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Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions

Wengui Yang *

Ministry of Public Education, Sanmenxia Polytechnic, Sanmenxia 472000, China

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ABSTRACT

In the this paper, we establish sufficient conditions for the existence and nonexistence of positive solutions to a general class of integral boundary value problems for a coupled system of fractional differential equations. The differential operator is taken in the Riemann–Liouville sense. Our analysis rely on Banach fixed point theorem, nonlinear differentiation of Leray–Schauder type and the fixed point theorems of cone expansion and compression of norm type. As applications, some examples are also provided to illustrate our main results.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. There are a large number of papers dealing with the existence or multiplicity of solutions or positive solutions of initial or boundary value problem for some nonlinear fractional differential equations. For details and examples, see [1–8] and the references therein. In [9–11], the authors have discussed the existence of positive solutions for boundary value problem of nonlinear fractional differential equations. In [12], Feng et al. studied the existence and multiplicity of positive solutions for the following higher-order singular boundary value problem of fractional differential equation:

$$\begin{cases} D^{\alpha}u(t)+g(t)f(t,u(t))=0, \quad 0<t<1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, u(1)=\displaystyle\int_0^1 h(t)x(t)dt, \end{cases}$$

where D is the standard Riemann–Liouville fractional derivative of order $n-1<\alpha\le n, n\ge 3, g\in C((0,1),[0,+\infty))$ and g may be singular at $t=0$ or / and at $t=1$, $h\in L^1[0,1]$ is nonnegative, and $f\in C([0,1]\times[0,+\infty),[0,+\infty))$.

Recently, many people have established the existence and uniqueness for solutions of some systems of nonlinear fractional differential equations, readers can see [13–20] and references cited therein. For example, Su [21] established sufficient conditions for the existence of solutions for a two-point boundary value problem for a coupled system of fractional

* Tel.: +86 15939854723.
E-mail address: wgyang0617@yahoo.com.

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differential equations:

$$\begin{cases} D^\alpha u(t) = f(t, v(t), D^\mu v(t)), D^\beta v(t) = f(t, u(t), D^\nu u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

where $1 < \alpha, \beta < 2, \mu, \nu > 0, \alpha - \nu \geq 1, \beta - \mu \geq 1, f, g \colon [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions, and D is the standard Riemann–Liouville fractional derivative. Ahmad and Nieto [22] extended the results of [21] to a three-point boundary value problem for the following coupled system of fractional differential equations:

$$\begin{cases} D^\alpha u(t) = f(t, v(t), D^\mu v(t)), D^\beta v(t) = f(t, u(t), D^\nu u(t)), \quad 0 < t < 1, \\ u(0) = 0, u(1) = \gamma u(\eta), v(0) = 0, v(1) = \gamma v(\eta), \end{cases}$$

where $1 < \alpha, \beta < 2, \mu, \nu, \gamma > 0, 0 < \eta < 1, \alpha - \nu \geq 1, \beta - \mu \geq 1, \gamma\eta^{\alpha-1} < 1, \gamma\eta^{\beta-1} < 1, f, g \colon [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and D is the standard Riemann–Liouville fractional derivative. Wang et al. [23] obtained the existence and uniqueness of positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations:

$$\begin{cases} D^\alpha u(t) = f(t, v(t)), D^\beta v(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0, u(1) = au(\xi), v(0) = 0, v(1) = bv(\xi), \end{cases}$$

where $1 < \alpha, \beta < 2, 0 \leq a, b \leq 1, 0 < \xi < 1, f, g \colon [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions, and D is the standard Riemann–Liouville fractional derivative.

Motivated by the above mentioned works, we consider the existence and nonexistence of positive solutions to boundary values problem for a coupled system of nonlinear fractional differential equations as follows:

$$\begin{cases} D^\alpha u(t) + a(t)f(t, v(t)) = 0, D^\beta v(t) + b(t)g(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = \displaystyle\int_0^1 \phi(t)u(t)dt, v(0) = 0, v(1) = \int_0^1 \psi(t)v(t)dt, \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta \leq 2$, $a, b \in C((0, 1), [0, +\infty))$, $\phi, \psi \in L^1[0, 1]$ are nonnegative and $f, g \in C([0, 1] \times [0, +\infty), [0, +\infty))$, and D is the standard Riemann–Liouville fractional derivative. By applying Banach fixed point theorem, nonlinear differentiation of Leray–Schauder type and the fixed point theorems of cone expansion and compression of norm type, sufficient conditions for the existence and nonexistence of positive solutions to a general class of integral boundary value problems for a coupled system of fractional differential equations are obtained. Furthermore, some example are also provided to illustrate our main results.

2. Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Definition 2.1 (*See [24,25]*)**.** The fractional integral of order q with the lower limit a for a function f is defined as

$$I^q_{a+}f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s)ds, \quad t > a, q > 0, \tag{2.1}$$

provided the right-hand side is pointwise defined on $[a, \infty)$, where $g \in C[a, b]$ and Γ is the gamma function. For $a = 0$, the fractional integral (2.1) can be written as $I^\alpha_{0+}h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \leq 0$.

Definition 2.2 (*See [24,25]*)**.** Riemann–Liouville derivative of order q with the lower limit $t0$ for a function $f \colon [a, \infty) \to \mathbb{R}$ can be written as

$$D^q_{a+}f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} f(s)ds, \quad t > a, n-1 < q < n.$$

Lemma 2.3 (*Nonlinear Differentiation of Leray–Schauder Type, See [26]*)**.** *Let E be a Banach space with $C \subseteq E$ closed and convex. Let U be a relatively open subset of C with $0 \in U$ and let $T \colon U \to C$ be a continuous and compact mapping. Then either*

(a) *the mapping T has a fixed point in U, or*
(b) *there exist $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda Tu$.*

Lemma 2.4 (*Fixed-Point Theorem of Cone Expansion and Compression of Norm Type, See [27]*)**.** *Let P be a cone of real Banach space E, and let Ω_1 and Ω_2 be two bounded open sets in E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let operator $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous operator. Suppose that one of the two conditions holds:*

(i) $\|Au\| \le \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_2$;
(ii) $\|Au\| \ge \|u\|$, *for all* $u \in P \cap \partial\Omega_1$; $\|Au\| \le \|u\|$, *for all* $u \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Now we present the Green's function for system associated with boundary value problem (1.1).

Lemma 2.5 (*See [12]*). *Assume that* $\int_0^1 \phi(t)t^{\alpha-1}dt \ne 1$. *Then for any* $\sigma \in C[0, 1]$, *the unique solution of boundary value problem*

$$\begin{cases} D^\alpha u(t) + \sigma(t) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \int_0^1 \phi(t)u(t)dt \end{cases}$$

is given by

$$u(t) = \int_0^1 G_{1\alpha}(t, s)\sigma(s)ds,$$

where

$$G_{1\alpha}(t, s) = G_{2\alpha}(t, s) + G_{3\alpha}(t, s), \tag{2.2}$$

$$G_{2\alpha}(t, s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_{3\alpha}(t, s) = \frac{t^{\alpha-1}}{1 - \int_0^1 \phi(t)t^{\alpha-1}dt} \int_0^1 \phi(t)G_{2\alpha}(t, s)dt.$$

We call $G(t, s) = (G_{1\alpha}(t, s), G_{1\beta}(t, s))$ the Green's functions of the boundary value problem (1.1).

Lemma 2.6 (*See [12]*). *If* $\int_0^1 \phi(t)t^{\alpha-1}dt \in [0, 1)$, *the function* $G_{1\alpha}(t, s)$ *defined by* (2.2) *satisfies*

(i) $G_{1\alpha}(t, s) \ge 0$ *is continuous for all* $t, s \in [0, 1]$, $G_{1\alpha}(t, s) > 0$ *for all* $t, s \in (0, 1)$;
(ii) $G_{1\alpha}(t, s) \le G_{1\alpha}(s)$ *for each* $t, s \in [0, 1]$, *and* $\min_{t\in[\theta,1-\theta]} G_{1\alpha}(t, s) \ge \gamma_\alpha G_{1\alpha}(s)$, *where* $\theta \in (0, 1/2)$ *and*

$$G_{1\alpha}(s) = G_{2\alpha}(s, s) + G_{3\alpha}(1, s), \qquad \gamma_\alpha = \theta^{\alpha-1}.$$

3. Existence and nonexistence of positive solutions

In this section, we will discuss the existence of positive solutions for boundary value problem (1.1).

First of all, we define the Banach space $X = \{u(t)|u(t) \in C[0, 1]\}$ endowed with the norm $\|u\|_X = \max_{t\in[0,1]} |u(t)|$, $Y = \{v(t)|v(t) \in C[0, 1]\}$ endowed with the norm $\|v\|_Y = \max_{t\in[0,1]} |v(t)|$. For $(u, v) \in X \times Y$, let $\|(u, v)\|_{X\times Y} = \max\{\|u\|_X, \|v\|_Y\}$. Clearly, $(X \times Y, \|(u, v)\|_{X\times Y})$ is a Banach space. Define $P = \{(u, v) \in X \times Y|u(t) \ge 0, v(t) \ge 0\}$, then the cone $P \subset X \times Y$. Let $J_\theta = [\theta, 1-\theta]$ for $\theta \in (0, 1/2)$ and

$$K = \left\{(u, v) \in P, \min_{t\in J_\theta} u(t) \ge \gamma_\alpha\|u\|, \min_{t\in J_\theta} v(t) \ge \gamma_\beta\|v\|\right\},$$

$$K_r = \{(u, v) \in K: \|(u, v)\| \le r\}, \qquad \partial K_r = \{(u, v) \in K: \|(u, v)\| = r\}.$$

From Lemma 2.5 in Section 2, we can obtain the following lemma.

Lemma 3.1. *Suppose that* $f(t, v)$ *and* $g(t, u)$ *are continuous, then* $(u, v) \in X \times Y$ *is a solution of BVP* (1.1) *if and only if* $(u, v) \in X \times Y$ *is a solution of the integral equations*

$$\begin{cases} u(t) = \int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds, \\ v(t) = \int_0^1 G_{1\beta}(t, s)b(s)g(s, u(s))ds. \end{cases}$$

Let $T: X \times Y \to X \times Y$ be the operator defined as

$$T(u, v)(t) = \left(\int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds, \int_0^1 G_{1\beta}(t, s)b(s)g(s, u(s))ds\right) =: (T_1u(t), T_2v(t)), \tag{3.1}$$

then by Lemma 3.1, the fixed point of operator T coincides with the solution of system (1.1).

Lemma 3.2. *Let* $f(t, v)$ *and* $g(t, u)$ *be continuous on* $[0, 1] \times [0, +\infty) \to [0, +\infty)$, *then* $T: P \to P$ *and* $T: K \to K$ *defined by* (3.1) *are completely continuous.*

Proof. Since Lemma 3.2 is similar to Lemma 3.2 in [22,23], we omit the proof Lemma 3.2. □

Theorem 3.3. *Assume that* $a(t)$ *and* $b(t)$ *are continuous on* $(0, 1) \to [0, +\infty)$ *and* $f(t, u)$ *and* $g(t, v)$ *are continuous on* $[0, 1] \times [0, +\infty) \to [0, +\infty)$, *and there exist two positive functions* $m(t)$, $n(t)$ *that satisfy*

(A_1) $|f(t, v_2) - f(t, v_1)| \leq m(t)|v_2 - v_1|$, *for* $t \in [0, 1]$, $v_1, v_2 \in [0, \infty)$,
(A_2) $|g(t, u_2) - f(t, u_1)| \leq n(t)|u_2 - u_1|$, *for* $t \in [0, 1]$, $u_1, u_2 \in [0, \infty)$.

Then system (1.1) *has a unique positive solution if*

$$\rho = \int_0^1 G_{1\alpha}(s)a(s)m(s)ds < 1, \qquad \theta = \int_0^1 G_{1\beta}(s)b(s)n(s)ds < 1. \tag{3.2}$$

Proof. For all $(u, v) \in P$, by the nonnegativeness of $G(t, s)$ and $a(t), b(t), f(t, v), g(t, u)$, we have $T(u, v)(t) \geq 0$. Hence, $T(P) \subset P$. From Lemma 2.6, we obtain

$$\begin{aligned}\|T_1v_2 - T_1v_1\| &= \max_{t\in[0,1]} |T_1v_2 - T_1v_1| \\ &= \max_{t\in[0,1]} \left|\int_0^1 G_{1\alpha}(t, s)a(s)[f(s, v_2(s)) - f(s, v_1(s))]ds\right| \\ &\leq \int_0^1 G_{1\alpha}(s)a(s)m(s)ds\|v_2 - v_1\| = \rho\|v_2 - v_1\|. \end{aligned} \tag{3.3}$$

Similarly,

$$\|T_2u_2 - T_2u_1\| \leq \theta\|u_2 - u_1\|. \tag{3.4}$$

From (3.3) to (3.4), we get

$$\|T(u_2, v_2) - T(u_1, v_1)\| \leq \max(\rho, \theta)\|(u_2, v_2) - (u_1, v_1)\|.$$

From Lemma 3.2, T is completely continuous, by Banach fixed point theorem, the operator T has a unique fixed point in P, which is the unique positive solution of system (1.1). This completes the proof. □

Theorem 3.4. *Assume that* $a(t)$ *and* $b(t)$ *are continuous on* $(0, 1) \to [0, +\infty)$ *and* $f(t, v)$ *and* $g(t, u)$ *are continuous on* $[0, 1] \times [0, +\infty) \to [0, +\infty)$, *and satisfy*

(A_3) $|f(t, v(t))| \leq c_1(t) + c_2(t)|v(t)|$,
(A_4) $|g(t, u(t))| \leq d_1(t) + d_2(t)|u(t)|$,
(A_5) $C_1 = \int_0^1 G_{1\alpha}(s)a(s)c_2(s)ds < 1$, $D_1 = \int_0^1 G_{1\alpha}(s)a(s)c_1(s)ds < \infty$,
(A_6) $C_2 = \int_0^1 G_{1\beta}(s)b(s)d_2(s)ds < 1$, $D_2 = \int_0^1 G_{1\beta}(s)b(s)d_1(s)ds < \infty$.

Then the system (1.1) *has at least one positive solution* (u, v) *in*

$$Q = \left\{(u, v) \in P | \|(u, v)\| < \min\left(\frac{D_1}{1 - C_1}, \frac{D_2}{1 - C_2}\right)\right\}.$$

Proof. Let $Q = \{(u, v) \in P | \|(u, v)\| < r\}$ with $r = \min(D_1/(1 - C_1), D_2/(1 - C_2))$, define the operator $T: Q \to P$ as (3.1). Let $(u, v) \in Q$, that is, $\|(u, v)\| < r$. Then

$$\begin{aligned}\|T_1v\| &= \max_{t\in[0,1]} \left|\int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds\right| \\ &\leq \int_0^1 G_{1\alpha}(s)a(s)(c_1(t) + c_2(t)|v(t)|)ds \\ &\leq \int_0^1 G_{1\alpha}(s)a(s)c_1(t)ds + \int_0^1 G_1(s)a(s)c_2(t)ds\|v\| \\ &= D_1 + C_1\|v\| \leq r.\end{aligned}$$

Similarly, $\|T_2u\| \leq r$, so $\|T(u, v)\| \leq r$, $T(u, v) \subseteq \overline{Q}$. From Lemma 3.2, we have $\overline{T: Q \to \overline{Q}}$ is completely continuous.

Consider the eigenvalue problem

$$(u, v) = \lambda T(u, v), \quad \lambda \in (0, 1). \tag{3.5}$$

Under the assumption that (u, v) is a solution of (3.5) for a $\lambda \in (0, 1)$, we have

$$\begin{aligned}\|u\| &= \|\lambda T_1 v\| = \lambda \max_{t\in[0,1]} \left|\int_0^1 G_{1\alpha}(t,s)a(s)f(s,v(s))ds\right| \\ &< \int_0^1 G_{1\alpha}(s)a(s)(c_1(t)+c_2(t)|v(t)|)ds \\ &\le \int_0^1 G_{1\alpha}(s)a(s)c_1(t)ds + \int_0^1 G_{1\alpha}(s)a(s)c_2(t)ds\|v\| \\ &= D_1 + C_1\|v\| \le r.\end{aligned}$$

Similarly, $\|v\| = \|\lambda T_2 u\| < r$, so $\|(u, v)\| < r$, which shows that $(u, v) \notin \partial Q$. By Lemma 2.3, T has a fixed point in $\overline{Q}$. We complete the proof of Theorem 3.4. □

In the following we need the following assumptions and some notations:

(B_1) $a, b \in C((0, 1), [0, +\infty))$, $a(t) \not\equiv 0$ and $b(t) \not\equiv 0$ on any subinterval of (0, 1), and $0 < \int_0^1 G_{1\alpha}(s)a(s)ds < \infty$ and $0 < \int_0^1 G_{1\beta}(s)b(s)ds < \infty$, where $G_{1\alpha}(s)$ and $G_{1\beta}(s)$ are defined in Lemma 2.6;
(B_2) $f, g \in C([0, 1] \times [0, +\infty), [0, +\infty))$, and $f(t, 0) = 0$ and $g(t, 0) = 0$ uniformly with respect to t on [0, 1];
(B_3) $\mu, \nu \in [0, 1)$, where μ, ν is defined as follows:

$$\mu = \int_0^1 \phi(t)t^{\alpha-1}dt \quad \text{and} \quad \nu = \int_0^1 \psi(t)t^{\beta-1}dt.$$

Let

$$f^\delta = \limsup_{u\to\delta} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \qquad f_\delta = \liminf_{u\to\delta} \min_{t\in[0,1]} \frac{f(t,u)}{u},$$

where δ denotes 0 or ∞, and

$$\sigma_1 = \int_0^1 G_{1\alpha}(s)a(s)ds, \qquad \sigma_2 = \int_0^1 G_{1\beta}(s)b(s)ds.$$

Theorem 3.5. *Assume that* (B_1) –(B_3) *hold. And supposes that one of the following conditions is satisfied:*

(H_1) $f_0 > 1/\left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$ *and* $f^\infty < 1/\sigma_1$ *(particularly,* $f_0 = \infty$ *and* $f^\infty = 0$*);*
$g_0 > 1/\left(\gamma_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$ *and* $g^\infty < 1/\sigma_2$ *(particularly,* $g_0 = \infty$ *and* $g^\infty = 0$*).*
(H_2) *there exist two constants* r_2, R_2 *with* $0 < r_2 \le R_2$ *such that* $f(t, \cdot)$ *and* $g(t, \cdot)$ *are nondecreasing on* $[0, R_2]$ *for all* $t \in [0, 1]$, $f(t, \gamma_\alpha r_2) \ge r_2/\left(\gamma_\alpha \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$, $g(t, \gamma_\beta r_2) \ge r_2/\left(\gamma_\beta \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$, *and* $f(t, R_2) \le R_2/\sigma_1$, $g(t, R_2) \le R_2/\sigma_2$ *for all* $t \in [0, 1]$.

Then boundary value problem (1.1) has at least one positive solution.

Proof. Let T be cone preserving completely continuous that is defined by (3.1).

Case 1. The condition (H_1) holds. Considering $f_0 > 1/\left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$, there exists $r_1 > 0$ such that $f(t, v) \ge (f_0 - \varepsilon_1)v$, for all $t \in [0, 1]$, $v \in [0, r_1]$, where $\varepsilon_1 > 0$, satisfies $(f_0 - \varepsilon_1)\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds \ge 1$. Then, for $t \in [0, 1]$, $(u, v) \in \partial K_{r_1}$, we get

$$\begin{aligned}T_1 v(t) &= \int_0^1 G_{1\alpha}(t,s)a(s)f(s,v(s))ds \\ &\ge \gamma_\alpha \int_0^1 G_{1\alpha}(s)a(s)f(s,v(s))ds \\ &\ge \gamma_\alpha \int_0^1 G_{1\alpha}(s)a(s)(f_0-\varepsilon_1)v(s)ds \\ &\ge (f_0-\varepsilon_1)\gamma_\alpha^2 \int_0^1 G_{1\alpha}(s)a(s)ds\|v\| \ge \|v\|.\end{aligned}$$

Similarly, we have $T_2 u(t) \ge \|u\|$, that is $(u, v) \in \partial K_{r_1}$ implies that

$$\|T(u,v)\| \ge \|(u,v)\|. \tag{3.6}$$

On the other hand, for $f^\infty < 1/\sigma_1$, there exists $\overline{R}_1 > 0$ such that $f(t, v) \le (f^\infty + \varepsilon_2)v$, for $t \in [0, 1]$, $v \in (\overline{R}_1, +\infty)$, where $\varepsilon_2 > 0$ satisfies $\sigma_1(f^\infty + \varepsilon_2) \le 1$. Set $M = \max_{t\in[0,1], v\in[0,\overline{R}_1]} f(t, v)$, then $f(t, v) \le M + (f^\infty + \varepsilon_2)v$.

Choose $R_1 > \max\{r_1, \overline{R}_1, M\sigma_1(1 - \sigma_1(f^\infty + \varepsilon_2))^{-1}\}$. Then, for $t \in [0, 1]$, $(u, v) \in \partial K_{R_1}$, we get

$$\begin{aligned} T_1v(t) &= \int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds \\ &\le \int_0^1 G_{1\alpha}(s)a(s)f(s, v(s))ds \\ &\le \int_0^1 G_{1\alpha}(s)a(s)(M + (f^\infty + \varepsilon_2)v(s))ds \\ &\le M\int_0^1 G_{1\alpha}(s)a(s)ds + (f^\infty + \varepsilon_2)\int_0^1 G_{1\alpha}(s)a(s)ds\|v\| \\ &< R_1 - \sigma_1(f^\infty + \varepsilon_2)R_1 + (f^\infty + \varepsilon_2)\sigma_1\|v\| \le R_1. \end{aligned}$$

Similarly, we have $T_2u(t) < R_1$, that is $(u, v) \in \partial K_{R_1}$ implies that

$$\|T(u, v)\| < \|(u, v)\|. \tag{3.7}$$

Case 2. The condition (H_2) holds. For $(u, v) \in K$, from the definition of K, we obtain that $\min_{t\in J_\theta} u(t) \ge \gamma_\alpha\|u\|$, $\min_{t\in J_\theta} v(t) \ge \gamma_\beta\|v\|$. Therefore, for $(u, v) \in \partial K_{r_2}$, we have $\|(u, v)\| = r_2$ for $t \in J_\theta$. From (H_2), we have

$$\begin{aligned} T_1v(t) &= \int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds \\ &\ge \gamma_\alpha\int_\theta^{1-\theta} G_{1\alpha}(s)a(s)f(s, v(s))ds \\ &\ge \gamma_\alpha\frac{r_2}{\gamma_\alpha\int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds}\int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds = r_2. \end{aligned}$$

Similarly, we have $T_2u(t) \ge r_2$, that is $(u, v) \in \partial K_{r_2}$ implies that

$$\|T(u, v)\| \ge \|(u, v)\|. \tag{3.8}$$

On the other hand, for $(u, v) \in \partial K_{R_2}$, we have that $(u, v) = R_2$ for $t \in [0, 1]$, from (H_2), we obtain

$$\begin{aligned} T_1v(t) &= \int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds \\ &\le \int_0^1 G_{1\alpha}(s)a(s)f(s, v(s))ds \\ &\le \frac{R_2}{\sigma_1}\int_0^1 G_{1\alpha}(s)a(s)ds = R_2. \end{aligned}$$

Similarly, we have $T_2u(t) \le R_2$, that is $(u, v) \in \partial K_{R_2}$ implies that

$$\|T(u, v)\| \le \|(u, v)\|. \tag{3.9}$$

Applying Lemma 2.4 to (3.6) and (3.7), or (3.8) and (3.9), yields that T has a fixed point $(\overline{u}, \overline{v}) \in \overline{K}_{r,R}$ or $(\overline{u}, \overline{v}) \in \overline{K}_{r_i,R_i} (i = 1, 2)$ with $\overline{u}(t) \ge \gamma_\alpha\|\overline{u}\| > 0$ and $\overline{v}(t) \ge \gamma_\beta\|\overline{v}\| > 0$, $t \in [0, 1]$. Thus it follows that boundary value problems (1.1) has a positive solution $(\overline{u}, \overline{v})$. We complete the proof of Theorem 3.5. □

Similarly, we have the following result.

Theorem 3.6. *Assume that* (B_1) –(B_3) *hold. And supposes that the following conditions is satisfied:*

(H_3) $f^0 < 1/\sigma_1$ *and* $f_\infty > 1/\left(\gamma_\alpha^2\int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$ *(particularly,* $f^0 = 0$ *and* $f_\infty = \infty$*);*

$g^0 < 1/\sigma_2$ *and* $g_\infty > 1/\left(\gamma_\beta^2\int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$ *(particularly,* $g^0 = 0$ *and* $g_\infty = \infty$*).*

Then boundary value problem (1.1) has at least one positive solution.

Theorem 3.7. *Assume that* (B_1)–(B_3) *hold. And supposes that the following two conditions are satisfied:*

(H_4) $f_0 > 1/\left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$ *and* $f_\infty > 1/\left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$ *(particularly,* $f_0 = f_\infty = \infty$*);*
$g_0 > 1/\left(\gamma_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$ *and* $g_\infty > 1/\left(\gamma_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$ *(particularly,* $g_0 = g_\infty = \infty$*).*

(H_5) *there exists* $b > 0$ *such that* $\max_{t\in[0,1],(u,v)\in\partial K_b} f(t, v) < b/\sigma_1$ *and* $\max_{t\in[0,1],(u,v)\in\partial K_b} f(t, u) < b/\sigma_2$.

Then boundary value problem (1.1) has at least two positive solutions (u_1, v_1), (u_2, v_2)*, which satisfy*

$$0 < \|(u_1, v_1)\| < b < \|(u_2, v_2)\|. \tag{3.10}$$

Proof. We consider condition (H_4). Choose r, R with $0 < r < b < R$. If $f_0 > 1/\left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$ and $g_0 > 1/\left(\gamma_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$, then similar to the proof of (3.6), we have

$$\|T(u, v)\| \geq \|(u, v)\|, \quad \text{for } (u, v) \in \partial K_r. \tag{3.11}$$

If $f_\infty > 1/\left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$ and $g_\infty > 1/\left(\gamma_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$, then similar to the proof of (3.6), we have

$$\|T(u, v)\| \geq \|(u, v)\|, \quad \text{for } (u, v) \in \partial K_R. \tag{3.12}$$

On the other hand, together with (H_5), $(u, v) \in \partial K_b$, we have

$$\begin{aligned} T_1 v(t) &= \int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds \\ &\leq \int_0^1 G_{1\alpha}(s)a(s)f(s, v(s))ds \\ &< \frac{b}{\sigma_1} \int_0^1 G_{1\alpha}(s)a(s)ds = b. \end{aligned}$$

Similarly, we have $T_2 u(t) < b$, that is $(u, v) \in \partial K_b$ implies that

$$\|T(u, v)\| < \|(u, v)\|. \tag{3.13}$$

Applying Lemma 2.4 to (3.11)–(3.13) yields that T has a fixed point $(u_1, v_1) \in \partial \overline{K}_{r,b}$, and a fixed point $(u_2, v_2) \in \partial \overline{K}_{b,R}$. Thus it follows that boundary value problem (1.1) has at least two positive solutions (u_1, v_1) and (u_2, v_2). Noticing (3.13), we have $\|(u_1, v_1)\| \neq b$ and $\|(u_2, v_2)\| \neq b$. Therefore (3.10) holds, and the proof is complete. □

Similarly, we have the following results.

Theorem 3.8. *Assume that* (B_1)–(B_3) *hold. And supposes that the following conditions is satisfied:*

(H_6) $f^0 < 1/\sigma_1$ *and* $f^\infty < 1/\sigma_1$; $g^0 < 1/\sigma_2$ *and* $g^\infty < 1/\sigma_2$.

(H_7) *there exists* $B > 0$ *such that*

$$\max_{t\in J_\theta,(u,v)\in\partial K_B} f(t, v) > B/\left(\gamma_\alpha \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right) \quad \textit{and} \quad \max_{t\in J_\theta,(u,v)\in\partial K_B} g(t, u) > B/\left(\gamma_\beta \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right).$$

Then boundary value problem (1.1) has at least two positive solutions (u_1, v_1), (u_2, v_2)*, which satisfy*

$$0 < \|(u_1, v_1)\| < B < \|(u_2, v_2)\|.$$

Theorem 3.9. *Assume that* (B_1)–(B_3) *hold. If there exist* $2l$ *positive numbers* d_k, D_k, $k = 1, 2, \ldots, l$ *with* $d_1 < \gamma_\alpha D_1 < D_1 < d_2 < \gamma_\alpha D_2 < D_2 < \cdots < d_l < \gamma_\alpha D_l < D_l$ *and* $d_1 < \gamma_\beta D_1 < D_1 < d_2 < \gamma_\beta D_2 < D_2 < \cdots < d_l < \gamma_\beta D_l < D_l$ *such that*

(H_8) $f(t, v) \geq d_k/\left(\gamma_\alpha \int_0^1 G_{1\alpha}(s)a(s)ds\right)$ *for* $(t, v) \in [0, 1] \times [\gamma_\alpha d_k, d_k]$ *and* $f(t, v) \leq \sigma_1^{-1} D_k$ *for* $(t, v) \in [0, 1] \times [\gamma_\alpha D_k, D_k]$, $k = 1, 2, \ldots, l$;
$g(t, u) \geq d_k/\left(\gamma_\beta \int_0^1 G_{1\beta}(s)b(s)ds\right)$ *for* $(t, u) \in [0, 1] \times [\gamma_\beta d_k, d_k]$ *and* $g(t, u) \leq \sigma_2^{-1} D_k$ *for* $(t, u) \in [0, 1] \times [\gamma_\beta D_k, D_k]$, $k = 1, 2, \ldots, l$.

Then boundary value problem (1.1) has at least l *positive solutions* (u_k, v_k) *satisfying* $d_k \leq \|(u_k, v_k)\| \leq D_k$, $k = 1, 2, \ldots, l$.

Theorem 3.10. *Assume that* (B_1)–(B_3) *hold. If there exist* $2l$ *positive numbers* d_k, D_k, $k = 1, 2, \ldots, l$ *with* $d_1 < D_1 < d_2 < D_2 < \cdots < d_l < D_l$ *such that*

(H_9) $f(t, \cdot)$ *and* $g(t, \cdot)$ *are nondecreasing on* $[0, D_l]$ *for all* $t \in [0, 1]$.

(H_{10}) $f(t, \gamma_\alpha d_k) \geq d_k / \left(\gamma_\alpha \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right)$, *and* $f(t, D_k) \leq \sigma_1^{-1} D_k$, $k = 1, 2, \ldots, l$;

$g(t, \gamma_\beta d_k) \geq d_k / \left(\gamma_\beta \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$, *and* $g(t, D_k) \leq \sigma_2^{-1} D_k$, $k = 1, 2, \ldots, l$.

Then boundary value problem (1.1) has at least l *positive solutions* (u_k, v_k) *satisfying* $d_k \leq \|(u_k, v_k)\| \leq D_k$, $k = 1, 2, \ldots, l$.

Now the nonexistence of positive solutions for boundary value problem (1.1).

Theorem 3.11. *Assume that* (B_1) –(B_3) *hold, and* $f(t, v) < \sigma_1^{-1} v$ *and* $g(t, u) < \sigma_2^{-1} u$ *for all* $t \in [0, 1]$, $u > 0$, $v > 0$, *then boundary value problem (1.1) has no positive solution.*

Proof. Assume to the contrary that (u, v) is a positive solution of the boundary value problem (1.1). Then $(u, v) \in K$, $u(t) > 0$ and $v(t) > 0$ for $t \in (0, 1)$, and

$$\begin{aligned}
\|u\| &= \max_{t\in[0,1]} |u(t)| = \max_{t\in[0,1]} \int_0^1 G_{1\alpha}(t, s)a(s)f(s, v(s))ds \\
&\leq \int_0^1 G_{1\alpha}(s)a(s)f(s, v(s))ds \\
&< \int_0^1 G_{1\alpha}(s)a(s)\frac{\|v\|}{\sigma_1}ds = \frac{1}{\sigma_1}\int_0^1 G_{1\alpha}(s)a(s)ds\|v\| = \|v\|.
\end{aligned}$$

Similarly, $\|v\| < \|u\|$, which is a contradiction, and Theorem is received. □

Similarly, we have the following result.

Theorem 3.12. *Assume that* (B_1) –(B_3) *hold, and*

$$f(t, v) > v / \left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right) \quad \text{and} \quad g(t, u) > u / \left(\gamma_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right)$$

for all $t \in [0, 1]$, $u > 0$, $v > 0$, *then boundary value problem (1.1) has no positive solution.*

4. Some examples

Example 4.1. Consider the system of nonlinear fractional differential equations:

$$\begin{cases} D^{\frac{7}{4}}u(t) + \dfrac{t}{1+t}|\sin v(t)| = 0, D^{\frac{7}{4}}v(t) + \dfrac{t}{1+t}|\sin u(t)| = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = \displaystyle\int_0^1 tu(t)dt, v(0) = 0, v(1) = \int_0^1 tv(t)dt. \end{cases} \tag{4.1}$$

Set $x(t), y(t) \in [0, +\infty)$ and $t \in [0, 1]$, then we have

$$\left|\frac{t}{1+t}|\sin x(t)| - \frac{t}{1+t}|\sin y(t)|\right| \leq \frac{t}{1+t}|x(t) - y(t)|.$$

Therefore,

$$\begin{aligned}
\rho &= \int_0^1 G_{1\alpha}(s)a(s)m(s)ds \leq \int_0^1 G_{1\alpha}(s)ds \approx 0.892377 < 1, \\
\theta &= \int_0^1 G_{1\beta}(s)b(s)n(s)ds \leq \int_0^1 G_{1\beta}(s)ds \approx 0.892377 < 1.
\end{aligned}$$

With the use of Theorem 3.3, BVP (4.1) has a unique positive solution.

Example 4.2. Consider the system of nonlinear fractional differential equations:

$$\begin{cases} D^{\frac{7}{4}}u(t) + [v(t)]^a = 0, D^{\frac{7}{4}}v(t) + [u(t)]^b = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = \displaystyle\int_0^1 tu(t)dt, v(0) = 0, v(1) = \int_0^1 tv(t)dt. \end{cases} \tag{4.2}$$

Let $f(t, v) = v^a$ and $g(t, u) = u^b$, $0 < a, b < 1$. It is easy to see that (B_1)–(B_3) hold. By simple computation, we have $f_0 = g_0 = \infty$ and $f^\infty = g^\infty = 0$. Thus it follows that problem (4.2) has a positive solution by (H_1).

Example 4.3. Consider the system of nonlinear fractional differential equations:

$$\begin{cases} D^{\frac{7}{4}}u(t) + [v(t)]^c = 0, D^{\frac{7}{4}}v(t) + [u(t)]^d = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = \int_0^1 tu(t)dt, v(0) = 0, v(1) = \int_0^1 tv(t)dt. \end{cases} \tag{4.3}$$

Let $f(t, v) = v^c$ and $g(t, u) = u^d$, $1 < c, d < \infty$. It is easy to see that (B_1)–(B_3) hold. By simple computation, we have $f^0 = g^0 = 0$ and $f_\infty = g_\infty = \infty$. Thus it follows that problem (4.3) has a positive solution by (H_3).

Example 4.4. Consider the system of nonlinear fractional differential equations:

$$\begin{cases} D^{\frac{7}{4}}u(t) + a(t)\dfrac{(2[v(t)]^2 + v(t))(20 + \sin v(t))}{v(t) + 1} = 0, D^{\frac{7}{4}}v(t) + b(t)\dfrac{(2[u(t)]^2 + u(t))(20 + \sin u(t))}{u(t) + 1} = 0, \\ \quad 0 < t < 1, \\ u(0) = 0, u(1) = \int_0^1 tu(t)dt, v(0) = 0, v(1) = \int_0^1 tv(t)dt. \end{cases} \tag{4.4}$$

Let $f(t, v) = (2v^2 + v)(20 + \sin v)/(v + 1)$ and $g(t, u) = (2u^2 + u)(20 + \sin u)/(u + 1)$. It is easy to see that (B_1)–(B_3) hold. By simple computation, we have $f^0 = g^0 = f_0 = g_0 = 20, f^\infty = g^\infty = 43, f_\infty = g_\infty = 38$, and $20v < f(t, v) < 43v$, $20u < g(t, u) < 43u$.

(i) Let $a(t) = b(t) = 1/40$, from Example 4.1, we have $\sigma_1 = \sigma_2 \approx 0.0223094, f(t, v) < 43v < \sigma_1^{-1}v \approx 44.8241v$ and $g(t, u) < 43u\sigma_2^{-1}u \approx 44.8241u$ Thus, by Theorem 3.11, the boundary value problem (4.4) has no positive solution.

(ii) Choose $\theta = 1/3$ and $a(t) = b(t) = 1$, then $f(t, v) > 20v > v/\left(\gamma_\alpha^2 \int_\theta^{1-\theta} G_{1\alpha}(s)a(s)ds\right) \approx 15.8688v$ and $g(t, u) > 20u > u/\left(\gamma_\beta^2 \int_\theta^{1-\theta} G_{1\beta}(s)b(s)ds\right) \approx 15.8688u$. Thus, by Theorem 3.12, the boundary value problem (4.4) has no positive solution.

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