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On boundary-value problems for the laplacian in bounded and in unbounded domains with perforated boundaries☆

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Abstract

In this paper we consider boundary-value problems in domains with perforated boundaries. We use the classification of homogenized (limit) problems depending on the ratio of small parameters, which characterize the diameter of the holes and the distance between them. We study the analogue of the Helmholtz resonator for domains with a perforated boundary.

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0. Introduction

The main goal of this paper is to study problems in unbounded domains with perforated boundaries. We find the ratio of the small parameters characterizing the diameter and the distance between holes, that implies in the limit the decomposition of the original problem into a pair of independent problems. One of them is set in the bounded domain and the other is set in the unbounded complement. We show that this decomposition leads to the appearance of poles (scattering frequencies) with a small imaginary part of the analytical continuation of solutions to the original problem. It is well known that such poles for the Helmholtz resonator do exist (see, for instance, Arsen'ev [3], Beale [4], Hislop and Martínez [26], Gadyl'shin [20–22]). Exactly, these poles generate a resonance in the Helmholtz resonator (Arsen'ev [3], Gadyl'shin [20–22]). Recall that the classical Helmholtz resonator is modelled by a boundary-value problem for the Helmholtz equation in an unbounded domain outside the surface with a small aperture (Lord Rayleigh [39], Miles [34]). A model two-dimensional analogue of the Helmholtz resonator in a periodically perforated domain was considered by Gadyl'shin [23,25]. The author discovered the resonances for this analogue of the Helmholtz resonator. The two-dimensional analogue of the Helmholtz resonator is a particular case of such physical models. In this paper we consider the most natural situation of three-dimensional analogue of the Helmholtz resonator in homogenization theory. The main difference from the papers mentioned above is that in this paper we do not have a locally periodic microstructure along the whole boundary. We deal with a number of holes situated only on a part of the boundary. Models of this kind can be found in everyday life. For instance, the construction of a classical microwave assumes that the door of the microwave on the one hand must be transparent and on the other must protect people against electromagnetic radiation. For this purpose engineers install a wire net inside the door. Hence, we deal with the Helmholtz resonator perforated along a part of the boundary. Here we investigate exactly such a problem. By studying this problem we prove the existence of the scattering frequencies with a small imaginary part.

The technique of the proof assumes that we have to consider various problems in bounded domains with perforated boundaries (similar problems were considered by Marchenko and Khruslov [33], Sánchez-Palencia [42], Cioranescu, Murat [15,35], Allaire [1,2], Jäger et al. [28], Lobo et al. [31] and Belyaev et al. [6]) and uses the results close to problems with frequently alternating type of boundary conditions. These problems have been attracting the attention of mathematicians for almost 40 years (from the mid of 1960s) see, for instance, Marchenko and Khruslov [33], Sánchez-Palencia [42], Cioranescu and Murat [15], Murat [35], Allaire [1,2], Jäger et al. [28], Lobo et al. [31], Damlamian and L. Ta-Tsien [16], Lobo and Pérez [32], Chechkin [10–12], Brillard et al. [9], Friedman et al. [19], Oleinik and Chechkin [37], Chechkin and Gadyl'shin [13], Beliaev and Chechkin [5], Chechkin and Doronina [14], Borisov [8]. Such problems appear in physics and engineering sciences, when one studies, for example, the scattering of acoustic waves on the small periodic obstacles, the behavior of partially fastened membranes and many others. The engineering applications of such problems could also be found in the construction of atomic power stations, in space antennas. One can study the problem of permeation of fuel through the walls of a plastic tank. In order to reduce permeation of fuel, the inner boundary of the container is coated with a thin barrier layer of fluorine by a blow-molding process. The resulting thin layer, however, typically has flaws: it leaves many small patches uncovered. This model is described in more detail in Rosi and Nulman [40] and Friedman [18].

It should be noted that in problems with frequently alternating type of boundary conditions (as well as in problems in domains with perforated boundaries) the convergence of solutions was established by Friedman et al. [19] in a general situation on the basis of different methods. A nonperiodic boundary structure was also considered by Beliaev and Chechkin [5] and Oleinik and Chechkin [37]. On the other hand, the direct combination of the approaches from Chechkin [12] and Gadyl'shin [23], [25] gives an opportunity to obtain the estimate for the rate of convergence for solutions in the periodic situation (see also Chechkin and Gadyl'shin [13]) in the most appropriate form. Roughly speaking, the combination of the homogenization methods (Bensoussan et al. [7], Sánchez-Palencia [41], Oleinik [36], Jikov et al. [29]) and the method of matching asymptotic expansions (Van Dyke [43], Il'in [27]) give a chance to study the problems of this kind in the most reasonable and shortest way.

Let us describe briefly the contents of this paper. In Section 1 we introduce notation, describe the domains, set the problems and formulate main theorems. In Section 2 we construct the analytic continuation of the solutions. Auxiliary problems on convergence of solutions of singularly perturbed problems in bounded domains are considered in Sections 3 and 4. In Section 5 the proofs of the main theorems are proved. We give the conclusion remarks in Section 6.

1. Statements

Let Ω be a bounded domain in $\mathbb{R}^3$ with a C^∞-boundary Γ. We suppose that Ω lies in the half-space $x_3 < 0$, $\Gamma_1 = \operatorname{int}(\{x : x_3 = 0\} \cap \Gamma)$, $\operatorname{mes}_2 \Gamma_1 \neq 0$. Denote by ω a two-dimensional bounded domain with a smooth boundary, on the plane $x_3 = 0$. We suppose that $0 < \varepsilon, \delta \ll 1$ are small parameters. Introduce the following notation: $\omega_\varepsilon = \{x : x\varepsilon^{-1} \in \omega\}$, $\Pi_\varepsilon = \{x : x = (2n, 2m, 0) + x',\ x' \in \omega_\varepsilon,\ n, m \in \mathbb{Z}\}$, $\Pi_\varepsilon^\delta = \{x : \delta^{-1}x \in \Pi_\varepsilon\}$, $\Gamma_2 = \Gamma \backslash \overline{\Gamma_1}$, $\Gamma_{\varepsilon,\delta}^D = \Gamma_1 \cap \Pi_\varepsilon^\delta$, and $\Gamma_{\varepsilon,\delta}^S = \Gamma_1 \backslash \overline{\Gamma_{\varepsilon,\delta}^D}$ (see Fig. 1).

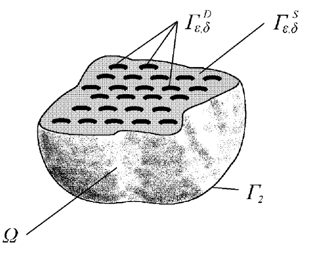

Fig. 1. Domain with fine-grained boundary.

We consider the case when $\delta = \delta(\varepsilon)$ depends on ε and

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} = p, \qquad p = 0 \quad \text{and} \quad p = \infty.$$

Assume that F is a function from $L_2(\mathbb{R}^3)$ with bounded support. We consider the following boundary-value problems in unbounded domains:

$$\begin{cases} \left(\Delta + k^2\right) u_{\varepsilon,\delta} = F, & x \in \mathbb{R}^3 \backslash \overline{\Gamma^D_{\varepsilon,\delta} \cup \Gamma_2}, \\ u_{\varepsilon,\delta} = 0, & x \in \Gamma^D_{\varepsilon,\delta} \cup \Gamma_2, \end{cases} \tag{1.1}$$

$$\begin{cases} \left(\Delta + k^2\right) u_{\varepsilon,\delta} = F, \quad x \in \mathbb{R}^3 \backslash \overline{\Gamma^S_{\varepsilon,\delta} \cup \Gamma_2}, \\ u_{\varepsilon,\delta} = 0, \quad x \in \Gamma_2, \qquad \dfrac{\partial u_{\varepsilon,\delta}}{\partial x_3} = 0, \quad x \in \Gamma^S_{\varepsilon,\delta}, \end{cases} \tag{1.2}$$

with the radiation condition

$$u_{\varepsilon,\delta} = O(r^{-1}), \qquad \frac{\partial u_{\varepsilon,\delta}}{\partial r} - iku_{\varepsilon,\delta} = o(r^{-1}), \qquad r \to \infty, \tag{1.3}$$

where k satisfies the inequality $\operatorname{Im} k \geqslant 0$ and $r = |x|$.

Our goal is to prove the following two auxiliary statements:

Theorem 1.1. *Let $p = \infty$. Suppose also that f and $\widetilde{f}$ are the restrictions of F in Ω and in $\mathbb{R}^3 \backslash \overline{\Omega}$, respectively. Then the solution of problems* (1.1), (1.3) *converges to the function*

$$u(x) = \begin{cases} u_0(x), & x \in \Omega, \\ \widetilde{u}_0(x), & x \in \mathbb{R}^3 \backslash \overline{\Omega} \end{cases} \tag{1.4}$$

strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ as $\varepsilon \to 0$, where $u_0(x)$ is a solution to the boundary-value problem

$$\begin{cases} -\Delta u_0 = k^2 u_0 - f, & x \in \Omega, \\ \quad u_0 = 0, & x \in \Gamma, \end{cases} \tag{1.5}$$

and $\widetilde{u}_0(x)$ is a solution to the boundary-value problem

$$\begin{cases} \left(\Delta + k^2\right) \widetilde{u}_0 = \widetilde{f}, & x \in \mathbb{R}^3 \backslash \overline{\Omega}, \\ \widetilde{u}_0 = 0, & x \in \Gamma, \end{cases} \tag{1.6}$$

satisfying the radiation condition

$$\widetilde{u}_0 = O(r^{-1}), \qquad \frac{\partial \widetilde{u}_0}{\partial r} - ik\widetilde{u}_0 = o(r^{-1}), \qquad r \to \infty. \tag{1.7}$$

It is assumed here that k^2 is not an eigenvalue of problem (1.5).

If $k^2 = k_0^2$ is an eigenvalue to problem (1.5), *then there exists a pole $\tau_{\varepsilon,\delta(\varepsilon)}$ of the analytic continuation of the solution of* (1.1), (1.3) *in the half plane* $\operatorname{Im} k < 0$, *converging to k_0 as $\varepsilon \to 0$.*

Theorem 1.2. *Let $p = 0$, f and $\widetilde{f}$ be the restrictions of F in Ω and in $\mathbb{R}^3 \backslash \overline{\Omega}$, respectively. Then the solution to problems* (1.2), (1.3) *converges strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3 \backslash \overline{\Gamma}_1)$ as $\varepsilon \to 0$ to function* (1.4), *where $u_0(x)$ is a solution to the boundary-value problem*

$$\begin{cases} -\Delta u_0 = k^2 u_0 - f, & x \in \Omega, \\ u_0 = 0, & x \in \Gamma_2, \\ \dfrac{\partial u_0}{\partial x_3} = 0, & x \in \Gamma_1 \end{cases} \tag{1.8}$$

and $\widetilde{u}_0(x)$ is a solution to the boundary-value problem

$$\begin{cases} \left(\Delta + k^2\right) \widetilde{u}_0 = \widetilde{f}, \quad x \in \mathbb{R}^3 \backslash \overline{\Omega}, \\ \widetilde{u}_0 = 0, \quad x \in \Gamma_2, \qquad \dfrac{\partial \widetilde{u}_0}{\partial x_3} = 0, \quad x \in \Gamma_1 \end{cases} \tag{1.9}$$

with the radiation condition (1.7).

Here it is assumed that k^2 is not an eigenvalue of problem (1.8).

If $k^2 = k_0^2$ is an eigenvalue of problem (1.8), *then there is a pole $\tau_{\varepsilon,\delta(\varepsilon)}$ of the analytic continuation of the solution to* (1.2), (1.3) *in the half-plane* $\operatorname{Im} k < 0$, *converging to k_0 as $\varepsilon \to 0$.*

The notions of an analytic continuation and of a convergence in H^m_{loc} are classical and we shall give all the necessary definitions in Section 2.

2. Construction of analytic continuations of solutions

In this section we construct the solutions to problems (1.1)–(1.9) and their analytic continuations. This construction is similar to the standard construction of Sánchez-Palencia [41, Chapter 16, § 4], employed in Gadyl'shin [24] for the Helmholtz resonator and in Gadyl'shin [23,25] for its two-dimensional analogue in homogenization.

Given a Banach space X (for instance, $X = L_2$), we put $X_{\mathrm{loc}}(D) \stackrel{\mathrm{def}}{=} \{u : u \in X(D \cap S(R)) \quad \forall R\}$, where $S(R)$ is the open ball of radius R centered at the origin. We

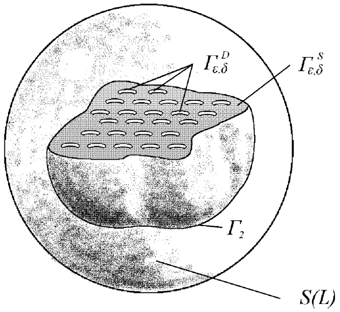

Fig. 2. Domain with perforated boundary.

say that a sequence converges in $X_{\text{loc}}(D)$, if it converges in $X(D\cap S(R))$ for all R. Let $\mathcal{B}(X,Y)$ be the Banach space of bounded linear operators, mapping the Banach space X into the Banach space Y, $\mathcal{B}(X)\stackrel{\text{def}}{=}\mathcal{B}(X,X)$, $\mathcal{B}(Y,X_{\text{loc}}(D))$ be the set of maps $\mathcal{A}:Y\to X_{\text{loc}}(D)$ such that $\mathcal{A}\in\mathcal{B}(Y,X(D\cap S(R)))$ for all R. We denote by $\mathcal{B}^h(X,Y)$ (by $\mathcal{B}^m(X,Y)$) the set of holomorphic (meromorphic) operator-valued functions whose values belong to $\mathcal{B}(X,Y)$; $\mathcal{B}^{h(m)}(X,X)\stackrel{\text{def}}{=}\mathcal{B}^{h(m)}(X)$, $\mathcal{B}^{h(m)}(X,Y_{\text{loc}}(D))\stackrel{\text{def}}{=}\{\mathcal{A}:\ \mathcal{A}\in\mathcal{B}^{h(m)}(X,Y(D\cap S(R)))\quad\forall R\}$.

The following proposition is well known (see, for example, [41]).

Proposition 2.1. *The map defined by*

$$(\Delta+k^2)^{-1}g\stackrel{\text{def}}{=}-\frac{1}{4\pi}\int_{S(L)}\frac{\mathrm{e}^{ik|x-y|}}{|x-y|}\,g(y)\,dy,\qquad x\in\mathbb{R}^3,$$

$g\in L_2(S(L))$, *belongs to* $\mathcal{B}^h(L_2(S(L),H^2_{\text{loc}}(\mathbb{R}^3))$.

The function $U=(\Delta+k^2)^{-1}g$ *satisfies the equation* $(\Delta+k^2)U=g$ *in* $\mathbb{R}^3$ *and for* $\operatorname{Im}k\geqslant 0$ *it also satisfies the radiation conditions*

$$U=O(r^{-1}),\qquad \frac{\partial U}{\partial r}-ikU=o(r^{-1}),\qquad r\to\infty.$$

Hereinafter we use the same notation for a function from $L_2(S(L))$ and its continuation by zero outside $S(L)$, regarding the latter as a function from $L_2(\mathbb{R}^3)$ (Fig. 2).

Denote $\Omega^{(1)}_{\varepsilon,\delta}=S(L)\backslash\overline{\Gamma^D_{\varepsilon,\delta}\cup\Gamma_2}$, $\Omega^{(2)}_{\varepsilon,\delta}=S(L)\backslash\overline{\Gamma^S_{\varepsilon,\delta}\cup\Gamma_2}$ (see Fig. 2). We assume that $L>0$ is such that $\overline{\Omega}\subset S\left(\frac{L}{3}\right)$. Consider two families of boundary-value problems in bounded domains:

$$\begin{cases}\big(\Delta-1\big)\,u_{\varepsilon,\delta}=\big(\Delta-1\big)\,w, & x\in\Omega^{(1)}_{\varepsilon,\delta},\\ u_{\varepsilon,\delta}=0,\quad x\in\Gamma^D_{\varepsilon,\delta}\cup\Gamma_2, & u_{\varepsilon,\delta}=w,\quad x\in\partial S(L),\end{cases}\tag{2.1}$$

and

$$\begin{cases} \big(\Delta - 1\big)\, u_{\varepsilon,\delta} = \big(\Delta - 1\big)\, w, \quad x \in \Omega^{(2)}_{\varepsilon,\delta}, \\ u_{\varepsilon,\delta} = 0, \quad x \in \Gamma_2, \qquad u_{\varepsilon,\delta} = w, \quad x \in \partial S(L), \\ \dfrac{\partial u_{\varepsilon,\delta}}{\partial x_3} = 0, \quad x \in \Gamma^S_{\varepsilon,\delta}, \end{cases} \tag{2.2}$$

where $w \in H^2(S(L))$. Denote by $\sigma^{(1)}_{\varepsilon,\delta}$ an operator whose value on $w \in H^2(S(L))$ is the solution $u_{\varepsilon,\delta} \in H^1(\Omega^{(1)}_{\varepsilon,\delta})$ of problem (2.1) and denote by $\sigma^{(2)}_{\varepsilon,\delta}$ an operator whose value on $w \in H^2(S(L))$ is the solution $u_{\varepsilon,\delta} \in H^1(\Omega^{(2)}_{\varepsilon,\delta})$ of problem (2.2). Similarly, denote by $\sigma^{(1)}_{0,0}$ an operator whose value on $w \in H^2(S(L))$ is the pair of the solutions $u_0 \in H^1(\Omega)$ and $\widetilde{u}_0 \in H^1(S(L)\backslash\overline{\Omega})$ to the problems

$$\begin{cases} \big(\Delta - 1\big)\, u_0 = \big(\Delta - 1\big)\, w, \quad x \in \Omega, \\ u_0 = 0, \quad x \in \Gamma \end{cases}$$

and

$$\begin{cases} \big(\Delta - 1\big)\, \widetilde{u}_0 = \big(\Delta - 1\big)\, w, \quad x \in S(L)\backslash\overline{\Omega}, \\ \widetilde{u}_0 = 0, \quad x \in \Gamma, \qquad \widetilde{u}_0 = w, \quad x \in \partial S(L), \end{cases}$$

respectively, and denote by $\sigma^{(2)}_{0,0}$ an operator whose value on $w \in H^2(S(L))$ is the pair of the solutions $u_0 \in H^1(\Omega)$ and $\widetilde{u}_0 \in H^1(S(L)\backslash\overline{\Omega})$ to the problems

$$\begin{cases} (\Delta - 1)u_0 = (\Delta - 1)w, \quad x \in \Omega, \\ u_0 = 0, \quad x \in \Gamma_2, \\ \dfrac{\partial u_0}{\partial x_3} = 0, \quad x \in \Gamma_1, \end{cases}$$

$$\begin{cases} (\Delta - 1)\widetilde{u}_0 = (\Delta - 1)w, \quad x \in S(L)\backslash\overline{\Omega}, \\ \widetilde{u}_0 = 0, \quad x \in \Gamma_2, \qquad \widetilde{u}_0 = w, \quad x \in \partial S(L), \\ \dfrac{\partial \widetilde{u}_0}{\partial x_3} = 0, \quad x \in \Gamma_1, \end{cases}$$

respectively.

Using the general theory of boundary value problems one can obtain the following Proposition.

Proposition 2.2. *The operator* $\sigma^{(1)}_{0,0}$ *belongs to* $\mathcal{B}(H^2(S(L)), H^1(S(L)))$ *and to* $\mathcal{B}(H^2(S(L)), H^2(S(L)\backslash\Gamma))$.

The operator $\sigma^{(2)}_{0,0}$ *belongs to* $\mathcal{B}(H^2(S(L)), H^1(S(L)\setminus\overline{\Gamma_1}))$ *and to* $\mathcal{B}(H^2(S(L)), H^2(Q))$ *for any domain* $Q \subset S(L)\setminus\Gamma$ *separated from* $\partial\Gamma_1$.

For any fixed ε *and* δ *the operator* $\sigma^{(1)}_{\varepsilon,\delta}$ *belongs to* $\mathcal{B}(H^2(S(L)), H^1(S(L)))$, *the operator* $\sigma^{(2)}_{\varepsilon,\delta}$ *belongs to* $\mathcal{B}(H^2(S(L)), H^1(S(L)\setminus\overline{\Gamma_1}))$, *and, in addition,* $\sigma^{(i)}_{\varepsilon,\delta}$ *belongs to* $\mathcal{B}(H^2(S(L)), H^2(Q))$ *for any domain* $Q \subset \Omega^{(i)}_{\varepsilon,\delta}$ *separated from* $\partial\Gamma^S_{\varepsilon,\delta}$.

Denote by $\chi(t)$ a smooth cut-off function that disappears for $t < 1/3$ and equals to zero for $t > 2/3$ and by p_L the operator of the restriction from $\mathbb{R}^3$ to $S(L)$,

$$
\begin{aligned}
A^{(j)}_{\varepsilon,\delta}(k) &\stackrel{\text{def}}{=} \left(1-\chi\left(\frac{r}{L}\right)\right)(\Delta+k^2)^{-1} + \chi\left(\frac{r}{L}\right)\sigma^{(j)}_{\varepsilon,\delta} p_L(\Delta+k^2)^{-1},\\
A^{(j)}_{0,0}(k) &\stackrel{\text{def}}{=} \left(1-\chi\left(\frac{r}{L}\right)\right)(\Delta+k^2)^{-1} + \chi\left(\frac{r}{L}\right)\sigma^{(j)}_{0,0} p_L(\Delta+k^2)^{-1},\\
T^{(j)}_{\varepsilon,\delta}(k)g &\stackrel{\text{def}}{=} \Bigg(\left(\Delta+k^2\right)\left(\chi\left(\frac{r}{L}\right)\right)\left(\left(1-\sigma^{(j)}_{\varepsilon,\delta}\,p_L\right)\left(\Delta+k^2\right)^{-1}\right)\\
&\quad + 2\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\chi\left(\frac{r}{L}\right)\right)\frac{\partial}{\partial x_i}\left(\left(1-\sigma^{(j)}_{\varepsilon,\delta}\,p_L\right)\left(\Delta+k^2\right)^{-1}\right)\Bigg)g,\\
T^{(j)}_{0,0}(k)g &\stackrel{\text{def}}{=} \Bigg(\left(\Delta+k^2\right)\left(\chi\left(\frac{r}{L}\right)\right)\left(\left(1-\sigma^{(j)}_{0,0}\,p_L\right)\left(\Delta+k^2\right)^{-1}\right)\\
&\quad + 2\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(\chi\left(\frac{r}{L}\right)\right)\frac{\partial}{\partial x_i}\left(\left(1-\sigma^{(j)}_{0,0}\,p_L\right)\left(\Delta+k^2\right)^{-1}\right)\Bigg)g,\\
B^{(j)}_{\varepsilon,\delta}(k) &= I - T^{(j)}_{\varepsilon,\delta}(k), \qquad B^{(j)}_{0,0}(k) = I - T^{(j)}_{0,0}(k),
\end{aligned}
$$

where I is the identity mapping.

From the definitions of $T^{(j)}_{\mu,\nu}(k)$ it follows that for $g \in L_2(S(L))$ the function $T^{(j)}_{\mu,\nu}(k)g \in L_2(\mathbb{R}^3)$ and $\operatorname{supp} T^{(j)}_{\mu,\nu}(k)g \subset \overline{S(L)}$. For this reason, the maps $T^{(j)}_{\mu,\nu}(k)$ and $B^{(j)}_{\mu,\nu}(k)$ can be considered as operators from $L_2(S(L))$ into $L_2(S(L))$. Under this interpretation, from the definitions of $A^{(j)}_{\mu,\nu}(k)$ and $T^{(j)}_{\mu,\nu}(k)$ and from Propositions 2.1, 2.2 the following statements hold.

Lemma 2.1. *For* $k \in \mathbb{C}$

$$
\begin{aligned}
&A^{(1)}_{\mu,\nu}(k) \in \mathcal{B}^h\left(L_2(S(L)), H^1_{\text{loc}}(\mathbb{R}^3)\right), \quad A^{(2)}_{\mu,\nu}(k) \in \mathcal{B}^h\left(L_2(S(L)), H^1_{\text{loc}}(\mathbb{R}^3\setminus\overline{\Gamma_1})\right),\\
&T^{(j)}_{\mu,\nu}(k) \in \mathcal{B}^h\left(L_2(S(L))\right), \quad B^{(j)}_{\mu,\nu}(k) \in \mathcal{B}^h\left(L_2(S(L))\right),
\end{aligned}
$$

and, for any fixed k, μ *and* ν, $T^{(j)}_{\mu,\nu}(k)$ *is a compact operator in* $L_2(S(L))$; *here* (μ,ν) *take the value* (ε,δ) *or* $(0,0)$.

One can see that

$$
\begin{aligned}
(\Delta + k^2)A^{(1)}_{\varepsilon,\delta}(k)g &= B^{(1)}_{\varepsilon,\delta}(k)g \ \text{ on } \mathbb{R}^3\backslash\overline{\Gamma^D_{\varepsilon,\delta}\cup\Gamma_2},\\
(\Delta + k^2)A^{(2)}_{\varepsilon,\delta}(k)g &= B^{(2)}_{\varepsilon,\delta}(k)g \ \text{ on } \mathbb{R}^3\backslash\overline{\Gamma^S_{\varepsilon,\delta}\cup\Gamma_2},\\
(\Delta + k^2)A^{(j)}_{0,0}(k)g &= B^{(j)}_{0,0}(k)g \ \text{ on } \mathbb{R}^3\backslash\Gamma.
\end{aligned}
$$

Hence, due to the definitions of $A^{(j)}_{\mu,\nu}(k)$ and Proposition 2.1 we have

Lemma 2.2. *For* $k \in \mathbb{C}$,

(a) *the function* $u_{\varepsilon,\delta} = A^{(1)}_{\varepsilon,\delta}(k)g$ *satisfies* (1.1) *for* $F = B^{(1)}_{\varepsilon,\delta}(k)g$;

(b) *the function* $u_{\varepsilon,\delta} = A^{(2)}_{\varepsilon,\delta}(k)g$ *satisfies* (1.2) *for* $F = B^{(2)}_{\varepsilon,\delta}(k)g$;

(c) *the restriction* u_0 *of* $A^{(1)}_{0,0}(k)g$ *to* Ω *satisfies* (1.5), *where* f *is equal to the restriction of* $B^{(1)}_{0,0}(k)g$ *to* Ω;

(d) *the restriction* $\widetilde{u}_0$ *of* $A^{(1)}_{0,0}(k)g$ *to* $\mathbb{R}^3\backslash\overline{\Omega}$ *satisfies* (1.6), *where* $\widetilde{f}$ *is equal to the restriction of* $B^{(1)}_{0,0}(k)g$ *to* $S(L)\backslash\overline{\Omega}$;

(e) *the restriction* u_0 *of* $A^{(2)}_{0,0}(k)g$ *to* Ω *satisfies* (1.8), *where* f *is equal to the restriction of* $B^{(2)}_{0,0}(k)g$ *to* Ω;

(f) *the restriction* $\widetilde{u}_0$ *of* $A^{(2)}_{0,0}(k)g$ *to* $\mathbb{R}^3\backslash\overline{\Omega}$ *satisfies* (1.9), *where* $\widetilde{f}$ *equals to the restriction of* $B^{(2)}_{0,0}(k)g$ *to* $S(L)\backslash\overline{\Omega}$; *and*

(g) *for* $\operatorname{Im} k \geqslant 0$, *the functions* $u_{\varepsilon,\delta}$ *and* $\widetilde{u}_0$ *satisfy the radiation conditions* (1.3), (1.7).

The square root of the eigenvalue is called *the eigenfrequency* of the boundary-value problem. Denote by $\Sigma^{(1)}$ and $\Sigma^{(2)}$ the sets of eigenfrequencies of boundary-value problems (1.5) and (1.8), respectively.

Proposition 2.3. *If* $\operatorname{Im} k \geqslant 0$ *then the solutions of the perturbed problems* (1.1) ((1.2)), (1.3) *and the limit external problems* (1.6) ((1.9)), (1.7) *are unique. If* $k \notin \Sigma^{(1)}$ ($k \notin \Sigma^{(2)}$) *then the solution of the limit internal problem* (1.5) ((1.8)) *is unique.*

The proof of this statement is well known (for the three-dimensional external Neumann problem outside nonclosed surfaces see, for instance Gadyl'shin [24]).

Lemma 2.3. *If* $g \in L_2(S(L))$ *and* $g \neq 0$, *then* $A^{(j)}_{\mu,\nu}(k)g \neq 0$.

Proof. If $B^{(j)}_{\mu,\nu}(k)g \neq 0$, then the statement of the lemma is obvious. Let $B^{(j)}_{\mu,\nu}(k)g = 0$, and let $u_{\mu,\nu} = A^{(j)}_{\mu,\nu}(k)g = 0$. The definition of $A^{(j)}_{\mu,\nu}(k)$ implies that

$$
u_{\mu,\nu}(x) = w(x) - \chi\left(\frac{r}{L}\right)(w(x) - v_{\mu,\nu}(x)),
$$

where $w = (\Delta + k^2)^{-1} g$ and $v_{\mu,\nu} = \sigma^{(j)}_{\mu,\nu} p_L(S(L))w$. The above formula implies that

$$0 = w(x) - \chi\left(\frac{r}{L}\right)(w(x) - v_{\mu,\nu}(x)), \tag{2.3}$$

whence $w = 0$ outside $S(\frac{2L}{3})$, and $v_{\mu,\nu} = 0$ in $S(\frac{L}{3})$. Hence, $U_{\mu,\nu} = w - v_{\mu,\nu}$ belongs to $H^2(S(L))$ and is a solution of the problem

$$(\Delta - 1)U_{\mu,\nu} = 0, \quad x \in S(L), \qquad U_{\mu,\nu} = 0, \quad x \in \partial S(L). \tag{2.4}$$

It is obvious that the solution of (2.4) is trivial. Formula (2.3) implies that $w = 0$ in $\mathbb{R}^3$. The definition of w gives $g = 0$. We have come to a contradiction, which completes the proof of the lemma. □

Lemma 2.4. *If* $\operatorname{Im} k > 0$ *or* $k > 0$, *and* $k \notin \Sigma^{(j)}$, *then there exists the operator* $\left(B^{(j)}_{\mu,\nu}\right)^{-1}(k) \in \mathcal{B}(L_2(S(L)))$.

Proof. Since $B^{(j)}_{\mu,\nu}$ is the Fredholm operator of the second kind, then it is sufficient to show that the equation

$$B^{(j)}_{\mu,\nu} g = 0$$

has a trivial solution only. Assume that there exists a nontrivial g, which satisfies the equation; then, due to Lemma 2.2 the function $A^{(j)}_{\mu,\nu} g$ is a nontrivial solution to the corresponding boundary value problem. Moreover, this function is nonzero due to Lemma 2.3. It contradicts Proposition 2.3. □

In further analysis we shall use the following statement from Sánchez-Palencia [41, Chapter 15, § 7, Theorem 7.1].

Proposition 2.4. *Suppose that* D *is a connected domain of the complex plane,* $T(k)$, $(k \in D)$ *is a holomorphic family of compact operators in a Banach space* $\mathcal{X}$ *and there exists a point* $k_* \in D$, *such that* $(I - T(k_*))^{-1} \in \mathcal{B}(\mathcal{X})$. *Then* $(I - T(k))^{-1}$ *is a meromorphic function in* D *with values in* $\mathcal{B}(\mathcal{X})$.

From Proposition 2.4 and Lemma 2.4 the following lemma follows:

Lemma 2.5. *There exists* $\left(B^{(j)}_{\mu,\nu}\right)^{-1}(k)$ *belonging to* $\mathcal{B}^m(L_2(S(L)))$ *for* $k \in \mathbb{C}$.

Denote $\mathcal{A}^{(j)}_{\mu,\nu}(k) \overset{\text{def}}{=} A^{(j)}_{\mu,\nu}(k)\left(B^{(j)}_{\mu,\nu}\right)^{-1}(k)$.

Theorem 2.1. *For* $k \in \mathbb{C}$

(a)

$$\mathcal{A}^{(1)}_{\mu,\nu}(k) \in \mathcal{B}^m\left(L_2(S(L)), H^1_{\mathrm{loc}}(\mathbb{R}^3)\right),$$
$$\mathcal{A}^{(2)}_{\mu,\nu}(k) \in \mathcal{B}^m\left(L_2(S(L)), H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\Gamma_1})\right);$$

(b) *the function* $u_{\varepsilon,\delta(\varepsilon)} = \mathcal{A}^{(1)}_{\varepsilon,\delta(\varepsilon)}(k)F$ *satisfies* (1.1), *the function* $u_{\varepsilon,\delta(\varepsilon)} = \mathcal{A}^{(2)}_{\varepsilon,\delta(\varepsilon)}(k)F$ *satisfies* (1.2), *the restriction* u_0 ($\widetilde{u}_0$) *of* $\mathcal{A}^{(1)}_{0,0}(k)F$ *to* Ω (*to* $\mathbb{R}^3\backslash\overline{\Omega}$) *satisfies* (1.5) ((1.6)), *where* f ($\widetilde{f}$) *is the restriction of* F *to* Ω (*to* $S(L)\backslash\overline{\Omega}$), *the restriction* u_0 ($\widetilde{u}_0$) *of* $\mathcal{A}^{(2)}_{0,0}(k)F$ *to* Ω (*to* $\mathbb{R}^3\backslash\overline{\Omega}$) *satisfy* (1.8) ((1.9)), *where* f ($\widetilde{f}$) *is the restriction of* F *to* Ω (*to* $S(L)\backslash\overline{\Omega}$), *and, for* $\mathrm{Im}\, k \geqslant 0$, *the functions* $u_{\varepsilon,\delta(\varepsilon)}$ *and* $\widetilde{u}_0$ *satisfy the radiation conditions* (1.3), (1.7);

(c) *if* $\mathrm{supp}\, F \subset S(T)$, *then the function* $\mathcal{A}^{(j)}_{\mu,\nu}(k)F$ *does not depend on* $L \geqslant T$; $j = 1, 2$;

(d) *the set of poles of the operators* $\mathcal{A}^{(j)}_{\mu,\nu}(k)$ *and* $\left(B^{(j)}_{\mu,\nu}(k)\right)^{-1}$ *coincide for fixed* j; $j = 1, 2$; *and*

(e) *the set of poles of the operator* $\mathcal{A}^{(j)}_{\mu,\nu}(k)$ *does not depend on* L; $j = 1, 2$.

Proof. Statement (a) follows from Lemmas 2.1 and 2.5. Statement (b) follows from Lemmas 2.2. Statement (c) follows from Proposition 2.3 and the uniqueness of the analytic continuation. Statement (d) follows from Lemma 2.3 and the definition of the operators $\mathcal{A}^{(j)}_{\mu,\nu}(k)$; $j = 1, 2$. Let us prove statement (e) for $j = 1$. Denote by $\mathcal{A}^{(1)}_{\mu,\nu,t}(k)$ the operator $\mathcal{A}^{(1)}_{\mu,\nu}(k)$ defined for $L = t$. Suppose $a > b$. It is obvious that the set of poles of $\mathcal{A}^{(1)}_{\nu,\mu,b}(k)$ is a subset of the set of poles of $\mathcal{A}^{(1)}_{\mu,\nu,a}(k)$. Now we show the inverse inclusion. Suppose that $\mathrm{supp} F \subset S(a)$ and assume that

$$W = (1 - \chi(rb^{-1}))(\Delta + k^2)^{-1}F.$$

Since the support $\mathrm{supp}(F - (\Delta + k^2)W) \subset \overline{S(b)}$, then due to (b) the solution of the perturbed problems (1.5) and (1.3) for $\mu, \nu > 0$ and of the limit problems (1.1) and (1.6), (1.3) for $\mu = \nu = 0$ can be defined by one of the following formulae:

$$u_{\mu,\nu} = \mathcal{A}^{(1)}_{\mu,\nu,a}(k)F, \qquad u_{\mu,\nu} = \mathcal{A}^{(1)}_{\mu,\nu,a}(k)(F - (\Delta + k^2)W) + W.$$

Since W is holomorphic, the set of poles of $\mathcal{A}^{(1)}_{\varepsilon,a}$ is a subset of the set of poles of $\mathcal{A}^{(1)}_{\varepsilon,b}$. The case $j = 2$ can be proved in an analogues way. The theorem is proved. □

3. Convergence of the operator $\sigma^{(1)}_{\varepsilon,\delta}$

The goal of this section is to prove the following statement.

Theorem 3.1. *If* $p=\infty$, *then*

$$\|\sigma^{(1)}_{\varepsilon,\delta(\varepsilon)}-\sigma^{(1)}_{0,0}\|_{\mathcal{B}(H^2(S(L)),H^1(S(L)))}\underset{\varepsilon\to 0}{\to}0.$$

The proof of the following lemma is carried out along the main lines of the proof of a similar two-dimensional lemma from [12].

Lemma 3.1. *For any* $v\in H^1(S(L))$ *such that* $v=0$ *on* $\partial S(L)\cup\Gamma^D_{\varepsilon,\delta}\cup\Gamma_2$ *the following estimates*

$$\|v\|_{L_2(\Gamma)}\leqslant C\left(\frac{\delta}{\varepsilon}\right)^{1/2}\|v\|_{H^1(\Omega)}$$

and

$$\|v\|_{L_2(\Gamma)}\leqslant C\left(\frac{\delta}{\varepsilon}\right)^{1/2}\|v\|_{H^1(S(L)\setminus\Omega)}$$

are valid.

From Lemma 3.1 one can deduce the following statement (in the same way as it was proved in two-dimensional case in [12]; see also [10]).

Lemma 3.2. *Let* $p=\infty$. *Suppose also that* $F,\ F_\varepsilon\in L_2(S(L))$, f *and* $\widetilde{f}$ *are the restrictions of* F *in* Ω *and in* $S(L)\setminus\overline{\Omega}$, *respectively, and* $F_\varepsilon\rightharpoonup F$ *as* $\varepsilon\to 0$ *weakly in* $L_2(S(L))$. *Then the solution of the boundary-value problem*

$$\begin{cases}(\Delta-1)u_{\varepsilon,\delta(\varepsilon)}=F_\varepsilon, & x\in S(L)\setminus\overline{\Gamma^D_{\varepsilon,\delta(\varepsilon)}\cup\Gamma_2},\\ u_{\varepsilon,\delta(\varepsilon)}=0, & x\in\Gamma^D_{\varepsilon,\delta(\varepsilon)}\cup\Gamma_2\cup\partial S(L),\end{cases}\tag{3.1}$$

converges to the function

$$u(x)=\begin{cases}u_0(x), & x\in\Omega,\\ \widetilde{u}_0(x), & x\in S(L)\setminus\overline{\Omega}\end{cases}\tag{3.2}$$

weakly in $H^1(S(L))$ *as* $\varepsilon\to 0$, *where* $u_0(x)$ *satisfies the boundary-value problem*

$$\begin{cases}(\Delta-1)u_0=f, & x\in\Omega,\\ u_0=0, & x\in\Gamma\end{cases}\tag{3.3}$$

and $\widetilde{u}_0(x)$ *satisfies the boundary-value problem*

$$\begin{cases} (\Delta - 1)\widetilde{u}_0 = \widetilde{f}, & x \in S(L)\backslash\overline{\Omega}, \\ \widetilde{u}_0 = 0, & x \in \Gamma \cup \partial S(L). \end{cases} \tag{3.4}$$

Lemma 3.3. *Under the conditions of Lemma* 3.2 *the convergence of the solutions* $u_{\varepsilon,\delta}$ *is strong in the Sobolev space* $H^1(S(L))$.

Proof. Using the integral identities of problems (3.1), (3.3) and (3.4), we obtain estimates

$$\|u_{\varepsilon,\delta(\varepsilon)}\|^2_{H^1(S(L))} = \int_{S(L)} F_\varepsilon u_{\varepsilon,\delta(\varepsilon)}\, dx, \quad \|u\|^2_{H^1(S(L))} = \int_{S(L)} Fu\, dx.$$

Since $u_{\varepsilon,\delta(\varepsilon)} \longrightarrow u$ strongly in $L_2(S(L))$ and $F_\varepsilon \rightharpoonup F$ weakly in $L_2(S(L))$, it follows that

$$\int_{S(L)} F_\varepsilon u_{\varepsilon,\delta(\varepsilon)}\, dx \longrightarrow \int_{S(L)} Fu\, dx.$$

Hence,

$$\|u_{\varepsilon,\delta(\varepsilon)}\|^2_{H^1(S(L))} \longrightarrow \|u\|^2_{H^1(S(L))}.$$

Taking into account the weak convergence of $u_{\varepsilon,\delta(\varepsilon)}$ in $H^1(S(L))$, we complete the proof. □

We use the notation $\mathcal{L}^{(1)}_\varepsilon$ for the operator mapping a function F into the solution of (3.1) and the notation $\mathcal{L}^{(1)}_0$ for the operator mapping a function F into the pair $(u_0, \widetilde{u}_0)$ of solutions of (3.3) and (3.4), where f and $\widetilde{f}$ are the restrictions of F in Ω and in $S(L)\backslash\overline{\Omega}$, respectively.

Lemma 3.4. *Let* $p = \infty$. *Then*

$$\|\mathcal{L}^{(1)}_\varepsilon - \mathcal{L}^{(1)}_0\|_{\mathcal{B}(H^2(S(L)),H^1(S(L)))} \to 0 \quad as\ \varepsilon \to 0. \tag{3.5}$$

Proof. We prove (3.5) by arguing by contradiction. If (3.5) is wrong, then there exist constant $c > 0$, and sequences $\varepsilon_i \to 0$ as $i \to \infty$ and $\widetilde{F}_i \in L_2(S(L))$, such that $\|\widetilde{F}_i\|_{L_2(S(L))} = 1$ and

$$\| \left(\mathcal{L}^{(1)}_{\varepsilon_i} - \mathcal{L}^{(1)}_0\right) \widetilde{F}_i\|_{H^1(S(L))} > c, \tag{3.6}$$

$$\widetilde{F}_i \rightharpoonup F_0 \quad \text{weakly in } L_2(S(L)). \tag{3.7}$$

On the other hand, due to Lemma 3.3 and (3.7) we have

$$\begin{aligned}\left\| \left(\mathcal{L}^{(1)}_{\varepsilon_i} - \mathcal{L}^{(1)}_0\right) \widetilde{F}_i \right\|_{H^1(S(L))} \leqslant\ & \|\mathcal{L}^{(1)}_{\varepsilon_i}(\widetilde{F}_i - F_0)\|_{H^1(S(L))} + \|(\mathcal{L}^{(1)}_{\varepsilon_i} - \mathcal{L}^{(1)}_0)F_0\|_{H^1(S(L))} \\ & + \|\mathcal{L}^{(1)}_0(F_0 - \widetilde{F}_i)\|_{H^1(S(L))} \to 0 \quad \text{as } i \to \infty.\end{aligned}$$

This contradiction with (3.6) completes the proof. $\square$

Theorem 3.1 is a direct consequence of Lemma 3.4 and the definition of $\sigma^{(1)}_{\varepsilon,\delta}$.

4. Convergence of the operator $\boldsymbol{\sigma^{(2)}_{\varepsilon,\delta}}$

The goal of this section is to prove the following.

Theorem 4.1. *If* $p = 0$, *then*

$$\|\sigma^{(2)}_{\varepsilon,\delta(\varepsilon)} - \sigma^{(2)}_{0,0}\|_{\mathcal{B}(H^2(S(L)),H^1((S(L)\setminus\overline{\Gamma_1})))} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

It is known (see, for instance Eskin [17, Chapter 6, § 22]) that there exists a harmonic in $\mathbb{R}^3_- = \{x : x_3 < 0\}$ function $X_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3_-) \cap C^\infty(\overline{\mathbb{R}^3_-}\setminus\partial\omega)$ which disappears at infinity and satisfies the boundary conditions $X_0 = 1$ in ω and $\partial X_0/\partial x_3 = 0$ on $\gamma = \{x : x_3 = 0,\ (x_1, x_2) \notin \overline{\omega}\}$. In addition, the function X_0 has the differentiable asymptotics

$$X_0(x) = c_\omega r^{-1} + O(r^{-2}) \qquad \text{as } r \to \infty,$$

where $c_\omega > 0$ is the capacity of ω (see, [30, Chapter II, § 3], [38, Chapter I]).

Let $\Sigma = (-1, 1) \times (-1, 1) \times (0, -\infty)$. Denote by $W^\varepsilon(x)$ the even continuation of the function

$$1 - \chi(r\varepsilon^{-1/2})X_0(x\varepsilon^{-1})$$

defined in Σ, with respect to x_3. We employ the same notation for the $(-1, 1)\times(-1, 1)$-periodic translation of the function $W^\varepsilon(x)$ on the plane $x_3 = 0$. Denote $W^{\varepsilon,\delta(\varepsilon)}(x) = W^\varepsilon\left(\frac{x}{\delta(\varepsilon)}\right)$. Taking into account the definition of X_0 one can obtain the following statement.

Lemma 4.1. *Let* $p = 0$. *Then* $W^{\varepsilon,\delta(\varepsilon)} \in H^1\left(S(L)\backslash\overline{\Gamma^S_{\varepsilon,\delta(\varepsilon)}}\right)$, $W^{\varepsilon,\delta(\varepsilon)}(x) = 0$ *on* $\Gamma^D_{\varepsilon,\delta(\varepsilon)}$,

$$\|W^{\varepsilon,\delta(\varepsilon)} - 1\|_{H^1(S(L)\backslash\overline{\Gamma_1})} \underset{\varepsilon\to 0}{\to} 0.$$

Lemma 4.2. *Let* $p = 0$. *Suppose also that* F, $F_\varepsilon \in L_2(S(L))$, f *and* $\widetilde{f}$ *are the restrictions of* F *in* Ω *and in* $S(L)\backslash\overline{\Omega}$, *respectively, and* $F_\varepsilon \rightharpoonup F$ *as* $\varepsilon \to 0$ *weakly in* $L_2(S(L))$. *Then the solution of the boundary-value problem*

$$\begin{cases} (\Delta - 1)u_{\varepsilon,\delta(\varepsilon)} = F_\varepsilon, \quad x \in S(L)\backslash\overline{\Gamma^S_{\varepsilon,\delta(\varepsilon)} \cup \Gamma_2}, \\ u_{\varepsilon,\delta(\varepsilon)} = 0, \quad x \in \Gamma_2 \cup \partial S(L), \qquad \dfrac{\partial u_{\varepsilon,\delta(\varepsilon)}}{\partial x_3} = 0, \quad x \in \Gamma^S_{\varepsilon,\delta(\varepsilon)} \end{cases} \tag{4.1}$$

converges strongly in $H^1\left(S(L)\backslash\overline{\Gamma_1}\right)$ *as* $\varepsilon \to 0$ *to function* (3.2), *where* $u_0(x)$ *satisfies the boundary-value problem*

$$\begin{cases} (\Delta - 1)u_0 = f, \quad x \in \Omega, \\ u_0 = 0, \quad x \in \Gamma_2, \qquad \dfrac{\partial u_0}{\partial x_3} = 0, \quad x \in \Gamma_1 \end{cases} \tag{4.2}$$

and $\widetilde{u}_0(x)$ *satisfies the boundary-value problem*

$$\begin{cases} (\Delta - 1)\widetilde{u}_0 = \widetilde{f}, \quad x \in S(L)\backslash\overline{\Omega}, \\ \widetilde{u}_0 = 0, \quad x \in \Gamma_2 \cup \partial S(L), \qquad \dfrac{\partial \widetilde{u}_0}{\partial x_3} = 0, \quad x \in \Gamma_1. \end{cases} \tag{4.3}$$

Proof. From the integral identity of problem (4.1) we deduce the uniform boundedness of the function $u_{\varepsilon,\delta(\varepsilon)}$ in $H^1(S(L)\backslash\overline{\Gamma_1})$. Let $\{\varepsilon_n\}$ be a sequence which tends to zero as $n \to \infty$. Due to the embedding theorems, weak compactness of the bounded set of functions in $H^1(S(L))$, there exists a subsequence $\{\varepsilon'_n\}$, such that $u_{\varepsilon,\delta(\varepsilon)} \rightharpoonup u_*$ weakly in $H^1(\Omega)$ and strongly in $L_2(\Omega)$ as $\varepsilon'_n \to 0$, $u_* \in H^1(\Omega)$, $u_* = 0$ on Γ_2, and $u_{\varepsilon,\delta(\varepsilon)} \rightharpoonup \widetilde{u}_*$ weakly in $H^1(S(L)\backslash\overline{\Omega})$ and strongly in $L_2(S(L)\backslash\overline{\Omega})$ as $\varepsilon'_n \to 0$, $\widetilde{u}_* \in H^1(S(L)\backslash\overline{\Omega})$, $\widetilde{u}_* = 0$ on $\partial S(L) \cup \Gamma_2$.

Let V be any function from $C^\infty(\overline{\Omega})$ such that $V = 0$ on Γ_2, and $\widetilde{V}$ be any function from $C^\infty(\overline{S(L)}\backslash\Omega)$ such that $\widetilde{V} = 0$ on $\partial S(L) \cup \Gamma_2$. Then, due to Lemma 4.1, first,

$$\|V - W^{\varepsilon,\delta(\varepsilon)}V\|_{H^1(\Omega)} + \|\widetilde{V} - W^{\varepsilon,\delta(\varepsilon)}\widetilde{V}\|_{H^1(S(L)\backslash\overline{\Omega})} \underset{\varepsilon\to 0}{\to} 0, \tag{4.4}$$

and, second, the continuation of $W^{\varepsilon,\delta(\varepsilon)}V$ into $S(L)\backslash\overline{\Omega}$ by zero and the continuation of $W^{\varepsilon,\delta(\varepsilon)}\widetilde{V}$ into Ω by zero are functions from the space $H^1(S(L)\backslash\overline{\Gamma^S_{\varepsilon,\delta(\varepsilon)}})$ which equal zero on $\partial S(L) \cup \Gamma_2$. Hence, substituting $v = W^{\varepsilon,\delta(\varepsilon)}V$ and $v = W^{\varepsilon,\delta(\varepsilon)}\widetilde{V}$ into the integral identity of problem (4.1), we have

$$\int_\Omega \left(\nabla u_{\varepsilon,\delta(\varepsilon)} \nabla \left(W^{\varepsilon,\delta(\varepsilon)}V\right) + u_{\varepsilon,\delta(\varepsilon)} \left(W^{\varepsilon,\delta(\varepsilon)}V\right)\right) dx = \int_\Omega f_\varepsilon W^{\varepsilon,\delta(\varepsilon)} V \, dx \tag{4.5}$$

and

$$\int_{S(L)\setminus\overline{\Omega}} \left(\nabla u_{\varepsilon,\delta(\varepsilon)} \nabla \left(W^{\varepsilon,\delta(\varepsilon)}\widetilde{V}\right) + u_{\varepsilon,\delta(\varepsilon)} \left(W^{\varepsilon,\delta(\varepsilon)}\widetilde{V}\right)\right) dx$$
$$= -\int_{S(L)\setminus\overline{\Omega}} \widetilde{f}_\varepsilon W^{\varepsilon,\delta(\varepsilon)} \widetilde{V} \, dx, \tag{4.6}$$

where f_ε and $\widetilde{f}_\varepsilon$ are the restrictions of F_ε on Ω and $S(L)\setminus\overline{\Omega}$, respectively.

Keeping in mind (4.4) and passing to the limit in (4.5), and (4.6) as $\varepsilon'_n \to 0$, we obtain

$$\int_\Omega (\nabla u_* \nabla V + u_* V) \, dx = \int_\Omega f V \, dx$$

and

$$\int_{S(L)\setminus\overline{\Omega}} \left(\nabla \widetilde{u}_* \nabla \widetilde{V} + \widetilde{u}_* \widetilde{V}\right) dx = \int_{S(L)\setminus\overline{\Omega}} \widetilde{f}\widetilde{V} \, dx,$$

respectively. Due to the uniqueness of solutions to problems (4.2) and (4.3) we conclude that $u_* \equiv u_0$ and $\widetilde{u}_* \equiv \widetilde{u}_0$.

On the other hand, owing to the freedom in choosing the sequence $\{\varepsilon_n\}$, we obtain that $u_{\varepsilon,\delta(\varepsilon)} \rightharpoonup u$ weakly in $H^1\left(S(L)\setminus\overline{\Gamma_1}\right)$ and strongly in $L_2(S(L))$ as $\varepsilon \to 0$.

The proof of the strong convergence of $u_{\varepsilon,\delta(\varepsilon)} \to u$ in $H^1\left(S(L)\setminus\overline{\Gamma_1}\right)$ is similar to that in the Lemma 3.3. The lemma is proved. □

We use the notation $\mathcal{L}_\varepsilon^{(2)}$ for the operator mapping a function F into the solution of (3.1) and the notation $\mathcal{L}_0^{(2)}$ for the operator mapping a function F into the pair $(u_0, \widetilde{u}_0)$ of solutions of (4.2) and (4.3), where f and $\widetilde{f}$ are the restrictions of F in Ω and in $S(L)\setminus\overline{\Omega}$, respectively.

The proof of the following lemma is analogues to the proof of Lemma 3.4, keeping in mind Lemma 4.2.

Lemma 4.3. *Let* $p = 0$. *Then*

$$\|\mathcal{L}_\varepsilon^{(2)} - \mathcal{L}_0^{(2)}\|_{\mathcal{B}(H^2(S(L)),H^1((S(L)\setminus\overline{\Gamma_1})))} \to 0 \quad \text{as } \varepsilon \to 0.$$

Theorem 4.1 is a direct consequence of Lemma 4.3 and the definition of $\sigma_{\varepsilon,\delta}^{(2)}$.

5. Proof of Theorems 1.1 and 1.2

The next lemma follows from the definitions of operators $T^{(j)}_{\mu,\nu}(k)$, $B^{(j)}_{\mu,\nu}(k)$, Proposition 2.1, and Theorems 3.1 and 4.1.

Lemma 5.1. *Assume that K is an arbitrary compact set in $\mathbb{C}$. Then*

(a) *if $p=\infty$, then $T^{(1)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\to} T^{(1)}_{0,0}(k)$ and $B^{(1)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\to} B^{(1)}_{0,0}(k)$ in the norm $\mathcal{B}(L_2(S(L)))$ uniformly on $k\in K$; and*

(b) *if $p=0$, then $T^{(2)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\to} T^{(2)}_{0,0}(k)$ and $B^{(2)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\to} B^{(2)}_{0,0}(k)$ in the norm $\mathcal{B}(L_2(S(L)))$ uniformly on $k\in K$.*

In what follows, we shall use the following statement from Sánchez-Palencia [41, Chapter 15, § 7, Theorem 7.2]:

Proposition 5.1. *Suppose that D is a connected domain in the complex plane, $T(k,\mu)$ is a family of compact operators in Banach space $\mathcal{X}$, defined for $k\in D$ and $\mu\in[0,\mu_0]$, such that it is a holomorphic of k for each μ and continuous on $D\times[0,\mu_0]$ in the norm $\mathcal{B}(\mathcal{X})$. Furthermore, assume that there exists a point $k_0\in D$, such that $(I-T(k_0,\mu))^{-1}\in\mathcal{B}(\mathcal{X})$ for any $\mu\in(0,\mu_0)$. Then*

(a) *$(I-T(k,\mu))^{-1}$ (for any μ) is a meromorphic function in D with values in $\mathcal{B}(\mathcal{X})$;*

(b) *if k_* is not a pole $(I-T(k,\mu_*))^{-1}$, then the operator-values function $(I-T(k,\mu))^{-1}$ is continuous in the norm in a neighborhood of (k_*,μ_*);*

(c) *the poles $(I-T(k,\mu))^{-1}$ depend on μ continuously.*

Denote by $\Sigma^{(j)}_{\mu,\nu}$ the set of the poles of the operator $\mathcal{A}^{(j)}_{\mu,\nu}(k)$. Obviously, $\Sigma^{(j)}\subset\Sigma^{(j)}_{0,0}$. By Lemmas 2.5 and 5.3 the family $T^{(j)}_{\mu,\nu}(k)$ satisfies the conditions of Proposition 5.1. Then the following lemma holds true:

Lemma 5.2. (a) *Assume that $p=\infty$. If K is an arbitrary compact set in $\mathbb{C}$, such that $K\cap\Sigma^{(1)}_{0,0}=\emptyset$, then $\left(B^{(1)}_{\varepsilon,\delta(\varepsilon)}(k)\right)^{-1} \underset{\varepsilon\to 0}{\longrightarrow} \left(B^{(1)}_{0,0}(k)\right)^{-1}$ in the norm $\mathcal{B}(L_2(S(L)))$ uniformly on $k\in K$. If $\tau_0\in\Sigma^{(1)}_{0,0}$, then there exists the pole $\tau_{\varepsilon,\delta(\varepsilon)}\in\Sigma^{(1)}_{\varepsilon,\delta(\varepsilon)}$, converging to τ_0 as $\varepsilon\to 0$.*

(b) *Assume that $p=0$. If K is an arbitrary compact set in $\mathbb{C}$, such that $K\cap\Sigma^{(2)}_{0,0}=\emptyset$, then $\left(B^{(2)}_{\varepsilon,\delta(\varepsilon)}(k)\right)^{-1} \underset{\varepsilon\to 0}{\longrightarrow} \left(B^{(2)}_{0,0}(k)\right)^{-1}$ in the norm $\mathcal{B}(L_2(S(L)))$ uniformly on $k\in K$. If $\tau_0\in\Sigma^{(2)}_{0,0}$, then there exists the pole $\tau_{\varepsilon,\delta(\varepsilon)}\in\Sigma^{(2)}_{\varepsilon,\delta(\varepsilon)}$, converging to τ_0 as $\varepsilon\to 0$.*

Definition 5.1. Suppose that $\mathcal{D}_\varepsilon$ is a family of operators acting from $L_2(S(L))$ in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ (in $H^1_{\mathrm{loc}}(\mathbb{R}^3\setminus\overline{\Gamma_1})$).

We say that $\mathcal{D}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{D}_0$ in the topology of $\mathcal{B}(L_2(S(L)),H^1_{\mathrm{loc}}(\mathbb{R}^3))$ (of $\mathcal{B}(L_2(S(L)),H^1_{\mathrm{loc}}(\mathbb{R}^3\setminus\overline{\Gamma_1})))$, if $\mathcal{D}_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{D}_0$ in $\mathcal{B}(L_2(S(L)),H^1(S(R)))$ (in $\mathcal{B}(L_2(S(L)),H^1(S(R)\setminus\overline{\Gamma_1})))$ for any $R>0$.

From the definitions of operators $A^{(j)}_{\mu,\nu}(k)$, and Lemma 5.1 it follows that:

Lemma 5.3. *Assume that K is an arbitrary compact set in $\mathbb{C}$. Then*
(a) *if $p=\infty$, then*

$$A^{(1)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\longrightarrow} A^{(1)}_{0,0}(k)$$

in the topology of $\mathcal{B}(L_2(S(L)), H^1_{\mathrm{loc}}(\mathbb{R}^3))$ uniformly on $k\in K$;
(b) *if $p=0$, then*

$$A^{(2)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\longrightarrow} A^{(2)}_{0,0}(k)$$

in the topology of $\mathcal{B}(L_2(S(L)), H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\Gamma_1}))$ uniformly on $k\in K$.

Finally from Lemmas 5.3, 5.2 and the definitions of operators $\mathcal{A}^{(j)}_{\mu,\delta}(k)$ we deduce

Theorem 5.1. (a) *Suppose that $p=\infty$. If K is an arbitrary compact set in $\mathbb{C}$, such that $K\cap\Sigma^{(1)}_{0,0}=\emptyset$, then*

$$\mathcal{A}^{(1)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\to} \mathcal{A}^{(1)}_{0,0}(k)$$

in the topology of $\mathcal{B}(L_2(S(L)), H^1_{\mathrm{loc}}(\mathbb{R}^3))$ uniformly on $k\in K$. If $\tau_0\in\Sigma^{(1)}_{0,0}$, then there exists a pole $\tau_{\varepsilon,\delta(\varepsilon)}\in\Sigma^{(1)}_{\varepsilon,\delta(\varepsilon)}$, converging to τ_0 as $\varepsilon\to 0$.

(b) *Suppose that $p=0$. If K is an arbitrary compact set in $\mathbb{C}$, such that $K\cap\Sigma^{(2)}_{0,0}=\emptyset$, then*

$$\mathcal{A}^{(2)}_{\varepsilon,\delta(\varepsilon)}(k) \underset{\varepsilon\to 0}{\to} \mathcal{A}^{(2)}_{0,0}(k)$$

in the topology of $\mathcal{B}(L_2(S(L)), H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\Gamma_1}))$ uniformly on $k\in K$. If $\tau_0\in\Sigma^{(2)}_{0,0}$, then there exists a pole $\tau_{\varepsilon,\delta(\varepsilon)}\in\Sigma^{(2)}_{\varepsilon,\delta(\varepsilon)}$, converging to τ_0 as $\varepsilon\to 0$.

Since $\Sigma^{(j)}\subset\Sigma^{(j)}_{0,0}$, then Theorems 1.1 and 1.2 are the implication of Theorems 5.1 and 2.1.

6. Conclusion remarks

If $k^2=k_0^2$ is a simple eigenvalue of the limit problem (1.5) (or (1.8)), by analogy with [25] one can show that there exists a unique pole $\tau_{\varepsilon,\delta(\varepsilon)}$ of the analytic continuation of the solution to the perturbed problems (1.1), (1.3) (or (1.2), (1.3)), converging to k_0 as $\varepsilon \to 0$. In addition, $\tau_{\varepsilon,\delta(\varepsilon)}$ is a pole of the first order and the associated residue is "one dimensional".

Note that in the case $0 < p < +\infty$ the perturbed problem does not decompose into two domains in the limit, one of which has a discrete spectrum. In this case the limit problem is a problem in $\mathbb{R}^3$ with compatibility conditions on the surface Γ. Hence, this limit problem does not have a discrete spectrum and there are no poles of the analytic continuation of solutions with small imaginary parts, which give rise to resonances.

In our three-dimensional case the construction of a complete asymptotic expansion of $\tau_{\varepsilon,\delta(\varepsilon)}$ is impossible in contrast to the two-dimensional case considered in [25]. It was assumed in the two-dimensional case that the whole boundary has a locally periodic structure. In our case we consider the domain with a partially perforated boundary and the absence of a periodic structure does not allow us to construct complete asymptotics by the method of matching of asymptotic expansions [43], [27] used in [25].

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