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An Observation on the Positive Real Lemma

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We give a necessary and a sufficient condition that the transfer function of an exponentially stable linear finite-dimensional system be a real positive matrix. The condition does not assume controllability–observability properties. © 2001 Academic Press

1. INTRODUCTION AND REFERENCES

The positive real lemma is an important tool in systems and circuit theory, see, for example, [1]. The lemma can be described as follows. Let $T(z)$ be a rational transfer function, which is the transfer function of the exponentially stable linear time invariant system $[A, B, C', D]$, with **real** matrices ($'$ is used to denote transposition),

$$\dot{x} = Ax + Bu, \qquad y = C'x + Du.$$

We assume that $x \in \mathbb{R}^n$, u, $y \in \mathbb{R}^m$ so that the transfer function $T(z)$ is a square $m \times m$ matrix,

$$T(z) = D + C'(zI - A)^{-1}B.$$

The number z is complex. Hence, for the computation of $T(z)$, we complexify the linear spaces of dimensions n and m.

We assumed that the system is exponentially stable so that the eigenvalues of A belong to $\Re e\, z < 0$.

For future reference, we collect the crucial assumptions:

Assumption (H). $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$; the matrices A, B, C, D have consistent dimensions, and are real; every eigenvalue of the matrix A has a negative real part.

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By definition, the transfer function $T(z)$ is *positive real* when

$$y^*[T^*(z) + T(z)]y \geq 0, \qquad \forall y \in \mathbb{C}^m,\ \forall z \in \mathbb{C}, \Re e\, z > 0,$$

(* denotes transposition followed by conjugation).

It is possible to prove that a positive real rational matrix does not have unstable poles; but it may have poles on the imaginary axis. In this article, we assume that the matrix A is stable; hence the poles of $T(z)$ have negative real parts.

We put

$$\begin{aligned}\Pi(i\omega) &= T(i\omega) + T^*(i\omega)\\ &= C'(i\omega I - A)^{-1}B - B'(i\omega I + A)^{-1}C + (D + D'),\end{aligned}$$

and we note that this matrix has the following rational extension to the complex plane:

$$\Pi(z) = C'(zI - A)^{-1}B - B'(zI + A)^{-1}C + (D + D'). \tag{1}$$

We have:

THEOREM 1. *Let Assumption* (H) *hold. The matrix $T(z)$ is positive real if and only if $y^*[T^*(i\omega) + T(i\omega)]y = y^*\Pi(i\omega)y \geq 0$ for every $\omega \in \mathbb{R}$ and $y \in \mathbb{C}^m$.*

See [1, p. 53] for the proof.

The positive real property, under suitable controllability–observability assumptions, is equivalent to the solvability of a particular set of equations:

THEOREM 2 (Positive Real Lemma). *Let $[A, B, C', D]$ be a* minimal *realization of $T(z)$, which satisfies Assumption* (H). *Then, the matrix $T(z)$ is positive real if and only if there exist matrices Q, W, and symmetric* positive definite *P which satisfy the following system*:

$$A'P + PA = -QQ', \qquad PB = C - QW, \qquad W'W = D + D'. \tag{2}$$

The problem described by Eq. (2) is important also in the stability analysis of systems affected by feedback nonlinearities, a problem known as the Lur'e problem.

Equation (2) is a special instance of the linear operator inequality.

Several proofs of this theorem, or related results, appeared in the literature. In particular, the sufficiency part is obtained by elementary algebraic manipulations, while the necessity is more involved. See [1] for several different proofs.

In spite of the fact that results related to Theorem 2 are quite old (see [8, 6, 4]), new proofs still appear, see, for example, [7] for a nice proof based on convex analysis (in this context, convex analysis was used previously, see [5]). However, at what extent the minimality assumption is really crucial in the previous theorem is not yet completely clarified. For example, in [7] controllability of the pair (A, B) is assumed, but not observability. Even more, minimality is not used in [1] to prove that solvability of (2) implies $\Pi(i\omega) \geq 0$ and an examination of the derivation in [1] shows that $\Pi(i\omega) \geq 0$ even if (2) is solvable with P only positive semidefinite. Instead, we see that minimality is crucial if we want the matrix P to be positive definite (instead then semidefinite).

The scope of this article is to examine the solvability of Eq. (2) when controllability and observability properties are not assumed.

Now, we introduce the results and the notations that we use in our key result, Theorem 4.

We make use of the following factorization theorem: if $\Pi(z)$ is a rational matrix which is bounded on the imaginary axis, and such that $\Pi(i\omega) \geq 0$, then it is possible to find a matrix $N(z)$ which is bounded in $\Re e\, z > 0$ and such that

$$\Pi(i\omega) = N'(-i\omega)N(i\omega)\,.$$

See [9] where the existence of a *spectral* factorization is proved; i.e., a factorization such that $N(i\omega)$ has a rational right inverse which is regular in $\Re e\, z > 0$, of course when $\Pi(i\omega)$ is not identically zero. The matrix $N(z)$ of a spectral factorization (known as a spectral factor of $\Pi(z)$) has as many rows as the normal rank of $\Pi(z)$.

We use the factorization of $\Pi(i\omega)$ as follows. We consider an eigenvalue z_0 of the matrix A and a Jordan chain of z_0, i.e., a finite sequence of vectors such that

$$Av_0 = z_0 v_0, \qquad Av_i = z_0 v_i + v_{i-1}, \quad 0 < i \leq r-1.$$

(The number r is the *length* of the Jordan chain).

The matrix e^{At} has the following property:

$$e^{At}v_0 = e^{z_0 t}v_0, \qquad e^{At}v_k = e^{z_0 t}\sum_{i=0}^{k}\frac{t^i}{i!}v_{k-i}.$$

In general, the eigenvalue z_0 has a finite number of Jordan chains. We enumerate them in any order and we denote $J_{z_0,\nu}$ the νth Jordan chain of z_0 in the chosen order.

We use the factor $N(z)$ and the Jordan chain $J_{z_0,\nu} = \{v_0, v_1, \dots, v_{r-1}\}$ of z_0 to construct the block matrix,

$$N_{z_0,\nu} = \begin{bmatrix} N_0 & 0 & 0 & \cdots & 0 \\ N_1 & N_0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \\ N_{r-1} & N_{r-2} & N_{r-3} & \cdots & N_0 \end{bmatrix}, \tag{3}$$

where r is the length of the Jordan chain that we are considering and

$$N_s = \frac{1}{s!}\frac{d^s}{\mathrm{d}z^s}N'(-z_0) = \left[\frac{1}{s!}\frac{d^s}{dz^s}N'(-z)\right]_{|_{z_0}}.$$

Remark 3. Ambiguity with respect to the signs in front of the derivative forces us to repeat explicitly: $s!N_s$ is the sth derivative of the function $\Psi(z) = N'(-z)$, computed at $z = z_0$.

With these notations we have:

THEOREM 4. *Let the matrices $N_{z_0,\nu}$ be constructed from any spectral factor of $\Phi(z)$ and let assumption* (H) *hold. If the transfer function $T(z)$ is positive, then there exist matrices Q, W and $P = P^* \geq 0$ which solve system* (2) *if and only if the following conditions hold for every Jordan chain $J_{z_0,\nu}$ of the matrix A*:

$$\mathrm{col}\left[C'v_0, C'v_1, \ldots, C'v_{r-1}\right] \in \mathrm{im}\, N_{z_0,\nu}. \tag{4}$$

Minimality is not assumed; but in this case in general P will not be **positive** defined: it will only be positive **semi**definite, see Example 7.

It is clear that the previous theorem is quite implicit and the necessity condition is most interesting, since it gives a new property that the spectral factors of $\Pi(z)$ must enjoy, when system (2) is solvable. But also sufficiency is interesting since we can derive an easier test for solvability from it; i.e., we can derive the Churilov condition (see [3]): Eq. (2) can be solved if there exists ω_0 such that $\det \Pi(i\omega_0) \neq 0$. In fact, in this case $\det N(z)$ has at most isolated zeros with positive real parts. The fact that $N(z)$ has a regular right inverse shows that it has no zero. The matrix (3) has a surjective matrix on the diagonal: it is itself surjective so that condition (4) holds.

A second important case is expressed by the following corollary:

COROLLARY 5. *If every Jordan chain of the matrix A has length* 1 *and the transfer function $T(z)$ is positive real*, *system* (2) *is solvable if and only if*

$$C'v_k \in \mathrm{im}\, N'(-z_k)$$

for every eigenvector v_k (*whose eigenvalue is z_k*) *of the matrix A.*

For most clarity, we prove the theorem in the next section. The idea of the proof is borrowed from the article [2].

We present now some preliminary observations.

First of all we note the following formula, which will be repeatedly used: if $L(t)$ and $R(t)$ are integrable matrix valued functions which are supported on $[0, +\infty)$, then the Fourier transformation of the function,

$$\int_0^{+\infty} L(s)R(t+s)\,ds, \qquad t>0, \qquad \int_0^{+\infty} L(s-t)R(s)\,ds, \qquad t<0 \tag{5}$$

is the function $\hat{L}(-i\omega)\hat{R}(i\omega)$.

We note now that if $T(z)$ is positive real then $D+D' \geq 0$ so that there exist matrices W which solve the equation $W'W = D+D'$. The idea of the proof is as follows: We fix one of these matrices W and we try to solve the remaining equations, i.e.,

$$A'P + PA = -QQ', \qquad PB = C - QW. \tag{6}$$

The idea, from [2], is to consider the matrix Q as a parameter. When Q varies, we get a family of matrices P given by

$$P = \int_0^{+\infty} e^{A's}QQ'e^{As}\,ds. \tag{7}$$

These matrices satisfy the first equation in (6). The problem is so reduced to find a certain parameter Q such that $PB = C - QW$, i.e., such that

$$B'P = \int_0^{+\infty} \{B'e^{A's}Q\}Q'e^{As}\,\mathrm{d}s = C' - W'Q'. \tag{8}$$

We put braces in the formula (8), around a term that we are going to examine more closely now. We introduce

$$M'(t) = \begin{cases} B'e^{A't}Q & \text{if } t>0, \\ 0 & \text{if } t<0. \end{cases}$$

With this notation, we write

$$C' = B'P + W'Q' = \int_0^{+\infty} M'(s)Q'e^{As}\,ds + W'Q',$$

and we define the function

$$C'e^{At}B = \int_0^{+\infty} M'(s)Q'e^{A(t+s)}B\,ds + W'Q'e^{At}B, \qquad \text{if } t>0,$$

$$B'e^{-A't}C = \int_0^{+\infty} B'e^{A'(s-t)}QM(s)\,ds + B'e^{-A't}QW, \qquad \text{if } t<0. \tag{9}$$

Hence, we get the equality

$$\begin{matrix} C'e^{At}B & \text{if } t>0 \\ B'e^{-A't}C & \text{if } t<0 \end{matrix} = \begin{cases} \int_0^{+\infty} M'(s)M(t+s)\,ds + W'M(t) & \text{if } t>0, \\ \int_0^{+\infty} M'(s-t)M(s)\,ds + M'(-t)W & \text{if } t<0. \end{cases}$$

The Fourier transformation of the left-hand side is

$$C'(i\omega I - A)^{-1}B - B'(i\omega I + A')^{-1}C\,.$$

We compute the Fourier transformation of the right-hand side and we get the equality,

$$C'(i\omega I - A)^{-1}B - B'(i\omega I + A')^{-1}C = \hat{M}'(-i\omega)\hat{M}(i\omega) + W'\hat{M}(i\omega) + \hat{M}'(-i\omega)W,$$

so that, summing $W'W$ to both sides we obtain the factorization

$$\Pi(i\omega) = \left[\hat{M}'(-i\omega) + W'\right]\left[\hat{M}(i\omega) + W\right] = \left[\hat{M}(-i\omega) + W\right]'\left[\hat{M}(i\omega) + W\right]. \tag{10}$$

We put

$$N(i\omega) = \hat{M}(i\omega) + W\,. \tag{11}$$

Consequently, we have the following factorization result, which is in fact well known:

THEOREM 6. *If there exist matrices Q and $P = P^* \geq 0$ which solve* (6), *then the function $N(z) = [\hat{M}(z) + W]$ is holomorphic and bounded in a half plane $\Re e\, z > -\sigma$, with $\sigma > 0$ and satisfies*

$$\Pi(i\omega) = N'(-i\omega)N(i\omega).$$

Of course,

$$\operatorname{im}\Pi(i\omega) = \operatorname{im}[\hat{M}'(-i\omega) + W'] = \operatorname{im}N'(i\omega).$$

2. THE PROOF OF THE MAIN THEOREM

Before that we proceed to the proof, it is interesting to discuss the following example and to prove a lemma that it suggests.

EXAMPLE 7. Let $C = 0$, $B \neq 0$, $D = 0$. In this case, $\Pi(i\omega) = 0$ and problem (6) **is solvable:** a solution is $Q = 0$, $P = 0$.

Instead, let $B = 0$ and $C \neq 0$. Also, in this case $\Pi(i\omega)$ is zero; but, now Eq. (6) is not solvable: $0 = PB = C' \neq 0$ has no solution.

Remark 8. The case $C = 0$, $B \neq 0$, and $n \geq 1$ corresponds to a nonminimal system. The problem (6) is solvable, but P is not coercive.

LEMMA 9. *If the function $\Pi(i\omega)$ is zero, then system* (6) *is solvable if and only if $C = 0$.*

Proof. If $C = 0$ then $Q = 0$, $P = 0$ solves the problem. Conversely, let $\Pi(i\omega) = 0$. In this case, we have also $W = 0$. Let P, Q solve problem (6). We have

$$C' = \int_0^{+\infty} \{B'e^{A't}Q\}Q'e^{At}\,dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{M}'(-i\omega)Q'(i\omega I - A)^{-1}\,d\omega, \qquad (12)$$

(the second equality from Parseval identity). We noted that $M'(i\omega) = N'(i\omega)$ (since $W = 0$) is a factor of $\Pi(i\omega)$: it is zero if $\Pi(i\omega)$ is zero so that $C = 0$ too. ∎

The previous lemma proves Theorem 4 in the case that $\Pi(i\omega)$ is zero. Hence, we can confine our analysis to the case of **nonzero** $\Pi(i\omega)$. We have the following proof of the "only if" part:

Proof of Theorem 4, *necessity.* We saw that if Eqs. (6) are solvable then there exists Q such that

$$C' = \int_0^{+\infty} \{B'e^{A't}Q\}Q'e^{At}\,dt + W'Q'. \qquad (13)$$

Let us consider a Jordan chain of A. Let $v_0, v_1, \dots, v_{r-1}$ be its elements and let z_0 be the eigenvalue. We multiply both sides of (13) by v_0. We get

$$C'v_0 = \int_0^{+\infty} \{B'e^{A't}Q\}Q'e^{z_0 t}v_0\,dt + W'Q'v_0$$
$$= \int_0^{+\infty} M'(t)Q'e^{z_0 t}v_0\,dt + W'Q'v_0$$
$$= \hat{M}'(-z_0)Q'v_0 + W'Q'v_0 = N'(-z_0)Qv_0.$$

Now, we repeat the previous computation for $v_1, v_2, \dots$. We get the equalities,

$$C'v_1 = \int_0^{+\infty} M'(t)Q'te^{z_0 t}v_0\,dt + \int_0^{+\infty} M'(t)Q'e^{z_0 t}v_1\,dt + W'Q'v_1$$
$$= \frac{d}{dz}\hat{M}'(-z_0)Q'v_0 + \hat{N}(-z_0)Q'v_1 = \frac{d}{dz}N'(-z_0)Q'v_0 + N(-z_0)Q'v_1.$$

The convention that we explicitly stated in Remark 3 comes from the previous and the next computation.

We proceed as above untill the end of the Jordan chain and we get

$$C'v_i = \sum_{s=0}^{i}\left[\frac{1}{s!}\frac{d^s}{dz^s}N'(-z_0)\right]Qv_{i-s} = \sum_{s=0}^{i} N_s Q v_{i-s}.$$

We collect the previous equalities and we get condition (4) for the matrix $N(z)$.

We sum up: we proved that the condition of the theorem holds for the matrix $N(z)$; but the proof is not yet finished since we are not asserting that the factor $N(z)$ that we introduced is a spectral factor, while the theorem states that condition (4) should hold for a spectral factor. We complete the proof by proving that if condition (4) holds for a certain factor, it also holds for a spectral factor. We relay on [9, Corollary 1]: if $N(z)$ is any factor of $\Pi(z)$ then there exists a spectral factor $\Phi(z)$ and a rational matrix $V(z)$ such that $N(z) = V(z)\Phi(z)$. The matrix $V(z)$ has a certain structure that is not relevant to the following argument; but it has no poles on the right half plane since $\Phi(z)$ is regular and of full rank in the right half plane, and the matrix $N(z)$ which we introduced above is regular in the right half plane.

It is easy to compute the block in position (k, r) of the matrix $N_{z_0,\nu}$: with $(1/s!) = 0$ when $s < 0$ and D to denote the derivative, this block is equal to

$$\frac{1}{s!}D^s N'(-z_0) = \frac{1}{s!}\sum_{i=0}^{s}\binom{s}{i}D^{(s-i)}\Phi'(-z_0)D^{(i)}V'(-z_0)$$
$$= \sum_{i=0}^{s}\left[\frac{1}{(s-i)!}D^{s-i}\Phi'(-z_0)\right]\left[\frac{1}{i!}D^i V'(-z_0)\right].$$

Let $\Phi_{z_0,\nu}$ be the matrix in (3), constructed from the spectral factor $\Phi(z)$. We see from the previous computation that

$$N_{z_0,\nu} = \Phi_{z_0,\nu}K(z_0)$$

for a certain matrix $K(z_0)$ which depends on $V(z)$. Hence, if condition (4) holds for $N(z)$ it holds for $\Phi(z)$ too. **This ends the proof of the only if part of Theorem 4**. ∎

We prove now the "if" part of Theorem 4.

Proof of Theorem 4, *Sufficiency.* The if part was proved already in the special case $\Pi(i\omega) = 0$, see Example 7 and Theorem 9. Hence, we consider the case $\Pi(i\omega) \neq 0$.

We noted already that the function $\Pi(i\omega)$ has a rational extension to the complex plane, see (1). Moreover, we assume positivity; i.e., we assume $\Pi(i\omega) \geq 0$ for each ω. Hence, there exists a spectral factor of $\Pi(i\omega)$, which we call $N(z)$: it is possible to represent

$$\Pi(z) = N'(-z)N(z),$$

where, we recall, $N(z)$ is a matrix which is holomorphic on $\Re e\, z \geq 0$ and which has a holomorphic right inverse $J(z)$ in $\Re e\, z > 0$.

We observe that the rational function $N(z)$ is bounded, so that $N(\infty)$ exists. We choose $W = N(\infty)$ so that $W'W = D' + D$.

The function $E'(i\omega) = N'(i\omega) - W'$ tends to zero for $|\omega| \to +\infty$; It is a rational function, so that it is square integrable on the imaginary axis. Moreover, it is holomorphic in $\Re e\, z > 0$ so that it has an inverse Fourier transform $\check{E}'(t)$ which is zero for negative times. The function $\check{E}'(t)$ is, for $t > 0$, a combination of polynomials and decaying exponential functions.

To prove the theorem, we construct a matrix Q such that

$$C' - W'Q' = \int_0^{+\infty} \check{E}'(t)Q'e^{At}\, dt \tag{14}$$

To construct the matrix Q, we distinguish two cases:

Case (a). The eigenvectors v_k of A are n, hence a complete system in $\mathbb{R}^n$. In this case, we can construct

$$(C' - W'Q')v_k = \int_0^{+\infty} \check{E}'(t)Q'v_k e^{z_k t}\, dt = E'(-z_k)Q'v_k.$$

Consequently, we are looking for a solution q_k of

$$C'v_k = N'(\xi_k)q_k \qquad \text{where } \xi_k = -z_k, \Re e\, \xi_k > 0. \tag{15}$$

This equation can be solved for the vector $q_k = Q'v_k$ thanks to the condition (4). In fact, if the Jordan chain has length 1 then the corresponding matrix (3) has only one block.

This completes the construction of the matrix Q in this case.

Case (b). The case that the matrix A admits Jordan chains of length larger then one.

We fix our attention to a Jordan chain $v_0, v_1, \ldots, v_{r-1}$ of A. Let z_0 be the eigenvalue.

Multiplication of both sides of the required equality (14) by the vectors v_i gives the system of equations,

$$C'v_i = \sum_{s=0}^{i} N_s Q v_{i-s}$$

see the corresponding computation in the proof of the only if part.

The assumed condition (4) shows that these equations can be solved.

We proceed analogously for every Jordan chain and we get a solution Q of (14).

This ends the construction of the matrix Q which satisfies

$$C' - W'Q = \int_0^{+\infty} \check{E}'(t)Q'e^{At}\, dt.$$

We construct the matrix P as in (7) and, finally, we show that this matrix satisfies $B'P = C' - W'Q'$. It is sufficient for this that we note the following lemma which is known at least in the case $W = 0$. We present the proof for completeness.

LEMMA 10. *We have*: $\check{E}'(t) = B'e^{A't}Q$.

Proof. We know

$$\Pi(i\omega) = N'(-i\omega)N(i\omega) = [E'(-i\omega) + W'][E(i\omega) + W].$$

Hence.

$$C'(i\omega I - A)^{-1}B - B'(i\omega I + A')^{-1}C$$
$$= E'(-i\omega)E(i\omega) + E'(-i\omega)W + W'E(i\omega).$$

It follows from (5) that, for $t > 0$,

$$C'e^{At}B = \int_0^{+\infty} \check{E}'(s)\check{E}(t+s)\,ds + W'\check{E}(t), \tag{16}$$

and, from (14),

$$C'e^{At}B = \int_0^{+\infty} \check{E}'(s)Q'e^{A(t+s)}B\,ds + W'Q'e^{At}B. \tag{17}$$

We put $F(t) = Q'e^{At}B - \check{E}'(t)$ for $t \geq 0$, $F(t) = 0$ for $t < 0$. We subtract (16) from (17). We get

$$W'F(t) = -\int_0^{+\infty} \check{E}'(s)F(t+s)\,ds = -\int_t^{+\infty} \check{E}'(r-t)F(r)\,dr$$
$$= -\int_0^{+\infty} \check{E}'(r-t)F(r)\,dr = -\int_{-\infty}^{+\infty} \check{E}'(r-t)F(r)\,dr.$$

The last row follows since we know that $E(s) = 0$, $F(s) = 0$ if $s < 0$.

The previous equality holds for $t \geq 0$ so that the function,

$$U(t) = W'F(t) + \int_{-\infty}^{+\infty} \check{E}'(r-t)F(r)\,dr$$

is square integrable on $\mathbb{R}$ and it is zero for $t \geq 0$. It follows that there exists an extension $\hat{U}(z)$ of the Fourier transformation of $U(t)$, which is holomorphic in $\Re e\, z < 0$.

A simple computation shows that

$$\hat{U}(z) = N'(-z)\hat{F}(z),$$

and we note that $\hat{F}(z)$, if not zero, must have its poles in $\Re e\, z < 0$. Hence, if $F(t)$ were not zero, the function $N'(-z)$ should cancel the poles of $F(z)$ in $\Re e\, z < 0$. This is not possible, since the matrix $N'(-z)$ has a holomorphic left inverse. Consequently, $F(t) = 0$, as wanted. ∎

This lemma completes the proof of Theorem 4.

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