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On the resonances of the Laplacian on waveguides

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Abstract

The resonances for the Dirichlet and Neumann Laplacian are studied on compactly perturbed waveguides. In the absence of resonances, an upper bound is proven for the localised resolvent. This is then used to prove that the existence of a quasimode whose asymptotics is bounded away from the thresholds implies the existence of resonances converging to the real axis. The following upper bound to the number of resonances is also proven:

$$\#\left\{k_j \in \mathrm{Res}(\Delta),\ \mathrm{dist}(k_j, \text{physical plane}) < 1 + \sqrt{|k_j|}/2,\ |k_j| < r\right\} < Cr^{3+\epsilon}.$$

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1. Introduction

Resonances of the Laplacian have been the object of study in a wide variety of geometric settings (see surveys in [33,37,39]). Resonances, which are essentially equivalent to poles of the scattering matrix, have been related to long-lived waves ("metastable states" in the quantum mechanics literature) and also arise naturally in studying the long time behaviour of evolution equations, particularly

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the wave equation (see [24], where this connection is pointed out in the context of Schrödinger operators).

Despite the physical significance of resonances, very little is understood about the resonances associated to perturbations of waveguides. The only work known to this author that explicitly studies resonances is by Aslanyan et al. [1], where the authors estimate the complex part of resonances arising as perturbations of trapped modes. The authors also use numerical methods to count the resonances that appear at low frequencies.

Several other works on the scattering theory of waveguides indirectly apply to resonances. Christiansen and Zworski [6], and independently Parnovski [14], have computed the asymptotics for the embedded eigenvalue counting function added to the scattering phase for boundariless manifolds which are asymptotic to a cylinder. They also proved a sharp upper bound on the eigenvalue counting function, improving on an earlier work by Donnelly [7]. Melrose [12] studied the properties of the resolvent for asymptotically perturbed cylinders, including the nature of the singularity of the resolvent at the thresholds. Weidenmuller [34] studied various scattering theoretic properties of Laplacian on the perturbed strip with Dirichlet boundary conditions.

Also related is the extensive literature on the existence of L^2-eigenvalues. With our definition of resonance, any L^2-eigenvalue will be considered a resonance. The existence of L^2-eigenvalues for waveguides has drawn much study, motivated both by their association with standing waves or “trapped modes” (see [8,9] and references therein) and with their role in forcing equations ([17,18] and references therein). However, it is generally believed that L^2-eigenvalues do not exist for generic perturbations of the strip.

In this paper, we consider the resonances associated to the Laplacian on waveguides with either Dirichlet or Neumann boundary conditions. We prove a resolvent estimate from which it follows that the existence of a quasimode with certain asymptotics implies the existence of a sequence of resonances approaching the real axis. We also prove an upper bound on the number of resonances in a neighbourhood of the physical plane.

We define resonances as poles of the meromorphic continuation of the resolvent away from the thresholds, in union with any L^2 eigenvalues occurring at the thresholds. In particular, let $\Omega \subset \mathbf{R}^2$ be a domain with smooth boundary which to the exterior of some compact set coincides with the strip $(-\infty, \infty) \times (0, \pi)$. Let Δ be either the Dirichlet or Neumann Laplacian, with signs arranged so that the operator is positive semi-definite. It is well known that the Dirichlet Laplacian has essential spectrum $[1, \infty)$, with thresholds at $\{j^2\}_{j=1}^{\infty}$. For the Neumann Laplacian, the essential spectrum is $[0, \infty)$ with thresholds at $\{j^2\}_{j=0}^{\infty}$. Let $\chi \in C_0^{\infty}(\mathbf{R}^2)$, i.e., a smooth function of compact support. We show in Section 2 that $\chi(\Delta - k)^{-1}\chi$, an analytic function in k with values in the bounded operators on $L^2(\Omega)$, extends meromorphically from $\mathbf{C} - [0, \infty)$ to an infinitely branched Riemann surface S, with the branch points occurring at the thresholds.

The geometry of S was studied in [34], where it is proven that S is not simply connected. This, and the infinitely many branch points, are probably the main reasons that the resonances on waveguides are less well understood than for the corresponding problem for exterior domains (for exterior domains the corresponding Riemann surface is the surface associated to $\sqrt{z}$ for odd dimensions, and $\ln z$ for even dimensions). Also, the tools of complex scaling as in [22] have not been established in this setting.

Let $\Pi : S \to \mathbf{C}$ be the canonical projection. The Riemannian metric induced by Π naturally induces a distance function on S, which we denote dist. We prove the following resolvent estimate:

Theorem 1. *Let $\Omega \subset \mathbf{R}^2$ be a smooth domain which to the exterior of a bounded set equals the strip $\{(x, y)\colon x \in (-\infty, \infty), y \in (0, \pi)\}$. Fix $\alpha > 0$. Then for integer $p \geqslant 3$, $q \in [0, p/2)$, there exists $M_{p,q} > 0$ such that if $m > M_{p,q}$, the following property holds: if there are no thresholds in the interval $(m - \alpha - 2m^{-q}, m + \alpha + 2m^{-q})$ and no resonances in the open set*

$$\left\{k \in S\colon \operatorname{dist}\left(k, [m - 2m^{-q}, m + 2m^{-q}]\right) < 2(m - 2)^{-p}\right\},$$

then for $k \in [m - m^{-q}, m + m^{-q}]$, we have

$$\left\|\chi(\Delta - k)^{-1}\chi\right\|_{L^2 \to L^2} \leqslant C_p k^{2p}.$$

Here C_p is a positive constant depending only on p and Ω.

We also prove some upper bounds in a neighbourhood of the thresholds; see Lemma 7. A consequence of this theorem is that the existence of periodic billiard trajectories in Ω satisfying certain geometric hypotheses will imply the existence of a sequence of resonances converging to the real axis. To be specific we must first define localised quasimodes.

We define a pair of sequences (u_j, λ_j), with u_j in the operator domain of Δ and $\lambda_j \in \mathbf{R}$, to be a quasimode if the u_j are uniformly compactly supported with $\|u_j\| = 1$, $\lambda_j \to \infty$, and

$$\left\|(\Delta - \lambda_j)u_j\right\|_{L^2(\Omega)} = O\left(\lambda_j^{-\infty}\right).$$

Quasimodes that are supported in a neighbourhood of a stable periodic billiard trajectory and for which the asymptotics of λ_j are fully determined by the local geometry have been constructed by a number of authors [2,3,11]. In Section 4 of this paper, we briefly present an example due to Buldyrev.

Corollary 1. *Suppose there exists a quasimode such that there exists $\alpha > 0$ such that*

$$\left|\lambda_j - n^2\right| > \alpha, \quad \forall n, j \in \mathbf{Z}.$$

Then there exists an infinite sequence $\{k_j\}$ *of resonances of* Δ *such that for any* $N > 0$,

$$|\lambda_j - k_j| \leqslant C_N \lambda_j^{-N}.$$

Corollary 1 follows from Theorem 1 by direct application of the arguments appearing in [29]. Such "quasimode-to-resonance" results, based on *a priori* resolvent estimates, have previously been proven for other geometric settings, [25–29].

If the quasimodes also satisfy certain spacing hypotheses, then Theorem 1 would also imply that the resonance counting function is bounded below by the quasimode counting function (see [29]).

Using estimates proven in Theorem 1 along with Jensen's formula, one can also obtain an upper bound on the number of resonances near the physical plane.

Theorem 2. *Let* $\{k_j\}$ *be the resonances of* Δ, *counted with multiplicity. Define*

$$N(r) = \left\{ k_j \colon \operatorname{dist}(k_j, \textit{physical plane}) < 1 + \frac{1}{2}\sqrt{|k|},\ |k_j| < r \right\}.$$

Then for any $\epsilon > 0$, *there exists a positive constant* C *such that*

$$N(r) < Cr^{3+\epsilon}.$$

Using the methods of this paper, one could also obtain a global upper bounds on the number of resonances.

Upper bounds for the number of resonances proven in other geometries suggest that the sharp upper bound for $N(r)$ should of the form Cr. It should also be noted that Christiansen and Zworski in [6] proved that the sharp upper bound for the embedded eigenvalue counting function is Cr.

We now give a sketch of the proof, which is based on the Fredholm determinant method. Let χ_1, χ_2 be smooth cutoff functions of bounded support. Then using a well-known procedure (see, e.g., [23]), we show that $\chi_1(\Delta - k)^{-1}\chi_2$ extends meromorphically to the Riemann surface S. It is well known that the (non-threshold) poles of the resolvent are among the zeros of a certain Fredholm determinant which is analytic on S away from the thresholds. We use estimates for the Green's function for the unperturbed strip and adapt arguments previously used to study resonances for the exterior problem [12,31,35] to obtain an upper bound on the Fredholm determinant. Using the minimodulus theorem of Cartan together with an adaptation of a minimodulus theorem for sectors found in [4], we obtain a lower bound on the Fredholm determinant.

Arguing as in [36], we then obtain an *a priori* estimate on the extended resolvent in an open set away from the thresholds and away from the resonances. Theorem 1 is then proven using an application of the maximum principle inspired by one used in [29], where the argument is given in a semi-classical framework.

The use of the Cartan theorem in the study of spectral and scattering theory was initiated in [16,38], and the use of the minimodulus result for sectors by [15]. For a different proof of the minimodulus theorem for sectors see [21].

Theorem 2 follows by applying Jensen's formula for meromorphic functions [4], together with the upper and lower bounds on the Fredholm determinant, to obtain upper bounds on the number of resonances on disks, the union of which covers the positive real axis. Remark: for the exterior problem in odd dimensions, Jensen's formula was used to obtain global, sharp upper bounds on the number of resonances [35].

We conclude this section by observing that the methods used in this paper could easily be applied to prove analogous results for perturbations of more general cylinders, in particular the cylinder in $\mathbf{R}^3$:

$$\bigl\{(x, y, z)\colon\ x^2 + y^2 = 1,\ z \in (-\infty, \infty)\bigr\}.$$

Note. Since the submission of this paper, we have received a preprint from T. Christiansen [5] in which the sharp estimate $N(r) < Cr$ is proven.

2. Preliminaries

We prove our results for the Neumann Laplacian, leaving it to the reader to make the simple modifications necessary for Dirichlet boundary conditions.

Let

$$\Omega_0 = \bigl\{(x, y)\colon\ x \in (-\infty, \infty),\ y \in (0, \pi)\bigr\}.$$

Let Ω be a domain with smooth boundary such that there exists $M > 0$ so that

$$\Omega - \Bigl\{\sqrt{x^2 + y^2} > M\Bigr\} = \Omega_0 - \Bigl\{\sqrt{x^2 + y^2} > M\Bigr\}. \tag{1}$$

On such a domain, we define the Neumann Laplacian, Δ, as the operator living on $L^2(\Omega)$ with

$$\Delta u \equiv -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2},$$

and with operator core

$$\left\{u \in L^2(\Omega),\ \Delta u \in L^2(\Omega),\ \frac{\partial u}{\partial \eta} = 0\right\}.$$

Here $\partial/\partial\eta$ is the normal derivative at the boundary of Ω. (For the Dirichlet Laplacian, the operator core is $C_0^\infty(\Omega)$.)

Denote by $L^2(\Omega)$ the set of square integrable functions on Ω, and the set of bounded operators on $L^2(\Omega)$ by $\mathcal{L}(L^2(\Omega))$. Denote by $B(a, r)$ the ball centered at a of radius r, intersected with Ω when appropriate. Denote the Neumann

Laplacian on Ω (respectively, Ω_0) by Δ (respectively, Δ_0). Define the Sobolev spaces $H^i(\Omega)$ as the operator domains of $(\Delta+1)^{i/2}$. We define a smooth partition of unity $\chi_1+\chi_2=1$ such that $\chi_i \geqslant 0$, $\mathrm{supp}(\chi_1) \subset B(0, M+2)$, and $\chi_1 = 1$ on $B(0, M+1)$. We also define smooth cutoff functions $\tau_i \geqslant 0$ such that $\tau_1 = 1$ on $\mathrm{supp}(\chi_1)$ and $\mathrm{supp}(\tau_1) \subset B(0, M+3)$, and $\tau_2 = 1$ on $\mathrm{supp}(\chi_2)$ and $\tau_2 = 0$ on $B(0, M)$. Finally, we define a smooth cutoff function ρ such that the $\mathrm{supp}(\rho) \subset B(0, M+4)$ and

$$\rho|_{B(0,M+3)} = 1. \tag{2}$$

Denote the associated resolvent $(\Delta_0 - k)^{-1}$ by $R_0(k)$. Denote $(\Delta - k)^{-1}$ by $R(k)$. Then it is well known that the Green function for the operator $\Delta_0 - k$, i.e., the Schwartz kernel for $R_0(k)$, is given by

$$\begin{aligned} G_k(x, y, x', y') = {} & \frac{1}{\pi\sqrt{-k}} e^{-\sqrt{-k}|x-x'|} \\ & + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-k}} e^{-\sqrt{n^2-k}\,|x-x'|} \cos(ny)\cos(ny'). \end{aligned} \tag{3}$$

In the formula above, the branch lines for the functions $k \to \sqrt{n^2-k}$ are assumed to be $[n^2, \infty)$. In what follows, let $\arg_n$ be the argument associated to the branch point n^2. For each square-root function, the set $\{k\colon \arg_n(k) \in (0, 2\pi)\}$ will be referred to as the "physical branch," and the set $\{k\colon \arg_n(k) \in [2\pi, 0]\}$ the non-physical branch.

Let S be the infinitely branched Riemann surface associated to G_k. Thus G_k extends pointwise to S. The geometry of S was studied in [34]. A point $k \in S$ will be on the "physical branch of S" if

$$\arg_n(k) \in (0, 2\pi), \quad \forall n;$$

thus the physical branch of S can be identified with the subset $\mathbf{C} - [0, \infty)$ of the complex plane. For $k \in S$, we denote by $\Lambda(k)$ the finite multi-index $(n_1, \ldots, n_P)$ such that k is on the "non-physical branch" precisely for the functions $\{\sqrt{n_i^2 - k}\}_{i=1}^{P}$.

Let Π be the canonical projection of S onto $\mathbf{C}$. The induced Riemannian metric on S induces a distance function that we will denote dist. It will also be convenient to define the following functions on S: $|k| \equiv |\Pi k|$, $\Re k \equiv \Re \Pi k$, and $\Im k \equiv \Im \Pi k$. Note that since Π is not a global isometry, $\mathrm{dist}(z_1, z_2)$ is not always equal to $|z_1 - z_2|$.

Since we will be concerned with the behaviour of the resolvent only in a neighbourhood of the physical plane, we shall define the following subsets of S:

$$\tilde{S} = \bigl\{k \in S\colon \mathrm{dist}(k, \text{physical plane}) < 1 + \sqrt{|k|}\bigr\}.$$

It will also be convenient to study $\tilde{S}$ away from the thresholds, hence for $\alpha > 0$

$$\tilde{S}_\alpha = \bigl\{k \in \tilde{S}\colon \mathrm{dist}(k, n^2) > \alpha,\ \forall n \in \mathbf{Z}\bigr\}.$$

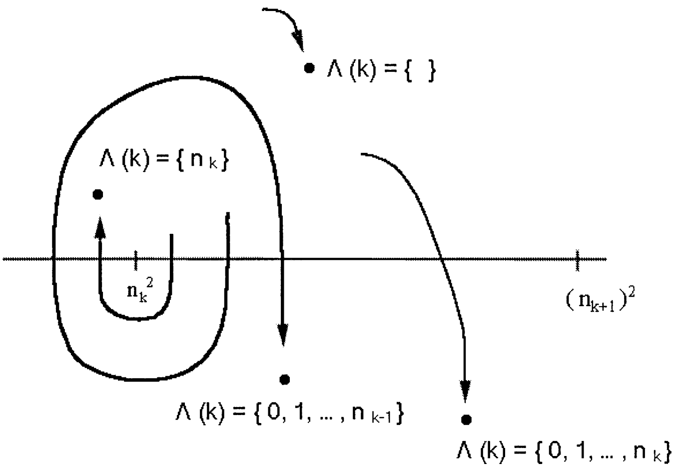

Fig. 1. Possible values for $\Lambda(k)$.

For $k \in \tilde{S}$, define n_k to the the greatest integer such that k is on the non-physical plane for $\sqrt{n_k^2 - k}$. Note that (see Fig. 1)

$$k \in \tilde{S} \quad \Rightarrow \quad \Lambda(k) = \begin{cases} \{\,\}, & \text{or} \\ \{0, 1, \ldots, n\}, & \text{or} \\ \{n\}. \end{cases}$$

Denote $\Lambda(k)^c$ to be the complement of $\Lambda(k)$ within the integers. We note for future reference the following formula:

$$\Re\sqrt{n^2 - k} = \pm \frac{((\Im k)^2 + (n^2 - \Re(k))^2)^{1/4}}{2} \times \left(1 - \frac{1}{\sqrt{1 + (\Im k/(n^2 - \Re k))^2}}\right), \tag{4}$$

with the plus (respectively, minus) sign appearing when $n \in \Lambda(k)^c$ (respectively, $n \in \Lambda(k)$).

Lemma 1. *Let ψ_1, ψ_2 be smooth functions of bounded support on Ω that depend on x alone, and with values in $[0, 1]$. Then the mapping from $\mathbf{C} - [0, \infty)$ to $\mathcal{L}(L^2(\Omega))$ given by*

$$k \to \psi_2 R_0(k) \psi_1$$

extends to a meromorphic function on S. Also, the same is true for the mappings

$$k \to \frac{\partial}{\partial x} \psi_2 R_0(k) \psi_1, \qquad k \to \frac{\partial}{\partial y} \psi_2 R_0(k) \psi_1.$$

Furthermore, for $k \in \tilde{S}_\alpha$ on the physical sheet,

$$\left\| \frac{\partial^i}{\partial x^i} \psi_2 R_0(k) \psi_1 \right\|_{L^2 \to L^2} \leqslant C \ln(|k|)(1+|k|)^{i/2}, \quad i = 0, 2, \tag{5}$$

where C depends only on α, ψ_1, ψ_2, and

$$\left\| \frac{\partial^i}{\partial x^i} \psi_2 R_0(k) \psi_1 \right\|_{L^2 \to L^2} \leqslant C(1+|k|)^{i/2}, \quad i = 1, 3, \tag{6}$$

where C depends only on α, ψ_1, ψ_2.

Proof. The analytic extension of $\psi_2 R_0(k) \psi_1$ and its first partial derivatives follow immediately from the compactness of the support of $\psi_1 G_k \psi_2$, together with the pointwise meromorphicity in k of G_k.

To prove Eqs. (5) and (6), fix k on the physical plane. If $|\Im k| > 1$, then Eqs. (5) and (6) hold be the Spectral Theorem and interpolation. Thus in what follows, we assume $|\Im k| \leqslant 1$.

Let P be the orthogonal projection of $L^2(\Omega_0)$ onto the closed subspace spanned by $\{f(x)\cos(ny)\colon\ f \in L^2(\mathbf{R}),\ 0 \leqslant n \leqslant n_k\}$. Thus $(I-P)\Delta_0(I-P)$ is a self-adjoint operator whose spectrum is $[(n_k+1)^2, \infty)$. Since k is on the physical branch of all of the square-root functions associated to the Schwartz kernel of $(I-P)\Delta_0(I-P)$, it follows that $((I-P)\Delta_0(I-P)-k)^{-1}$ is a bounded operator on $L^2(\Omega_0)$ and the following estimate holds by the Spectral Theorem:

$$\begin{aligned} \left\|(I-P)R_0(k)(I-P)\right\| &= \left\|\bigl((I-P)\Delta_0(I-P)-k\bigr)^{-1}\right\| \\ &\leqslant \left|k-(n_k+1)^2\right|^{-1} \leqslant \alpha^{-1}. \end{aligned}$$

The last inequality follows from the assumption that $|k-n^2| > \alpha$ for all n. Hence

$$\left\|\psi_1(I-P)R_0(k)(I-P)\psi_2\right\| \leqslant \alpha^{-1}. \tag{7}$$

We now estimate the norm of the operator $\psi_1 P R_0(k) P \psi_2$, whose Schwartz kernel is

$$\psi_1(x)\psi_2(x')\frac{1}{\pi\sqrt{-k}} e^{-\sqrt{-k}|x-x'|} + \psi_1(x)\psi_2(x')\sum_{n=1}^{n_k} \frac{1}{\sqrt{n^2-k}} e^{-\sqrt{n^2-k}|x-x'|}\cos(ny)\cos(ny').$$

In what follows, C will denote various positive constants that are independent of k, n. Then, since $|\Im k| < 1$,

$$\begin{aligned} \left\|\psi_1 P R_0(k) P \psi_2\right\| &\leqslant \frac{1}{\pi\sqrt{|k|}} + \sum_{n=1}^{n_k} \frac{1}{|\sqrt{n^2-k}|} \\ &\leqslant \sum_{n=0}^{n_k} \frac{1}{|\sqrt{n^2-k}|} \leqslant C \ln(\Re k). \end{aligned} \tag{8}$$

Since P commutes with $R_0(k)$, it follows that

$$\psi_1(I-P)R_0(k)(I-P)\psi_2 + \psi_1 P R_0(k) P\psi_2 = \psi_1 R_0(k)\psi_2.$$

Hence, by combining Eqs. (7) and (8) we obtain that as $k \to \infty$ with $k \in \tilde{S}_\alpha$, we have

$$\|\psi_1 R_0(k)\psi_2\| \leqslant C\ln(|k|).$$

Thus Eq. (5) has been proven for $i=0$. The proof of Eq. (6), with $i=1$, is similar. For $i=2$, we write

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\psi_2 R_0(k)\psi_1 &= \frac{\partial^2}{\partial x^2}(\Delta_0+1)^{-1}[\Delta_0,\psi_2]R_0(k)\psi_1\\ &\quad + (k+1)\frac{\partial^2}{\partial x^2}(\Delta_0+1)^{-1}\psi_2 R_0(k)\psi_1\\ &\quad + \frac{\partial^2}{\partial x^2}(\Delta_0+1)^{-1}\psi_2\psi_1.\end{aligned}$$

Then the desired estimate follows from the estimates for $i=0,1$.

The proof for $i=3$ is similar. □

We now prove the existence of a meromorphic extension of $R(k)$. The argument follows closely along the lines of the corresponding result for exterior domains found in [23]. For a proof of this result for more general perturbations of the cylinder, see [13].

Proposition 1. *Let $\chi \in C_0^\infty(\Omega)$. Then the mapping from $\mathbf{C}-[0,\infty)$ to $\mathcal{L}(L^2(\Omega))$ given by*

$$k \to \chi(\Delta-k)^{-1}\chi$$

extends to a meromorphic function in S. At each pole k_0, the coefficients of the negative powers of $(k-k_0)$ in the Laurent series are finite rank operators.

Proof. We define an approximation of $R(k)$ as follows. Assume for the moment that k is on the physical branch of S. Let

$$R_a(k) = \tau_1 R(k_0)\chi_1 + \tau_2 R_0(k)\chi_2. \tag{9}$$

Here k_0 is a parameter to be chosen below.

We have

$$(\Delta-k)R_a(k) = I + K, \tag{10}$$

with

$$K = (k_0-k)\tau_1 R(k_0)\chi_1 + [\Delta,\tau_1]R(k_0)\chi_1 + [\Delta,\tau_2]R_0(k)\chi_2. \tag{11}$$

By Eq. (10) we have for $k \in \mathbf{C} - [0, \infty)$:

$$R_a = (\Delta - k)^{-1}(I + K).$$

By Eqs. (11) and (2) we have $\rho K = K$, hence

$$R_a \rho = (\Delta - k)^{-1}\rho(I + K\rho).$$

For $k = k_0$ and $\Im(k_0) \gg 0$, we have by the Spectral Theorem that $\|K\rho\|_{L^2 \to L^2} < 1$ and hence we can write

$$\rho R_a(k)\rho(I + K\rho)^{-1} = \rho R(k)\rho. \tag{12}$$

Fix such a k_0.

Next we observe from Eq. (9) that $k \to \rho R_a(k)\rho$ extends meromorphically to $S - \{0^2, 1^2, \ldots\}$, with values in $\mathcal{L}(L^2(\Omega))$. For the terms involving $R_0(k)$, this follows from Lemma 1, while for the term involving $R(k_0)$, note that the function $k \to (k_0 - k)$ extends to the function $k \to (k_0 - \Pi k)$, which is analytic on S. It follows that the meromorphy of $\rho R(k)\rho$ is equivalent to meromorphy of $(I + K\rho)^{-1}$.

On the other hand, since χ_1 and ρ are compactly supported, it follows that $K\rho$ is an analytic compact operator-valued function of k on $S - \{0^2, 1^2, \ldots\}$. Thus $\rho R(k)\rho$ is a finitely-meromorphic Fredholm family in k [30], and meromorphic Fredholm theory thus implies $\rho R(k)\rho$ is meromorphic for $k \in S - \{0^2, 1^2, \ldots\}$.

To prove $\rho R(k)\rho$ is meromorphic in a neighbourhood of the threshold L^2, $L \in \mathbf{Z}$, one applies the argument above to the function $z \to \rho R(L^2 - z^2)\rho$ in a neighbourhood of $z = 0$.

Finally, it is easy to see that the function ρ can be replaced by any smooth cutoff function. This completes the proof. □

Next, we note the following result due to Melrose [13, Proposition 6.28]. As $z \to 0$ for integer L,

$$\rho(\Delta - L^2 - z^2)^{-1}\rho = \frac{A}{z^2} + \frac{B}{z} + C, \tag{13}$$

where A is the orthogonal projection onto the $L^2(\Omega)$ eigenspace associated to the energy level L^2, B is a projection operator related to the generalised (non-$L^2(\Omega)$) eigenfunctions associated to the energy level L^2, and C is an operator bounded near $z = 0$. We will use this asymptotic formula to provide upper bounds on the resolvent in Lemma 7.

We now define the resonances of Δ to be the poles $\rho R(k)\rho$ in $S - \{L^2,\ L \in \mathbf{Z}\}$, in union with any L^2 eigenvalues occurring at the thresholds. We define the multiplicity of a non-threshold resonance k_j as the rank of the projection

$$\int_\gamma \rho R(k)\rho \, dk$$

for a sufficiently small contour γ about k_j. The multiplicity of the any resonance occurring at a threshold is defined to be the dimension of the corresponding eigenspace.

3. Estimates on Fredholm determinant

In what follows, let C be various positive constants. Let K be as in the proof of Proposition 1. A simple argument shows that $(I + K\rho)$ is invertible if and only if $(I + (K\rho)^3)$ is invertible. On the other hand, since $K\rho$ is a pseudodifferential operator of order -1 in $\mathcal{L}(L^2(\Omega))$, with compactly supported Schwartz kernel, it follows that $(K\rho)^3$ is trace class. Thus the Fredholm determinant $\det(I + (K\rho)^3)$ is entire on S, and furthermore:

Lemma 2. *The non-threshold resonances of Δ (counted with their multiplicities) are among the zeros of the function*

$$k \to h(k) \equiv \det\bigl(I + (K\rho)^3(k)\bigr),$$

counted with their multiplicities.

The reader is referred to [32] for a proof of this result.

The rest of this section is used to prove:

Proposition 2. *For $k \in \tilde{S}$, we have the estimate*

$$|h(k)| \leqslant \frac{C \exp C|k|^{3/2}}{\operatorname{dist}(k, \{L^2,\ L \in \mathbf{Z}\})^3}, \tag{14}$$

with C a positive constant independent of k.

Proof. Assume in what follows that $k \in \tilde{S}$, $k \neq L^2$ for $L \in \mathbf{Z}$. We apply the theory of characteristic values developed in [10], and adapted to exterior problems in [12,31,35]. The characteristic values $\mu_j(A)$ of a compact operator A are the eigenvalues, listed in decreasing order and counting multiplicities, of the operator $|A|$. We recall the following inequalities from [10]: $\mu_{j+k-1}(AB) \leqslant \mu_j(A)\mu_k(B)$, $\mu_{j+k-1}(A + B) \leqslant \mu_j(A) + \mu_k(B)$, $\mu_j(AB) \leqslant \|A\|\mu_j(B)$.

We write $K\rho = K_1 + K_2$, with $K_2 = [\Delta, \tau_2]R_0(k)\chi_2\rho$.

Applying inequalities on Fredholm determinants appearing in [10], we get

$$\begin{aligned}\bigl|\det\bigl(I + (K\rho)^3\bigr)\bigr| &\leqslant \det\bigl(I + 4|K_1|^3\bigr)^6 \det\bigl(I + 4|K_2|^3\bigr)^6 \\ &\leqslant \left(\prod_{j=1}^{\infty}\bigl(1 + 4\mu_j(|K_1|)^3\bigr)\right)^6 \left(\prod_{j=1}^{\infty}\bigl(1 + 4\mu_j(|K_2|)^3\bigr)\right)^6.\end{aligned} \tag{15}$$

We shall estimate the terms on the right-hand side of the last equation with a series of lemmas. We estimate first the term involving K_1. Recall that for $k \in S$

$$K_1 = (k_0 - \Pi k)\tau_1 R(k_0)\chi_1\rho + [\Delta, \tau_1]R(k_0)\chi_1\rho. \tag{16}$$

Lemma 3.

$$\prod_{j=1}^{\infty}\bigl(1 + 4\mu_j(|K_1|)^3\bigr) \leqslant Ce^{C|k|\ln|k|}, \quad \forall k \in S.$$

Proof. The argument here follows [35]. Since τ_1, χ_1 are compactly supported, it follows by standard eigenvalue asymptotics for pseudodifferential operators [20] that

$$\mu_j\bigl(\bigl|\tau_1 R(k_0)\chi_1\rho\bigr|\bigr) \sim Cj^{-1}$$

and

$$\mu_j\bigl(\bigl|[\Delta, \tau_1]R(k_0)\chi_1\rho\bigr|\bigr) \sim Cj^{-1/2}.$$

It follows that, denoting the largest integer below x by $\lfloor x\rfloor$,

$$\mu_{j-1}(|K_1|) \leqslant C|k|\lfloor j/2\rfloor^{-1} + C\lfloor j/2\rfloor^{-1/2}.$$

Hence we get

$$\begin{aligned}\mu_j\bigl(|K_1|^3\bigr) &\leqslant \bigl(\mu_{(\lfloor j/3\rfloor+1)}(|K_1|)\bigr)^3\\ &\leqslant \bigl(C|k|\lfloor j/6+2\rfloor^{-1} + C(\lfloor j/6\rfloor+2)^{-1/2}\bigr)^3\\ &\leqslant C|k|^3 j^{-3} + Cj^{-3/2}.\end{aligned}$$

Note that $|k|^2 < j$ is equivalent to $|k|^3 j^{-3} < j^{-3/2}$. Thus

$$\prod_{j=1}^{\infty}\bigl(1 + 4\mu_j(|K_1|)^3\bigr) \leqslant \prod_{j\leqslant|k|^2}\bigl(1 + C|k/j|^3\bigr)\prod_{j>|k|^2}\bigl(1 + Cj^{-3/2}\bigr)$$

These two factors are bounded as in [35]; we sketch the argument. The first factor is bounded by comparing it to

$$\exp\left(\int_1^{|k|^2}\ln\bigl(1 + C|k/x|^3\bigr)\,dx\right). \tag{17}$$

Note that

$$\begin{aligned}\int_1^{|k|^2}\ln\bigl(1 + C|k/x|^3\bigr)\,dx &\leqslant \int_1^{C|k|}\ln\bigl(1 + C|k/x|^3\bigr)\,dx + 2\int_{C|k|}^{|k|^2}C|k/x|^3\,dx\\ &\leqslant C|k|\ln(|k|).\end{aligned}$$

Thus Eq. (17) is bounded by $\exp(C|k|\ln|k|)$. The second factor is treated similarly. Thus

$$\prod_{j=1}^{\infty}\bigl(1+4\mu_j(|K_1|)^3\bigr)\leqslant e^{C|k|\ln|k|}. \tag{18}$$

Note that this estimate holds for all $k\in S$. □

Next, we estimate the terms involving K_2 away from the thresholds.

Lemma 4. *Suppose $k\in\tilde S_\alpha$. Then $\prod_{j=1}^{\infty}(1+4\mu_j(|K_2|)^3)<e^{C|k|^{3/2}}$, where C is some positive constant.*

Proof. The proof is an adaptation of the "good half plane–bad half plane" argument found in [31,35].

First, we assume $k\in\tilde S_\alpha$ is on the physical sheet, and assume without loss of generality that $|k|$ is large. We have, by Lemma 1,

$$\begin{aligned}\mu_j(K_2)&=\mu_j\bigl(\rho(I+\Delta)^{-1}(I+\Delta)K_2\bigr)\\&\leqslant\mu_j\bigl(\rho(I+\Delta)^{-1}\bigr)\bigl\|(I+\Delta)K_2\bigr\|\leqslant Cj^{-1}|k|^{3/2}.\end{aligned}$$

Now the arguments leading to Eq. (18) are easily adapted to this case. In fact,

$$\prod_{j=1}^{\infty}\bigl(1+4\mu_j(|K_2|)^3\bigr)\leqslant\prod_{j=1}^{\infty}\bigl(1+C|k|^{9/2}/j^3\bigr).$$

This last product is estimated as follows. First, one proves

$$\exp\left(\int_1^{|k|^{3/2}}\ln\bigl(1+C|k|^{9/2}/x^3\bigr)\,dx\right)\leqslant e^{C|k|^{3/2}} \tag{19}$$

as follows:

$$\begin{aligned}&\int_1^{|k|^{3/2}}\ln\bigl(1+C|k|^{9/2}/x^3\bigr)\,dx\\&\quad=\left(\int_1^{|k|^{3/2}/\ln|k|}+\int_{|k|^{3/2}/\ln|k|}^{|k|^{3/2}/10}+\int_{|k|^{3/2}/10}^{|k|^{3/2}}\right)\ln\bigl(1+C|k|^{9/2}/x^3\bigr)\,dx.\end{aligned}$$

The first and third integrals on the right-hand side are easily shown to be $O(|k|^{3/2})$, and the second integral is estimated as

$$\int_{|k|^{3/2}/\ln|k|}^{|k|^{3/2}/10} \ln\bigl(1+C|k|^{9/2}/x^3\bigr)\,dx \sim \int_{|k|^{3/2}/\ln|k|}^{|k|^{3/2}/10} \ln\bigl(C|k|^{9/2}/x^3\bigr)\,dx \leqslant C|k|^{3/2}.$$

Also it is easy to show that

$$\exp\Biggl(\int_{|k|^{3/2}}^{\infty} \ln\bigl(1+C|k^{3/2}/x|^3\bigr)\,dx\Biggr) \leqslant e^{C|k|^{3/2}}. \tag{20}$$

By Eqs. (19) and (20), the lemma holds for k on the physical sheet.

Now suppose $k \in \tilde{S}_\alpha$ is on the non-physical sheet. There are two possible cases: $\Lambda(k) = \{0, 1, \ldots, n_k^2\}$ or $\Lambda(k) = \{n_k^2\}$ (see Fig. 1). Suppose for now the first case. We write

$$\begin{aligned} G_k(x,x',y,y') &= \frac{1}{\pi\sqrt{-k}}\bigl(e^{-\sqrt{-k}|x-x'|} + e^{\sqrt{-k}|x-x'|}\bigr) \\ &\quad + \sum_{n=1}^{n_k} \frac{1}{\sqrt{n^2-k}}\bigl(e^{-\sqrt{n^2-k}|x-x'|} + e^{\sqrt{n^2-k}|x-x'|}\bigr)\cos(ny)\cos(ny') \\ &\quad - \frac{1}{\pi\sqrt{-k}} e^{\sqrt{-k}|x-x'|} - \sum_{n=1}^{n_k} \frac{1}{\sqrt{n^2-k}} e^{\sqrt{n^2-k}|x-x'|}\cos(ny)\cos(ny') \\ &\quad + \sum_{n=n_k+1}^{\infty} \frac{1}{\sqrt{n^2-k}} e^{-\sqrt{n^2-k}|x-x'|}\cos(ny)\cos(ny'). \end{aligned}$$

Note first that

$$e^{-\sqrt{n^2-k}|x-x'|} + e^{\sqrt{n^2-k}|x-x'|} = e^{-\sqrt{n^2-k}(x-x')} + e^{\sqrt{n^2-k}(x-x')},$$

and hence the operator A_1, whose Schwartz kernel is

$$\begin{aligned} &\frac{1}{\pi\sqrt{-k}}\bigl(e^{-\sqrt{-k}|x-x'|} + e^{\sqrt{-k}|x-x'|}\bigr) \\ &\quad + \sum_{n=1}^{n_k} \frac{1}{\sqrt{n^2-k}}\bigl(e^{-\sqrt{n^2-k}|x-x'|} + e^{\sqrt{n^2-k}|x-x'|}\bigr)\cos(ny)\cos(ny'), \end{aligned}$$

will have rank $2n_k + 2$. Thus the operator $[\Delta, \tau_2]A_1\chi_2\rho$ will also have rank $2n_k + 2$. It follows now from Eq. (4) that for $k \in \tilde{S}_\alpha$

$$\mu_j\bigl([\Delta,\tau_2]A_1\chi_2\rho\bigr) \leqslant \begin{cases} e^{C|k|^{1/2}}, & j \leqslant 2n_k + 2, \\ 0, & j > 2n_k + 2. \end{cases}$$

Also, observe that the operator A_2, whose Schwartz kernel is given by

$$-\frac{1}{\pi\sqrt{-k}}e^{\sqrt{-k}|x-x'|}-\sum_{n=1}^{n_k}\frac{1}{\sqrt{n^2-k}}e^{\sqrt{n^2-k}|x-x'|}\cos(ny)\cos(ny')$$
$$+\sum_{n=n_k+1}^{\infty}\frac{1}{\sqrt{n^2-k}}e^{-\sqrt{n^2-k}|x-x'|}\cos(ny)\cos(ny'),$$

equals $R_0(\Pi k)$, i.e., R_0 evaluated on the physical sheet. Hence

$$\begin{aligned}
&\det\left(I+4\left|K_2(k)\right|^3\right)\\
&\quad\leqslant \det\left(I+16\left|[\Delta,\tau_2]A_1\chi_2\rho\right|^3\right)^6\det\left(I+16\left|[\Delta,\tau_2]A_2\chi_2\rho\right|^3\right)^6\\
&\quad\leqslant \prod_{j=1}^{2n_k+2}\left(1+e^{C|k|^{1/2}}\right)e^{C|k|^{3/2}} &(21)\\
&\quad\leqslant e^{C|k|^{3/2}}. &(22)
\end{aligned}$$

The last inequality holds because $n_k|k|^{1/2}\leqslant C|k|$.

For the case where $\Lambda(k)=\{n_k^2\}$, we write

$$\begin{aligned}
G_k(x,x',y,y')&=\frac{1}{\pi\sqrt{-k}}e^{-\sqrt{-k}|x-x'|}\\
&\quad+\sum_{n=1}^{n_k-1}\frac{1}{\sqrt{n^2-k}}e^{-\sqrt{n^2-k}|x-x'|}\cos(ny)\cos(ny')\\
&\quad+\sum_{n=n_k+1}^{\infty}\frac{1}{\sqrt{n^2-k}}e^{-\sqrt{n^2-k}|x-x'|}\cos(ny)\cos(ny')\\
&\quad+\frac{1}{\sqrt{n_k^2-k}}\left(e^{-\sqrt{n_k^2-k}|x-x'|}-e^{\sqrt{n_k^2-k}|x-x'|}\right)\cos(ny)\cos(ny').
\end{aligned}$$

The argument in this case is similar to the one for the case previous. The details are left to the reader. □

Proposition 2, for k away from the thresholds, now follows from Lemmas 3, 4 and Eq. (15). We now prove bounds on the determinant near the thresholds.

Lemma 5. *Let L be any integer. Then for $k\in S$, $\operatorname{dist}(k,L^2)\leqslant\alpha$,*

$$\left|\det\left(I+(K\rho)^3\right)\right|\leqslant\frac{Ce^{C|k|^{3/2}}}{\operatorname{dist}(L^2,k)^{3/2}},$$

with C independent of k, L.

Proof. The key observation is that the pole for K at L^2 is simple with rank one residue. In particular, note that

$$[\Delta, \tau_2]\frac{1}{\sqrt{L^2-k}}e^{\sqrt{L^2-k}|x'-x|}$$
$$= \tau''\frac{1}{\sqrt{L^2-k}} + \tau''\frac{e^{\sqrt{L^2-k}|x'-x|}-1}{\sqrt{L^2-k}} + 2\tau' \operatorname{sgn}(x'-x)e^{\sqrt{L^2-k}|x'-x|},$$

where $\operatorname{sgn}(t) = 1$ for $t > 0$, $\operatorname{sgn}(t) = -1$ for $t < 0$. Thus, we may write

$$K\rho = K_3 + K_4,$$

where K_3 is a rank one operator with Schwartz kernel

$$\frac{1}{\sqrt{L^2-k}}\tau_2''(x)\rho(x')\chi_2(x')\cos(ny)\cos(ny'), \tag{23}$$

and K_4 has pointwise bounded Schwartz kernel in a neighbourhood of the L^2. In the argument that follows we use the inequality

$$\mu_j\big(|A+B|^2\big) \leqslant 2\mu_j\big(|A|^2 + |B|^2\big),$$

which follows from the quadratic form inequality $|A+B|^2 \leqslant 2|A|^2 + 2|B|^2$ and a minimax argument. Fix $\epsilon \in (0, 1/2)$.

$$\begin{aligned}\mu_j\big((K\rho)^3\big) &= \mu_j\big(|K\rho|^2\big)^{3/2} \\ &\leqslant \mu_j\big(|K_3+K_4|^2\big)^{3/2} \leqslant 2^{3/2}\mu_j\big(|K_3|^2+|K_4|^2\big)^{3/2} \\ &\leqslant 2^{3/2}\big(\mu_{\lceil(1-\epsilon)j\rceil}(|K_3|)^2 + \mu_{\lfloor\epsilon j\rfloor+1}(|K_4|)^2\big)^{3/2}.\end{aligned}$$

It follows from Eq. (23) that

$$\mu_{\lceil(1-\epsilon)j\rceil}(|K_3|) = \begin{cases} C/\sqrt{L^2-k}, & j = 1, \\ 0, & j > 1. \end{cases}$$

Thus

$$\begin{aligned}\big|\det\big(I + (K\rho)^3\big)\big| &\leqslant \prod_{j=1}^{\infty}\big(1 + \mu_j\big((K\rho)^3\big)\big) \\ &\leqslant \left(1 + 2^{3/2}\left(\frac{C}{(L^2-k)^{1/2}} + \mu_1(K_4)\right)^{3/2}\right) \\ &\quad \times \prod_{j=2}^{\infty}\big(1 + 2^{3/2}\mu_{\lfloor\epsilon j\rfloor+1}(K_4)^3\big).\end{aligned} \tag{24}$$

Now we analyse $\mu_j(K_4)$. Let P be the orthogonal projection onto the orthogonal complement of the subspace of $L^2(\Omega_0)$:

$$\left\{ f(x)\cos(Ly);\ \int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty \right\}.$$

Then we have $K_4 = K_1 + K_2 P + K_5$, where K_1 and K_2 are as in the proof of Proposition 2, and where K_5 has Schwartz kernel

$$\rho(x)\left(\tau''(x)\frac{e^{\sqrt{L^2-k}|x'-x|}-1}{\sqrt{L^2-k}} + 2\tau' \operatorname{sgn}(x'-x)e^{\sqrt{L^2-k}|x'-x|}\right) \times \chi_2(x')\rho(x')\cos(Ly)\cos(Ly'). \tag{25}$$

Thus

$$\begin{aligned}\mu_n(K_4)^3 &\leqslant \bigl(\mu_{\lfloor n/3\rfloor+2}(K_1) + \mu_{\lfloor n/3\rfloor}(K_2P) + \mu_{\lceil n/3\rceil}(K_5)\bigr)^3 \\ &\leqslant C\bigl(\mu_{\lfloor n/3\rfloor+2}(K_1)^3 + \mu_{\lfloor n/3\rfloor}(K_2P)^3 + \mu_{\lceil n/3\rceil}(K_5)^3\bigr).\end{aligned}$$

Hence, by applying inequalities for Fredholm determinants as in Eq. (15),

$$\begin{aligned}\prod_{j=1}^{\infty}\bigl(1+2^{3/2}\mu_{\lfloor\epsilon j-1\rfloor}(K_4)^3\bigr) &\leqslant \left(\prod_{j=1}^{\infty}\bigl(1+C\mu_{\lfloor\epsilon j/3\rfloor+1}(K_1)^3\bigr)\right)^6 \\ &\quad\times\left(\prod_{j=1}^{\infty}\bigl(1+C\mu_{\lfloor\epsilon j/3\rfloor}(K_2P)^3\bigr)\right)^{36}\left(\prod_{j=1}^{\infty}\bigl(1+C\mu_{\lfloor\epsilon j/3\rfloor}(K_5)^3\bigr)\right)^{36}.\end{aligned}$$

The first of these products is estimated exactly as in the proof of Lemma 3, and the estimate for the second of these products is derived similarly to the estimate for $\det(I + 4|K_2|^3)$ away from the thresholds (noting that $\mu_n(K_2P) \leqslant \mu_n(K_2)$). Finally,

$$\begin{aligned}\prod_{j=1}^{\infty}\bigl(1+C\mu_j(K_5)\bigr)^3 &= \exp\left(\sum\Bigl(\ln\bigl(1+\mu_j\bigl((K_5)^3\bigr)\bigr)\Bigr)\right) \\ &\sim \exp\left(\sum\mu_j\bigl((K_5)^3\bigr)\right) \\ &\leqslant \exp\left(\left(\sum\mu_j\bigl(|K_5|^2\bigr)^2\right)^{1/2}\left(\sum\mu_j\bigl((K_5)^2\bigr)\right)^{1/2}\right) \\ &\leqslant C.\end{aligned}$$

The last inequality holds because, by Eq. (25), the Hilbert–Schmidt norms (see, e.g., [19, Vol. 1]) of K_5 and K_5^2 are bounded by a bound independent of k, for k in a small neighbourhood of L^2. Thus

$$\prod_{j=1}^{\infty}\bigl(1+2^{3/2}\mu_{\lfloor\epsilon j-1\rfloor}(|K_4|)^3\bigr) \leqslant Ce^{C|k|^{3/2}}. \qquad \square$$

Proposition 2 now follows from Lemmas 3–5. □

4. Resolvent estimate and bounds on number of resonances

Proving both Theorems 1 and 2 requires lower bounds on the Fredholm determinant studied in the previous section. For this we use the following lemma, which is an adaptation of an argument found in Cartwright [4, pp. 89–91]. The Cartwright result has previously been used in the scattering theoretic context by [15].

Lemma 6. *Let $k = re^{i\theta}$. Suppose the function g is analytic in the sector $\{\theta \in (0,\pi)\}$, and satisfies*

$$|g(k)| < C\exp\bigl(C|k|^{3/2}\bigr).$$

Let $\phi:\mathbf{R}\to\mathbf{R}$ be any increasing, real-valued function such that $\lim_{x\to\infty}\phi(x) = \infty$. Then, for any $M > 0$, there exists $R_2 = R_2(M)$ such that for each $r > R_2$

$$|g(k)| > \exp\bigl(-|k|^{5/2}\phi(|k|)\bigr),$$

except perhaps in a set of θ, denoted θ_r, with $|\theta_r| < 1/(Mr^{1/2})$. Here $|A|$ denotes the Lebesgue measure of A.

Proof. By Carleman's formula [11], we have

$$\sum_{1\leqslant r_n\leqslant R}\frac{\sin\theta_n}{r_n}\left(1-\frac{r_n^2}{R^2}\right) = \frac{1}{\pi R}\int_0^{\pi}\ln\bigl|g(Re^{i\theta})\bigr|\sin\theta\,d\theta$$
$$+\frac{1}{2\pi}\int_1^R\bigl(\ln|g(y)|+\ln|g(-y)|\bigr)\left(\frac{1}{y^2}-\frac{1}{R^2}\right)dy+\chi(R),$$

where $\chi(R) = O(1)$ as $R\to\infty$, and $r_ne^{i\theta_n}$ are the zeros of $g(z)$ in upper half space. Set $\ln_+(x) = \max(\ln(x),0)$. Then it follows that

$$\frac{1}{\pi R}\int_0^{\pi}\ln\bigl|g(Re^{i\theta})\bigr|\sin\theta\,d\theta$$
$$\geqslant -\frac{1}{2\pi}\int_1^R\bigl(\ln_+|g(y)|+\ln_+ g|(-y)|\bigr)\frac{1}{y^2}\,dy - O(1)$$
$$\geqslant -C\int_1^R y^{3/2}\frac{1}{y^2}\,dy - O(1) \geqslant -CR^{1/2}-O(1)$$
$$\geqslant -CR^{1/2},\quad R > R_0, \tag{26}$$

for some constant R_0. Now fix $M > 0$ and suppose now that for each r, there exists a set θ_r of measure at least $1/Mr^{1/2}$ such that for $\theta \in \theta_r$

$$|g(k)| < \exp\bigl(-|k|^{5/2}\phi(|k|)\bigr).$$

Then we have

$$\begin{aligned}
\frac{1}{\pi R}\int_0^{\pi} \ln\bigl|g(Re^{i\theta})\bigr| \sin\theta\, d\theta &\leqslant CR^{1/2} - 2R^{3/2}\phi(R) \int_0^{1/(2MR^{1/2})} \sin\theta\, d\theta \\
&\leqslant CR^{1/2} - 2R^{3/2}\phi(R)\left(1-\cos\left(\frac{1}{2MR^{1/2}}\right)\right) \\
&\leqslant CR^{1/2} - R^{1/2}\phi(R)/(4M^2). \qquad (27)
\end{aligned}$$

Comparing Eqs. (26) and (27), we derive a contradiction for $R > R_1$ for some $R_1 = R_1(M)$. Setting $R_2 = \max(R_0, R_1)$, we obtain for $|k| > R_2$

$$|g(k)| > \exp\bigl(|k|^{5/2}\phi(|k|)\bigr),$$

except for $\arg(k) \in \theta_{|k|}$, with $|\theta_{|k|}| < 1/(M|k|^{1/2})$. The lemma is proven. □

Proof of Theorem 2. We can, without loss of generality, assume $|k| > R_2$, where R_2 will be determined by Lemma 6. The proof will apply Jensen's formula for meromorphic functions [4, p. 9] to the function

$$h(k) = \det\bigl(I + (K\rho)^3\bigr)$$

to obtain upper bounds on the number of resonances in a set of disks that form a cover for the following subset of $\tilde{S}$:

$$\left\{k \in S\colon\ \mathrm{dist}(k, \text{physical plane}) < 1 + \frac{1}{2}\sqrt{|k|},\ |k| > R_2\right\}.$$

We consider separately the two cases:

(A) the part of $\tilde{S}$ which is a continuation from the upper half of the physical plane,
(B) the part of $\tilde{S}$ which is a continuation from the lower half of the physical plane.

We treat case (A); the argument for case (B) is similar. To obtain the necessary lower bounds on $h(k)$, we first apply Lemma 6 and Proposition 2 to a sector that is slightly shifted away from the positive real axis. Setting $M = 100$ and $\phi(x) = \ln x$ in Lemma 6, we get for $|k| > R_2$

$$|h(k)| > \exp\bigl(-|k|^{5/2}\ln|k|\bigr),$$

except for $\arg(k) \in \theta_{|k|}$, with $|\theta_{|k|}| < 1/(100|k|^{1/2})$. Fix a positive integer L with $L^2 \geqslant R_2$. Thus setting $\Re(k_0) = L^2 + L$, one can choose $\Im k_0 < L/99$ such that

$$|h(k_0)| > \exp\bigl(-|k_0|^{5/2}\ln|k_0|\bigr). \tag{28}$$

We now apply the Minimodulus Theorem of Cartan to obtain a lower bound on $|h(k)|$ near the real axis. The following version of the theorem can be easily deduced from the arguments found in [11, pp. 21–22]: if g is analytic in $B(0, R) \subset \mathbf{C}$ and $|g(0)| > 0$, then for any $r < R$ one has

$$|g(z)| > |g(0)|^{1+H}\Bigl(\max_{|z|=R}|g(z)|\Bigr)^{-H}, \quad H = \frac{2r}{R-r} + \frac{\ln(3e/2\eta)}{\ln(R/r)};$$

this estimate valid in $B(0,r)$ outside an exceptional set of disks whose summed radii is less that $4\eta R$. We set $R = L$, $r = L/(10)$, $\eta = 1/(4L)$. Applying the Cartan theorem and Proposition 2, there exists $k_1 \in B(L^2 + L, 1) \subset S$ such that

$$|h(k_1)| > \exp\bigl(-C|k_1|^{5/2}\ln|k_0|\ln|k_1|\bigr) > \exp\bigl(-C|k_1|^{5/2+\epsilon}\bigr). \tag{29}$$

Note that the disk $B(k_1, 0.9L)$ does not contain any thresholds. We apply Proposition 2 and Jensen's formula to $h(k)$ on the disk $B(k_1, 0.9L)$ to conclude that the number of zeros, counting multiplicities, in the disk $B_L \equiv B(k_1, 0.8L)$ is bounded by $C|k_1|^{5/2+\epsilon}$.

Next, we bound the number of zeros in a neighbourhood of the threshold L^2. Suppose first that $h(k)$ has a pole at $k = L^2$. Then clearly there exists $k_2 \in \tilde{S}$, with $\operatorname{dist}(k_2, L^2) < 1$, such that $|h(k_2)| > 1$. The function $z \to h(L^2 - z^2)$ is meromorphic in the disk

$$\bigl\{z:\ \bigl|z - \sqrt{L^2 - k_2}\bigr| < \sqrt{0.9(L+1)}\bigr\};$$

here we view the disk as lying in the complex plane and $|\cdot|$ is the standard absolute value function. The only pole for $z \to h(L^2 - z^2)$ in this disk is at $z = 0$, and by Proposition 2 the pole has order at most 3. Hence by Jensen's formula, the number of zeroes in the disk

$$\bigl\{z:\ \bigl|z - \sqrt{L^2 - k_2}\bigr| < \sqrt{0.8(L+1)}\bigr\} \tag{30}$$

is bounded by $C|k_2|^{3/2}$. We label the disk in S corresponding to Eq. (30) as $\tilde{B}_L$.

Next suppose that $h(k)$ has no pole at L^2. It follows that $z \to h(L^2 - z^2)$ is analytic in the disk $\{z:\ |z| < \sqrt{L}\}$. By Proposition 2 and Lemma 6, there exists z_1 with $|z_1| < \sqrt{L/100}$ such that $|h(z_1)| > e^{-|L^2|^{5/2}\ln|L^2|}$. Applying the Cartan theorem, we obtain z_2 such that $|z_2| < 1$ and $|h(z_2)| > e^{-C|L^2|^{5/2+\epsilon}}$. Now applying Jensen's formula as above, the number of zeros on the disk $\{z:\ |z - z_2| < \sqrt{0.8(L+1)}\}$ is bounded by $CL^{5/2+\epsilon}$. Again in this case we label the corresponding disk in S as $\tilde{B}_L$.

Theorem 2 now follows by noting

$$\biggl\{k \in \tilde{S},\ \operatorname{dist}(k, \text{physical sheet}) < \frac{\sqrt{|k|}}{2} + 1,\ R_2 < |k| < r\biggr\}$$

$$\subset \bigcup_{L=\lfloor R_2 \rfloor^2}^{\lceil r \rceil^2} \big(B_L \cup \tilde{B}_L\big). \qquad \square$$

Lemma 7. (A) *For any $t, \epsilon, \alpha > 0$, there exists a constant C dependent on ϵ, α but independent of t such that*

$$\|\rho R(k)\rho\| \leqslant Ce^{Ct|k|^{5/2+\epsilon}}, \tag{31}$$

for all $k \in \tilde{S}_\alpha - \bigcup_{k_j} B(k_j, |k_j|^{-t})$ with $|\Im(k)| < |k|^{1/2}/2$, where k_j are among the resonances of Δ.

(B) *For L any integer, we have the following estimates at the threshold L^2 for any $\epsilon > 0$, $\delta \in (0, 1/2)$. If there exist no resonances in the disk $B(L^2, \delta)$, then*

$$\|\rho R(k)\rho\| \leqslant \frac{Ce^{C|k|^{5/2+\epsilon}\ln(1/\delta)}}{|L^2 - k|^{1/2}\delta^3}, \quad k \in B(L^2, \delta/2). \tag{32}$$

If the only resonances in the disk $B(L^2, \delta)$ are precisely at $k = L^2$, then

$$\|\rho R(k)\rho\| \leqslant \frac{Ce^{C|k|^{5/2+\epsilon}\ln(1/\delta)}}{|L^2 - k|\delta^{5/2}}, \quad k \in B(L^2, \delta/2). \tag{33}$$

Here C is independent of k, L, δ.

Proof. As in [36], we bound the resolvent in terms of Fredholm determinants. In what follows, C will denote various positive constants. For simplicity we set $\alpha = 1$. We recall:

$$\rho R_a(k)\rho(I + K\rho)^{-1} = \rho R(k)\rho. \tag{34}$$

We begin by estimating the resolvent away from the thresholds, so our analysis will be conducted on $\tilde{S}_\alpha$.

It follows from Eq. (9) and Lemma 1 that for $k \in \tilde{S}_\alpha$

$$\|\rho R_a \rho\|_{L^2 \to L^2} \leqslant C|k|^{1/2}. \tag{35}$$

To bound $(1 + K\rho)^{-1}$, we proceed as follows: from [10, Theorem 5.1], we have

$$\big\|(I + K\rho)^{-1}\big\|_{L^2 \to L^2} \leqslant \big|\det\big(I + (K\rho)^3\big)\big|^{-1} \det\big(I + |K\rho|^3\big)^3. \tag{36}$$

By the proof of Proposition 2, we have

$$\det\big(I + |K\rho|^3\big)^3 \leqslant e^{C|k|^{3/2}}, \quad k \in \tilde{S}_\alpha. \tag{37}$$

We now obtain a lower bound on $|h(k)| = |\det(I + (K\rho)^3)|$. Set $M = 1/100$ and fix $R_2 = R_2(M)$ in Lemma 6. In what follows, we assume without loss of generality that $|k| > R_2$. We will also prove the result only for the portion of the non-physical branch in $\tilde{S}$ that is reached by a path from the upper half space.

Arguing as in the proof of Theorem 2, for L any non-negative integer, we have that there exists $k_1 \in B(L^2 + L, 1)$ such that $|h(k_1)| > \exp(-|k_1|^{5/2} \ln |k_1|)$. Applying the Cartan theorem with $R = 0.9L$, $r = 0.8L$, and $\eta = (2L)^{-2t-1}/4$, we get

$$|h(k)| > \exp\bigl(-(2t+1)C|k|^{5/2+\epsilon}\bigr), \tag{38}$$

for k in $B(k_1, 0.8L)$ but outside an exceptional set of disks of radius no larger than $(2L)^{-2t}$. We decompose the system of disks into the union $\bigcup U_j$, where U_j are connected and mutually disjoint. We can assume that each U_j contains a resonance, which we label k_j. For if not, then Eq. (31) holds on U_j by the Maximum Principle. Using the inequality $(2L)^{-2t} < |k_j|^{-t}$, it then follows that for each j, $U_j \subset B(k_j, |k_j|^{-t})$.

We now obtain lower bounds on h in a neighbourhood of the thresholds. Choose k_1 with $\Re k_1 = L^2$ and $\Im k_1 \in (1, (L+1)/100)$, such that

$$|h(k_1)| > \exp\bigl(-|k_1|^{5/2} \ln |k_1|\bigr).$$

Suppose first that $h(k)$ has a pole of order j at L^2, with $j = 1, 2$ or 3. Applying the Cartan theorem to the function $z \to z^j h(L^2 - z^2)$, in the disk $\{|z - \sqrt{L^2 - k_1}| < R\}$ with $R = 1.2\sqrt{L}$, $r = \sqrt{L}$, and $\eta = 1/(4R(2L)^{2t})$ we obtain

$$|h(k)| > \frac{e^{-Ct|k|^{5/2+\epsilon}}}{|L^2 - k|^{j/2}}, \tag{39}$$

for k in $B(L^2, 0.9L)$ but outside a union of disks with summed radii no greater than $(2L)^{-2t}$. On the other hand, if $z \to h(L^2 - z^2)$ is regular at $z = 0$, then we can apply the Cartan theorem directly to obtain Eq. (39) holding in $B(L^2, 0.9L)$. In either case, the inequality appearing in Eq. (31) now holds in $\tilde{S}_1$ in the complement of the system of disks. Arguing as above, we can assume that the union of disks is of the form $\bigcup B(k_j, |k_j|^{-t})$. Part (A) of the lemma has been proven.

We now prove Eq. (32). Thus suppose there exists no resonance in the disk $B(L^2, \delta)$. Using Eq. (3) and Proposition 2, and using the Cartan theorem as above, we have

$$\begin{aligned}
&\bigl\|\rho(\Delta - L^2 - z^2)^{-1}\rho\bigr\| \\
&\quad < \bigl\|\rho R_a(k)\rho\bigr\| \, \bigl|\det\bigl(I + |K\rho|^3\bigr)\bigr|^3 \bigl|\det\bigl(I + (K\rho)^3\bigr)\bigr|^{-1} \\
&\quad \leqslant \frac{C}{\sqrt{\delta}} \frac{e^{C|k|^{3/2}}}{\delta^3} e^{C|k|^{5/2+\epsilon} \ln(1/\delta)} \\
&\quad \leqslant \frac{Ce^{C|k|^{5/2+\epsilon} \ln(1/\delta)}}{\delta^{7/2}}, \qquad |z| \in \bigl(\sqrt{\delta/3}, \sqrt{2\delta/3}\bigr).
\end{aligned}$$

Also, by Eq. (13), the function $z \to z\rho(\Delta - L^2 - z^2)^{-1}\rho$ is analytic in $\{z\colon |z| < \sqrt{2\delta/3}\}$. By the previous inequality we have

$$\left\| z\rho(\Delta - L^2 - z^2)^{-1}\rho \right\| < \frac{Ce^{C|k|^{5/2+\epsilon}\ln(1/\delta)}}{\delta^3}, \quad |z| = \sqrt{\delta/2}.$$

Hence Eq. (32) holds by the Maximum Principle. The proof for Eq. (33) is similar. □

Lemma 8. *Let p be an integer with $p > 2$ and $q \in [0, p/2)$. Then there exists a positive constants $M_{p,q}, C_p$ such that if $m > M_{p,q}$, and if $f(k)$ is an analytic function in a region Γ in* $\mathbf{C}$, *with*

$$\Gamma \equiv \left\{ \Re(k) \in [m - 2m^{-q}, m + 2m^{-q}],\ -\frac{1}{(\Re k)^p} \leqslant \Im k \leqslant \frac{1}{(\Re k)^{2p}} \right\},$$

and if f satisfies the estimates

(A) $|f(k)| \leqslant e^{|k|^p}$,
(B) $|f(k)| \leqslant 1/\Im(k)$ *for* $\Im k > 0$,

then for $k \in [m - m^{-q}, m + m^{-q}]$ we have

$$|f(k)| \leqslant C_p |k|^{2p}. \tag{40}$$

Proof. In what follows, C_p will denote various constants independent of m, k, while C will various constants independent of m, k, p. We use an argument based on the Maximum Principle. Below, we will construct a family of functions $F_\alpha(k)$ parametrised by α such that

(1) F_α is analytic on Γ,
(2) $|F_\alpha| < e$ on Γ,
(3) on the interval $[m - m^{-q}, m + m^{-q}]$, $|F_\alpha| > 1/2$,
(4) on $\{z \in \Gamma\colon |z - m| \geqslant (3/2)m^{-q}\}$, we have $|F_\alpha| \leqslant C|k|^p \exp(-C|k|^{2p-2q})$.

Assuming such F_α exist, consider the function on Γ:

$$h(k) \equiv f(k) F_\alpha(k) \exp\bigl(-ik^{2p+1}\bigr).$$

On the curve $\Im k = -1/(\Re k)^p$, there exists a positive constant M_1 such that $\Re k > M_1$ implies $\Im(k^{2p+1}) < -(\Re k)^p$ and $|k|^p - (\Re k)^p < C$. Thus we have

$$|h(k)| \leqslant \exp\bigl(|k|^p\bigr) e \exp\bigl(-(\Re k)^p\bigr) \leqslant C.$$

On the curve $\Im k = 1/(\Re k)^{2p}$, we have for $\Re k > M_1$

$$|h(k)| \leqslant (\Re k)^{2p} e |\exp(C_p)| \leqslant C_p (\Re k)^{2p}.$$

On the curve $\Re k = m + 2m^{-q}$, we have

$$\begin{aligned}|h(k)| &\leqslant \exp\bigl((m+2m^{-q})^p\bigr)C|k|^p \exp\bigl(-C|k|^{2p-2q}\bigr)|\exp(C_p)| \\ &\leqslant C_p|k|^p \exp\bigl(|k|^{p-2q}\bigr).\end{aligned}$$

Noting that $p > 2q$, it follows that there exists $M_2 > 0$ such that if $m > M_2$, then $|h(k)| < C_p$. Similarly, on the curve $\Re k = m - 2m^{-q}$ we have (assuming $m > M_2$)

$$|h(k)| \leqslant C_p.$$

Choose M_3 such that $\Re k > M_3 - 2$ implies $C_p(\Re k)^{2p} > C$ with C the maximum of the various C's above. Setting $M_{p,q} = \max(M_1, M_2, M_3)$, we have that for $m > M_{p,q}$ it follows by the Maximum Principle that $|h(k)| \leqslant C_p(m+2m^{-q})^{2p}$ on Γ. Since $|F_\alpha \exp(-ik^{2p+1})| > 1/2$ on the interval $[m - m^{-q}, m + m^{-q}]$, Eq. (40) follows.

It remains to prove the existence of F_α. Let $\psi \in C_0^\infty(\mathbf{R})$ by defined so that $\psi = 1$ on $[m - 1.1m^{-q}, m + 1.1m^{-q}]$, and $\psi = 0$ on $(-\infty, m - 1.2m^{-q}] \cup [m + 1.2m^{-q}, \infty)$. Define

$$F_\alpha(z) = (\pi\alpha^{-2})^{-1/2} \int_{\mathbf{R}} \exp\left(\frac{-(x-z)^2}{\alpha^2}\right)\psi(x)\,dx.$$

The analyticity of F_α follows immediately. To prove property (2), note first that

$$(\pi\alpha^{-2})^{-1/2} \int_{\mathbf{R}} \exp\left(\frac{-x^2}{\alpha^2}\right)dx = 1. \tag{41}$$

Thus, setting $z = u + iv$, with $u, v \in \mathbf{R}$, it is easy to see that

$$|F_\alpha(z)| \leqslant \exp|v^2/\alpha^2|.$$

Setting $\alpha = (m + 2m^{-q})^{-p}$, property (2) follows.

To prove property (3), suppose $z \in [m - m^{-q}, m + m^{-q}]$. Thus

$$\begin{aligned}|F_\alpha(z) - 1| &= \pi^{-1/2} \int_{\mathbf{R}} e^{-y^2}|\psi(\alpha y + z) - 1|\,dy \\ &\leqslant \pi^{-1/2} \int_{|y|>0.1m^{-q}/\alpha} e^{-y^2}\,dy \leqslant 1/2\end{aligned}$$

since m^{-q}/α is large.

For property (4), assume $z = u + iv \in \Gamma \cap \{\zeta\colon |\zeta - m| > 3/2m^{-q}\}$. Then

$$|F_\alpha(z)| \leqslant (\pi\alpha^{-2})^{-1/2}\bigl|\exp(v^2/\alpha^2)\bigr| \int_{\mathbf{R}} \exp\left(\frac{-(x-u)^2}{\alpha^2}\right)\psi(x)\,dx$$

$$\leqslant (\pi\alpha^{-2})^{-1/2} e \int\limits_{[m-1.2m^{-q}, m+1.2m^{-q}]} \exp\left(\frac{-(x-u)^2}{\alpha^2}\right) dx$$
$$\leqslant C\alpha^{-1} \exp\left(-(0.3m^{-q})^2\alpha^{-2}\right) \leqslant C|k|^p \exp\left(-C|k|^{2p-2q}\right).$$

The proof of Lemma 5 is complete. □

Proof of Theorem 1. In view of Lemmas 7 and 8, we set $p = 3$. We can (increasing $M_{p,q}$ if necessary) suppose $\|\rho R(k)\rho\| \leqslant e^{|k|^3}$ away from $\bigcup(k_j, |k_j|^{-t})$. Let $q \in [0, p/2)$, and let $M_{p,q}$ be as in Lemma 7. Assume the hypotheses of the theorem; hence the operator valued function $\rho R(k)\rho$ is analytic on

$$\left\{k \in S\colon \operatorname{dist}\left(k, [m - 2m^{-q}, m + 2m^{-q}]\right) < 2(m-2)^{-3}\right\}.$$

Setting $t = 2p$ in Lemma 7, we obtain

$$\|\rho R(k)\rho\| \leqslant e^{|k|^3}$$

in the region

$$\mathcal{G} \equiv \left\{k \in S\colon \operatorname{dist}\left(k, [m - 2m^{-q}, m + 2m^{-q}]\right) < (m-2)^{-3}\right\}.$$

Let $\tilde{\Gamma} \subset \mathcal{G}$ be an open subset such that the projection Π, restricted to $\tilde{\Gamma}$, is an isometry onto Γ, with Γ as in Lemma 8. Thus $\tilde{\Gamma}$ lies on one of the branches of $\mathcal{G}$, and its intersection with the physical plane will be non-empty and consist of one of the two sets

$$k\colon\ \Gamma_- \equiv \left\{\Re(k) \in [m - 2m^{-q}, m + 2m^{-q}],\ -\frac{1}{(\Re k)^3} \leqslant \Im k \leqslant 0\right\},$$

or

$$k\colon\ \Gamma_+ \equiv \left\{\Re(k) \in [m - 2m^{-q}, m + 2m^{-q}],\ 0 \leqslant \Im k \leqslant \frac{1}{(\Re k)^6}\right\}.$$

Assume for the moment that the intersection is Γ_+. Then we have shown that estimate A of the previous lemma holds for $\rho R(k)\rho$, and estimate B holds by the Spectral Theorem. The conclusion of Theorem 1 follows. The case of Γ_- is proven in the same way, using the obvious adaptation of Lemma 8. The theorem now follows from Lemma 8. □

5. Example of quasimode construction

We present an example due to Buldyrev [3], in which a quasimode is constructed for the Dirichlet Laplacian. Figure 2 is the union of two circular arcs of radii r_1, r_2. Under an assumption (see p. 20 in [3]) that will be satisfied for

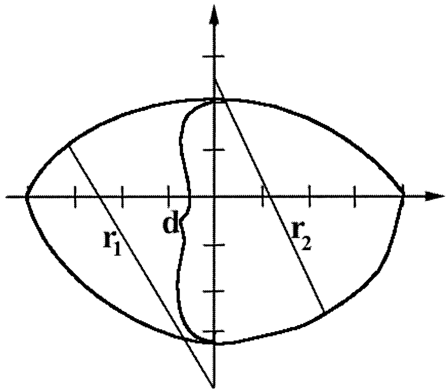

Fig. 2. Circular arcs.

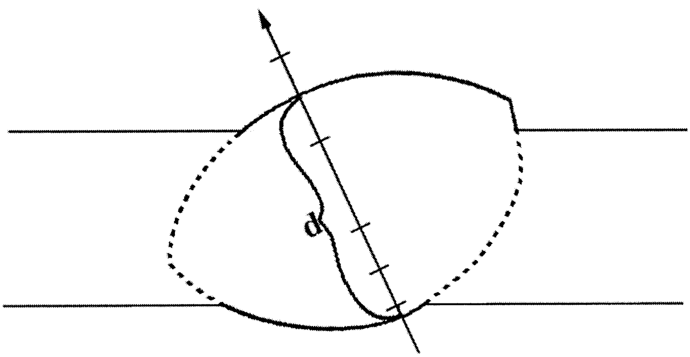

Fig. 3. Waveguide with resonances.

generic r_1, r_2, d, Buldyrev then constructs a quasimode concentrated on the periodic billiard trajectory of period $2d$ which lies along the y-axis. The associated frequencies are $w_{p,q}^2$, with

$$w_{p,q} = \frac{1}{2d}\left(\pi p + \left(q + \frac{1}{2}\right)\arccos\sqrt{\left(1 - \frac{2d}{r_1}\right)\left(1 - \frac{2d}{r_2}\right)} + O\left(\frac{1}{p}\right)\right);$$

here p, q are arbitrary positive integers. It is easy see that for any r_1, r_2, d and any fixed q, the sequence $w_{p,q}$ will satisfy the asymptotics required in the hypothesis of Corollary 1.

It should be remarked that under weaker—and easier to verify—hypotheses on r_1, r_2, d, Buldyrev's construction yields a sequence of functions u_j such that $\|(\Delta - \lambda_j)u_j\|_{L^2(\Omega)} = O(\lambda_j^m)$ with $m < \infty$, and this would enable one to prove a weaker version of Corollary 1.

Fig. 3 shows one of the ways in which the circular arcs in Fig. 2 can be placed in a portion of a waveguide (actually only the portion near the periodic billiard trajectory is necessary for the quasimode construction).

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