Computers and Mathematics with Applications 59 (2010) 1253–1265

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On initial and boundary value problems for fractional order mixed type functional differential inclusions

Mohamed Abdalla Darwish [a], S.K. Ntouyas [b,*]

[a] Department of Mathematics, Faculty of Science, Alexandria University at Damanhour, 22511 Damanhour, Egypt

[b] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

ARTICLE INFO

Keywords:
Functional differential inclusions
Initial value problems
Boundary value problems
Fractional derivatives
Retarded argument
Advanced argument
Existence

ABSTRACT

In this paper we prove some existence results for initial and boundary value problems for functional differential inclusions of fractional order with both retarded and advanced arguments. The Banach fixed point theorem, the nonlinear alternative of the Leray–Schauder type and the Covitz–Nadler fixed point theorem are the main tools in deriving our proofs.

1. Introduction

The first serious attempt to give a logical definition of a fractional derivative is due to Liouville. Now, the fractional calculus topic is attracting growing interest from scientists and engineers; see [1–5] and references therein. Differential equations of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics and other fields can be described with the help of fractional differential equations; see [1,5–9] and references therein. The theory of differential equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. There are numerous research papers and monographs devoted to fractional differential equations; for example see [2,4,10–12]. On the other hand, functional differential equations arise in a variety of areas of biological, physical, and engineering applications; see, for example, the books of Kolmanovskii and Myshkis [13] and Hale and Verduyn Lunel [14], and the references therein.

Recently the authors in [15] have established existence results for an initial value problem for fractional functional differential equations of mixed type. More precisely, the initial value problem

$$D^{\beta}x(t) = f(t, x^t), \quad t \in J,\ 0 < \beta < 1, \tag{1.1}$$

$$x(t) = \phi(t), \quad -r_1 \le t \le 0, \tag{1.2}$$

$$x(t) = \psi(t), \quad b \le t \le b + r_2 \tag{1.3}$$

was studied, where D^{β} is the standard Riemann–Liouville fractional derivative, $f : J \times C([-r_1, r_2], \mathbb{R}) \to \mathbb{R}$ is a given function, $\phi \in C([-r_1, 0], \mathbb{R})$ with $\phi(0) = 0$, and $\psi \in C([b, b + r_2], \mathbb{R})$ with $\psi(b) = 0$. For any function x defined on $[-r_1, b + r_2]$ and any $t \in J$, we denote by x^t the element of $C([-r_1, r_2], \mathbb{R})$ defined by $x^t(\theta) = x(t + \theta)$ for $-r_1 \le \theta \le r_2$, where $r_1,\ r_2 \ge 0$ are constants.

As remarked in [15]:

- If $r_1 = r_2 = 0$ then we have an ordinary differential equation of fractional order.
- If $r_1 > 0$ and $r_2 = 0$ then we have a retarded functional differential equation of fractional order.
- If $r_1 = 0$ and $r_2 > 0$ then we have an advanced differential equation of fractional order.
- If $r_1 > 0$ and $r_2 > 0$ then we have a mixed differential equation of fractional order.

* Corresponding author.
E-mail addresses: mdarwish@ictp.it, darwishma@yahoo.com (M.A. Darwish), sntouyas@cc.uoi.gr, sntouyas@uoi.gr (S.K. Ntouyas).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2009.05.006

Our purpose here is to extend the results of [15] to the case of initial value problems for a mixed type functional differential inclusion of fractional order. Furthermore, to encompass the full scope of our paper, we study the boundary value problems for a mixed type fractional functional differential inclusion.

The structure of this paper is as follows. In Section 2, we collect some definitions and results which will be needed throughout the paper. In Section 3, we consider the following initial value problem (IVP for short) for a mixed type functional differential inclusion of fractional order, namely

$$D^{\beta}x(t) \in f(t, x^t), t \in J, \quad 0 < \beta < 1, \tag{1.4}$$

$$x(t) = \phi(t), \quad -r_1 \le t \le 0, \tag{1.5}$$

$$x(t) = \psi(t), \quad b \le t \le b + r_2, \tag{1.6}$$

where D^{β} is the standard Riemann–Liouville fractional derivative, $F : J \times C([-r_1, r_2], \mathbb{R}) \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ is a multi-valued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and ϕ and ψ are as in problem (1.1)–(1.3).

There are few papers in which the authors consider the Dirichlet-type problem for linear and nonlinear ordinary differential equations of fractional order; see for example [16–19]. In Section 4, we prove the existence of solutions to boundary value problems (BVP for short) of a fractional functional differential equation of mixed type

$$-D^{\beta}x(t) \in F(t, x^t), \quad 0 \le t \le 1, \ 1 < \beta < 2, \tag{1.7}$$

$$x(t) = \phi(t), \quad -r_1 \le t \le 0, \tag{1.8}$$

$$x(t) = \psi(t), \quad 1 \le t \le 1 + r_2, \tag{1.9}$$

where D^{β} is the standard Riemann–Liouville fractional derivative. Here, $F : [0, 1] \times C([-r_1, r_2], \mathbb{R})\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$, $\phi \in C([-r_1, 0], \mathbb{R})$ with $\phi(0) = 0$ and $\psi \in C([1, 1 + r_2], \mathbb{R})$ with $\psi(1) = 0$. For any function x defined on $[-r_1, 1 + r_2]$ and any $t \in J$, we denote by x^t the element of $C([-r_1, r_2], \mathbb{R})$ defined by $x^t(\theta) = x(t + \theta)$ for $-r_1 \le \theta \le r_2$, where $r_1, \ r_2 \ge 0$ are constants.

The Banach fixed point theorem, the nonlinear alternative of the Leray–Schauder type and the Covitz and Nadler fixed point theorem are the main tools to obtain our results.

2. Auxiliary facts and results

This section is devoted to collecting some definitions and results which will be needed throughout this paper.

By $C := C([-r_1, r_2], \mathbb{R})$ we denote the Banach space of all continuous functions from $[-r_1, r_2]$ into E equipped with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r_1 \le \theta \le r_2\}$$

and $C(J, \mathbb{R})$ is endowed with norm $\|x\|_0 = \sup\{|x(t)| : t \in J\}$. Also, let

$$\|x\|_{r_1,r_2} = \max\{\sup_{-r_1 \le t \le 0} |x(t)|, \ \|x\|_0, \ \sup_{b \le t \le b+r_2} |x(t)|\}.$$

We recall some facts from multi-valued analysis. Let $(X, \|\cdot\|)$ be a Banach space. Let $\mathcal{P}(X) = \{Y \subset X : Y \ne \emptyset\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, $\mathcal{P}_{c,cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex and compact}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X, i.e., $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < (\infty)$. The map G is called *upper semicontinuous (u.s.c.)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Also, G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, \ y_n \to y_*, \ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

A multi-valued map $G : J \to \mathcal{P}_{cl}(X)$ is said to be *measurable* if for each $x \in E$, the function $Y : J \to X$ defined by

$$Y(t) = \text{dist}(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\},$$

is Lebesgue measurable.

Definition 2.1. A multi-valued map $F : J \times C([-r_1, r_2], \mathbb{R}) \to \mathcal{P}_{c,cp}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \mapsto F(t, x)$ is measurable for each $x \in C([-r_1, r_2], \mathbb{R})$,
(ii) $x \mapsto F(t, x)$ is upper semicontinuous for almost all $t \in J$, and
(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \le h_\rho(t), \quad \text{a.e. } t \in J$$

for all $u \in C([-r_1, r_2], \mathbb{R})$ with $\|u\| \le \rho$.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} d(a, \mathcal{B}), \sup_{b\in\mathcal{B}} d(\mathcal{A}, b)\right\},$$

where $d(\mathcal{A}, b) = \inf_{a\in\mathcal{A}} d(a, b)$, $d(a, \mathcal{B}) = \inf_{b\in\mathcal{B}} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space.

Definition 2.2. A multi-valued operator $G : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x, y), \quad \text{for each } x, \ y \in X;$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

For more details on multi-valued maps we refer to the books of Deimling [20], Górniewicz [21], Hu and Papageorgiou [22] and Tolstonogov [23].

Now, we recall some definitions and facts about fractional derivatives and fractional integrals of arbitrary orders; see [2–5].

Definition 2.3. The fractional primitive of order $\beta > 0$ of a function $g : (0, b] \to \mathbb{R}$ is defined by

$$I_0^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)\mathrm{d}s,$$

provided the right hand side is pointwise defined on $(0, b]$, where Γ is the gamma function.

Note that $I^\beta g$ exists for all $\beta > 0$ and $g \in C((0, b], \mathbb{R}) \cap L^1((0, b], \mathbb{R})$. Also, when $g \in C(J, \mathbb{R})$ then $I^\beta g \in C(J, \mathbb{R})$ and $I^\beta g(0) = 0$.

Definition 2.4. The fractional derivative of order $\beta > 0$ of a continuous function $g : (0, b] \to \mathbb{R}$ is defined by

$$\begin{aligned} D^\beta g(t) \equiv \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} g(t) &= \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-s)^{-\beta} g(s)\mathrm{d}s \\ &= \frac{\mathrm{d}}{\mathrm{d}t} I_a^{1-\beta} g(t). \end{aligned}$$

Lemma 2.1 (*[16]*). *Given $h \in C[0, 1]$ and $1 < \alpha \leq 2$, the unique solution of*

$$D^\beta u(t) + g(t) = 0, \quad 0 < t < 1 \tag{2.1}$$

$$u(0) = u(1) = 0, \tag{2.2}$$

is

$$u(t) = \int_0^1 G(t, s) g(s)\mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} \dfrac{[t(1-s)]^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)}, & \text{if } 0 \leq s \leq t \leq 1, \\ \dfrac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Making use of (2.3) the unique solution u of (2.1) and (2.2) may be written as

$$u(t) = -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)\mathrm{d}s + \frac{1}{\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-s)^{\beta-1} g(s)\mathrm{d}s, \quad t \in J.$$

The considerations of this paper are based on the following fixed point theorems.

Theorem 2.1 (*Nonlinear Alternative for Single-Valued Maps [24]*). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.*

Theorem 2.2 (*Nonlinear Alternative for Kakutani Maps [24]*). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is an upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$.*

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.3 (*[25]*). *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multi-valued operator which is lower semicontinuous (l.s.c.) and has nonempty closed and decomposable values. Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $f : Y \to L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in Y$.*

The next fixed point theorem is the well-known Covitz and Nadler fixed point theorem for multi-valued contractions [26] (see also Deimling, [20] Theorem 11.1).

Lemma 2.2 (*Covitz and Nadler [26]*). *Let (X, d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3. Initial value problems for fractional functional differential inclusions

In this section we are concerned with the existence of solutions for the problem (1.4)–(1.6). We first give the definition of its solution. For this section $J = [0, b]$.

Definition 3.1. A function $x \in C([-r_1, b + r_2], \mathbb{R})$ is said to be a solution of (1.4)–(1.6) if $x(t) = \phi(t)$, $t \in [-r_1, 0]$, $x(t) = \psi(t)$ on $[b, b + r_2]$ and there exists a function $v \in L^1(J, \mathbb{R})$ such that

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} v(s) \mathrm{d}s, \quad t \in J.$$

Theorem 3.1. *Assume that the following conditions are satisfied:*

(H1) *$F : J \times C([-r_1, r_2], \mathbb{R}) \to \mathcal{P}_{c,cp}(\mathbb{R})$ is an L^1-Carathéodory multi-valued map;*
(H2) *there exist $p \in C(J, \mathbb{R})$ and $\Omega : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that*

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \le p(t)\Omega(\|u\|)$$

for almost all $t \in J$ and all $u \in C([-r_1, r_2], \mathbb{R})$;
(H3) *there exists $l \in L^1(J, \mathbb{R})$, with $I^\beta l < \infty$ such that*

$$H_d(F(t, u), F(t, \bar{u})) \le l(t)\|u - \bar{u}\| \quad \text{for every } u, \bar{u} \in C([-r_1, r_2], \mathbb{R}),$$

and

$$d(0, F(0, u)) \le l(t), \quad \text{for a.e. } t \in J;$$

(H4) *there exists a number $K_0 > 0$ such that*

$$\frac{K_0}{\frac{\|p\|_0 b^\beta}{\Gamma(\beta+1)} \Omega(K_0 + \max\{\|\phi\|, \|\psi\|\})} > 1.$$

Then the IVP (1.4)–(1.6) has at least one solution on $[-r_1, b + r_2]$.

Proof. We transform the problem (1.4)–(1.6) into a fixed point problem. A solution to (1.4)–(1.6) is a fixed point of the operator $\mathcal{G} : C([-r_1, b + r_2], \mathbb{R}) \longrightarrow \mathcal{P}(C([-r_1, b + r_2], \mathbb{R}))$ defined by

$$\mathcal{G}(x) := \left\{ \begin{array}{l} h \in C([-r_1, b + r_2], \mathbb{R}) : \\ h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r_1, 0], \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t - s)^{\beta-1} v(s) \mathrm{d}s, & \text{if } t \in J, \\ \psi(t), & \text{if } t \in [b, b + r_2], \end{cases} \end{array} \right\}$$

where

$$v \in S_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y^t) \text{ for a.e. } t \in J\}.$$

Let $u : [-r_1, b + r_2] \to \mathbb{R}$ be a function defined by

$$u(t) = \begin{cases} \phi(t), & \text{if } t \in [-r_1, 0], \\ 0, & \text{if } t \in J, \\ \psi(t), & \text{if } t \in [b, b + r_2]. \end{cases}$$

For each $y \in C(J, \mathbb{R})$ with $y(0) = 0$ we denote by z the function defined by

$$z(t) = \begin{cases} 0, & \text{if } t \in [-r_1, 0], \\ y(t), & \text{if } t \in J, \\ 0, & \text{if } t \in [b, b + r_2]. \end{cases}$$

If the function x satisfies the integral equation

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) \mathrm{d}s$$

we can decompose the function x as $x(t) = y(t) + u(t)$ for $t \in J$. This implies that $x^t = y^t + u^t$ for every $t \in J$ and the function y satisfies

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) \mathrm{d}s.$$

In what follows, let $B = \{y \in C([-r_1, b + r_2], \mathbb{R}) : y_0 = 0\}$ and let $G : B \to B$ be defined by

$$G(y) := \left\{ \begin{aligned} & h \in C([-r_1, b + r_2], \mathbb{R}) : \\ & h(t) = \begin{cases} 0, & \text{if } t \in [-r_1, 0], \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} v(s) \mathrm{d}s, & \text{if } t \in J, \\ 0, & \text{if } t \in [b, b + r_2]. \end{cases} \end{aligned} \right\}$$

Then the operator $\mathcal{G}$ having a fixed point is equivalent to the operator G having a fixed point. So we turn to proving that G has a fixed point which is a solution of the problem (1.4)–(1.6).

Claim 1: *$G(y)$ is convex for each $y \in C([-r_1, b + r_2], \mathbb{R})$.*

Indeed, if h_1, h_2 belongs to $G(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v_i(s) \mathrm{d}s, \quad i = 1, 2.$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$(\mathrm{d}h_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [\mathrm{d}v_1(s) + (1-d)v_2(s)] \mathrm{d}s.$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$\mathrm{d}h_1 + (1-d)h_2 \in G(y).$$

Claim 2: *G sends bounded sets into bounded sets in $C([-r_1, b + r_2], \mathbb{R})$.*

It is enough to show that, for any $\alpha > 0$, there exists a positive constant $\hat{L}$ such that, for each

$$y \in B_\alpha = \{y \in B : \|y\|_{[-r_1, b+r_2]} \leq \alpha\},$$

we have $\|h\|_0 \leq \hat{L}$. For $y \in B$ and $s \in J$ we have

$$\|y^s\|_{[-r_1, r_2]} = \max_{\theta \in [-r_1, r_2]} |y(s + \theta)| \leq \max_{[-r_1, b+r_2]} |y(t)| = \|y\|_{[-r_1, b+r_2]} = \|y\|_{r_1, r_2}$$

and

$$\|y^s + u^s\| \leq \|y^s\| + \|u^s\| \leq \|y\|_{r_1, r_2} + \max\{\|\phi\|, \|\psi\|\}.$$

Let $y \in B_\alpha$. Then for each $h \in G(y)$ there exists $v \in S_{F,y}$ such that

$$h(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds, \quad t \in J.$$

By (H2) we have for each $t \in J$

$$\begin{aligned} |h(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) \mathrm{d}s \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t p(s) \Omega(\|y^s + u^s\|)(t-s)^{\beta-1} \mathrm{d}s \\ &\leq \frac{1}{\Gamma(\beta)} \Omega(\alpha + \max\{\|\phi\|, \|\psi\|\}) \int_0^t p(s)(t-s)^{\beta-1} \mathrm{d}s \end{aligned}$$

$$\leq \frac{\|p\|_0}{\Gamma(\beta)}\Omega(\alpha+\max\{\|\phi\|,\|\psi\|\})\int_0^t (t-s)^{\beta-1}\mathrm{d}s$$
$$\leq \frac{b^\beta\|p\|_0}{\Gamma(\beta+1)}\Omega(\alpha+\max\{\|\phi\|,\|\psi\|\}).$$

Thus

$$\|h\|_0 \leq \frac{b^\beta\|p\|_0}{\Gamma(\beta+1)}\Omega(\alpha+\max\{\|\phi\|,\|\psi\|\}) := \hat{L}.$$

Consequently, G maps bounded sets into bounded sets in B.

Claim 3: *G sends bounded sets in $C([-r_1,b+r_2],\mathbb{R})$ into equicontinuous sets.*

We consider B_α as in Claim 2 and let $h\in G(y)$ for $y\in B_\alpha$. Let $\epsilon>0$ be given. Now let $\tau_1,\tau_2\in J$ with $\tau_2>\tau_1$. We consider two cases $\tau_1>\epsilon$ and $\tau_1\leq\epsilon$.

Case 1. If $\tau_1>\epsilon$ then

$$\begin{aligned}|h(t_2)-h(t_1)| &\leq \frac{1}{\Gamma(\beta)}\int_0^{t_1-\epsilon}\left|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right||v(s)|\mathrm{d}s\\ &\quad+\frac{1}{\Gamma(\beta)}\int_{t_1-\epsilon}^{t_1}\left|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right||v(s)|\mathrm{d}s+\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}|v(s)|\mathrm{d}s\\ &\leq \frac{\Omega(\alpha+\max\{\|\phi\|,\|\psi\|\})}{\Gamma(\beta)}\left(\left|\int_0^{t_1-\epsilon}[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]h_\rho(s)\mathrm{d}s\right|\right.\\ &\quad\left.+\left|\int_{t_1-\epsilon}^{t_1}\left[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right]h_\rho(s)\,\mathrm{d}s\right|+\left|\int_{t_1}^{t_2}(t_2-s)^{\beta-1}h_\rho(s)\mathrm{d}s\right|\right).\end{aligned}$$

Case 2. Let $\tau_1\leq\epsilon$. For $\tau_2-\tau_1<\epsilon$ we get

$$\begin{aligned}|h(t_2)-h(t_1)| &\leq \frac{1}{\Gamma(\beta)}\left|\int_0^{t_2}(t_2-s)^{\beta-1}v(s)\mathrm{d}s-\int_0^{t_1}(t_2-s)^{\beta-1}v(s)\mathrm{d}s\right|\\ &\leq \frac{\Omega(\alpha+\max\{\|\phi\|,\|\psi\|\})}{\Gamma(\beta)}\left(\int_0^{2\epsilon}(t_2-s)^{\beta-1}h_\rho(s)\mathrm{d}s+\int_0^{\epsilon}(t_1-s)^{\beta-1}h_\rho(s)\mathrm{d}s\right).\end{aligned}$$

As a consequence of Claims 2, 3 and the Arzelá–Ascoli theorem we can conclude that $G:B\longrightarrow\mathcal{P}(B)$ is completely continuous.

Claim 4: *G has closed graph.*

Let $y_n\longrightarrow y_*$, $h_n\in G(y_n)$ and $h_n\longrightarrow h_*$. We shall prove that $h_*\in G(y_*)$. Now $h_n\in G(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that

$$h_n(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v_n(s)\mathrm{d}s,\quad t\in J.$$

We must prove that there exists $v_*\in S_{F,y_*}$ such that

$$h_*(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v_*(s)\mathrm{d}s,\quad t\in J.$$

Since $F(t,\cdot)$ is upper semicontinuous, then for every $\epsilon>0$, there exists $n_0(\epsilon)\geq 0$ such that for every $n\geq n_0$, we have

$$v_n(t)\in F(t,y_n^t+u^t)\subset F(t,y_*^t+u^t)+\epsilon B(0,1),\quad \text{a.e. } t\in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot)\to v_*(\cdot)\quad \text{as } m\to\infty$$

and

$$v_*(t)\in F(t,y_*^t+u^t),\quad \text{a.e. } t\in J.$$

For every $w\in F(t,y^t+u^t)$, we have

$$|v_{n_m}(t)-v_*(t)|\leq|v_{n_m}(t)-w|+|w-v_*(t)|.$$

Then

$$|v_{n_m}(t)-v_*(t)|\leq d(v_{n_m}(t),F(t,y_*^t+u^t)).$$

By an analogous relation, obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n^t + u^t), F(t, y_*^t + u^t)) \leq l(t)\|y_n^t - y_*^t\|.$$

Then

$$\begin{aligned} |h_n(t) - h_*(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |v_{n_m}(s) - v_*(s)| \mathrm{d}s \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s) \|y_n - y_*\|_0 \mathrm{d}s. \end{aligned}$$

Hence

$$\|h_n - h_*\| \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s) \|y_n - y_*\|_0 \mathrm{d}s \to 0, \quad \text{as } m \to \infty.$$

Claim 5: *Now, we show that the set*

$$\mathcal{M} := \{y \in C([-r_1, b + r_2], E) : y \in \lambda G(y) \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{M}$ be such that $y \in \lambda G(y)$ for some $\lambda < 1$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) \mathrm{d}s, \quad t \in J.$$

This implies by our assumptions that for each $t \in J$ we have

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t p(s) \Omega(\|y^s + u^s\|)(t-s)^{\beta-1} \mathrm{d}s \\ &\leq \frac{\|p\|_0 b^\beta}{\Gamma(\beta+1)} \Omega(\|y\|_{r_1,r_2} + \max\{\|\phi\|, \|\psi\|\}), \quad t \in J. \end{aligned}$$

Then

$$\frac{\|y\|_{r_1,r_2}}{\frac{\|p\|_0 b^\beta}{\Gamma(\beta+1)} \Omega(\|y\|_{r_1,r_2} + \max\{\|\phi\|, \|\psi\|\})} \leq 1.$$

By (H4), there exists M_* such that $\|y\|_{r_1,r_2} \neq M_*$.

Set

$$U = \{y \in C([-r_1, b + r_2], \mathbb{R}) : \|y\|_{r_1,r_2} < M_* + 1\}.$$

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda G(y)$ for $\lambda \in (0, 1)$. As a consequence of the Leray–Schauder Alternative for Kakutani maps [24] we deduce that G has a fixed point and therefore the problem (1.4)–(1.6) has at least one solution. □

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the Leray–Schauder Alternative for single-valued maps combined with a selection theorem due to Bressan and Colombo [25] for lower semicontinuous multi-valued operators with decomposable values.

Theorem 3.2. *Suppose that:*

(h1) $F : J \times C([-r_1, r_2], \mathbb{R}) \longrightarrow \mathcal{P}(E)$ *is a nonempty, compact-valued, multi-valued map such that:*
  (a) $(t, u) \mapsto F(t, u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
  (b) $u \mapsto F(t, u)$ *is lower semicontinuous for a.e.* $t \in J$;

(h2) *for each* $\rho > 0$, *there exists a function* $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq \varphi_\rho(t) \quad \textit{for a.e. } t \in J$$

and for $u \in C([-r_1, r_2], E)$ *with* $\|u\| \leq \rho$.

In addition suppose (H1) *and* (H4) *are satisfied. Then the IVP* (1.4)–(1.6) *has at least one solution.*

Proof. Assumptions (h1) and (h2) imply that F is of lower semicontinuous type. Then there exists [25] a continuous function $p : C(J, E) \to L^1(J, \mathbb{R})$ such that $p(y) \in \mathcal{F}(y)$ for all $y \in C(J, E)$, where $\mathcal{F}$ is the Nemitsky operator defined by

$$\mathcal{F}(y) = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, y^t) \quad \text{for a.e. } t \in J\}.$$

Consider the problem

$$D^{\beta}y(t) = p(y)(t), t \in J, \quad 0 < \beta < 1, \tag{3.1}$$

$$y(t) = \phi(t), \quad -r_1 \le t \le 0, \tag{3.2}$$

$$y(t) = \psi(t), \quad b \le t \le b + r_2. \tag{3.3}$$

It is obvious that if $y \in C(J, E)$ is a solution of the problem (3.1)–(3.3), then y is a solution to the problem (1.4)–(1.6).

Transform the problem (3.1) and (3.2) into a fixed point problem considering the operator $N : C([-r_1, b + r_2], \mathbb{R}) \to C([-r_1, b + r_2], \mathbb{R})$ defined by:

$$N(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r_1, 0] \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} p(y)(s)\mathrm{d}s, & \text{if } t \in J, \\ \psi(t), & \text{if } t \in [b, b + r_2]. \end{cases}$$

We prove that $N : C([-r_1, b + r_2], \mathbb{R}) \to C([-r_1, b + r_2], \mathbb{R})$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C([-r_1, b + r_2], \mathbb{R})$. Then there is an integer q such that $\|y_n\| \le q$ for all $n \in \mathbb{N}$ and $\|y\|_{r_1,r_2} \le q$; so $y_n \in B_q$ and $y \in B_q$. We have then by the dominated convergence theorem

$$\|N(y_n) - N(y)\| \le \frac{1}{\Gamma(\beta)} \sup_{t\in J} \left[\int_0^t (t-s)^{\beta-1} |p(y_n) - p(y)| \mathrm{d}s\right] \longrightarrow 0.$$

Thus N is continuous. Next we prove that N is completely continuous by proving, as in Theorem 3.1, that *N maps bounded sets into bounded sets in $C([-r_1, b + r_2], \mathbb{R})$ and N maps bounded sets into equicontinuous sets of $C([-r_1, b + r_2], \mathbb{R})$.*

Finally, as in Theorem 3.1 we can show that the set

$$\mathcal{E}(N) := \{y \in C([-r_1, b + r_2], \mathbb{R}) : y = \lambda G(y), \quad \text{for some } 0 < \lambda < 1\}$$

is bounded. As a consequence of the Leray–Schauder Alternative for single-valued maps we deduce that N has a fixed point y which is a solution to problem (3.1)–(3.3). Then y is a solution to the IVP (1.4)–(1.6). □

We present now a result for the problem (1.4)–(1.6) with a nonconvex valued right hand side, by using the Covitz and Nadler fixed point theorem.

Theorem 3.3. *Suppose* (H3) *and the following hypothesis hold:*

(h3) $F : J \times C([-r_1, r_2], \mathbb{R}) \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ *has the property that* $F(\cdot, y) : J \longmapsto \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $y \in C([-r_1, r_2], \mathbb{R})$.

If

$$\|I^{\beta}p\|_0 < 1;$$

then the IVP (1.4)–(1.6) *has at least one solution.*

Remark 3.1. For each $y \in C(J, E)$ the set $S_{F(y)}$ is nonempty since by (h3) F has a measurable selection (see [27], Theorem III.6).

Proof. Transform the problem (1.4)–(1.6) into a fixed point problem. Let the multi-valued operator $G : B \to \mathcal{P}(B)$ be defined as in Theorem 3.1. We shall show that G satisfies the assumptions of Lemma 2.2. The proof will be given in two steps.

Step 1: *$G(y) \in \mathcal{P}_{cl}(B)$ for each $y \in B$.*

Indeed, let $(y_n)_{n\ge0} \in G(y)$ such that $y_n \longrightarrow \tilde{y}$ in B. Then $\tilde{y} \in B$ and there exists $g_n \in S_{F,y}$ such that for each $t \in J$

$$y_n(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g_n(s)\mathrm{d}s.$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence if necessary to get that g_n converges weakly to g in $L^1(J, \mathbb{R})$. An application of Mazur's theorem implies that g_n converges strongly to g and hence $g \in S_{F,y}$. Then for each $t \in J$

$$y_n(t) \longrightarrow \tilde{y}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)\mathrm{d}s.$$

So $\tilde{y} \in G(y)$.

Step 2: *There exists $\gamma < 1$, such that*

$$H_d(G(y), G(\overline{y})) \le \gamma \|y - \overline{y}\|_{r_1,r_2} \quad \text{for each } y, \overline{y} \in B.$$

Let $y, \overline{y} \in B$ and $h \in G(y)$. Then there exists $g(t) \in F(t, y^t + u^t)$ such that for each $t \in J$

$$h(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) \mathrm{d}s.$$

From (H3) it follows that

$$H_d(F(t, y^t + u^t), F(t, \overline{y}^t + u^t)) \leq l(t)\|y - \overline{y}\|, \quad t \in J.$$

Hence there is $w \in F(t, \overline{y}_t + u^t)$ such that

$$|g(t) - w| \leq l(t)\|y - \overline{y}\|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |g(t) - w| \leq l(t)\|y - \overline{y}\|\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t, \overline{y}^t + u^t)$ is measurable (see Proposition III.4 in [27]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t, \overline{y}^t + u^t)$ and

$$|g(t) - \overline{g}(t)| \leq l(t)\|y - \overline{y}\|, \quad \text{for each } t \in J.$$

Let us define for each $t \in J$

$$\overline{h}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \overline{g}(s) \mathrm{d}s.$$

Then for $t \in J$

$$\begin{aligned} |h(t) - \overline{h}(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |g(s) - \overline{g}(s)| \mathrm{d}s \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)\|y - \overline{y}\| \mathrm{d}s \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t (b-s)^{\beta-1} l(s)\|y - \overline{y}\|_{r_1,r_2} \mathrm{d}s \\ &\leq \|I^\beta l\|_0 \|y - \overline{y}\|_{r_1,r_2}. \end{aligned}$$

Then

$$\|h - \overline{h}\|_0 \leq \|I^\beta l\|_0 \|y - \overline{y}\|_{r_1,r_2}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(G(y), G(\overline{y})) \leq \|I^\beta l\|_0 \|y - \overline{y}\|_{r_1,r_2}.$$

So, G is a contraction and thus, by Lemma 2.2, G has a fixed point y, which is a solution to (1.4)–(1.6). □

4. Boundary value problems for fractional functional differential inclusions

By a solution of (1.7)–(1.9) we mean a function $x \in C([-r_1, 1 + r_2], \mathbb{R})$ with $x(t) = \phi(t)$, $\phi(0) = 0$ on $[-r_1, 0]$, $x(t) = \psi(t)$, $\psi(1) = 0$ on $[1, b + r_2]$ and there exists a function $v \in L^1([0, 1], \mathbb{R})$ such that

$$x(t) = \frac{1}{\Gamma(\beta)} \int_0^1 G(t, s) v(s) \mathrm{d}s.$$

We mention that for this section $J = [0, 1]$.

Theorem 4.1. *Suppose* (H1) *and* (H3) *hold. Moreover assume that the following conditions hold:*

(A2) *there exist $p \in L^1(J, \mathbb{R})$ and $\Omega : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that*

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\Omega(\|u\|)$$

for almost all $t \in J$ and all $u \in C([-r_1, r_2], \mathbb{R})$;

(A3) *there exists a number $K_0 > 0$ such that*

$$\frac{K_0}{\frac{2}{\Gamma(\beta)} \Omega(K_0 + \max\{\|\phi\|, \|\psi\|\}) \|p\|_{L^1}} > 1.$$

Then the BVP (1.7)–(1.9) has at least one solution on the interval $[-r_1, 1 + r_2]$.

Proof. We transform the problem (1.7)–(1.9) into a fixed point problem. A solution to (1.7)–(1.9) is a fixed point of the operator $\mathcal{N} : C([-r_1, 1+r_2], \mathbb{R}) \longrightarrow \mathcal{P}(C([-r_1, 1+r_2], \mathbb{R}))$ defined by

$$\mathcal{N}(x) := \left\{ \begin{array}{l} h \in C([-r_1, 1+r_2], \mathbb{R}) : \\ h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r_1, 0], \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t G(t,s)v(s)\mathrm{d}s, & \text{if } t \in J, \\ \psi(t), & \text{if } t \in [1, 1+r_2], \end{cases} \end{array} \right\}$$

where

$$v \in S_{F,y} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, y^t) \text{ for a.e. } t \in J\}.$$

Using the decomposition of Theorem 3.1, in what follows, let $B = \{y \in C([-r_1, 1+r_2], \mathbb{R}) : y_0 = 0\}$ and let $N : B \to B$ be defined by

$$N(y) := \left\{ \begin{array}{l} h \in C([-r_1, 1+r_2], \mathbb{R}) : \\ h(t) = \begin{cases} 0, & \text{if } t \in [-r_1, 0], \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t G(t,s)v(s)\mathrm{d}s, & \text{if } t \in J, \\ 0, & \text{if } t \in [1, 1+r_2]. \end{cases} \end{array} \right\}$$

Then the operator $\mathcal{N}$ having a fixed point is equivalent to the operator N having a fixed point. So we turn to proving that N has a fixed point which is a solution of the problem (1.7)–(1.9).

Claim 1: *$N(y)$ is convex for each $y \in C([-r_1, 1+r_2], \mathbb{R})$.*

This claim is obvious, since F has convex values (see also the proof of Step 1 of Theorem 3.1).

Claim 2: *N sends bounded sets into bounded sets in $C([-r_1, 1+r_2], \mathbb{R})$.*

It is enough to show that, for any $\alpha > 0$, there exists a positive constant $\hat{L}$ such that, for each

$$y \in B_\alpha = \{y \in B : \|y\|_{[-r_1, 1+r_2]} \le \alpha\},$$

we have $\|h\|_0 \le \hat{L}$. For $y \in B$ and $s \in J$ we have

$$\|y^s\|_{[-r_1, r_2]} = \max_{\theta \in [-r_1, r_2]} |y(s+\theta)| \le \max_{[-r_1, 1+r_2]} |y(t)| = \|y\|_{[-r_1, 1+r_2]} = \|y\|_{[r_1, r_2]}.$$

Let $y \in B_\alpha$. Then for each $h \in N(y)$ there exists $v \in S_{F,y}$ such that

$$h(t) = \int_0^1 G(t,s)v(s)\mathrm{d}s, \quad t \in J.$$

We have for each $t \in J$

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1}|v(s)|\mathrm{d}s + \frac{1}{\Gamma(\beta)} \int_0^1 |1-s|^{\beta-1}|v(s)|\mathrm{d}s \\ &\le \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1}|h_\rho(s)|\mathrm{d}s + \frac{1}{\Gamma(\beta)} \int_0^1 |1-s|^{\beta-1}h_\rho(s)\mathrm{d}s \\ &\le \frac{1}{\Gamma(\beta)} \int_0^1 |h_\rho(s)|\mathrm{d}s + \frac{1}{\Gamma(\beta)} \int_0^1 h_\rho(s)\mathrm{d}s = \frac{2}{\Gamma(\beta)} \int_0^1 h_\rho(s)\mathrm{d}s. \end{aligned}$$

Thus

$$\|h\|_0 \le \frac{2}{\Gamma(\beta)} \|h_\rho\|_{L^1} := \hat{L}.$$

Consequently, N maps bounded sets into bounded sets in B.

Claim 3: *N sends bounded sets in $C([-r_1, 1+r_2], \mathbb{R})$ into equicontinuous sets.*

We consider B_α as in Claim 2 and let $h \in N(y)$ for $y \in B_\alpha$. Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. Then we have

$$\begin{aligned} |h(t_2) - h(t_1)| &= \left| \int_0^1 [G(t_2,s) - G(t_1,s)]v(s)\mathrm{d}s \right| \\ &\le \int_0^{t_1} [G(t_2,s) - G(t_1,s)]v(s)\mathrm{d}s + \int_{t_1}^{t_2} [G(t_2,s) - G(t_1,s)]v(s)\mathrm{d}s + \int_{t_2}^1 [G(t_2,s) - G(t_1,s)]v(s)\mathrm{d}s \\ &\le \frac{t_2^{\beta-1} - t_1^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}|v(s)|\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] |v(s)| \mathrm{d}s \right| + \left| \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} |v(s)| \mathrm{d}s \right| \\
&\le \frac{t_2^{\beta-1} - t_1^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1 - s)^{\beta-1} h_\rho(s) \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} |(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}| h_\rho(s) \mathrm{d}s + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} h_\rho(s) \mathrm{d}s.
\end{aligned}
$$

As $t_1 \to t_2$ the right hand side of the last inequality tends to zero.

As a consequence of Claims 2, 3 and the Arzelá–Ascoli theorem we can conclude that $N : B \longrightarrow \mathcal{P}(B)$ is completely continuous.

Claim 4: *N has closed graph.*

This can be proven as in Step 4 of Theorem 3.1, and thus the details are omitted.

Claim 5: *Now, we show that the set*

$$\mathcal{M} := \{y \in C([-r_1, 1 + r_2], E) : y \in \lambda N(y) \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$ be such that $y \in \lambda N(y)$ for some $\lambda < 1$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda \int_0^1 G(t, s)v(s)\mathrm{d}s, \quad t \in J.$$

This implies by our assumptions that for each $t \in J$ we have

$$
\begin{aligned}
|y(t)| &\le \int_0^1 |G(t, s)||v(s)|\mathrm{d}s \\
&\le \int_0^1 \sup\{|G(t, s)| : (t, s) \in J \times J\} p(s)\, \Omega(\|y^s + u^s\|)\mathrm{d}s \\
&\le \frac{2}{\Gamma(\beta)} \Omega(\|y\|_{r_1,r_2} + \max\{\|\phi\|, \|\psi\|\})\|p\|_{L^1}.
\end{aligned}
$$

Then

$$\frac{\|y\|_{r_1,r_2}}{\frac{2}{\Gamma(\beta)} \Omega(\|y\|_{r_1,r_2} + \max\{\|\phi\|, \|\psi\|\})\|p\|_{L^1}} \le 1.$$

By (A3), there exists M_* such that $\|y\|_{r_1,r_2} \neq M_*$. Set

$$U = \{y \in C([-r_1, 1 + r_2], \mathbb{R}) : \|y\|_{r_1,r_2} < M_* + 1\}.$$

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for $\lambda \in (0, 1)$. As a consequence of the Leray–Schauder Alternative for Kakutani maps [24] we deduce that N has a fixed point and therefore the BVP (1.7)–(1.9) has at least one solution. □

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the Leray–Schauder Alternative for single-valued maps combined with a selection theorem due to Bressan and Colombo [25] for lower semicontinuous multi-valued operators with decomposable values.

Theorem 4.2. *Suppose that* (H1), (h1), (h2) *and* (A3) *hold. Then the BVP* (1.7)–(1.9) *has at least one solution.*

Proof. Assumptions (h1) and (h2) imply that F is of lower semicontinuous type. Then there exists [25] a continuous function $p : C(J, E) \to L^1(J, \mathbb{R})$ such that $p(y) \in \mathcal{F}(y)$ for all $y \in C(J, E)$, where $\mathcal{F}$ is the Nemitsky operator defined by

$$\mathcal{F}(y) = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, y^t) \text{ for a.e. } t \in J\}.$$

Consider the problem

$$-D^\beta y(t) = p(y)(t), \quad t \in J,\ 1 < \beta < 2, \tag{4.1}$$

$$y(t) = \phi(t), \quad -r_1 \le t \le 0, \tag{4.2}$$

$$y(t) = \psi(t), \quad b \le t \le 1 + r_2. \tag{4.3}$$

It is obvious that if $y \in C(J, \mathbb{R})$ is a solution of the problem (4.1)–(4.3), then y is a solution to the problem (1.7)–(1.9).

Transform the problem (4.1)–(4.3) into a fixed point problem considering the operator $N_1 : C([-r_1, 1+r_2], \mathbb{R}) \to C([-r_1, 1+r_2], \mathbb{R})$ defined by:

$$N_1(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r_1, 0] \\ \displaystyle\int_0^1 G(t,s)p(y)(s)\mathrm{d}s, & \text{if } t \in J, \\ \psi(t), & \text{if } t \in [1, 1+r_2]. \end{cases}$$

We prove that $N_1 : C([-r_1, 1+r_2], \mathbb{R}) \to C([-r_1, 1+r_2], \mathbb{R})$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C([-r_1, 1+r_2], \mathbb{R})$. Then there is an integer q such that $\|y_n\| \le q$ for all $n \in \mathbb{N}$ and $\|y\|_{r_1,r_2} \le q$; so $y_n \in B_q$ and $y \in B_q$. We have then by the dominated convergence theorem

$$\begin{aligned} |N_1(y_n)(t) - N_1(y)(t)| &\le \int_0^1 |G(t,s)||p(y_n(s)) - p(y(s))|\mathrm{d}s \\ &\le \frac{2}{\Gamma(\beta)} \int_0^1 |p(y_n(s)) - p(y(s))|\mathrm{d}s. \end{aligned}$$

Hence

$$\|N_1(y_n) - N_1(y)\|_0 \to 0 \quad \text{as } n \to \infty.$$

Thus N_1 is continuous. Next we prove that N_1 is completely continuous by proving, as in Theorem 4.1, that N_1 *maps bounded sets into bounded sets in* $C([-r_1, 1+r_2], \mathbb{R})$ and N_1 *maps bounded sets into equicontinuous sets of* $C([-r_1, 1+r_2], \mathbb{R})$.

Finally, as in Theorem 4.1 we can show that the set

$$\mathcal{E}(N_1) := \{y \in C([-r_1, 1+r_2], \mathbb{R}) : y = \lambda N_1(y), \quad \text{for some } 0 < \lambda < 1\}$$

is bounded. As a consequence of the Leray–Schauder Alternative for single-valued maps we deduce that N_1 has a fixed point y which is a solution to problem (4.1)–(4.3). Then y is a solution to the BVP (1.7)–(1.9). □

We present now a result for the problem (1.7)–(1.9) with a nonconvex valued right hand side, by using the Covitz and Nadler fixed point theorem.

Theorem 4.3. *Suppose that* (h3) *and* (H3) *hold. If*

$$\frac{2}{\Gamma(\beta)}\|l\|_{L^1} < 1,$$

then the BVP (1.7)–(1.9) *has at least one solution.*

Proof. Transform the problem (1.7)–(1.9) into a fixed point problem. Let the multi-valued operator $N : B \to \mathcal{P}(B)$ be defined as in Theorem 4.1. We shall show that N satisfies the assumptions of Lemma 2.2. The proof will be given in two steps.

Step 1: $N(y) \in \mathcal{P}_{cl}(B)$ for each $y \in B$.

Indeed, let $(y_n)_{n\ge 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in B. Then $\tilde{y} \in B$ and there exists $g_n \in S_{F,y}$ such that for each $t \in J$

$$y_n(t) = \int_0^1 G(t,s)g_n(s)\mathrm{d}s.$$

Using the fact that F has compact values and from (A2), we may pass to a subsequence if necessary to get that g_n converges weakly to g in $L^1([0.1], \mathbb{R})$. An application of Mazur's theorem implies that g_n converges strongly to g and hence $g \in S_{F(y)}$. Then for each $t \in J$

$$y_n(t) \longrightarrow \tilde{y}(t) = \int_0^1 G(t,s)g(s)\mathrm{d}s.$$

So $\tilde{y} \in N(y)$.

Step 2: *There exists* $\gamma < 1$, *such that*

$$H_d(N(y), N(\overline{y})) \le \gamma\|y - \overline{y}\|_{r_1,r_2} \quad \text{for each } y, \overline{y} \in B.$$

Let $y, \overline{y} \in B$ and $h \in N(y)$. Then there exists $g(t) \in F(t, y^t + u^t)$ such that for each $t \in J$

$$h(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s)\mathrm{d}s.$$

From (A2) it follows that

$$H_d(F(t, y^t + u^t)), (F(t, \overline{y}^t + u^t)) \le l(t)\|y - \overline{y}\|, \quad t \in J.$$

Hence there is $w \in F(t, \bar{y}^t + u^t)$ such that

$$|g(t) - w| \le l(t)\|y - \bar{y}\|, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : |g(t) - w| \le l(t)\|y - \bar{y}\|\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t, \bar{y}^t + u^t)$ is measurable (see Proposition III.4 in [27]), there exists a function $\bar{g}(t)$, which is a measurable selection for V. So, $\bar{g}(t) \in F(t, \bar{y}^t + u^t)$ and

$$|g(t) - \bar{g}(t)| \le l(t)\|y - \bar{y}\|, \quad \text{for each } t \in J.$$

Let us define for each $t \in J$

$$\bar{h}(t) = \int_0^1 G(t,s)\bar{g}(s)\mathrm{d}s.$$

Then for $t \in J$

$$\begin{aligned} |h(t) - \bar{h}(t)| &\le \int_0^1 |G(t,s)||g(s) - \bar{g}(s)|\mathrm{d}s \\ &\le \int_0^1 |G(t,s)|l(s)\|y - \bar{y}\|\mathrm{d}s \\ &\le \frac{2}{\Gamma(\beta)}\|l\|_{L^1}\|y - \bar{y}\|_{r_1,r_2}. \end{aligned}$$

Then

$$\|h - \bar{h}\|_0 \le \frac{2}{\Gamma(\beta)}\|l\|_{L^1}\|y - \bar{y}\|_{r_1,r_2}.$$

By an analogous relation, obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N(y), N(\bar{y})) \le \frac{2}{\Gamma(\beta)}\|l\|_{L^1}\|y - \bar{y}\|_{r_1,r_2}.$$

So, N is a contraction and thus, by Lemma 2.2, N has a fixed point y, which is a solution to (1.7)–(1.9). □

References

[1] R. Hilfer, Applications of Fractional calculus in Physics, World Scientific, Singapore, 2000.
[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.
[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[4] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[5] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science Publs, Amsterdam, 1993, (Russian Edition 1987).
[6] A.M.A. El-Sayed, F.M. Gaafar, Fractional calculus and some intermediate physical processes, Appl. Math. Comput. 144(1) (2003) 117–126.
[7] M. Garh, A. Rao, S.L. Kalla, Fractional generalization of temperature fields problems in oil strata, Mat. Bilten 30 (2006) 71–84.
[8] R.K. Saxena, S.L. Kalla, On a fractional generalization of free electron laser equation, Appl. Math. Comput. 143 (2003) 89–97.
[9] R.K. Saxena, A.M. Mathai, H.L. Haubold, On generalized fractional kinetic equations, Physica A 344 (2004) 657–664.
[10] O.P. Agrawal, Analytical schemes for a new class of fractional differential equations, J. Phys. A 40 (21) (2007) 5469–5477.
[11] V. Lakshmikantham, J.V. Devi, Theory of fractional differential equations in a Banach space, Eur. J. Pure Appl. Math. 1(1) (2008) 38–45.
[12] Ch. Yu, G. Gao, Existence of fractional differential equations, J. Math. Anal. Appl. 310 (2005) 26–29.
[13] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional-Differential Equations, in: Mathematics and its Applications, vol. 463, Kluwer Academic Publishers, Dordrecht, 1999.
[14] J. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, in: Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993.
[15] M.A. Darwich, S.K. Ntouyas, Existence results for a fractional functional differential equation of mixed type, Comm. Appl. Nonlinear Anal. 15 (2008) 47–55.
[16] Z. Bai, H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005) 495–505.
[17] M. El-Shahed, Positive solutions for boundary value problem of nonlinear fractional differential equation, Abstr. Appl. Anal. (2007) 1–8.
[18] A.A. Kilbas, J.J. Trujillo, Differential equations of fractional order: Methods, results and problems II, Appl. Anal. 81 (2002) 435–493.
[19] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, Electron. J. Differential Equations 36 (2006) 1–12.
[20] K. Deimling, Multivalued Differential Equations, Walter De Gruyter, Berlin, New York, 1992.
[21] L. Gorniewicz, Topological Fixed Point Theory of Multivalued Mappings, in: Mathematics and its Applications, vol. 495, Kluwer Academic Publishers, Dordrecht, 1999.
[22] Sh. Hu, N. Papageorgiou, Theory, in: Handbook of Multivalued Analysis, vol. I, Kluwer, Dordrecht, Boston, London, 1997.
[23] A.A. Tolstonogov, Differential Inclusions in a Banach Space, Kluwer Academic Publishers, Dordrecht, 2000.
[24] A. Granas, J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.
[25] A. Bressan, G. Colombo, Extensions and selections of maps with decomposable values, Studia Math. 90 (1988) 69–86.
[26] H. Covitz, S.B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, Israel J. Math. 8 (1970) 5–11.
[27] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, in: Lecture Notes in Mathematics, vol. 580, Springer-Verlag, Berlin, Heidelberg, New York, 1977.