Expositiones Mathematicae 29 (2011) 415–419

Contents lists available at SciVerse ScienceDirect

Expositiones Mathematicae

journal homepage: www.elsevier.de/exmath

Posner's second theorem for Jordan ideals in rings with involution

Lahcen Oukhtite

Université Moulay Ismaïl, Faculté des Sciences et Techniques, Département de Mathématiques, Groupe d'Algèbre et Applications, B.P. 509 Boutalamine, Errachidia, Morocco

ARTICLE INFO

Article history:
Received 8 March 2011
Received in revised form
13 April 2011

2010 Mathematics Subject Classifications:
16W10
16W25
16U80

Keywords:
Rings with involution
$*$-prime rings
Jordan ideals
Derivations

ABSTRACT

Posner's second theorem states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. In this paper we extend this result to Jordan ideals of rings with involution. Moreover, some related results are also discussed.

1. Introduction

Throughout this paper, R denotes an associative not necessarily unital ring with center $Z(R)$. We will write for all $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. R is 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R admits an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is in general not true. Indeed, if R^o denotes the ring opposite to a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.

An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. A Jordan ideal J which satisfies $J^* = J$ is called a $*$-Jordan ideal. An additive mapping $d : R \longrightarrow R$

E-mail address: oukhtitel@hotmail.com.

0723-0869/$ – see front matter © 2011 Elsevier GmbH. All rights reserved.
doi:10.1016/j.exmath.2011.07.002

is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all x, y in R. A mapping $F : R \longrightarrow R$ is said to be centralizing on a subset S of R if $[F(s), s] \in Z(R)$ for all $s \in S$. In particular, if $[F(s), s] = 0$ for all $s \in S$, then F is commuting on S. The history of commuting and centralizing mappings goes back to 1955 when Divinsky [2] proved that a simple artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later, Posner [6] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (*Posner's second theorem*). Several authors have proved commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations which are centralizing or commuting on an appropriate subset of the ring. Recently, Oukhtite et al. generalized Posner's second theorem to rings with involution in the case of characteristic not 2 as follows. *Let R be a 2-torsion free$*$-prime ring and let U be a square closed$*$-Lie ideal. If R admits a nonzero derivation d centralizing on U, then $U \subseteq Z(R)$* [5, Theorem 1]. In the present paper we shall attempt to generalize Posner's second theorem to Jordan ideals in rings with involution.

2. Derivations centralizing on Jordan ideals

Throughout, $(R, *)$ will be a 2-torsion free ring with involution and $Sa_*(R) := \{r \in R/r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R. We shall use without explicit mention the fact that if J is a nonzero Jordan ideal of a ring R, then $2[R, R]J \subseteq J$; and $2J[R, R] \subseteq J$ [7, Lemma 2.4].

In order to prove our main theorem, we shall need the following lemmas.

Lemma 1 (*[4, Lemma 2]*). *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $aJb = a^*Jb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2 (*[4, Lemma 4]*). *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If d is a derivation of R such that $d(J) = 0$, then $d = 0$ or $J \subseteq Z(R)$.*

Lemma 3. *Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $J \subseteq Z(R)$, then R is commutative.*

Proof. From $J \subseteq Z(R)$ it follows that $2rj = j \circ r \in J$ for all $r \in R$ and $j \in J$ and thus

$$2rsj = r(2sj) = 2sjr = 2srj \quad \text{for all } r, s \in R \text{ and } j \in J. \tag{1}$$

Using 2-torsion freeness, Eq. (1) yields

$$[r, s]j = 0 \quad \text{for all } r, s \in R \text{ and } j \in J. \tag{2}$$

Replacing s by st in (2), where $t \in R$, we get $[r, s]tj = 0$; thereby,

$$[r, s]Rj = 0 \quad \text{for all } r, s \in R \text{ and } j \in J. \tag{3}$$

Since J is a $*$-ideal, then (3) implies that

$$[r, s]Rj^* = 0 \quad \text{for all } r, s \in R \text{ and } j \in J. \tag{4}$$

In view of Lemma 1, because of $0 \neq J$, Eq. (3) together with (4) forces $[r, s] = 0$ for all $r, s \in R$. Accordingly, R is commutative. $\square$

The main result of the present paper is the following theorem.

Theorem 1. *Let $(R, *)$ be a 2-torsion free ring with involution. Let J be a nonzero $*$-Jordan ideal of R and d be a nonzero derivation centralizing on J. If R is $*$-prime, then R is commutative.*

Proof. Linearizing $[d(x), x] \in Z(R)$, for all $x \in J$, we obtain

$$[d(x), y] + [d(y), x] \in Z(R) \quad \text{for all } x \in J. \tag{5}$$

Replacing y by $2x^2$ in (5), as char$R \neq 2$, we find that $x[d(x), x] \in Z(R)$. Therefore, $[d(x), x[d(x), x]] = 0$ which leads to $[d(x), x]^2 = 0$ for all $x \in J$. Since $[d(x), x] \in Z(R)$, then

$$[d(x), x]R[d(x), x][d(x), x]^* = 0 \quad \text{for all } x \in J,$$

and the $*$-primeness of R assures $[d(x),x]=0$ or $[d(x),x][d(x),x]^*=0$. If $[d(x),x][d(x),x]^*=0$, then $[d(x),x]R[d(x),x]^*=0$. Since $[d(x),x]R[d(x),x]=0$, we conclude that

$$[d(x),x]=0 \quad \text{for all } x\in J. \tag{6}$$

Linearizing (6), we obtain

$$[d(x),y]+[d(y),x]=0 \quad \text{for all } x,y\in J. \tag{7}$$

Writing $2x[r,s]$ instead of y in (7), where $r,s\in R$, we find that

$$x[d(x),[r,s]]+d(x)[[r,s],x]+x[d([r,s]),x]=0 \quad \text{for all } x\in J,\ r,s\in R. \tag{8}$$

Replacing s by $2uv$ in (8), where $u,v\in J$, since $[r,2uv]=2u[r,v]+2[r,u]v\in J$, then (8) becomes

$$d(x)[[r,uv],x]=0 \quad \text{for all } u,v,x\in J,\ r\in R. \tag{9}$$

Substituting xt for r in (9) we find that

$$d(x)x[[t,uv],x]+d(x)[x,uv][t,x]=0 \quad \text{for all } u,v,x\in J,\ t\in R. \tag{10}$$

Since $d(x)x=xd(x)$, then $d(x)x[[t,uv],x]=xd(x)[[t,uv],x]=0$ by (9) and (10) assures that

$$d(x)[x,uv][t,x]=0 \quad \text{for all } u,v,x\in J,\ t\in R. \tag{11}$$

Writing rt instead of t in (11) we get $d(x)[x,uv]r[t,x]=0$ and thus

$$d(x)[x,uv]R[t,x]=0 \quad \text{for all } u,v,x\in J,\ t\in R. \tag{12}$$

Suppose that $x_0\in J\cap Sa_*(R)$; from (12) it follows that $d(x_0)[x_0,uv]R[t,x_0]^*=0$, so either $x_0\in Z(R)$ or $d(x_0)[x_0,uv]=0$ for all $u,v\in J$.

Suppose that

$$d(x_0)[x_0,uv]=0 \quad \text{for all } u,v\in J. \tag{13}$$

Replacing v by $2v[r,s]$ in (13), where $r,s\in R$, we find that

$$d(x_0)uv[x_0,[r,s]]=0 \quad \text{for all } u,v\in J,\ r,s\in R$$

and therefore

$$d(x_0)uJ[x_0,[r,s]]=0 \quad \text{for all } u\in J,\ r,s\in R. \tag{14}$$

Since $x_0\in Sa_*(R)$, then (14) assures that

$$d(x_0)uJ([x_0,[r,s]])^*=0 \quad \text{for all } u\in J,\ r,s\in R. \tag{15}$$

In view of Eqs. (14) and (15), Lemma 1 forces $[x_0,[r,s]]=0$ for all $r,s\in R$ or $d(x_0)u=0$ for all $u\in J$, in which case $d(x_0)J=0$.

If $d(x_0)J=0$ then $d(x_0)Jd(x_0)=0=d(x_0)J(d(x_0))^*$ and once again using Lemma 1 we arrive at $d(x_0)=0$. Now assume that

$$[x_0,[r,s]]=0 \quad \text{for all } r,s\in R. \tag{16}$$

Substituting x_0r for r in (16) we obtain

$$[x_0,s][x_0,r]=0 \quad \text{for all } r,s\in R. \tag{17}$$

Replacing s by st in (17), where $t\in R$, we find that

$$[x_0,s]t[x_0,r]=0$$

and thus

$$[x_0,s]R[x_0,r]=0 \quad \text{for all } r,s\in R. \tag{18}$$

Once again using the fact that $x_0 \in Sa_*(R)$, from (18) it follows that

$$[x_0, s]R[x_0, r]^* = 0 \quad \text{for all } r, s \in R. \tag{19}$$

Since R is $*$-prime, then (18) together with (19) assures that $x_0 \in Z(R)$.

In conclusion,

$$d(x_0) = 0 \quad \text{or } x_0 \in Z(R) \text{ for all } x_0 \in J \cap Sa_*(R). \tag{20}$$

Suppose that $x \in J$; as $x^* - x \in J \cap Sa_*(R)$, in view of (20) we obtain $d(x^* - x) = 0$ or $x^* - x \in Z(R)$.

(i) Assume that $d(x^* - x) = 0$. Similarly, the fact that $x^* + x \in J \cap Sa_*(R)$ implies that $d(x^* + x) = 0$ or $x^* + x \in Z(R)$.

If $d(x^* + x) = 0$ then $2d(x) = 0$ and 2-torsion freeness forces $d(x) = 0$.

If $x^* + x \in Z(R)$, then $[t, x] = -[t, x^*]$ for all $t \in R$. Hence, (12) implies that

$$d(x)[x, uv]R[t, x^*] = 0 \quad \text{for all } u, v \in J,\ t \in R$$

and therefore

$$d(x)[x, uv]R[t, x]^* = 0 \quad \text{for all } u, v \in J,\ t \in R. \tag{21}$$

Using (12) together with (21), because of the $*$-primeness of R, we get $x \in Z(R)$ or $d(x)[x, uv] = 0$ for all $u, v \in J$.

Suppose that

$$d(x)[x, uv] = 0 \quad \text{for all } u, v \in J. \tag{22}$$

Replacing v by $2v[r, s]$ in (22), where $r, s \in R$, and reasoning as in (13), the fact that $[x, [r, s]] = -[x^*, [r, s]]$ yields $d(x) = 0$ or $x \in Z(R)$.

(ii) Now assume that $x^* - x \in Z(R)$. Thus $[t, x] = [t, x^*]$ for all $t \in R$ and in view of (12) we get

$$d(x)[x, uv]R[t, x]^* = 0 \quad \text{for all } u, v \in J, t \in R. \tag{23}$$

Since R is $*$-prime, from Eqs. (23) and (12) we obtain $x \in Z(R)$ or $d(x)[x, uv] = 0$ for all $u, v \in J$. Since $[t, x] = [t, x^*]$ for all $t \in R$, arguing as above we find that $d(x) = 0$ or $x \in Z(R)$. Accordingly, in all cases we have

$$d(x) = 0 \quad \text{or } x \in Z(R) \text{ for all } x \in J. \tag{24}$$

From (24) it follows that J is a union of two additive subgroups G_1 and G_2, where

$$G_1 = \{x \in J \text{ such that } d(x) = 0\} \quad \text{and} \quad G_2 = \{x \in J \text{ such that } x \in Z(R)\}.$$

Since a group cannot be a union of two of its proper subgroups, we are forced to have $J = G_1$ or $J = G_2$. If $J = G_1$, then $d(J) = 0$ and Lemma 2 yields $J \subseteq Z(R)$. If $J = G_2$, then $J \subseteq Z(R)$. Hence, in both the cases we find that $J \subseteq Z(R)$ and Lemma 3 assures that R is commutative. $\square$

The following example proves that the $*$-primeness hypothesis in Theorem 1 is not superfluous.

Example. Suppose that $R = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} | x, y, z \in \mathbb{Z} \right\}$ where $\mathbb{Z}$ is the ring of integers. Let us consider $d\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$ and $\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}^* = \begin{pmatrix} z & 0 \\ -y & x \end{pmatrix}$. It is easy to check that R is a non-$*$-prime ring and d is a nonzero derivation of R. Moreover, if we set $J = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} | y \in \mathbb{Z} \right\}$, then J is a nonzero $*$-Jordan ideal and d is centralizing on J, but R is not commutative.

Corollary 1 ([3, Theorem 2]). *Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a nonzero derivation centralizing on I, then R is commutative.*

Proof. Assume that d is a nonzero derivation of R centralizing on I. Let $\mathcal{D}$ be the additive mapping defined on $\mathcal{R} = R \times R^0$ by $\mathcal{D}(x, y) = (d(x), 0)$. Clearly, $\mathcal{D}$ is a nonzero derivation of $\mathcal{R}$. Moreover, if we set $W = I \times I$, then W is a $*_{ex}$-Jordan ideal of $\mathcal{R}$. As d is centralizing on I, it is easy to check that $\mathcal{D}$ is centralizing on W. Since $\mathcal{R}$ is a $*_{ex}$-prime ring, in view of Theorem 1 we deduce that $\mathcal{R}$ is commutative and a fortiori R is commutative. $\square$

Awtar in [1, Theorem 3] proved that if a 2-torsion free prime ring R admits a nonzero derivation d centralizing on a nonzero Jordan ideal J, then $J \subset Z(R)$. However, the conclusion is less precise and incomplete. Indeed, arguing as in the proof of Lemma 3, if $J \neq 0$ then the condition $J \subset Z(R)$ forces R to be commutative. Application of Theorem 1 yields the following result which can be viewed as a short proof of [1, Theorem 3].

Theorem 2. *Let R be a prime ring of characteristic not equal to 2. Let d be a nonzero derivation of R and J be a nonzero Jordan ideal of R such that $[x, d(x)] \in Z(R)$ for all $x \in J$. Then R is commutative.*

Proof. Let us consider the nonzero derivation $\mathcal{D}$ defined on $\mathcal{R} = R \times R^0$ by $\mathcal{D}(x, y) = (d(x), 0)$. If we set $\mathcal{J} = J \times J$, then $\mathcal{J}$ is a $*_{ex}$-Jordan ideal of $\mathcal{R}$. Moreover, the fact that d is centralizing on J implies that $\mathcal{D}$ is centralizing on $\mathcal{J}$. Since $\mathcal{R}$ is a $*_{ex}$-prime ring, application of Theorem 1 assures that $\mathcal{R}$ is commutative. Accordingly, R is commutative. $\square$

References

[1] R. Awtar, Lie and Jordan structure in prime rings with derivations, Proc. Amer. Math. Soc. 41 (1973) 67–74.
[2] N. Divinsky, On commuting automorphisms of rings, Trans. R. Soc. Can. Sect. III 3 (49) (1955) 19–22.
[3] M. Mathieu, Posner's second theorem deduced from the first, Proc. Amer. Math. Soc. 114 (1992) 601–602.
[4] L. Oukhtite, On Jordan ideals and derivations in rings with involution, Comment. Math. Univ. Carolin. 51 (3) (2010) 389–395.
[5] L. Oukhtite, S. Salhi, L. Taoufiq, Commutativity conditions on derivations and Lie ideals in σ-prime rings, Beiträge Algebra Geom. 51 (1) (2010) 275–282.
[6] E.C. Posner, Derivations in prime rings, Proc. Amer. Math. Soc. 8 (1957) 1093–1100.
[7] S.M.A. Zaidi, M. Ashraf, S. Ali, On Jordan ideals and left (θ, θ)-derivations in prime rings, Int. J. Math. Math. Sci. 2004 (37-40) (2004) 1957–1964.