0
\]
at \((u_c, v_c)\). This means that \((u_c, v_c)\) satisfies
\[
\begin{cases}
\alpha_1(-\Delta u + V_1(x)u - 2\mu_1 u^3 - \beta uv^2) - \alpha_2 \beta uv^2 = 0, \\
\alpha_1 \beta u^2 v + \alpha_2(-\Delta v + V_2(x)v - 2\mu_2 v^3 - \beta u^2 v) = 0.
\end{cases}
\]

Writing \((u_c, v_c)\) as \((u, v)\) and multiplying \(u\) and \(v\) on both sides of the above equations, we get
\[
\begin{cases}
\alpha_1 \int |\nabla u|^2 + V_1(x)u^2 - 2\mu_1 u^4 - \beta u^2 v^2 - \alpha_2 \beta u^2 v^2 = 0, \\
\alpha_1 \beta u^2 v^2 + \alpha_2 \int |\nabla v|^2 + V_2(x)v^2 - 2\mu_2 v^4 - \beta u^2 v^2 = 0
\end{cases}
\]
i.e.
\[
\begin{cases}
\alpha_1 \int \mu_1 u^4 + \alpha_2 \int \beta u^2 v^2 = 0, \\
\alpha_1 \beta u^2 v^2 + \alpha_2 \int \mu_1 v^4 = 0.
\end{cases}
\]

Since \(\left(\frac{\mu_1 \int u^4}{\beta \int u^2 v^2}, \frac{\beta \int u^2 v^2}{\mu_1 \int v^4}\right) > 0\), we have \(\alpha_1 = 0, \alpha_2 = 0\). Therefore, \(\nabla I(u_c, v_c) = 0\). \(\square\)

The next problem is on the existence of ground states of (2) when the coupling constant \(\beta\) is sufficiently large. For simplicity, we only consider the simple case
\[
\begin{cases}
\Delta u - V(x)u + u^3 + \beta uv^2 = 0 \\
\Delta v - V(x)v + v^3 + \beta u^2 v = 0
\end{cases}
\] in \(\mathbb{R}^N\). \(\tag{6}\)

For general case, the proof is similar but more complicated.

Similarly, corresponding spaces \(E\) and \(T\) can be defined for \(V_1 = V_2 = V\) and \(\mu_1 = \mu_2 = 1\) as in Section 2 and here we use the same notations.

Define
\[
c_* = \inf_{(u,v) \in M_1} I(u, v),
\]
where
\[
T_1 = \{ (u, v) \in E \mid u \neq 0 \text{ or } v \neq 0 \},
\]
\[
M_1 = \{ (u, v) \in T_1 \mid \int |\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2 = \int u^4 + v^4 + 2\beta u^2 v^2 \}.
\]

**Theorem 3.2.** Suppose \(V(x)\) satisfies condition \((V)\) and \(\beta > 1\). Then problem (6) has a ground state.

**Proof.** The energy functional for (6) is
\[
I(u, v) = \int \frac{1}{2} |\nabla u|^2 + \frac{V(x)}{2} u^2 - \frac{1}{4} u^4 + \frac{1}{2} |\nabla v|^2 + \frac{V(x)}{2} v^2 - \frac{1}{4} v^4 - \frac{\beta}{2} u^2 v^2.
\]

Set
\[
\Gamma = \{ \gamma \in C([0, 1], E) \mid \gamma(0) = 0, \ I(\gamma(1)) < 0 \}.
\]
and 
\[ c^* = \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} I(\gamma(\theta)). \]

**Claim 1.** \( I \) satisfies (PS) condition: if \( I(u_n, v_n) \to C, I'(u_n, v_n) \to 0 \), then \((u_n, v_n)\) has a convergent subsequence.

The proof is standard and we omit it.

**Claim 2.** \( c^* = c_* \).

As in the proof of Lemma 2.2, for any \((u, v) \in T_1\), there exists a unique \( \theta = \theta(u, v) > 0 \) such that \( \max_{\theta > 0} I(\sqrt{\theta}u, \sqrt{\theta}v) = I(\sqrt{\theta}u, \sqrt{\theta}v) \) and \( (\sqrt{\theta}u, \sqrt{\theta}v) \in M_1 \). For any \((u, v) \in M_1\), choose \( \gamma_0(\theta) = (\theta Au, \theta Av) \) for sufficiently large \( A \) so that \( I(\gamma_0(\theta)) < 0 \). Then \( \gamma_0 \in \Gamma \) and 
\[ c^* \leq \max_{\theta > 0} I(\gamma_0(\theta)) = \max_{\theta > 0} I(\theta u, \theta v) = I(u, v). \]

Therefore, \( c^* \leq c_* \).

To show \( c^* \geq c_* \), we only need to prove that for any \( \gamma \in \Gamma \), \( \gamma([0,1]) \cap M_1 \neq \emptyset \). When \( 0 < \| (u, v) \| \ll 1 \),
\[ \int |\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2 > \int u^4 + v^4 + 2\beta u^2 v^2. \]
Since \( I(\gamma(0)) = 0 \), \( \gamma \) is continuous in \( \theta \in [0,1] \), \( I(\gamma(\theta)) \) is continuous in \( \theta \) and \( I(\gamma(\theta)) > 0 \) for \( 0 < \theta \ll 1 \). Thus, if set \( \gamma(\theta) = (u_\theta, v_\theta) \),
\[ \int |\nabla u_\theta|^2 + V(x)u_\theta^2 + |\nabla v_\theta|^2 + V(x)v_\theta^2 > \int u_\theta^4 + v_\theta^4 + 2\beta u_\theta^2 v_\theta^2 \]
holds for sufficiently small \( \theta \). By the continuity of \( I(\gamma(\theta)) \), if \( \gamma \in \Gamma \), \( \gamma([0,1]) \cap M_1 = \emptyset \), then
\[ \int |\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2 > \int u^4 + v^4 + 2\beta u^2 v^2. \]
This is a contradiction with \( I(\gamma(1)) < 0 \).

**Claim 3.** \( u_c \neq 0 \) and \( v_c \neq 0 \).

Recall
\[ M = \left\{ (u, v) \in T : \int |\nabla u|^2 + V(x)u^2 = \int u^4 + \beta u^2 v^2 \right\}. \]
Since \( c = \inf_{(u, v) \in M} I(u, v) \). Then \( c \geq c_* = c^* \). Suppose \( W \) is a positive ground state of \( \Delta w - V(x)w + w^3 = 0 \) in \( \mathbb{R}^N \).

See [8,24] for the existence. Then the least energy
\[ J_0 := \frac{1}{2} \int |\nabla W|^2 + V(x)W^2 = \frac{1}{4} \int W^4. \]
Let \( u = aW, v = bW \), where \( a = b = \sqrt{\frac{1}{1+\beta}} \). Then \((u, v) \in M, a^2 + b^2 = \frac{2}{1+\beta} < 1 \), and
\[ I(u, v) = \frac{1}{4} \int |\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2 \]
\[ = \frac{a^2 + b^2}{4} \int |\nabla W|^2 + V(x)W^2 \]
\[ < J_0. \]

This implies that \( c^* \leq c < J_0 \). Therefore, \( u_c \neq 0 \) and \( v_c \neq 0 \). Indeed, if \( v_c \equiv 0 \), then \( c^* = I(u_c, 0) = J_0 \). This is a contradiction. \( \square \)

4. Concentration phenomena

In this section, we will consider the concentration phenomena of ground states of

\[
\begin{aligned}
    h^2 \Delta u - V_1(x)u + \mu_1 u^3 + \beta uv^2 &= 0 & \text{in} \ R^N, \quad (N \leq 3) \\
    h^2 \Delta v - V_2(x)v + \mu_2 v^3 + \beta u^2 v &= 0 \\
\end{aligned}
\]

as \( h \to 0^+ \) under the conditions of Theorem 3.1. The energy functional for (1) is

\[ I^h(u, v) = \frac{h^2}{2} |\nabla u|^2 + \frac{V_1(x)}{2} u^2 - \frac{\mu_1}{4} u^4 + \frac{h^2}{2} |\nabla v|^2 + \frac{V_2(x)}{2} v^2 - \frac{\mu_2}{4} v^4 - \frac{\beta}{2} u^2 v^2, \]

and the solution manifold is

\[ M^h = \left\{ (u, v) \in T : \begin{array}{l}
    \int h^2 |\nabla u|^2 + V_1(x)u^2 = \int \mu_1 u^4 + \beta u^2 v^2 \\
    \int h^2 |\nabla v|^2 + V_2(x)v^2 = \int \mu_2 v^4 + \beta u^2 v^2 \end{array} \right\}. \]

Suppose \( (u^h, v^h) \) is the ground state of problem (1) for \( h > 0 \). Define

\[ u^1_h(x) = u^h(x_0 + hx), \quad v^1_h(x) = v^h(x_0 + hx), \quad (T1) \]

then \( (u^1_h, v^1_h) \) satisfies

\[
\begin{aligned}
    \Delta u - V_1(x_0 + hx)u + \mu_1 u^3 + \beta uv^2 &= 0, \\
    \Delta v - V_2(x_0 + hx)v + \mu_2 v^3 + \beta u^2 v &= 0. \\
\end{aligned}
\]

Define

\[ u_h(x) = u^h(y_0 + hx), \quad v_h(x) = v^h(z_0 + hx), \quad (T2) \]

then \( (u_h, v_h) \) satisfies

\[
\begin{aligned}
    \Delta u - V_1(y_0 + hx)u + \mu_1 u^3 + \beta u(x)v^2 \left( x + \frac{y_0 - z_0}{h} \right) &= 0, \\
    \Delta v - V_2(z_0 + hx)v + \mu_2 v^3 + \beta u^2 \left( x + \frac{z_0 - y_0}{h} \right) v(x) &= 0. \\
\end{aligned}
\]

The solution manifolds and energy functionals for (7) and (8) are, respectively,

\[ M^1_h = \left\{ (u, v) \in T : \begin{array}{l}
    \int |\nabla u|^2 + V_1(x_0 + hx)u^2 = \int \mu_1 u^4 + \beta u^2 \nu^2 \\
    \int |\nabla v|^2 + V_2(x_0 + hx)v^2 = \int \mu_2 \nu^4 + \beta \nu^2 \nu^2 \end{array} \right\}, \]

\[ M^2_h = \left\{ (u, v) \in T : \begin{array}{l}
    \int |\nabla u|^2 + V_1(y_0 + hx)u^2 = \int \mu_1 u^4 + \beta u^2(x)v^2 \left( x + \frac{y_0 - z_0}{h} \right) \\
    \int |\nabla v|^2 + V_2(z_0 + hx)v^2 = \int \mu_2 \nu^4 + \beta \nu^2 \left( x + \frac{z_0 - y_0}{h} \right) \nu^2(x) \end{array} \right\}, \]

and
\[ I_h^1(u, v) = \int \frac{1}{2} |\nabla u|^2 + \frac{V_1(x_0 + hx)}{2} u^2 - \frac{\mu_1}{4} u^4 + \frac{1}{2} |\nabla v|^2 + \frac{V_2(x_0 + hx)}{2} v^2 - \frac{\mu_2}{4} v^4 - \frac{\beta}{2} u^2 v^2, \]

\[ I_h^2(u, v) = \int \frac{1}{2} |\nabla u|^2 + \frac{V_1(y_0 + hx)}{2} u^2 - \frac{\mu_1}{4} u^4 + \frac{1}{2} |\nabla v|^2 + \frac{V_2(z_0 + hx)}{2} v^2 - \frac{\mu_2}{4} v^4 - \frac{\beta}{2} u^2 v^2. \]

Define

\[ c^h = \inf_{(u, v) \in M_h} I^h(u, v), \]

\[ c^1_h = \inf_{(u, v) \in M^1_h} I^1_h(u, v), \]

\[ c^2_h = \inf_{(u, v) \in M^2_h} I^2_h(u, v). \]

4.1. \( V_1 \) and \( V_2 \) have the same unique global minimum point \( x_0 \)

For simplicity, in this subsection, we assume that \( x_0 = 0, \mu_1 = \mu_2 = 1, \ V_1(0) = V_2(0) = 1, \ -1 < \beta < 0. \) (A4.1)

For general case, the proof is similar. Then Eqs. (7) become

\[ \begin{align*}
\Delta u - V_1(hx)u + u^3 + \beta uv^2 &= 0, \\
\Delta v - V_2(hx)v + v^3 + \beta u^2 v &= 0
\end{align*} \]

and \( M^1_h \) becomes

\[ M^1_h = \left\{ (u, v) \in T : \begin{array}{l}
\int |\nabla u|^2 + V_1(hx) u^2 = \int u^4 + \beta u^2 v^2 \\
\int |\nabla v|^2 + V_2(hx) v^2 = \int v^4 + \beta u^2 v^2
\end{array} \right\}. \]

**Theorem 4.1.** Under the conditions of Theorem 3.1, there exists a sequence \( \{h_k\} \to 0 \) such that \( c^1_{h_k} \) is bounded from below and above and \( (u^1_{h_k}, v^1_{h_k}) \to (u_0, v_0) \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \), where \( u_0, v_0 \in H^1(\mathbb{R}^N) \) and satisfies

\[ \begin{align*}
\Delta u - u + u^3 + \beta uv^2 &= 0, \\
\Delta v - v + v^3 + \beta u^2 v &= 0.
\end{align*} \]

**Proof.** Let \( w \) be as in Lemma 2.3 and \( \phi_R \) be a cut-off function: \( \phi \in C^\infty_0(\mathbb{R}^N), 0 \leq \phi \leq 1, \phi_R = 1 \) for \( |x| \leq R, \phi_R = 0 \) for \( |x| \geq R + 1 \), and \( |\nabla \phi_R| < \frac{2}{R} \).
Set \( w_R = w_\phi R \). Define \( u_R = v_R = w_R \). From Lemma 2.2, there exists unique \( \theta_1 > 0, \theta_2 > 0 \) such that \((\theta_1 u_R, \theta_2 v_R) \in M_h^1 \), i.e.

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\int |\nabla u_R|^2 + V_1(\epsilon x)u_R^2 = \theta_1^2 \int u_R^4 + \theta_2^2 \beta \int u_R^2 v_R^2, \\
\int |\nabla v_R|^2 + V_2(\epsilon x)v_R^2 = \theta_2^2 \int v_R^4 + \theta_1^2 \beta \int u_R^2 v_R^2
\end{array} \right.
\end{aligned}
\]

or

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\int |\nabla w_R|^2 + V_1(\epsilon x)w_R^2 = (\theta_1^2 + \theta_2^2 \beta) \int w_R^4, \\
\int |\nabla w_R|^2 + V_2(\epsilon x)w_R^2 = (\theta_2^2 + \theta_1^2 \beta) \int w_R^4.
\end{array} \right.
\end{aligned}
\]

Solving the above algebraic equations,

\[
\begin{aligned}
\theta_1^2 + \theta_2^2 \beta &= \frac{\int u_R^4}{\int |\nabla w_R|^2 + V_1(\epsilon x)w_R^2}, \\
\theta_2^2 + \theta_1^2 \beta &= \frac{\int u_R^4}{\int |\nabla w_R|^2 + V_2(\epsilon x)w_R^2}.
\end{aligned}
\]

Since \( \int |\nabla w|^2 + w^2 = \int w^4 \) and \( w_R \to w \) in \( H^1(\mathbb{R}^N) \cap L^4(\mathbb{R}^N) \) as \( R \to \infty \),

\[
\lim_{R \to \infty} \lim_{h \to 0} \theta_1^2 + \theta_2^2 \beta = 1, \quad \lim_{R \to \infty} \lim_{h \to 0} \theta_2^2 + \theta_1^2 \beta = 1
\]

and

\[
\lim_{R \to \infty} \lim_{h \to 0} \theta_1^2 = \lim_{R \to \infty} \lim_{h \to 0} \theta_2^2 = \frac{1}{1 + \beta}.
\]

Note that

\[
c_h^1 = \inf_{(u, v) \in M_h^1} I_h^1(u, v) \leq I_h^1(\theta_1 u_R, \theta_2 v_R)
\]

\[
= \frac{1}{4} \int \theta_1^2 (|\nabla u_R|^2 + V_1(\epsilon x)u_R^2) + \theta_2^2 (|\nabla v_R|^2 + V_2(\epsilon x)v_R^2).
\]

Hence

\[
\lim_{h \to 0} c_h^1 \leq \lim_{R \to \infty} \lim_{h \to 0} \frac{1}{4} \int \theta_1^2 (|\nabla u_R|^2 + V_1(\epsilon x)u_R^2) + \theta_2^2 (|\nabla v_R|^2 + V_2(\epsilon x)v_R^2)
\]

\[
= \frac{2}{1 + \beta} \frac{1}{4} \int |\nabla w|^2 + w^2 = \frac{2}{1 + \beta} I_0
\]

where \( I_0 = \frac{1}{4} \int |\nabla w|^2 + w^2 = \inf_{u \neq 0} \max_{s > 0} \int \frac{s}{2} |\nabla u|^2 + \frac{s}{2} u^2 - \frac{s^2}{4} u^4 .
\]

On the other hand,

\[
c_h^1 = \inf_{(u, v) \in T} \max_{s > 0} I_h^1(\nabla s u, \sqrt{tv})
\]

\[
= \inf_{(u, v) \in T} \max_{s > 0, t > 0} \int \frac{s}{2} |\nabla u|^2 + \frac{s V_1(\epsilon x)}{2} u^2 - \frac{s^2}{4} u^4
\]

\[
+ \frac{t}{2} |\nabla v|^2 + \frac{t V_2(\epsilon x)}{2} v^2 - \frac{t^2}{4} v^4 - \frac{st \beta}{2} u^2 v^2
\]

\[
\geq \inf_{(u, v) \in T} \max_{s > 0, t > 0} \int \frac{s}{2} |\nabla u|^2 + \frac{s}{2} u^2 - \frac{s^2}{4} u^4 + \frac{t}{2} |\nabla v|^2 + \frac{t}{2} v^2 - \frac{t^2}{4} v^4
\]
\[
= 2 \inf_{u \neq 0} \max_{s > 0} \int \left( \frac{s}{2} |\nabla u|^2 + \frac{s}{2} u^2 - \frac{s^2}{4} u^4 \right)
= 2 I_0.
\]

Combining the above arguments, there exists a sequence \( \{h_k\} : h_k \to 0 \) such that \( \{(u_{h_k}^1, v_{h_k}^1)\} \) is bounded in \( H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \). Then there exists a subsequence, denoted by \((u_k, v_k)\), such that \((u_k, v_k) \to (u_0, v_0)\) in \( H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \) and \((u_k, v_k) \to (u_0, v_0)\) a.e. in \( \mathbb{R}^N \). From Lemma 2.2, \( u_0 \) and \( v_0 \) are nontrivial. By elliptic regularity, \((u_k, v_k) \to (u_0, v_0)\) in \( C_{\text{loc}}^2(\mathbb{R}^N) \times C_{\text{loc}}^2(\mathbb{R}^N) \) and \((u_0, v_0)\) satisfies (10). \( \square \)

**Remark 4.2.** \((u_0, v_0)\) is a bound state, but not a ground state, since there does not exist ground state when \( \beta < 0 \) (see [18]). It offers a method to prove the existence of bound states. By bound state we mean a strictly nontrivial solution with finite energy.

4.2. \( V_1 \) and \( V_2 \) have different unique global minimum points \( y_0 \) and \( z_0 \)

**Theorem 4.3.** Suppose the conditions of Theorem 3.1 hold. Then

1. \( \lim_{h \to 0} c_h = I_1 + I_2; \)
2. there exists a sequence \( \{h_k\} \to 0 \) such that \( u_{h_k}^1 \) concentrates at \( y_0 \) and \( v_{h_k}^1 \) concentrates at \( z_0 \); 
3. \((u_{h_k}^1, v_{h_k}^1) \to (u_0, v_0)\) in \( H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \) where \( u_0 \) and \( v_0 \) are, respectively, ground states of 

\[
\nabla u - V_1(y_0) u + \mu_1 u^3 = 0 \quad \text{in } \mathbb{R}^N \quad (11)
\]

and

\[
\nabla v - V_2(z_0) v + \mu_2 v^3 = 0 \quad \text{in } \mathbb{R}^N, \quad (12)
\]

and \( I_1, I_2 \) are, respectively, the ground state energies associated with (11) and (12).

**Proof.** Suppose \( w_1, w_2 \) are ground states of (11) and (12), respectively (see Lemma 2.3). Let \( u_R = w_1 \phi_R, v_R = w_2 \phi_R \), where \( \phi_R \) is the cut-off function defined as in the proof of Theorem 4.1. Since \(-\sqrt{\mu_1 \mu_2} < \beta < 0\), there exists unique \( \theta_1 > 0, \theta_2 > 0 \) such that \((\theta_1 u_R, \theta_2 v_R) \in M^2_{h}, \) i.e.

\[
\left\{ \begin{array}{l}
\int |\nabla u_R|^2 + V_1(y_0 + hx)u_R^2 = \theta_1^2 \int \mu_1 u_R^4 + \theta_2^2 \beta \int u_R^2(x)v_R^2 \left( x + \frac{y_0 - z_0}{h} \right), \\
\int |\nabla v_R|^2 + V_2(z_0 + hx)v_R^2 = \theta_2^2 \int \mu_2 v_R^4 + \theta_1^2 \beta \int u_R^2(x)v_R^2 \left( x + \frac{z_0 - y_0}{h} \right).
\end{array} \right.
\]

For fixed \( R \) and sufficiently small \( h \),

\[
u_R^2(x)v_R^2 \left( x + \frac{y_0 - z_0}{h} \right) = 0, \quad u_R^2(x)v_R^2 \left( x + \frac{z_0 - y_0}{h} \right)v_R^2(x) = 0.
\]

Since

\[
\int |\nabla w_1|^2 + V_1(y_0)w_1^2 = \mu_1 \int w_1^4, \quad \int |\nabla w_2|^2 + V_2(z_0)w_2^2 = \mu_2 \int w_2^4
\]

and \( u_R \to w_1, v_R \to w_2 \) in \( H^1(\mathbb{R}^N) \) as \( R \to \infty \) we have

\[
\lim_{R \to \infty} \lim_{h \to 0} \theta_1^2 = \lim_{R \to \infty} \lim_{h \to 0} \theta_2^2 = 1.
\]
Since
\[ c^2_h = \inf_{(u,v) \in M^2_h} I_h^2(u,v) \leq I_1^1(\theta_1 u_R, \theta_2 v_R) \]
\[ = \int \frac{\theta_1^2}{2} (|\nabla u_R|^2 + V_1(y_0 + hx)u_R^2) - \frac{\theta_1^4 \mu_1}{4} u_R^4 \]
\[ + \frac{\theta_2^2}{2} (|\nabla v_R|^2 + V_2(z_0 + hx)v_R^2) - \frac{\theta_2^4 \mu_2}{4} v_R^4 \]
\[ : = f(R, h), \]
then
\[ \lim_{R \to \infty} \lim_{h \to 0} f(R, h) = \int \frac{1}{2} (|\nabla w_1|^2 + V_1(y_0)w_1^2) - \frac{\mu_1}{4} w_1^4 + \int \frac{1}{2} (|\nabla w_2|^2 + V_2(y_0)w_2^2) - \frac{\mu_2}{4} w_2^4 \]
\[ : = I_1 + I_2. \]

Hence
\[ \limsup_{h \to 0} c^2_h \leq I_1 + I_2. \]

On the other hand,
\[ c^2_h = \inf_{(u,v) \in T_{s>0,t>0}} I_h^1(\sqrt{s}u, \sqrt{t}v) \]
\[ = \inf_{(u,v) \in T_{s>0,t>0}} \max \int \frac{s}{2} |\nabla u|^2 + \frac{sv_1(y_0 + hx)}{2} u^2 - \frac{s^2 \mu_1}{4} u^4 \]
\[ + \frac{t}{2} |\nabla v|^2 + \frac{tv_2(z_0 + hx)}{2} v^2 - \frac{t^2 \mu_2}{4} v^4 \]
\[ = \inf_{(u,v) \in T_{s>0,t>0}} \max \int \frac{s}{2} |\nabla u|^2 + \frac{sv_1(y_0)}{2} u^2 - \frac{s^2 \mu_1}{4} u^4 + \frac{t}{2} |\nabla v|^2 + \frac{tv_2(z_0)}{2} v^2 - \frac{t^2 \mu_2}{4} v^4 \]
\[ = I_1 + I_2. \]

Therefore,
\[ \lim_{h \to 0} c^2_h = I_1 + I_2. \]

Since
\[ c^2_h = \frac{1}{4} \int |\nabla u_h|^2 + V_1(y_0 + hx)u_h^2 + |\nabla v_h|^2 + V_2(z_0 + hx)v_h^2 \]
where \((u_h, v_h)\) is defined by (T2), there exists a sequence \(\{h_k\} : h_k \to 0\), such that \((u_{h_k}, v_{h_k}) \rightharpoonup (u_0, v_0)\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) and \((u_{h_k}, v_{h_k}) \to (u_0, v_0)\) a.e. in \(\mathbb{R}^N\).

Write \((u_{h_k}, v_{h_k})\) as \((u_k, v_k)\) and consider the asymptotic behaviors of \((u_k, v_k)\) as \(h_k \to 0\).

Since
\[ I_1 + I_2 = \frac{1}{4} \int |\nabla w_1|^2 + V_1(y_0)w_1^2 + |\nabla w_2|^2 + V_2(z_0)w_2^2 \]
\[ \leq \liminf_{k \to \infty} \frac{1}{4} \int |\nabla u_k|^2 + V_1(y_0)u_k^2 + |\nabla v_k|^2 + V_2(z_0)v_k^2 \]
\[ \leq \liminf_{k \to \infty} \frac{1}{4} \int |\nabla u_k|^2 + V_1(y_0 + h_k x)u_k^2 + |\nabla v_k|^2 + V_2(z_0 + h_k x)v_k^2 \]
\[ = \liminf_{k \to \infty} \frac{1}{h_k} \]
\[ = I_1 + I_2, \]

we have \( u_k \to u_0, \quad v_k \to v_0 \) in \( H^1(\mathbb{R}^N) \) and hence
\[ \int_{|x| \geq R} u_k^2 \to 0, \quad \int_{|x| \geq R} v_k^2 \to 0 \quad \text{as } R \to \infty \]
uniformly with respect to \( k \). From the one-sided Harnack inequality [28,29],
\[ \max_{B_1(Q)} u_k \leq C \left( \int_{B_2(Q)} u_k^2 \right)^{1/2}, \quad \max_{B_1(Q)} v_k \leq C \left( \int_{B_2(Q)} v_k^2 \right)^{1/2}, \]
where \( Q \) is an arbitrary point in \( \mathbb{R}^N \), \( C \) is a constant depending only on \( N \) and the bound of \( \|u_k\|_{L^2(B_2(Q))} \) and \( \|v_k\|_{L^2(B_2(Q))} \).

Let \( y_k \) be a local maximum point of \( u_k \) and \( z_k \) be a local maximum point of \( v_k \). From (8),
\[ \begin{cases} 
\Delta u_k(y_k) - V_1(y_0 + h_k y_k)u_k(y_k) + \mu_1 u_k^3(y_k) + \beta u_k(y_k)v_k^2 \left( y_k + \frac{y_0 - z_0}{h} \right) = 0, \\
\Delta v_k(z_k) - V_2(z_0 + h_k z_k)v_k(z_k) + \mu_2 v_k^3(z_k) + \beta u_k^2 \left( z_k + \frac{z_0 - y_0}{h} \right) v_k(z_k) = 0.
\end{cases} \]

Since \( \Delta u_k(y_k) \leq 0, \quad \Delta v_k(z_k) \leq 0, \quad \beta < 0, \quad y_0 \) and \( z_0 \) are minimum points of \( V_1 \) and \( V_2 \), we have
\[ u_k(y_k) \geq \frac{\sqrt{V_1(y_0)}}{\mu_1}, \quad v_k(z_k) \geq \frac{\sqrt{V_2(z_0)}}{\mu_2}. \]

From (T2), \( u^{h_k}(x) = u_k \left( \frac{x - y_0}{h_k} \right), \quad v^{h_k}(x) = v_k \left( \frac{x - z_0}{h_k} \right) \), it follows that \( u^{h_k} \) concentrates at \( y_0 \) and \( v^{h_k} \) concentrates at \( z_0 \). This means that there exists constant \( c_0 > 0 \) such that for any neighborhood \( O_1 \) of \( y_0 \) and any neighborhood \( O_2 \) of \( z_0 \), \( u^{h_k} \to 0 \) uniformly outside \( O_1 \) and \( v^{h_k} \to 0 \) uniformly outside \( O_2 \) as \( h_k \to 0 \), but \( \max_{O_1} u^{h_k} > c_0 \) and \( \max_{O_2} v^{h_k} > c_0 \).

Recall \( (u_h, v_h) \) satisfies
\[ \begin{cases} 
\Delta u - V_1(y_0 + hx)u + \mu_1 u^3 + \beta u(x)v^2 \left( x + \frac{y_0 - z_0}{h} \right) = 0, \\
\Delta v - V_2(z_0 + hx)v + \mu_2 v^3 + \beta u^2 \left( x + \frac{z_0 - y_0}{h} \right) v(x) = 0.
\end{cases} \] (8)

By elliptic regularity, \( (u^{h_k}, v^{h_k}) \to (u_0, v_0) \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \times C^2_{\text{loc}}(\mathbb{R}^N) \). Multiple test functions on both side of equations in (8) and integrate by parts. Let \( h_k \to 0 \), we get \( u_0 \) solves (11) and \( v_0 \) solves (12). \( \square \)
Acknowledgments

The author is very grateful to professor Jiaxing Hong for his helpful discussions and suggestions. He also thanks the referees for their careful reading of the manuscript and useful remarks.

References