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Global solvability for first order real linear partial differential operators

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ABSTRACT

F. Treves, in [17], using a notion of convexity of sets with respect to operators due to B. Malgrange and a theorem of C. Harvey, characterized globally solvable linear partial differential operators on $C^\infty(X)$, for an open subset X of $\mathbb{R}^n$.
Let $P = L + c$ be a linear partial differential operator with real coefficients on a C^∞ manifold X, where L is a vector field and c is a function. If L has no critical points, J. Duistermaat and L. Hörmander, in [2], proved five equivalent conditions for global solvability of P on $C^\infty(X)$.
Based on Harvey–Treves's result we prove sufficient conditions for the global solvability of P on $C^\infty(X)$, in the spirit of geometrical Duistermaat–Hörmander's characterizations, when L is zero at precisely one point. For this case, additional non-resonance type conditions on the value of c at the equilibrium point are necessary.

1. Introduction

Let X be a C^∞ manifold Hausdorff with a countable basis of open sets and $P : C^\infty(X) \to C^\infty(X)$ a linear partial differential operator. P is said to be *globally solvable*, or *solvable*, on $C^\infty(X)$ when $P(C^\infty(X)) = C^\infty(X)$.

B. Malgrange [9, p. 295] in 1955 introduced the notion of P-convexity and showed it to be equivalent to the global solvability of P on $C^\infty(X)$, when P has constant coefficients and X is an open

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subset of $\mathbb{R}^n$. When P has variable coefficients, he showed that P-convexity is a necessary condition for the global solvability of P on $C^\infty(X)$.

Let X be an n-dimensional C^∞ manifold Hausdorff space with countable basis. Take $\mathcal{F}$ to be a local coordinate system (X_κ, κ) for X. The space of distributions $\mathscr{D}'(X)$ is defined in the following way (see [7, p. 144]), for every κ consider a distribution $u_\kappa \in \mathscr{D}'(\kappa(X_\kappa))$ such that

$$u_{\kappa'} = u_\kappa \circ \left(\kappa \circ \kappa'^{-1}\right) \quad \text{in } \kappa'(X_\kappa \cap X_{\kappa'}),$$

in this case, (u_κ) is called a *distribution* on X. The set of all distributions in X is denoted by $\mathscr{D}'(X)$. Similarly we define the space of compact support distribution $\mathscr{E}'(X)$.

Denote $M \Subset X$ if M is a compact subset of X and ${}^tP$ the formal transpose of P. In this article $\operatorname{supp}(u)$ denotes the support and $\operatorname{singsupp}(u)$ denotes the singular support of the distribution u. We say that X is *P-convex for supports* if $\forall K \Subset X$, $\exists K' \Subset X$ such that

$$u \in \mathscr{E}'(X), \quad \operatorname{supp}\left({}^tPu\right) \subset K \quad \Rightarrow \quad \operatorname{supp}(u) \subset K'.$$

In a similar way we define the P-convexity for singular supports.

In 1967, F. Treves [17, p. 60] and C. Harvey [5, p. 700] using the P-convexity for supports, gave a general characterization of globally solvable linear partial differential operators on $C^\infty(X)$.

Unless otherwise mentioned, from now on $P = L + c$ will be a linear partial differential operator with real coefficients in $C^\infty(X)$, where L is a vector field and c is a function. In 1972, when L has no critical points, J. Duistermaat and L. Hörmander (see [2, p. 212]) gave five equivalent conditions for global solvability of P on $C^\infty(X)$. They used the notions of global transversal of L on X and of convexity of X with respect to the trajectories of L. In [6], J. Hounie extended one of these characterizations for L complex.

In order to state our main theorem we recall some definitions and results.

We say that X is *convex with respect to the trajectories* of L if $\forall K \Subset X, \exists K' \Subset X$ such that any compact interval of trajectory of L with endpoints in K, is contained in K' (see [2, p. 208]).

If L has a critical point at the origin and $c \in \mathbb{C}$, V. Guillemin and D. Schaeffer [3, p. 175] gave, in 1977, sufficient conditions for the equation $Pu = f$ to have a C^∞ solution in a neighborhood of zero, for an arbitrary $f \in C^\infty(\mathbb{R}^n)$ flat at the origin. We remark that in [3] and [11] results on propagation of singularities for operators of type $P = L + c$ are presented.

Suppose that x_0 is a critical point of L. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n'}, \lambda_{n'+1}, \ldots, \lambda_n$ be the eigenvalues of $DL(x_0)$, where $\lambda_1, \lambda_2, \ldots, \lambda_{n'}$ are the real eigenvalues and $\lambda_{n'+1}, \ldots, \lambda_n$ are non-real eigenvalues.

For $c = 0$, from S. Sternberg [15, p. 629], see also E. Nelson [10, p. 50] and V. Guillemin and D. Schaeffer [3, p. 175], we have: If

$$\lambda_j \neq \sum_{k=1}^n m_k \lambda_k, \quad j = 1, 2, \ldots, n, \ m_1, \ldots, m_n \in \mathbb{N}, \quad \sum_{k=1}^n m_k \geqslant 2, \tag{NRC 1}$$

then given $f \in C^\infty(\mathbb{R}^n)$ flat at x_0, $\exists u \in C^\infty(\mathbb{R}^n)$ such that $Pu = f$ in a neighborhood of x_0.

Observe that the condition (NRC 1) implies that every eigenvalue of $DL(x_0)$ has nonzero real part, that is, x_0 is a *hyperbolic critical point* for L.

If $c(x_0) = 0$ then, since $Lu(x_0) = 0$, we have $Pu(x_0) = 0$ hence the operator P is not C^∞-solvable at any neighborhood of x_0. Therefore we consider the following non-resonance condition

$$-c(x_0) \neq \sum_{j=1}^n m_j \operatorname{Re} \lambda_j, \quad \forall m_1, \ldots, m_{n'} \in \mathbb{N}, \ \forall m_{n'+1}, \ldots, m_n \in 2\mathbb{N}. \tag{NRC 2}$$

Our main result is:

Theorem 1. *Let $P = L + c$ be a first order differential operator with coefficients in $C^\infty(X, \mathbb{R})$ with a critical point at x_0. If*

(a) (NRC 1) *and* (NRC 2) *are valid,*
(b) *no orbit of L on $X \setminus \{x_0\}$ is relatively compact in X, and*
(c) *X is convex with respect to the trajectories of L*

then

$$P \text{ is solvable on } C^\infty(X).$$

Also in this paper we consider the relationship between P-convexity and convexity with respect to the trajectories of L for $P = L + c$, see Proposition 1.

This paper is organized in the following way. In Section 2 we present results concerning the relationship between P-convexity for supports, P-convexity for singular supports and convexity with respect to the trajectories of L when L is a real vector field. In Section 3 we prove Theorem 1.

2. *L*-convexity for supports, *L*-convexity for singular supports and convexity with respect to the trajectories

In this section we use propagation of singularities and of supports to characterize, in geometrical terms, the L-convexity for supports and singular supports. From these characterizations, we obtain in our setting the equivalence between those conditions.

The main result of this section is:

Proposition 1. *Let L be a real vector field on X. The following conditions are equivalent*:

(a) *X is L-convex for singular supports.*
(b) (b.1) *$\exists \widetilde{K} \Subset X$ such that no orbit of $L|_{X\setminus\widetilde{K}}$ is relatively compact, and*
(b.2) *X is convex with respect to the trajectories of L.*

Let L be a non-singular real vector field on X. If one of the following conditions holds:

(i) X is any open set of $\mathbb{R}^n$ and L has constant coefficients, or
(ii) X is a simply connected open subset of $\mathbb{R}^2$,

then condition (b.1) holds with $\widetilde{K} = \emptyset$, because the orbits are lines in case (i) and because of the Poincaré–Bendixson theorem in case (ii). Therefore, under conditions (i) or (ii) above, from Proposition 1 we have (a) $\Leftrightarrow$ (b.2).

Observe that if $L \equiv 0$ then every manifold X is convex with respect to the trajectories of L but X is not L-convex for singular supports. If $X \subset \mathbb{R}^2$ is not simply connected then (b.2) $\not\Rightarrow$ (a), for example take $X = \mathbb{R}^2 \setminus \{0\}$ and $L = x_2\partial_1 - x_1\partial_2$.

In [14], H. Seifert proposed the following question, which is known as Seifert's Conjecture: Does every smooth vector field on the 3-dimensional sphere have a periodic orbit? This conjecture was proved to be false for C^1 vector fields by P. Schweitzer (see [13]) and latter in the C^∞ case by K. Kuperberg (see [8]). In contrast with (ii), the second author in [16] starting from an example for which the statement of the conjecture is true, constructed a real non-singular vector field on $\mathbb{R}^3$ such that (b.2) $\not\Rightarrow$ (a).

2.1. Proof of Proposition 1

We will introduce some definitions concerning vector fields. Let L be a real vector field on a manifold X and γ the associated flow. For each $x \in X$, we denote the maximal interval of definition of the orbit passing through x by $I_x = (\omega_-(x), \omega_+(x))$ and the orbit (or trajectory) of x by $\Gamma_x = \{\gamma(t,x);\ t \in I_x\}$. Also denote $\Gamma_x^+ = \{\gamma(t,x);\ 0 \leqslant t < \omega_+(x)\}$ and $\Gamma_x^- = \{\gamma(t,x);\ \omega_-(x) < t \leqslant 0\}$.

When $\omega_+(x) = +\infty$ (resp. $\omega_-(x) = -\infty$) we define

$$\omega(x) = \{y \in X,\ \gamma(t_j, x) \to y \text{ for some sequence } t_j \to +\infty\}$$

(resp. $\alpha(x) = \{y \in X,\ \gamma(t_j, x) \to y$ for some sequence $t_j \to -\infty\}$.)

We say that $\{x_0\} \subset X$ is a *local attractor* of L when there exist a neighborhood U of x_0 such that $\lim_{t\to\omega_+(x)} \gamma(t,x) = x_0,\ \forall x \in U$. In this case, the *basin of attraction* of $\{x_0\}$ is defined by $\mathcal{B}(x_0) = \{x \in X;\ \lim_{t\to\omega_+(x)} \gamma(t,x) = x_0\}$. When $\mathcal{B}(x_0) = X$ we say that $\{x_0\}$ is a *global attractor*.

To prove Proposition 1 we will need some preliminary results, namely Lemma 1 to Lemma 3. Choose a sequence $\{K_j\}_{j=1}^{\infty}$ of compact subsets of X such that

$$\bigcup K_j = X, \quad K_j \subset K_{j+1}^{\circ}, \quad j = 1, 2, \ldots, \quad \text{and} \quad \forall K \Subset X,\ \exists j_0 \in \mathbb{N} \quad \text{such that} \quad K \subset K_{j_0}. \tag{1}$$

Here A° denotes the interior of the subset $A \subset X$.

If K is a compact subset of X then we denote by $C^{\infty}(K)$ the quotient of $C^{\infty}(X)$ by the space consisting of elements vanishing of infinite order on K. Then $C^{\infty}(K)$ is a Fréchet space and the family of seminorms given by

$$p_j(\dot{\phi}) = \inf_{\phi \in \dot{\phi}} \sum_{|\alpha| \leqslant j} \sup_{K_j} |\partial^{\alpha}\phi|, \quad \dot{\phi} \in C^{\infty}(K),\ j = 0, 1, 2, \ldots,$$

is a basis of continuous seminorms of $C^{\infty}(K)$. Here $\dot{\phi}$ denotes the class of $\phi \in C^{\infty}(X)$ in $C^{\infty}(K)$. Denote $B_{p_j} = \{\phi \in C^{\infty}(K);\ p_j(\dot{\phi}) < 1\}$. Then $\forall j \in \mathbb{N}, \exists C > 0$ such that

$$L\left(\frac{1}{C} B_{p_{j+1}}\right) \subset B_{p_j}. \tag{2}$$

This implies the continuity of L on $C^{\infty}(K)$.

We use the identification $(C^{\infty}(K))' = \mathscr{E}'(K)$, where $\mathscr{E}'(K)$ denotes the space of distributions on X with compact support contained in K. Using this identification we prove the following result, see Theorem 6.4.1 of [2].

Lemma 1. *If $K \Subset X$ and $\overline{L(C^{\infty}(K))} = C^{\infty}(K)$ then $\exists \phi \in C^{\infty}(X)$ such that $L^2\phi > 0$ on K.*

Proof. Choose $j \in \mathbb{N}$ such that $K \subset K_j$ and consider $\phi_1 \in C^{\infty}(X)$ satisfying $\phi_1 = 1$ on K. From the hypothesis it follows that there exist $\dot{\phi}_2, \dot{\phi} \in C^{\infty}(K)$ such that

$$L\dot{\phi}_2 - \dot{\phi}_1 \in \frac{1}{4} B_{p_j}, \tag{3}$$

and $L\dot{\phi} - \dot{\phi}_2 \in \frac{1}{4C} B_{p_{j+1}}$ (here $C > 0$ is given by (2)). From (2) we obtain

$$L(L\dot{\phi} - \dot{\phi}_2) \in \frac{1}{4} B_{p_j}. \tag{4}$$

Since $L^2\dot{\phi} - \dot{\phi}_1 = L(L\dot{\phi} - \dot{\phi}_2) + L\dot{\phi}_2 - \dot{\phi}_1$, from (3) and (4) we obtain $L^2\dot{\phi} - \dot{\phi}_1 \in \frac{1}{2} B_{p_j}$. Hence $\exists \psi \in L^2\dot{\phi} - \dot{\phi}_1$ such that $\sum_{|\alpha| \leqslant j} \sup_{K_j} |\partial^{\alpha}\psi| \leqslant \frac{3}{4}$, in particular $\sup_{K_j} |\psi| \leqslant \frac{3}{4}$.

But $K \subset K_j$ and $L^2\phi - \phi_1 = \psi$ on K, therefore $\sup_K |L^2\phi - \phi_1| \leqslant \frac{3}{4}$. Since $\phi_1 = 1$ on K it follows that $L^2\phi \geqslant \frac{1}{4}$ on K. $\square$

Denote $\mathscr{D}'(X)$ the space of the distributions on X.

Remark 1. Let L be a real non-singular vector field on X and $c \in C^\infty(X)$. If $u \in \mathscr{D}'(X)$ and $(L+c)u = 0$ by the Flow Box theorem it follows that $\operatorname{supp}(u)$ is invariant under the flow of L.

Lemma 2. *If Γ is a relatively compact orbit of the real vector field L then*

(i) $\exists u \in \mathscr{E}'(X)$ *such that* ${}^tLu = 0$ *and* $\operatorname{supp}(u) = \overline{\Gamma}$. *So* $\operatorname{singsupp}(u) = \Gamma$, *if Γ is a periodic orbit.*
(ii) *For each orbit Λ satisfying* $\Lambda \cap \partial\overline{\Gamma} \neq \emptyset$, $\exists u \in \mathscr{E}'(X)$ *such that* ${}^tLu = 0$ *and* $\operatorname{supp}(u) = \operatorname{singsupp}(u) = \overline{\Lambda} \subset \overline{\Gamma}$.

Proof. We will divide the proof in four steps. From Steps 1 and 2 we will have (i) and from Steps 3 and 4 will follow (ii).

Step 1. If Γ is a periodic orbit then $\exists u \in \mathscr{E}'(X)$ such that ${}^tLu = 0$ and $\operatorname{supp}(u) = \operatorname{singsupp}(u) = \Gamma$.

In fact, if Γ is a critical point then we may take u to be Dirac distribution. If Γ is a periodic orbit define

$$u(\phi) = \int_a^b \phi \circ \gamma(s)\, ds, \quad \phi \in C^\infty(X), \tag{5}$$

where $a \neq b$, $\gamma(a) = \gamma(b)$ and γ is the integral curve whose image is Γ. It is easy to see that $\operatorname{supp}(u) = \Gamma$. Since

$$WF(u) = \big\{(x,\xi) \in T^*(X);\ x \in \Gamma,\ \xi \neq 0 \text{ and } L(x,\xi) = 0\big\}$$

(see Example 8.2.5 of [7]) we have $\operatorname{singsupp}(u) = \Gamma$.

Step 2. If Γ is a non-periodic orbit then $\exists u \in \mathscr{E}'(X)$ such that ${}^tLu = 0$ and $\operatorname{supp}(u) = \overline{\Gamma}$.

In fact, from Lemma 1 and a result concerning solvability on compact subsets due to Duistermaat–Hörmander (see Theorem 6.4.1 of [2]) we have $\overline{L(C^\infty(\overline{\Gamma}))} \neq C^\infty(\overline{\Gamma})$. The Hahn–Banach theorem implies that there exists $0 \neq u \in \mathscr{E}'(\overline{\Gamma})$ such that $u = 0$ on $\overline{L(C^\infty(X))}$. Since ${}^tLu = 0$ and L is non-singular in a neighborhood of Γ, using Remark 1 we obtain $\operatorname{supp}(u) = \overline{\Gamma}$.

Step 3. If Λ is a non-periodic orbit then (ii) holds.

In fact, using the invariance of the sets $\alpha(x)$ and $\omega(x)$ under the flow and the hypothesis $\Lambda \cap \partial\overline{\Gamma} \neq \emptyset$ we obtain $\overline{\Lambda} \subset \overline{\Gamma}$. From (i) it follows that $\exists u \in \mathscr{E}'(X)$ such that ${}^tLu = 0$ and $\operatorname{supp}(u) = \overline{\Lambda}$. We will prove that $\operatorname{singsupp}(u) = \overline{\Lambda}$. From propagation of singularities (see Theorem 6.1.1 of [2]) it is sufficient to prove that

$$\Lambda \cap \operatorname{singsupp}(u) \neq \emptyset. \tag{6}$$

Let $\lambda : \mathbb{R} \to X$ be the integral curve whose image is Λ and $\psi \in C^\infty(X)$ such that $-{}^tL = L + \psi$. For each bounded interval $I \subset \mathbb{R}$, from Flow Box theorem $\exists \phi \in C^\infty(X)$ such that $L\phi = \psi$ in a neighborhood of $\lambda(I)$.

If $\Lambda \cap \operatorname{singsupp}(u) = \emptyset$ then u is a continuous function on Λ. Since $\operatorname{supp}(u) = \overline{\Lambda} \subset \overline{\Gamma}$ it follows that

$$u = 0 \quad \text{on } \partial\overline{\Gamma}. \tag{7}$$

Moreover, since u is a C^∞-function in a neighborhood of $\lambda(I)$ we have

$$\big((e^\phi u) \circ \lambda\big)'(s) = L\big(e^\phi u\big) \circ \lambda(s) = \big(e^\phi (L\phi)u + e^\phi Lu\big) \circ \lambda(s), \quad \forall s \in I.$$

But $L\phi = \psi$ in a neighborhood of $\lambda(I)$ and ${}^tLu = 0$, then

$$\big((e^{\phi}u)\circ\lambda\big)'(s) = 0, \quad \forall s \in I.$$

We proved that for any bounded interval $I \subset \mathbb{R}$, $\exists\phi \in C^\infty(X)$ such that $e^{\phi}u$ is a constant function on $\lambda(I)$. Since $\mathrm{supp}(u) = \overline{\Lambda}$ we obtain $u \neq 0$ on Λ. This is a contradiction with (7), since $\Lambda \cap \partial\overline{\Gamma} \neq \emptyset$. The proof of (6) is finished.

Step 4. If Λ is a periodic orbit then (ii) holds.

In fact, if Λ is a critical point then the result follows from Step 1. Otherwise, consider $a < b$ such that $\lambda(a) = \lambda(b)$. In this case, take $I = (a-\epsilon, b+\epsilon)$, where $\epsilon > 0$ is sufficiently small. The proof follows in the same way as the proof of Step 3. □

We say that $\Gamma := \gamma([a,b])$ is a *non-periodic interval of trajectory* of L when Γ is homeomorphic to the interval $[0,1] \subset \mathbb{R}$.

Lemma 3. *If $\Gamma = \gamma([a,b])$ is a non-periodic interval of trajectory of L then there exists $u \in \mathscr{E}'(X)$ such that*

$$\mathrm{supp}(u) = \mathrm{singsupp}(u) = \Gamma$$

and

$$\mathrm{supp}\big({}^tPu\big) = \mathrm{singsupp}\big({}^tPu\big) = \big\{\gamma(a), \gamma(b)\big\}.$$

Proof. As in (5) define

$$v(\phi) = \int_a^b \phi\circ\gamma(s)\,ds, \quad \phi \in C^\infty(X).$$

It is easy to see that $\mathrm{supp}(v) = \mathrm{singsupp}(v) = \Gamma$ and

$${}^tLv = \delta_{\gamma(b)} - \delta_{\gamma(a)}.$$

Here $\delta_{\gamma(a)}$, $\delta_{\gamma(b)}$ are the Dirac distributions supported on $\gamma(a)$ and $\gamma(b)$, respectively. Since $\gamma(a) \neq \gamma(b)$ we obtain

$$\mathrm{supp}\big({}^tLv\big) = \big\{\gamma(a), \gamma(b)\big\}. \tag{8}$$

From the Flow Box theorem, it follows that $\exists\phi \in C^\infty(X)$ such that $L\phi = c$ in a neighborhood Γ. Defining $u = e^{\phi}v$ we obtain ${}^tPu = e^{\phi}\cdot{}^tLv + e^{\phi}(c - L\phi)v$. Since $c = L\phi$ in a neighborhood Γ and $\mathrm{supp}(v) = \Gamma$ we have ${}^tPu = e^{\phi}\cdot{}^tLv$. From (8) we obtain the result. □

Proof of Proposition 1. For each $K \Subset X$ define

$$C_K = \{\Gamma;\ \Gamma \text{ is a compact interval of trajectory with endpoints in } K\}. \tag{9}$$

Let $\{K_j\}$ be a sequence of compact subsets of X with the properties (1).

Proof of (a) ⇒ (b.1). By taking $K = \emptyset$ in the definition of the P-convexity for singular supports we have that $\exists K' \Subset X$ with the following property:

$$u \in \mathscr{E}'(X), \quad {}^tLu = 0 \quad \Rightarrow \quad \mathrm{singsupp}(u) \subset K'. \tag{10}$$

We will prove that (b.1) holds with $\widetilde{K} = K'$. In fact, suppose that there exists an orbit Γ such that $\overline{\Gamma} \Subset X \setminus K'$. If Γ is a periodic orbit then from Lemma 2(i) there exists $u \in \mathscr{E}'(X)$ such that ${}^tLu = 0$ and $\operatorname{singsupp}(u) = \Gamma$. This contradicts (10). In case Γ is a non-periodic orbit then we have a contradiction with (10) because of Lemma 2(ii).

Proof of (a) $\Rightarrow$ (b.2). If (b.2) is false then $\exists K \Subset X$ and a sequence of integral curves $\gamma_j : [a_j, b_j] \to X$ such that $\Gamma_j := \gamma_j([a_j, b_j]) \in C_K$ but $\Gamma_j \not\subset K_j, \forall j \in \mathbb{N}$.

Choose an open subset V_K of X such that $K \subset V_K$ and $\overline{V_K} \Subset X$. Consider $j_0 \in \mathbb{N}$ such that $j \geqslant j_0 \Rightarrow \overline{V_K} \subset K_{j_0}$. Observe that Γ_j is not a critical point of L when $j \geqslant j_0$.

Suppose that $j \geqslant j_0$ and Γ_j is a periodic orbit of L. Since V_K is an open subset of X, $\exists c_j \in (a_j, b_j)$ such that $\gamma_j([a_j, c_j])$ is a non-periodic interval of trajectory, $\gamma_j([a_j, c_j]) \not\subset K_j$ and $\gamma_j(a_j), \gamma_j(c_j) \in V_K$.

For each $j \geqslant j_0$ define $\Gamma_j' = \Gamma_j$ if Γ_j is a non-periodic interval of trajectory and $\Gamma_j' = \gamma_j([a_j, c_j])$, otherwise. From Lemma 3, $\exists u_j \in \mathscr{E}'(X)$ such that $\operatorname{singsupp}({}^tLu_j) \subset V_K$ and $\operatorname{singsupp}(u_j) = \Gamma_j' \not\subset K_j$. Hence X is not convex for singular supports.

Proof of (b) $\Rightarrow$ (a). If X is not convex for singular supports then $\exists K \Subset X$ with the following property:

$$\forall K' \Subset X,\ \exists u \in \mathscr{E}'(X) \quad \text{such that} \quad \operatorname{singsupp}\big({}^tLu\big) \subset K \text{ but } \operatorname{singsupp}(u) \not\subset K'. \tag{11}$$

Let $\widetilde{K}$ be as in (b.1) and choose an open subset $V_{\widetilde{K}}$ of X such that $\widetilde{K} \subset V_{\widetilde{K}}$ and $\overline{V_{\widetilde{K}}} \Subset X$. Define $K_0 = K \cup \overline{V_{\widetilde{K}}}$. From (b.2) we have that $\exists K_0' \Subset X$ such that

$$\Gamma \in C_{K_0} \quad \Rightarrow \quad \Gamma \subset K_0'. \tag{12}$$

Property (11) implies there exist $u_0 \in \mathscr{E}'(X)$ and $x \in X$ such that

$$\operatorname{singsupp}\big({}^tLu_0\big) \subset K \tag{13}$$

and $x \in \operatorname{singsupp}(u_0) \setminus K_0'$. Hence $\Gamma_x^+ \cap K_0 = \emptyset$ or $\Gamma_x^- \cap K_0 = \emptyset$. In fact, if $\Gamma_x^+ \cap K_0 \neq \emptyset$ and $\Gamma_x^- \cap K_0 \neq \emptyset$ then, from (12), we have $x \in K_0'$. This is a contradiction. Then we may suppose that $K_0 \cap \Gamma_x^+ = \emptyset$. Since $K \subset K_0$ we obtain $K \cap \Gamma_x^+ = \emptyset$. Using (13) and propagation of singularities we obtain $\Gamma^+ \subset \operatorname{singsupp}(u_0)$. Hence $\overline{\Gamma_x^+} \Subset X$. But using (b.1) we have that Γ_x^+ is not relatively compact. □

Using the ideas of the proof of Proposition 1 we prove that the L-convexity for supports is equivalent to condition (b) of Proposition 1, when L is a real vector field. Then we have:

Remark 2. Let L be a real vector field on X. Then X is L-convex for supports if, and only if, X is L-convex for singular supports.

The proof of the following remark is analogous to the case $c \equiv 0$ proved in Proposition 1.

Remark 3. Let L be a real vector field on X and $c \in C^\infty(X)$. Define $P = L + c$. Consider the condition (b) of Proposition 1 and the following condition: (a′) X is P-convex for singular supports. Then (b) $\Rightarrow$ (a′) and (a′) $\Rightarrow$ (b.2). Moreover, if $c \in C_0^\infty(X)$ then (a′) $\Rightarrow$ (b.1).

3. Proof of Theorem 1

First we remark that any hyperbolic linear vector field on $\mathbb{R}^n$ satisfies the hypotheses (b) and (c) of Theorem 1. Since condition (NRC 1) implies that x_0 is a hyperbolic critical point of L, the following results imply Theorem 1.

Lemma 4. *With $X = \mathbb{R}^n$, suppose* (a) *holds. Then $\forall f \in C^\infty(\mathbb{R}^n)$, $\exists u \in C^\infty(\mathbb{R}^n)$ such that $Pu = f$ in a neighborhood of zero.*

Theorem 2. *Suppose that x_0 is a hyperbolic critical point. If* (b) *and* (c) *are true then $\forall f \in C^\infty(X)$ such that $f = 0$ in a neighborhood of x_0, $\exists u \in C^\infty(X)$, with $u = 0$ in a neighborhood of x_0, such that $Pu = f$.*

Observe that Theorem 2 holds for any smooth complex function c defined on X.

3.1. Proof of Lemma 4

Before the proof of Lemma 4 we will prove the following preliminary result:

Lemma 5. *Suppose that $X = \mathbb{R}^n$ and $x_0 = 0$. Condition* (NRC 2) *is equivalent to the property*: $\forall f \in C^\infty(\mathbb{R}^n)$, $\exists u \in C^\infty(\mathbb{R}^n)$ *such that $Pu - f$ is flat at the origin.*

Proof. We denote by $Pu \sim f$ when $Pu - f$ is flat at the origin. Write $L = \sum_{j=1}^n a_j \partial_j$ and consider formal Taylor expansions of u, a_j and c at $x = 0$:

$$\sum_\alpha \frac{\partial^\alpha u(0)}{\alpha!} x^\alpha,$$

$$\sum_\alpha \frac{\partial^\alpha a_j(0)}{\alpha!} x^\alpha, \quad j = 1, 2, \dots, n,$$

$$\sum_\alpha \frac{\partial^\alpha c(0)}{\alpha!} x^\alpha,$$

respectively. Then $Pu \sim f$ is equivalent to

$$\sum_{j,k} \alpha_k \partial_k a_j(0) \partial^{\alpha + e_j - e_k} u(0) + c(0)\partial^\alpha u(0) + R_\alpha = \partial^\alpha f(0), \quad \forall \alpha \in \mathbb{N}^n, \tag{14}$$

where e_j is the unit vector of $\mathbb{R}^n$ with 1 in the jth position. The term R_α depends only on the derivatives of u of order $\leqslant 1$ evaluated at the origin and has the following property: if $\partial^\beta u(0) = 0$, $\forall \beta \in \mathbb{N}^n$ such that $|\beta| \leqslant |\alpha| - 1$, then $R_\alpha = 0$, where $|\alpha| = \sum_{j=1}^n \alpha_j, \forall \alpha \in \mathbb{N}^n$.

$Pu \sim f$ is equivalent to a sequence of linear systems

$$\big(B^m + c(0)I\big)u^m = f^m + v^{m-1}, \quad m \in \mathbb{N}. \tag{15}$$

Consider $\Lambda_n^m = \{\alpha \in \mathbb{N}^n;\ |\alpha| = m\}$ and $M = \sharp \Lambda_n^m$. For each $m \in \mathbb{N}$, B^m is a real matrix $M \times M$ which depends on $DL(0)$ and on the choice of an ordering of Λ_n^m. The components of $u^m \in \mathbb{C}^M$ (resp. $f^m \in \mathbb{C}^M$) are the derivatives of u (resp. f) of order m evaluated at the origin. If $m \geqslant 1$ then the vector $v^{m-1} \in \mathbb{C}^M$ corresponds to the term R_α of (14). Define $v^0 = 0 \in \mathbb{R}$. The vector v^{m-1} depends only on the derivatives of u of order $\leqslant m - 1$ and this vector has the following property:

$$\partial^\alpha u(0) = 0, \quad \forall \alpha \in \mathbb{N}^n \text{ satisfying } |\alpha| \leqslant m - 1 \quad \Rightarrow \quad v^{m-1} = 0. \tag{16}$$

Using the real Jordan form for a choice of ordering of Λ_n^m we prove that

$$\operatorname{Spec} B^m \cap \mathbb{R} = \left\{ \sum_{j=1}^n m_j \operatorname{Re} \lambda_j;\ m_1, m_2, \dots, m_{n'} \in \mathbb{N} \text{ and } m_{n'+1}, m_{n'+2}, \dots, m_n \in 2\mathbb{N} \right\}. \tag{17}$$

Here Spec A denotes the set of the eigenvalues of the matrix A. Using (16) and (17) we conclude that the systems (15) can be solved recursively for $u^0, u^1, \ldots,$ if, and only if, (NRC 2) holds. □

Proof of Lemma 4. In view of Lemma 5 it is sufficient to prove that $\forall f \in C^\infty(\mathbb{R}^n)$ with f flat at the origin, $\exists u \in C^\infty(\mathbb{R}^n)$ such that $Pu = f$ in a neighborhood of the origin.

From (NRC 2) we obtain $c(0) \neq 0$. Define $P_1 = \frac{1}{c}P$ in a neighborhood of the origin. Then $P_1 = L_1 + 1$, where $L_1 = \frac{1}{c}L$. Since $L(0) = 0$ we have

$$DL_1(0) = \frac{1}{c(0)} DL(0).$$

Then (NRC 1) holds for L_1. From Sternberg's result there exists a change of coordinates which carries P_1 into P_2 corresponding to

$$\frac{1}{c(0)} DL(0) + 1.$$

From Guillemin–Schaeffer's result we conclude the proof of Lemma 4. □

3.2. Preliminaries for Theorem 2

Here, we will prove some preliminary results. Let L be a real vector field on $\mathbb{R}^2$. Suppose that the origin is a local attractor of L and $\{0\}$ is the unique critical point of L. Under these conditions, from Proposition 1 and since, for the case, convexity with respect of supports and singular support are the same, the result of dos Santos Filho [12, p. 263] can be written as, the origin is a global attractor of L if, and only if, $\mathbb{R}^2 \setminus \{0\}$ is convex with respect to the trajectories of L. We begin this section with a version of this result for an arbitrary manifold.

Lemma 6. *Suppose that X is a connected manifold and that $\{x_0\}$ is a local attractor of L. If*

(i) $\overline{\Gamma_x^+} \Subset X \Rightarrow \omega(x) = \{x_0\}$, *and*
(ii) *X is convex with respect to the trajectories of L*

then

$$\{x_0\} \textit{ is a global attractor of } L.$$

Proof. We will see that the boundary $\partial\mathcal{B}(x_0)$ of the basin of attraction $\mathcal{B}(x_0)$ is empty. Suppose there exists $x \in \partial\mathcal{B}(x_0)$. Since $\{x_0\}$ is a local attractor of L, $\mathcal{B}(x_0)$ is an open subset of X. Hence $\overline{\Gamma_x^+} \cap \mathcal{B}(x_0) = \emptyset$ then $x_0 \notin \overline{\Gamma_x^+}$. From (i) it follows that Γ_x^+ is not relatively compact orbit of L.

Consider neighborhoods U_x of x and U_{x_0} of x_0 such that $\overline{U_x}, \overline{U_{x_0}} \Subset X$. Take $K = \overline{U_{x_0}} \cup \overline{U_x}$. It is easy to see that for such K there is no compact K' satisfying the condition for convexity with respect to the trajectories of L, so (ii) is not true. □

If x_0 is a hyperbolic critical point local attractor for L, then the conditions (i) and (ii) of Lemma 6 are necessary for $\{x_0\}$ to be a global attractor of L.

Definition 1. A *global transversal* of L on X is a codimension one immersed submanifold Σ of X such that for all $x \in X$ there exists a unique $t \in \mathbb{R}$ such that $y = \gamma(t, x) \in \Sigma$ and $T_y(\Sigma) \oplus L(y) = T_y(X)$.

Here $T_x(M)$ denotes the tangent space of the manifold M at the point $x \in M$. Definition 1 is similar to the definition used in [1, p. 15]. Now, we state some simple remarks regarding this notion.

Remark 4. Let Σ be a global transversal of L on X.

(i) Let $\tau : X \to \mathbb{R}$ given by: for each $x \in X$, $\tau(x)$ is such that $\gamma(\tau(x), x) \in \Sigma$. Then $\tau \in C^\infty(X, \mathbb{R})$.
(ii) $M = \{(t, y);\ y \in \Sigma,\ t \in I_y\}$ is an open subset of $\mathbb{R} \times \Sigma$. $h : M \to X$ defined by $h(t, y) = \gamma(t, y)$ is a C^∞-diffeomorphism which carries $\frac{\partial}{\partial t}$ into L.

From Remark 4(ii) and Duistermaat–Hörmander's theorem (see Theorem 6.4.2 of [2]) we get that the existence of a global transversal of L on X is equivalent to the global solvability of L on $C^\infty(X)$. The next remark follows from Hartman's theorem (see Theorem 7.1 of [4]).

Remark 5. Let x_0 be a hyperbolic critical point of L. If $\{x_0\}$ is a global attractor of L then any global transversal of L on $X \setminus \{x_0\}$ is a compact subset of $X \setminus \{x_0\}$.

Sketch of the proof: Take a "sphere S centered at x_0" and contained at the neighborhood of x_0 preluded in Hartman's theorem. Then, we define the mapping T from S to Σ which takes any point of S to the unique point of Σ that belongs to the trajectory of L that passes through x_0. By continuous dependence, the injective mapping T is continuous. Therefore $T(S) \subset \Sigma$ is compact. But by the hypothesis of x_0 being a global attractor we have that, for any point y of Σ, the trajectory starting at y must go into the Hartman's neighborhood therefore must intercept S. Then T is onto, hence $\Sigma = T(S)$ is compact.

In the lemma below we construct a global transversal in the attractor case.

Lemma 7. *Let x_0 be a hyperbolic critical point of L. If $\{x_0\}$ is a global attractor of L then for all neighborhood V of x_0, there exists a global transversal Σ of L on $X \setminus \{x_0\}$ such that $\Sigma \subset V \setminus \{x_0\}$.*

Proof. Since $\{x_0\}$ is a global attractor, it follows that $\{x_0\}$ is the unique relatively compact orbit of L. From Hartman's theorem it follows that there exists a neighborhood U of x_0 such that $U \setminus \{x_0\}$ is convex with respect to the trajectories of L and $U \subset V$. Now, Duistermaat–Hörmander's theorem implies that exists a global transversal Σ of L on $U \setminus \{x_0\}$. Since $\{x_0\}$ is a global attractor of L then Σ is a global transversal of L on $X \setminus \{x_0\}$. □

The next result shows that an appropriated perturbation of a global transversal is still a global transversal.

Lemma 8. *Let Σ be a global transversal of L on X and $\chi \in C^\infty(\Sigma, \mathbb{R})$ such that $\omega_-(y) < \chi(y) < \omega_+(y)$, $\forall y \in \Sigma$. The image of the mapping $\sigma : \Sigma \to X$ given by $\sigma(y) = \gamma(\chi(y), y)$ is a global transversal of L on X.*

Proof. From Remark 4(ii) we may suppose that $X = M$ and $L = \frac{\partial}{\partial t}$. The result holds easily for this case. □

3.3. *Proof of Theorem 2*

Let s be the number of the eigenvalues of $DL(x_0)$ with negative real part. To prove Theorem 2 we consider two cases:

- Case A: $s \in \{0, n\}$ (attractor or repellent case).
- Case B: $s \notin \{0, n\}$ (saddle point case).

3.3.1. *Proof of Case A*

Suppose $s = n$ (the case $s = 0$ is analogous). From Lemma 6 it follows that $\{x_0\}$ is a global attractor of L. Let U be a neighborhood of x_0 such that $f = 0$ on U and

$$x \in U \quad \Rightarrow \quad \Gamma_x^+ \subset U. \tag{18}$$

Choose a neighborhood V of x_0 such that $\overline{V} \subset U$ and $\theta \in C^\infty(X)$ such that

$$\theta = 0 \quad \text{on } V \quad \text{and} \quad \theta = 1 \quad \text{on } \complement U. \tag{19}$$

From Remark 5 and Lemma 7 there exists a compact global transversal Σ of L on $X \setminus \{x_0\}$ contained in $V \setminus \{x_0\}$. From the Method of Characteristics it follows that $\exists \psi \in C^\infty(X \setminus \{x_0\})$ such that $L\psi = c\theta$ on $X \setminus \{x_0\}$ and $\psi = 0$ in a neighborhood of x_0. Then we may suppose $\psi \in C^\infty(X)$ and $L\psi = c\theta$ on X.

In the same way, using (18) we obtain $\phi \in C^\infty(X)$ such that $L\phi = e^\psi f$ on X and

$$\phi = 0 \quad \text{on } U. \tag{20}$$

Hence

$$P\big(\phi e^{-\psi}\big) = f + ce^{-\psi}\phi(1-\theta).$$

From (19) and (20) it follows $\phi(1-\theta) = 0$. Therefore, by taking $u = \phi e^{-\psi}$ we have $Pu = f$.

3.3.2. Preliminaries for Case B

We define the *stable* (resp. *unstable*) *manifold* of L at x_0 by

$$W^s(x_0) = \Big\{x \in X;\ \lim_{t \to \omega_+(x)} \gamma(t,x) = x_0\Big\}$$

(resp. $W^u(x_0) = \{x \in X;\ \lim_{t \to \omega_-(x)} \gamma(t,x) = x_0\}$), which is a C^∞ immersed submanifold of X. Take $X^s = X \setminus W^s(x_0)$ and $X^u = X \setminus W^u(x_0)$.

If Σ^s (resp. Σ^u) is a global transversal of L on X^s (resp. X^u), we denote $X^s_\pm(\Sigma^s) = \{\gamma(t,y);\ y \in \Sigma^s, \pm t > 0\}$ (resp. $X^u_\pm(\Sigma^u) = \{\gamma(t,y);\ y \in \Sigma^u,\ \pm t > 0\}$) subsets of X^s (resp. X^u).

The main result of this section is:

Proposition 2. *Let U_1 be a neighborhood of $\{x_0\}$. There exists a neighborhood U of $\{x_0\}$, with $U \subset U_1$, global transversal Σ^s_1 and Σ^s_2 of L on X^s, and global transversal Σ^u_1 and Σ^u_2 of L on X^u such that*:

(i) $\Sigma^u_2 \subset X^u_+(\Sigma^u_1)$ *and* $\Sigma^s_1 \subset X^s_+(\Sigma^s_2)$,
(ii) $X^u_+(\Sigma^u_1) \cup W^u(x_0) \subset X^s_+(\Sigma^s_1) \cup U$, *and*
(iii) $\forall f \in C^\infty(X)$ *such that* $f = 0$ *on* $X^s_-(\Sigma^s_2) \cup W^s(x_0) \cup U$ (*resp.* $X^u_+(\Sigma^u_1) \cup W^u(x_0)$), $\exists u \in C^\infty(X)$ *such that* $Lu = f$ *and* $u = 0$ *on* U (*resp.* $u = 0$ *on* $X^u_+(\Sigma^u_1) \cup W^u(0)$).

For the proof of Proposition 2, we do not use that $T_x(\Sigma^s_1) \oplus L(x) = T_x(X)$, $\forall x \in \Sigma^s_1$, similarly for Σ^u_1. In order to prove Proposition 2 we will use some preliminary results, here Lemma 9 to Lemma 13.

Lemma 9.

(i) $W^s(x_0) \cap W^u(x_0) = \{x_0\}$.
(ii) $W^s(x_0)$ (*resp.* $W^u(x_0)$) *is a closed subset of* X.

Proof. (i) If $x \in W^s(x_0) \cap W^u(x_0)$ then $\alpha(x) = \omega(x) = \{x_0\}$. Hence $\overline{\Gamma_x} \Subset X$. From (b) it follows that $x = x_0$.

(ii) If $W^s(x_0)$ is not closed in X then there exists a sequence $\{x_j\} \subset W^s(x_0)$ converging to some $x \in X \setminus W^s(x_0)$. Hence $x_0 \notin \omega(x)$. Since $\omega(x)$ is invariant under the flow, from (b) it follows that $\overline{\Gamma^+_x}$ is not relatively compact. Using the same arguments of the proof of Lemma 6 we obtain the result. □

From Lemma 9(ii) we have:

Remark 6. X^s (resp. X^u) is an open subset of X. Therefore $X^s_+(\Sigma^s)$ and $X^s_-(\Sigma^s)$ (resp. $X^u_+(\Sigma^u)$ and $X^u_-(\Sigma^u)$) are open subsets of X.

Moreover:

Lemma 10. *X^s (resp. X^u) is convex with respect to the trajectories of L.*

Proof. Suppose that X^s is not convex with respect to the trajectories of L, then there exist $K \Subset X$, a sequence $\{\Gamma_j\}$ of compact intervals of trajectories of L with endpoints in K and a sequence $\{x_j\}$ such that

$$x_j \in \Gamma_j \setminus K_j, \quad \forall j \in \mathbb{N}, \tag{21}$$

here $\{K_j\}$ is a sequence of compact subsets of X^s satisfying the properties (1). From hypothesis (c) of Theorem 2 it follows that $\exists K' \Subset \mathbb{R}^n$ such that $\{x_j\} \subset K'$. Hence there exist $x \in X$ and a subsequence $\{x_{j_k}\} \subset \{x_j\}$ such that $x_{j_k} \to x$. Without loss of generality, we may suppose that $x_j \to x$. Observe that from (21) we have

$$x \in W^s(x_0). \tag{22}$$

We will divide the rest of the proof in two cases.

Case $x \neq x_0$.

In this case take a sequence $\{C_k\}$ of compact subsets of X satisfying the properties (1). Since $x \neq x_0$, from (b) it follows that $\forall k \in \mathbb{N}$, $\exists y_k \in \Gamma_x \setminus C_k$. Using (22) we obtain $[x, y_k] \cap K = \emptyset$, then from Flow Box theorem there exists a neighborhood V_k of $[x, y_k]$ such that $L|_{V_k}$ is conjugated to ∂_1 and $V_k \cap K = \emptyset$. Since $x_j \to x$ it follows that $\exists j_k \in \mathbb{N}$ with the following property: $\forall j > j_k$, $\exists z_j \in \Gamma_j \setminus C_k$. Then (c) fails.

Case $x = x_0$.

From the proof of the previous case it is sufficient to prove that there exist $w \in W^s(x_0)$, with $w \neq x_0$, and a sequence $w_j \to w$ such that $w_j \in \Gamma_j$, $\forall j \in \mathbb{N}$.

Since $K \cap W^s(x_0) = \emptyset$ and $x_0 \in W^s(x_0)$ there exists a neighborhood V of x_0 satisfying $K \cap V = \emptyset$.

From Hartman's theorem we have there exists an open subset U of X such that $x_0 \in U \subset V$ and $U \setminus W^s(x_0)$ is convex with respect to the trajectories of L.

Consider a neighborhood W of x_0 such that $W \subset U$ and ∂W is homeomorphic to the sphere S^{n-1}. Choose $j_0 \in \mathbb{N}$ such that $j > j_0 \Rightarrow x_j \in W$. Since the endpoints of Γ_j are contained in K, from the continuity of Γ_j it follows that there exist $w_j, w'_j \in \Gamma_j \cap \partial W$ such that $x_j \in [w_j, w'_j]$. From a compactness argument there exist subsequences $\{w_{j_k}\} \subset \{w_j\}$ and $\{w'_{j_k}\} \subset \{w'_j\}$ such that $w_{j_k} \to w$ and $w'_{j_k} \to w'$. It is sufficient to prove that

$$w \in W^s(x_0) \quad \text{or} \quad w' \in W^s(x_0). \tag{23}$$

If $w \notin W^s(x_0)$ and $w' \notin W^s(x_0)$ then the sequences $\{w_{j_k}\}$ and $\{w'_{j_k}\}$ are contained in a compact subset of $\partial W \setminus W^s(x_0)$. Hence $U \setminus W^s(x_0)$ is not convex with respect to the trajectories of L. □

Using Lemma 10 we obtain:

Remark 7. $X^s_+(\Sigma^s)$ and $X^s_-(\Sigma^s)$ (resp. $X^u_+(\Sigma^u)$ and $X^u_-(\Sigma^u)$) are convex with respect to the trajectories of L.

Let Σ^s be a global transversal of L on X^s. Observe that $W^u(x_0)$ and Σ^s are immersed submanifold of X and Σ^s is transversal to $W^u(x_0)$. Then we have:

Remark 8. If Σ^s is a global transversal of L on X^s (resp. Σ^u is a global transversal of L on X^u) then $K := \Sigma^s \cap W^u(x_0)$ (resp. $K := \Sigma^u \cap W^s(x_0)$) is a global transversal of $L|_{W^u(x_0)}$ on $W^u(x_0) \setminus \{x_0\}$ (resp. of $L|_{W^s(x_0)}$ on $W^s(x_0) \setminus \{x_0\}$), furthermore $K \Subset X$.

Hartman's theorem is used to prove:

Lemma 11. *If Σ^s (resp. Σ^u) is a global transversal of L on X^s (resp. on X^u) then $X^s_-(\Sigma^s) \cup W^s(0)$ (resp. $X^u_+(\Sigma^u) \cup W^u(0)$) is an open subset of X.*

Proof. From Remark 6 is sufficient to prove that $\forall x \in W^s(x_0)$ there exists a neighborhood V_x of x such that $V_x \subset X^s_-(\Sigma^s) \cup W^s(x_0)$. In the other hand from the continuity of γ it is sufficient to prove that there exists a neighborhood V_0 of x_0 such that

$$V_0 \subset X^s_-\big(\Sigma^s\big) \cup W^s(x_0). \tag{24}$$

Consider the function $\tau : X^s \to \mathbb{R}$ given by Remark 4(i) and take $K = \Sigma^s \cap W^u(x_0)$. We will divide the rest of the proof in two steps.

Step 1. There exists an open subset U_0 of X such that $x_0 \in U_0$ and $U_0 \cap W^u(x_0) \setminus \{x_0\} \subset X^s_-(\Sigma^s)$.

In fact, since $K \Subset X$ (see Remark 8), there exists an open subset U_0 of X such that $x_0 \in U_0$, $U_0 \cap K = \emptyset$, U_0 satisfies the conclusion of Hartman's theorem and U_0 is convex with respect to the trajectories of L.

It is enough to prove that $\tau(y) > 0$, $\forall y \in U_0 \cap W^u(x_0) \setminus \{x_0\}$. From $U_0 \cap K = \emptyset$ we have $\tau(y) \neq 0$. Suppose that $\tau(y) < 0$. Since x_0 is a hyperbolic critical point of L and x_0 is a global attractor of $-L$ on $W^u(x_0)$, there exists an open subset A of $W^u(x_0)$, with $x_0 \in A \subset U_0 \cap W^u(x_0)$, such that

$$t \leqslant 0, \quad z \in A \quad \Rightarrow \quad \gamma(t, z) \in A. \tag{25}$$

Choose $t_0 < 0$ such that $\gamma(t_0, y) \in A$. If $\tau(y) \leqslant t_0$, from (25) it follows that $\gamma(\tau(y), y) \in U_0$. This is a contradiction, because $U_0 \cap K = \emptyset$. Hence $t_0 < \tau(y) < 0$. Since U_0 is convex with respect to the trajectories of L, these inequalities imply $\gamma(\tau(y), y) \in U_0$ and this is a contradiction with $K \cap U_0 = \emptyset$. Therefore we have $\tau(y) > 0$.

Step 2. There exists a neighborhood V_0 of x_0 with the property (24).

In fact, from Hartman's theorem there exists a subset Σ' of X such that $\Sigma' \subset U_0 \setminus \{x_0\}$ and Σ' is homeomorphic to S^{n-1}. Define $\Delta = \Sigma' \cap W^u(x_0)$. From Lemma 9(ii) we have $\Delta \Subset X$. From Step 1 it follows that there exists a neighborhood V_Δ of Δ such that

$$V_\Delta \subset X^s_-\big(\Sigma^s\big) \cap U_0. \tag{26}$$

Using (26), Hartman's theorem and the compactness of Δ we prove that there exists a neighborhood V_0 of x_0 such that $V_0 \setminus W^s(x_0) \subset X^s_-(\Sigma^s)$. This inclusion implies the statement of Step 2. □

From these lemmas we will construct global transversal of L on X^s with special properties. Denote $[x, y]$ the interval of trajectory of L with endpoints x and y.

Lemma 12. *Let U_1 be a neighborhood of $\{x_0\}$. Then there exists an open set U, with $x_0 \in U \subset U_1$, satisfying the conclusion of the Hartman's theorem with U convex with respect to the trajectories of L, and global transversal Σ^s_1 and Σ^s_2 of L on X^s such that:*

(i) $\Sigma^s_1 \cap W^u(x_0) \subset U$,
(ii) $\Sigma^s_1 \subset X^s_+(\Sigma^s_2)$, *and*
(iii) $x \in \Sigma^s_2$, $y \in \Gamma^+_x \cap U \Rightarrow [x, y] \subset U$.

Proof. From the hypothesis (b) of Theorem 2, Lemma 10 and Duistermaat–Hörmander's theorem it follows that there exists a global transversal Σ_0^s of L on X^s. From Lemma 11 there exists an open subset U of X, with $x_0 \in U \subset U_1$ such that: $U \subset X_-^s(\Sigma_0^s) \cup W^s(x_0), U$ satisfies the conclusion of Hartman's theorem and U is convex with respect to the trajectories of L. Observe that U has the additional property:

$$y \in \Sigma_0^s, \quad \gamma(t, y) \in U \quad \Rightarrow \quad t < 0. \tag{27}$$

We will divide the rest of the proof in four steps.

Step 1. There exist $T \in \mathbb{R}$ and an open subset W_0 of Σ_0^s, with $K \subset W_0$, such that

$$y \in W_0 \quad \Rightarrow \quad \omega_-(y) < T < 0 \tag{28}$$

and

$$y \in W_0 \quad \Rightarrow \quad \gamma(T, y), \gamma(T/2, y) \in U. \tag{29}$$

In fact, consider an open subset V of X such that $W^u(x_0) \subset V$ and $\omega_-(y) = -\infty$, $\forall y \in V$. Take $K = \Sigma_0^s \cap W^u(x_0)$. For each $y \in K$ take $t_y < 0$ such that $\gamma(t, y) \in U$, $\forall t \leqslant t_y$. From compactness of K there exists $T < 0$ such that $t \leqslant T \Rightarrow \gamma(t, y) \in U$, $\forall y \in K$. By continuity of γ it follows that there exists an open subset V_0 of X such that $K \subset V_0 \subset V$ and $\gamma(T, y), \gamma(T/2, y) \in U$, $\forall y \in V_0$. Set $W_0 = V_0 \cap \Sigma_0^s$.

Step 2. There exist a sequence $\{t_j\}_{j=1}^\infty \subset \mathbb{R}$ and a locally finite cover $\{W_j\}_{j=1}^\infty$ of Σ_0^s such that

$$y \in W_j \quad \Rightarrow \quad 0 < t_j < \omega_+(y). \tag{30}$$

In fact, for each $y \in \Sigma_0^s$ choose $t_y \in \mathbb{R}$ and a neighborhood V_y of y such that $0 < t_y < \omega_+(y)$, $\forall y \in V_y$. Consider a locally finite refinement $\{W_j\}_{j=1}^\infty$ of the cover $\{V_y \cap \Sigma_0^s\}_{y \in \Sigma_0^s}$. For each $j \geqslant 1$ choose V_y such that $W_j \subset V_y \cap \Sigma_0^s$ and define $t_j = t_y$. Hence Step 2 follows.

Consider $\mu_0 \in C^\infty(\Sigma_0^s, \mathbb{R})$ such that $0 \leqslant \mu_0 \leqslant 1$, $\mu_0 = 1$ in a neighborhood of K and $\operatorname{supp}(\mu_0) \subset W_0$. Let $\{\mu_j\}_{j=1}^\infty$ be a partition of unity subordinated to the cover $\{W_j\}_{j=1}^\infty$. Consider the functions $\chi_1, \chi_2 \in C^\infty(\Sigma_0^s, \mathbb{R})$ given by

$$\chi_1 = \frac{T}{2}\mu_0 + (1 - \mu_0)\sum_{j=1}^\infty t_j\mu_j \quad \text{and} \quad \chi_2 = T\mu_0.$$

Then we have the following result:

Step 3. For each $j = 1, 2$, the image Σ_j^s of the function

$$\begin{aligned} \sigma_j : \Sigma_0^s &\to X^s \\ y &\mapsto \gamma\big(\chi_j(y), y\big) \end{aligned}$$

is a global transversal of L on X^s.

In fact, from (28) it follows that $\omega_-(y) < \chi_2(y) < \omega_+(y)$, $y \in \Sigma_0^s$. In the same way, from (28) and (30) we have $\omega_-(y) < \chi_1(y) < \omega_+(y)$, $y \in \Sigma_0^s$. From Lemma 8 it follows that Σ_1^s and Σ_2^s are global transversal of L on X^s.

Step 4. The statements (i), (ii) and (iii) hold, if Σ_1^s and Σ_2^s are given as in Step 3.

In fact, to prove (i), observe that for each $x \in \Sigma_1^s \cap W^u(x_0)$, $\exists y \in K$ such that $x = \gamma(\chi_1(y), y)$ because Σ_0^s is a global transversal of L on X^s and $W^u(x_0)$ is invariant under the flow. Since $\mu_0(y) = 1$

and from (29) it follows that $x \in U$. So proof of (i) is concluded. Observe that (ii) follows from $\chi_2 < \chi_1$.

For (iii), first we observe that for each $x \in \Sigma_2^s$ and $y \in \Gamma_x^+ \cap U$, we can take $t \geqslant 0$ such that $\gamma(t, x) = y$. Since U is convex with respect to the trajectories of L, it is sufficient to prove that $x \in U$.

Choose $z \in \Sigma_0^s$ such that $\gamma(\chi_2(z), z) = x$. We will prove that $z \in W_0$. If $z \notin W_0$ then $\chi_2(z) = 0$. But $y = \gamma(t + \chi_2(z), z)$ we have $y = \gamma(t, z)$. Therefore from (27) it follows that $t < 0$. This is a contradiction. Then we have $z \in W_0$.

Since $T \leqslant \chi_2(z) \leqslant t + \chi_2(z)$ and U is convex with respect to the trajectories L, from (29) and $y \in U$ we have $x \in U$. $\square$

Also we have:

Lemma 13. *Let U be the neighborhood of x_0 and Σ_2^s the global transversal of L on X^s given by Lemma* 12. *There exist global transversal Σ_1^u and Σ_2^u of L on X^u such that*:

(i) $\Sigma_1^u \cap W^s(x_0) \subset U$,
(ii) $\Sigma_2^u \subset X_+^u(\Sigma_1^u)$, *and*
(iii) $\Sigma_1^u = \Sigma_1^s$ *on* $\complement U$.

Proof. In the same way as the proof of Lemma 12 we have that there exists a global transversal Σ_0^u of L on X^u such that $K := \Sigma_0^u \cap W^s(x_0) \subset U$. Consider the function $\tau : X^s \to \mathbb{R}$ given by $\gamma(\tau(y), y) \in \Sigma_1^s$. We will divide the rest of the proof in three steps.

Step 1. There exists an open subset W_0 of Σ_0^u such that $K \subset W_0 \subset U$ and

$$y \in W_0 \quad \Rightarrow \quad \gamma\bigl(\tau(y), y\bigr) \in U. \tag{31}$$

In fact, consider a subset Σ' of $U \setminus \{0\}$ homeomorphic to S^{n-1}. Here the homeomorphism is given by Hartman's theorem. Take $\Delta = \Sigma' \cap W^u(x_0)$. Using Lemma 12(i) it follows that there exists a neighborhood V_Δ of Δ such that

$$y \in V \quad \Rightarrow \quad \gamma\bigl(\tau(y), y\bigr) \in U. \tag{32}$$

Moreover, using the compactness of Δ and Hartman's theorem we prove that there exists a neighborhood V_0 of x_0 with the following property:

$$y \in V_0 \setminus W^s(0) \quad \Rightarrow \quad \exists t \in \mathbb{R} \quad \text{such that} \quad \gamma(t, y) \in V. \tag{33}$$

From (32), (33) and from the continuity of γ Step 1 follows.

Consider $\mu \in C^\infty(\Sigma_0^u, \mathbb{R})$ such that $0 \leqslant \mu \leqslant 1$, $\mu = 1$ in a neighborhood of K and $\operatorname{supp}(\mu) \subset W_0$. Since Σ_0^u is an immersed submanifold of X, we have $\tau|_{\Sigma_0^u \setminus K} \in C^\infty(\Sigma_0^u \setminus K)$. Let $\chi_1 : \Sigma_0^u \to X^u$ be the function given by $\chi_1 = (1 - \mu)\tau|_{\Sigma_0^u \setminus K}$. Then we have that $\chi_1 \in C^\infty(\Sigma_0^u, \mathbb{R})$.

Step 2. The image Σ_1^u of the function

$$\sigma_1 : \Sigma_0^u \to X^u$$
$$y \mapsto \gamma\bigl(\chi_1(y), y\bigr)$$

is a global transversal of L on X^u which satisfies (i).

In fact, from Lemma 8, Σ_1^u is a global transversal of L on X^u. Since $\mu = 1$ on K we have $\Sigma_1^u \cap W^s(0) = K$, hence $\Sigma_1^u \cap W^s(0) \subset U$. Then Step 2 follows.

The existence of Σ_2^u with the property is proved in the same way as in the proof of Lemma 12(iii).

Step 3. The statement (iii) holds.
In fact, we will prove that

$$\Sigma_1^u \cap \complement U \subset \Sigma_1^s \tag{34}$$

and

$$\Sigma_1^s \cap \complement U \subset \Sigma_1^u. \tag{35}$$

To prove (34), take $x \in \Sigma_1^u \cap \complement U$ and choose $y \in \Sigma_0^u$ such that $\gamma(\chi_1(y), y) = x$. If $y \in W_0$ then from (31) and $|\chi_1(y)| \leqslant |\tau(y)|$ result $x \in U$. This is a contradiction. From $y \notin W_0$ it follows that $\chi_1(y) = \tau(y)$. Hence $x \in \Sigma_1^s$ and the proof of (34) is finished. In the same way we prove (35). □

Proof of Proposition 2. Proof of (i). Use Lemma 12(ii) and Lemma 13(ii), respectively.
Proof of (ii). From Lemma 12(i) it follows that $W^u(x_0) \subset X_+^s(\Sigma_1^s) \cup U$, and Lemma 13(iii) implies $X_+^u(\Sigma_1^u) \subset X_+^s(\Sigma_1^s) \cup U$.
Proof of (iii). Use the Method of Characteristics, Lemma 12(iii) (resp. Lemma 13(ii)) and Lemma 11. □

3.3.3. Proof of Case B

Let U_1 be a neighborhood of x_0 such that $f = 0$ on U_1. With the notation of Proposition 2, we will prove Case B in two steps.

Step 1. $\forall f \in C^\infty(X)$ such that $f = 0$ on U, $\exists u_1 \in C^\infty(X)$ such that $Pu_1 = f$ on $U \cup X_+^s(\Sigma_1^s)$.
In fact, from Proposition 2(i) and Lemma 10 choose $\theta_1 \in C^\infty(X)$ such that

$$\theta_1 = 0 \quad \text{on } X_-^s\big(\Sigma_2^s\big) \cup W^s(x_0) \quad \text{and} \quad \theta_1 = 1 \quad \text{on } X_+^s\big(\Sigma_1^s\big). \tag{36}$$

By the Method of Characteristics and Lemma 11, $\exists \psi_1 C^\infty(X)$ such that $L\psi_1 = c\theta_1$. From Proposition 2(iii), $\exists \phi_1 C^\infty(X)$ such that $L\phi_1 = \theta_1 f e^{\psi_1}$ and $L\phi_1 = \theta_1 f e^{\psi_1}$ and

$$\phi_1 = 0 \quad \text{on } U. \tag{37}$$

Hence

$$P\big(\phi_1 e^{-\psi_1}\big) = \theta_1 f + c e^{-\psi_1} \phi_1 (1 - \theta_1).$$

Since $f = 0$ on U, from (36) and (37) it follows that on $X_+^s(\Sigma_1^s) \cup U$ we have

$$\phi_1(1 - \theta_1) = 0 \quad \text{and} \quad \theta_1 f = f.$$

Therefore, by taking $u_1 = \phi_1 e^{-\psi_1}$ Step 1 follows.

Step 2. $\forall f \in C^\infty(X)$ such that $f = 0$ on $U \cup X_+^s(\Sigma_1^s)$, $\exists u \in C^\infty(X)$ such that $Pu = f$ on X.
In fact, from Proposition 2(i) and Lemma 11, choose $\theta_2 \in C^\infty(X)$ such that

$$\theta_2 = 0 \quad \text{on } X_+^u\big(\Sigma_2^u\big) \cup W^u(x_0) \quad \text{and} \quad \theta_2 = 1 \quad \text{on } X_-^u\big(\Sigma_1^u\big).$$

Therefore, $\exists \psi_2 \in C^\infty(X)$ such that $L\psi_2 = c\theta_2$. Since $f = 0$ on $U \cup X_+^s(\Sigma_1^s)$, from Proposition 2(ii)–(iii) it follows that $\exists \phi_2 \in C^\infty(X)$ such that $L\phi_2 = f e^{\psi_2}$ and $\phi_2 = 0$ on $U \cup X_+^s(\Sigma_1^s)$.
Hence

$$P\big(\phi_2 e^{-\psi_2}\big) = f + c e^{-\psi_2} \phi_2 (1 - \theta_2),$$

and

$$\phi_2(1-\theta_2)=0 \quad \text{on } X_+^s\left(\Sigma_1^u\right)\cup U\cup X_-^u\left(\Sigma_1^u\right).$$

Therefore, taking $u=\phi_2 e^{-\psi_2}$ Step 2 follows.

Remark 9. The hypotheses (NRC 2) and (c) are necessary for global solvability of P on $C^\infty(X)$ from Lemma 5; and Remark 2, Theorem 4 of [9], respectively.

When L is a linear vector field on $\mathbb{R}^n$, it is easy to see that (b) and (c) of Theorem 1 are verified. In this case, the hypothesis of linearization (NRC 1) is dropped and we have that $P=L+c$ is globally solvable on $C^\infty(\mathbb{R}^n)$ if, and only if, (NRC 2) holds. In particular, the condition (NRC 1) is not necessary for global solvability.

Now, we present a family of operators for which the condition (b) is necessary for global solvability. Take $p(x)=\sum_{j=0}^n a_j x^j$, be a real polynomial. Let L be the vector field on $\mathbb{R}^2$ given by

$$L=x_1(1-x_1)\partial_1+x_2 g(x_1,x_2)\partial_2, \quad (x_1,x_2)\in\mathbb{R}^2,$$

where $g\in C^\infty(\mathbb{R}^2)$. Notice that $(0,0),(1,0)$ are critical points and $(0,1)\times\{0\}$ is a relatively compact orbit of L. Take the operator $P=L+c$ with $c\in C^\infty(\mathbb{R}^2)$ satisfying

$$c(x_1,0)=p(x_1), \quad x_1\in\mathbb{R}.$$

Under these hypotheses we have (see [16, p. 59]): If

$$a_0\notin\mathbb{Z} \quad \text{and} \quad a_j\notin\{1,2,\ldots\}, \quad j=1,2,\ldots,n,$$

then $\exists u\in\mathscr{E}'\in(\mathbb{R}^2)$ such that ${}^tPu=0$ and $\operatorname{supp}(u)=[0,1]\times\{0\}$. Hence P is not globally solvable on $C^\infty(\mathbb{R}^2)$.

References

[1] P.L. Dattori da Siva, Resolubilidade Global para Campos Reais, Master Dissertation, DM, UFSCar, 1999.
[2] J. Duistermaat, L. Hörmander, Fourier integral operators II, Acta Math. 128 (1972) 183–269.
[3] V. Guillemin, D. Schaeffer, On a certain class of Fuchsian partial differential equations, Duke Math. J. 44 (1977) 157–199.
[4] P. Hartman, Ordinary Differential Equations, John Wiley and Sons, 1964.
[5] C. Harvey, On domination estimates and global existence, J. Math. Mech. 16 (7) (1967) 675–702.
[6] J. Hounie, A note on global solvability of vector fields, Proc. Amer. Math. Soc. 94 (1) (1985) 61–64.
[7] L. Hörmander, The Analysis of Linear Partial Differential Equations, vol. 1, Springer-Verlag, 1983.
[8] K. Kuperberg, A smooth counterexample to the Seifert conjecture, Ann. of Math. 140 (1994) 723–732.
[9] B. Malgrange, Existence et approximation des solutions des équations aus dérivées partielles et des équations de convolution, Ann. Inst. Fourier (Grenoble) 6 (1955–1956) 271–355.
[10] E. Nelson, Topics in Dynamics I: Flows, Princeton University Press, Princeton, 1969.
[11] J.R. dos Santos Filho, Propagation of singularities through radial points, Comm. Partial Differential Equations 6 (1981) 399–435.
[12] J.R. dos Santos Filho, Injective mappings and solvable vector fields of Euclidean spaces, Topology Appl. 136 (2004) 261–274.
[13] P. Schweitzer, Counterexamples to the Seifert conjecture and opening closed leaves of foliations, Ann. of Math. 100 (1974) 386–400.
[14] H. Seifert, Closed integral curves in 3-spaces and isotopic two-dimensional deformation, Proc. Amer. Math. Soc. 1 (1950) 287–302.
[15] S. Sternberg, The structure of local homeomorphisms II, Amer. J. Math. 80 (1958) 623–632.
[16] M.F. da Silva, Resolubilidade Global para Operadores Diferenciais Reais de Ordem Um, PhD thesis in Mathematics, UFSCar, DM, 2006.
[17] F. Treves, Locally Convex Spaces and Linear Partial Differential Equations, Grundlehren Math. Wiss., vol. 146, Springer-Verlag, 1967.