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Phantom metrics with Killing spinors

W.A. Sabra

Centre for Advanced Mathematical Sciences and Physics Department, American University of Beirut, Lebanon

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ABSTRACT

We study metric solutions of Einstein–anti-Maxwell theory admitting Killing spinors. The analogue of the IWP metric which admits a space-like Killing vector is found and is expressed in terms of a complex function satisfying the wave equation in flat $(2+1)$-dimensional space–time. As examples, electric and magnetic Kasner spaces are constructed by allowing the solution to depend only on the time coordinate. Euclidean solutions are also presented.

1. Introduction

The first systematic classification of metrics admitting super-covariantly constant spinors in Einstein–Maxwell theory was performed many years ago by Tod in [1]. The analysis of Tod was to some extent motivated by the results of Gibbons and Hull [2]. The metrics found in [1] are bosonic solutions of minimal $N=2$ supergravity theory admitting half of the supersymmetry. In the context of the supergravity theory, the Killing spinor equation represents the vanishing of the gravitini supersymmetry transformation in a bosonic background. The metrics with a time-like Killing vector are the known Israel–Wilson–Perjés (IWP) metrics [3] with the static limit given by the Majumdar–Papapetrou (MP) metrics [4]. The second class of metrics with a null-Killing vector is given by plane-wave space–times [5]. In recent years, a considerable amount of research activities has been devoted to the understanding and the systematic classification of supersymmetric solutions in ungauged, gauged and fake (de Sitter) supergravity theories in various dimensions (see for example [6]). Fake de Sitter supergravity can be obtained by analytic continuation of anti de Sitter supergravity. We also note that de Sitter supergravities can also be obtained as genuine low energy effective theories of the so called * theories of [7]. For instance, a non-linear Kaluza Klein reduction arising of IIB* string theory and M* theory produce four and five-dimensional de Sitter supergravities with vector multiplets. However these theories have actions where some of the gauge fields kinetic terms have the non-conventional sign [8]. Black hole solutions with anti or phantom Maxwell fields[1] have been studied and analyzed in [9].

Black hole solutions with phantom fields and their relations to astrophysics and dark matter were also considered by many authors (see [10] and references therein). However, to our knowledge phantom solutions with Killing spinors have not yet been discussed.

In our present work, we shall study metrics admitting Killing spinors in gravitational theories with anti-Maxwell fields. We shall only focus on the simplest theory of four-dimensional Einstein gravity coupled to a Maxwell field as a first step for a future study of supergravity theories with many anti-Maxwell and scalar fields in various space–time dimensions. We will consider both the Lorentzian and the Euclidean theory. The action of the theory is given by

$$S=\int d^4x\sqrt{-g}\left(R+\kappa^2F_{\mu\nu}F^{\mu\nu}\right), \tag{1.1}$$

where $F_{\mu\nu}$ is the $U(1)$ gauge field strength. We have introduced a parameter κ which for $\kappa=i$, corresponds to the standard Einstein–Maxwell theory and for $\kappa=1$ corresponds to the Einstein–anti-Maxwell theory, i.e., where the Maxwell field kinetic term comes with the wrong sign. The signature of the metric is taken to be $(-,+,+,+)$. For $\kappa=1$, this action can be thought of as the bosonic part of a fake minimal $N=2$, $D=4$ supergravity. The Einstein and gauge field equations derived from (1.1) are

$$R_{\mu\nu}=-\kappa^2\left(2F_{\mu\rho}F_\nu{}^\rho-\frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right),$$
$$d*F=0\,. \tag{1.2}$$

Here F is the two form representing the gauge field strength $F_{\mu\nu}$. The Killing spinor equation is given by

$$\left(\partial_\mu+\frac{1}{4}\omega_{\mu,\nu_1\nu_2}\gamma^{\nu_1\nu_2}+\frac{\kappa}{4}F_{\nu_1\nu_2}\gamma^{\nu_1\nu_2}\gamma_\mu\right)\varepsilon=0. \tag{1.3}$$

E-mail address: waficsabra@gmail.com.

[1] There anti or phantom Maxwell field refers to an electromagnetic field of the opposite sign from usual.

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where ε is a non-zero Dirac Killing spinor and $\omega_{\mu,\nu_1\nu_2}$ are the spin connections components. For $\kappa = i$, the Killing spinor equation is simply the vanishing of the gravitini supersymmetry transformation in a bosonic background of minimal $N = 2$, $D = 4$ supergravity.

We note that if one interchanges F by its Hodge dual $* F$ in the Lorentzian Einstein–Maxwell equations, this simply maps solutions to solutions as Lorentzian Maxwell stress energy tensor is unchanged by this transformation. In the Euclidean case, however, this is no longer the case and the stress energy tensor picks up a minus sign. Therefore with Euclidean signatures, solutions for theory with the wrong sign of the coupling of Maxwell field are those for the theory with the "correct" sign of the coupling but with F and $* F$ interchanged [11]. This can also be seen from the inspection of the Killing spinor equations [12].

We shall use the spinorial geometry method which has proved to be a very powerful method in the classification of geometric backgrounds admitting various fractions of supersymmetry in supergravity theories. The isomorphism between Clifford algebras and exterior algebras allows one to express the Killing spinor in terms of differential forms. The canonical forms of the spinor are basically representatives up to gauge transformations which preserve the supercovariant connection (the reader can refer to [13] for spin geometry as well as supersymmetric black holes classifications).

Following [13], Dirac spinors in four space–time dimensions can be written as complexified forms on $\mathbb{R}^2$

$$\varepsilon = \lambda 1 + \mu_1 e^1 + \mu_2 e^2 + \sigma e^{12}, \tag{1.4}$$

where e^1, e^2 are 1-forms on $\mathbb{R}^2$ and $e^{12} = e^1 \wedge e^2$. The functions λ, μ_i and σ are complex functions. The action of γ-matrices on these forms is given by

$$\begin{aligned}
\gamma_0 &= -e^2 \wedge + i_{e^2}, \\
\gamma_1 &= e^1 \wedge + i_{e^1}, \\
\gamma_2 &= e^2 \wedge + i_{e^2}, \\
\gamma_3 &= i(e^1 \wedge - i_{e^1}) .
\end{aligned} \tag{1.5}$$

and γ_5 is defined by $\gamma_5 = i\gamma_{0123}$ and satisfies

$$\gamma_5 1 = 1, \qquad \gamma_5 e^{12} = e^{12}, \qquad \gamma_5 e^i = -e^i,\ i = 1, 2. \tag{1.6}$$

Following [14] we define

$$\begin{aligned}
\gamma_+ &= \frac{1}{\sqrt{2}}(\gamma_2 + \gamma_0) = \sqrt{2} i_{e^2}, \\
\gamma_- &= \frac{1}{\sqrt{2}}(\gamma_2 - \gamma_0) = \sqrt{2} e^2 \wedge, \\
\gamma_1 &= \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_3) = \sqrt{2} i_{e^1}, \\
\gamma_{\bar{1}} &= \frac{1}{\sqrt{2}}(\gamma_1 - i\gamma_3) = \sqrt{2} e^1 \wedge .
\end{aligned} \tag{1.7}$$

In this basis the non-zero metric components are given by $g_{+-} = 1$, $g_{1\bar{1}} = 1$.

As has been demonstrated in [14], using $Spin(3, 1)$ gauge transformations, one finds the three canonical orbits:

$$\varepsilon = 1 + \mu_2 e^2, \quad \varepsilon = 1 + \mu_1 e^1, \quad \varepsilon = e^2, \tag{1.8}$$

where μ_1 and μ_2 are complex functions. Note that the first orbit represents the Killing spinor for the IWP metric which has a time-like Killing vector. The other two orbits correspond to plane-waves with null Killing vector. In this letter we are interested in finding phantom solutions for the Killing spinor $\varepsilon = 1 + \mu e^2$. Similarly we consider the analogue solutions in the Euclidean case.

2. Phantom IWP solutions

For the orbit $\varepsilon = 1 + \mu e^2$, the integrability conditions of the Killing spinor equation are consistent with the equations of motion. Any solution of the Killing spinor equation in which the gauge field satisfies the Bianchi identity and Maxwell equation is automatically a solution of Einstein equations of motion.

Our solution can be written in the form

$$ds_4^2 = 2\mathbf{e}^+\mathbf{e}^- + 2\mathbf{e}^1\mathbf{e}^{\bar{1}}. \tag{2.1}$$

Plugging $\varepsilon = 1 + \mu e^2$ in (1.3) and using (1.7), the Killing spinor equations amounts to a set of sixteen algebraic and differential equations:

$$\begin{aligned}
-(\omega_{+,+-} + \omega_{+,1\bar{1}}) - \sqrt{2}\kappa\mu(F_{+-} + F_{1\bar{1}}) &= 0, \\
\omega_{+,-1} &= 0, \\
\partial_+\mu + \frac{\mu}{2}(\omega_{+,+-} - \omega_{+,1\bar{1}}) &= 0, \\
\omega_{+,+1} + \kappa\sqrt{2}\mu F_{+1} &= 0, \\
\omega_{-,+-} + \omega_{-,1\bar{1}} &= 0, \\
\mu\omega_{-,-1} + \kappa\sqrt{2}F_{-1} &= 0, \\
\partial_-\mu + \frac{\mu}{2}(\omega_{-,+-} - \omega_{-,1\bar{1}}) + \frac{\kappa}{\sqrt{2}}(F_{+-} - F_{1\bar{1}}) &= 0, \\
\omega_{-,+1} &= 0, \\
\omega_{1,+-} + \omega_{1,1\bar{1}} &= 0, \\
\omega_{1,-1} &= 0, \\
\partial_1\mu + \frac{\mu}{2}(\omega_{1,+-} - \omega_{1,1\bar{1}}) &= 0, \\
\omega_{1,+1} &= 0, \\
-\frac{1}{2}(\omega_{\bar{1},+-} + \omega_{\bar{1},1\bar{1}}) + \kappa\sqrt{2}\mu F_{-\bar{1}} &= 0, \\
\mu\omega_{\bar{1},-1} + \frac{\kappa}{\sqrt{2}}(F_{+-} - F_{1\bar{1}}) &= 0, \\
\partial_{\bar{1}}\mu + \frac{\mu}{2}(\omega_{\bar{1},+-} - \omega_{\bar{1},1\bar{1}}) + \kappa\sqrt{2}F_{+\bar{1}} &= 0, \\
\omega_{\bar{1},+1} - \frac{\kappa}{\sqrt{2}}\mu(F_{+-} + F_{1\bar{1}}) &= 0.
\end{aligned} \tag{2.2}$$

The analysis of this system of equations gives the following relations for the gauge field strength components

$$\begin{aligned}
F_{+-} &= -\frac{1}{\sqrt{2}}\partial_-\left(\kappa\bar{\mu} + \bar{\kappa}\mu\right), \\
F_{1\bar{1}} &= -\frac{1}{\sqrt{2}}\partial_-\left(\kappa\bar{\mu} - \bar{\kappa}\mu\right), \\
F_{-1} &= -\frac{\kappa}{\sqrt{2}|\mu|^2}\partial_1\mu, \\
F_{+1} &= -\frac{\kappa}{\sqrt{2}}\partial_1\bar{\mu},
\end{aligned} \tag{2.3}$$

together with the relation

$$\left(\partial_+ + \kappa^2|\mu|^2\partial_-\right)\mu = 0. \tag{2.4}$$

We also obtain for the spin connections

$$\omega_{+-} = \kappa^2\partial_-|\mu|^2\mathbf{e}^+ - \partial_1\log\mu\mathbf{e}^1 - \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}},$$
$$\omega_{1\bar{1}} = \kappa^2|\mu|^2\partial_-\log\frac{\bar{\mu}}{\mu}\mathbf{e}^+ + \partial_1\log\mu\mathbf{e}^1 - \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}},$$
$$\omega_{+1} = -\kappa^2\mu\left(\partial_-\bar{\mu}\mathbf{e}^{\bar{1}} - \partial_1\bar{\mu}\mathbf{e}^+\right),$$
$$\omega_{-1} = \frac{1}{\mu}\left(\frac{\kappa^2}{|\mu|^2}\partial_1\mu\mathbf{e}^- + \partial_-\mu\mathbf{e}^{\bar{1}}\right). \quad (2.5)$$

The equations (2.4) and (2.5) can be used to demonstrate that the vector V,

$$V = |\mu|^2\mathbf{e}^+ + \kappa^2\mathbf{e}^- = |\mu|^2\partial_- + \kappa^2\partial_+ \quad (2.6)$$

is a Killing vector which is space-like for $\kappa^2 = 1$ and time-like for $\kappa^2 = -1$.

Moreover, the vanishing of the torsion, i.e.,

$$d\mathbf{e}^a + \omega^a{}_b \wedge \mathbf{e}^b = 0, \quad (2.7)$$

implies the following relations

$$d\mathbf{e}^1 = -d(\log\bar{\mu}) \wedge \mathbf{e}^1, \quad (2.8)$$

and

$$d\mathbf{e}^+ = \left(\mathbf{e}^+ - \frac{\kappa^2}{|\mu|^2}\mathbf{e}^-\right) \wedge \left(\partial_1\log\mu\mathbf{e}^1 + \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}}\right) + \partial_-\log\frac{\mu}{\bar{\mu}}\mathbf{e}^1 \wedge \mathbf{e}^{\bar{1}}, \quad (2.9)$$

and

$$d\mathbf{e}^- = -\kappa^2\partial_-\left(|\mu|^2\right)\mathbf{e}^+ \wedge \mathbf{e}^- + \left(\partial_1\log\mu\mathbf{e}^1 + \partial_{\bar{1}}\log\bar{\mu}\mathbf{e}^{\bar{1}}\right) \wedge \mathbf{e}^- - \kappa^2\left(\mu\partial_-\bar{\mu} - \bar{\mu}\partial_-\mu\right)\mathbf{e}^1 \wedge \mathbf{e}^{\bar{1}} - \kappa^2\mathbf{e}^+ \wedge \left(\bar{\mu}\partial_{\bar{1}}\mu\mathbf{e}^{\bar{1}} + \mu\partial_1\bar{\mu}\mathbf{e}^1\right). \quad (2.10)$$

Using these relations, it can be shown that $\left(|\mu|^2\mathbf{e}^+ - \kappa^2\mathbf{e}^-\right)$ satisfies

$$d\left(|\mu|^2\mathbf{e}^+ - \kappa^2\mathbf{e}^-\right) = 0 \quad (2.11)$$

and thus is a total differential.[2]

The relations (2.6), (2.8) and (2.11) enable us to introduce the real coordinates (t, x, y, z), such that

$$\mathbf{e}^- = -\frac{1}{\sqrt{2}}\left(\kappa^2dz - |\mu|^2(dt + \phi)\right),$$
$$\mathbf{e}^+ = \frac{1}{\sqrt{2}|\mu|^2}\left(dz + \kappa^2|\mu|^2(dt + \phi)\right),$$
$$\mathbf{e}^1 = \frac{1}{\bar{\mu}\sqrt{2}}(dx + idy). \quad (2.12)$$

Here ϕ is a one form independent of the coordinate t and $\phi_t = 0$. Note that (2.4) implies that μ is independent of t. The metric is therefore given by

$$ds_4^2 = 2\mathbf{e}^+\mathbf{e}^- + 2\mathbf{e}^1\mathbf{e}^{\bar{1}} = \kappa^2|\mu|^2(dt + \phi)^2 + \frac{1}{|\mu|^2}ds_3^2 \quad (2.13)$$

where $ds_3^2 = \left(-\kappa^2dz^2 + dx^2 + dy^2\right)$. Moreover, substituting the relations (2.12) into (2.9) or (2.10), one can derive the relation

$$d\phi = \frac{i\kappa^2}{|\mu|^2} *_3 d\log\frac{\bar{\mu}}{\mu}, \quad (2.14)$$

where $*_3$ is the Hodge dual with metric ds_3^2.

Using (2.3), (2.12) and

$$\partial_+ = \frac{|\mu|^2}{\sqrt{2}}\partial_z, \qquad \partial_- = -\frac{\kappa^2}{\sqrt{2}}\partial_z, \qquad \partial_1 = \frac{\bar{\mu}}{\sqrt{2}}\left(\partial_x - i\partial_y\right), \quad (2.15)$$

we obtain for the gauge field strength two form

$$F = \frac{1}{2}d\left(\bar{\kappa}\bar{\mu} + \kappa\mu\right) \wedge (dt + \phi) - \frac{i}{2|\mu|^2} *_3 d\left(\bar{\kappa}\mu - \kappa\bar{\mu}\right). \quad (2.16)$$

The dual gauge field strength two form is given by

$$*F = \frac{i}{2}d\left(\bar{\kappa}\bar{\mu} - \kappa\mu\right) \wedge (dt + \phi) - \frac{1}{2|\mu|^2} * d\left(\bar{\kappa}\mu + \kappa\bar{\mu}\right).$$

Using (2.14), F and $*F$ can be rewritten in the form

$$F = \frac{1}{2}d\left[\left(\frac{\bar{\mu}}{\kappa} + \frac{\mu}{\bar{\kappa}}\right)(dt + \phi)\right] - \frac{i}{2} * d\left(\frac{\kappa}{\mu} - \frac{\bar{\kappa}}{\bar{\mu}}\right),$$
$$*F = \frac{i}{2}d\left[\left(\frac{\mu}{\kappa} - \frac{\bar{\mu}}{\bar{\kappa}}\right)(dt + \phi)\right] - \frac{1}{2} * d\left(\frac{\kappa}{\mu} + \frac{\bar{\kappa}}{\bar{\mu}}\right). \quad (2.17)$$

Then Bianchi identity together with Maxwell equation imply that

$$\nabla^2\left(\frac{\kappa}{\mu} - \frac{\bar{\kappa}}{\bar{\mu}}\right) = \nabla^2\left(\frac{\kappa}{\mu} + \frac{\bar{\kappa}}{\bar{\mu}}\right) = 0,$$
$$\nabla^2 = \left(\partial_x^2 + \partial_y^2 - \kappa^2\partial_z^2\right). \quad (2.18)$$

For $\kappa = i$, the solution obtained is the IWP metric [3] where the inverse of μ is a complex harmonic function. For $\kappa = 1$, we obtain the new solutions in which the inverse of μ satisfies the wave equation in flat $(2+1)$-space–time. For $\kappa = 1$, $\mu = \bar{\mu}$, we obtain the analogue of the electric MP solution [4]

$$ds^2 = \mu^2dt^2 + \frac{1}{\mu^2}\left(-dz^2 + dx^2 + dy^2\right),$$
$$A = \mu dt,$$
$$\left(\partial_x^2 + \partial_y^2 - \kappa^2\partial_z^2\right)\left(\frac{1}{\mu}\right) = 0, \quad (2.19)$$

where A is the gauge field one form. For $\mu = i\alpha$, we get the magnetic solution

$$ds^2 = \alpha^2dt^2 + \frac{1}{\alpha^2}\left(-dz^2 + dx^2 + dy^2\right)$$
$$F = - *_3 d\left(\frac{1}{\alpha}\right),$$
$$\left(\partial_x^2 + \partial_y^2 - \kappa^2\partial_z^2\right)\left(\frac{1}{\alpha}\right) = 0. \quad (2.20)$$

2.1. Charged Kasner universe

As special interesting phantom metric examples, we take as a solution to the wave equation in flat $(2+1)$-space–time,

$$\mu = \frac{q}{z} \quad (2.21)$$

with constant q. The metric and the gauge field strength then take the form

$$ds^2 = \frac{q^2}{z^2}dt^2 - \frac{z^2}{q^2}dz^2 + \frac{z^2}{q^2}\left(dx^2 + dy^2\right),$$
$$F = dA = -\frac{q}{z^2}dz \wedge dt. \quad (2.22)$$

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[2] Note that $\left(|\mu|^2\mathbf{e}^+ + \kappa^2\mathbf{e}^-\right)$ and $|\left(|\mu|^2\mathbf{e}^+ - \kappa^2\mathbf{e}^-\right)$ are related to Hermitian inner products by

$$\sqrt{2}\left(|\mu|^2\mathbf{e}^+ - \mathbf{e}^-\right) = \langle\gamma_0\varepsilon, \gamma_a\varepsilon\rangle; \qquad \sqrt{2}\left(|\mu|^2\mathbf{e}^+ + \mathbf{e}^-\right) = \langle\gamma_0\varepsilon, \gamma_5\gamma_a\varepsilon\rangle.$$

Introducing the new coordinates

$$\tau = \frac{1}{2q}z^2, \qquad x^1 = \sqrt{\frac{2}{q}}x, \qquad x^2 = \sqrt{\frac{2}{q}}y, \qquad x^3 = \sqrt{\frac{q}{2}}t, \tag{2.23}$$

then the metric takes the Kasner form [15]

$$ds^2 = -d\tau^2 + \sum_{j=1}^{3} \tau^{2p_j}\left(dx^j\right)^2,$$
$$F = -\frac{1}{2\tau^{3/2}}d\tau \wedge dx^3, \tag{2.24}$$

with the Kasner exponents

$$p_1 = p_2 = \frac{1}{2}, \quad p_3 = -\frac{1}{2}. \tag{2.25}$$

Note that here the Kasner exponents satisfy the conditions

$$\sum_{j=1}^{3} p_j = \frac{1}{2}, \qquad \sum_{j=1}^{3} p_j^2 = \frac{3}{4}, \tag{2.26}$$

while in vacuum they satisfy

$$\sum_{j=1}^{3} p_j = \sum_{j=1}^{3} p_j^2 = 1. \tag{2.27}$$

For $\mu = i\alpha = \frac{i}{pz}$, we get the solution

$$ds^2 = \left(\frac{1}{pz}\right)^2 dt^2 + (pz)^2\left(-dz^2 + dx^2 + dy^2\right),$$
$$F = p\,dx \wedge dy. \tag{2.28}$$

This metric takes the Kasner form

$$ds^2 = -d\tau^2 + \tau\left(dx^1\right)^2 + \tau\left(dx^2\right)^2 + \tau^{-2}\left(dx^3\right)^2,$$
$$F = dx^1 \wedge dx^2, \tag{2.29}$$

where we have introduced the coordinates

$$\tau = \frac{1}{2}pz^2, \quad x^1 = \sqrt{2p}x,$$
$$x^2 = \sqrt{2p}y, \qquad x^3 = \sqrt{\frac{1}{2p}}t. \tag{2.30}$$

3. Euclidean solutions

As already mentioned, one does not get new exotic solutions with phantom Euclidean Maxwell fields. The metric solution is independent of the sign of the coupling of the Maxwell field. For the sake of completeness, we briefly present the solutions for both couplings in a unified fashion. We take the metric to be of the form [12]

$$ds^2 = 2\mathbf{e}^1\mathbf{e}^{\bar{1}} + 2\mathbf{e}^2\mathbf{e}^{\bar{2}}. \tag{3.1}$$

Dirac spinor is taken to be a linear combination of the complexified space of forms on $\mathbb{R}^2$, with basis $\{1, e_1, e_2, e_{12} = e_1 \wedge e_2\}$. In this basis, the action of the Dirac matrices γ_m on the Dirac spinors is given by

$$\gamma_m = \sqrt{2}i_{e_m}, \qquad \gamma_{\bar{m}} = \sqrt{2}e_m\wedge \tag{3.2}$$

for $m = 1, 2$. We also define $\gamma_5 = \gamma_{1\bar{1}2\bar{2}}$. Euclidean version of the IWP metric were found in [12] for the case $\kappa = i$, and orbit

$$\epsilon = \lambda 1 + \sigma e_1, \tag{3.3}$$

with real λ and σ. Keeping κ as a parameter, then the analysis of the Killing spinor equation for the orbit (3.3) gives the geometric conditions

$$\omega_{1\bar{1}} = \partial_2 \log\frac{\lambda}{\sigma}\mathbf{e}^2 - \partial_1\log\sigma\lambda\mathbf{e}^1 - \partial_{\bar{2}}\log\frac{\lambda}{\sigma}\mathbf{e}^{\bar{2}} + \partial_{\bar{1}}\log\sigma\lambda\mathbf{e}^{\bar{1}},$$
$$\omega_{2\bar{2}} = \partial_2\log\lambda\sigma\,\mathbf{e}^2 - \partial_{\bar{2}}\log\lambda\sigma\,\mathbf{e}^{\bar{2}} + \partial_1\log\frac{\sigma}{\lambda}\mathbf{e}^1 - \partial_{\bar{1}}\log\frac{\sigma}{\lambda}\mathbf{e}^{\bar{1}},$$
$$\omega_{21} = -2\kappa^2\partial_2\log\lambda\mathbf{e}^1 - 2\partial_1\log\lambda\mathbf{e}^{\bar{2}},$$
$$\omega_{\bar{2}1} = -2\kappa^2\partial_{\bar{2}}\log\sigma\mathbf{e}^1 - 2\partial_1\log\sigma\mathbf{e}^2, \tag{3.4}$$

together with the condition

$$\left(\partial_1 + \kappa^2\partial_{\bar{1}}\right)\sigma = \left(\partial_1 + \kappa^2\partial_{\bar{1}}\right)\lambda = 0. \tag{3.5}$$

For the gauge field strength we get

$$F_{2\bar{2}} = \frac{1}{\sqrt{2}\lambda\sigma}\left[\partial_1\left(\frac{\lambda^2}{\kappa} + \frac{\sigma^2}{\bar{\kappa}}\right)\mathbf{e}^1 + \frac{\partial_2\sigma^2}{\bar{\kappa}}\mathbf{e}^2 + \frac{\partial_{\bar{2}}\lambda^2}{\kappa}\mathbf{e}^{\bar{2}}\right]\wedge\mathbf{e}^{\bar{1}}$$
$$+ \frac{1}{\sqrt{2}\lambda\sigma}\left[\frac{\partial_2\lambda^2}{\bar{\kappa}}\mathbf{e}^2 + \frac{\partial_{\bar{2}}\sigma^2}{\kappa}\mathbf{e}^{\bar{2}}\right]\wedge\mathbf{e}^1$$
$$+ \frac{1}{\sqrt{2}\lambda\sigma}\partial_1\left(\frac{\lambda^2}{\kappa} - \frac{\sigma^2}{\bar{\kappa}}\right)\mathbf{e}^2\wedge\mathbf{e}^{\bar{2}}. \tag{3.6}$$

For torsion free metric, the conditions (3.4) and (3.5) imply that $\lambda\sigma\left(\mathbf{e}^1 - \kappa^2\mathbf{e}^{\bar{1}}\right)$ is a total differential and that $\kappa\lambda\sigma\left(\mathbf{e}^1 + \kappa^2\mathbf{e}^{\bar{1}}\right)$ is a Killing vector. This enables us to introduce the coordinates (τ, x, y, z) and write

$$\mathbf{e}^1 = \frac{1}{\sqrt{2}}\left(-i\kappa\frac{dx}{\lambda\sigma} + \frac{1}{\kappa}\lambda\sigma\,(d\tau + \phi)\right), \quad \mathbf{e}^2 = \frac{1}{\sqrt{2}\lambda\sigma}(dy + idz),$$

and the solution is given by

$$ds^2 = (\lambda\sigma)^2(d\tau + \phi)^2 + \frac{1}{(\lambda\sigma)^2}\left(dx^2 + dy^2 + dz^2\right),$$
$$d\phi = \frac{2\kappa^2}{(\lambda\sigma)^2} * d\log\frac{\lambda}{\sigma},$$
$$F = \frac{1}{2}d\left[\left(\kappa^2\sigma^2 + \lambda^2\right)(d\tau + \phi)\right] + \frac{1}{2} * d\left(\frac{1}{\lambda^2} - \frac{\kappa^2}{\sigma^2}\right),$$
$$\tilde{F} = -\frac{1}{2}d\left[\left(\lambda^2 - \kappa^2\sigma^2\right)(d\tau + \phi)\right] + \frac{1}{2} * d\left(\frac{1}{\lambda^2} + \frac{\kappa^2}{\sigma^2}\right), \tag{3.7}$$

with λ and σ independent of τ. The Bianchi identity and Maxwell equation imply the equations

$$\nabla^2\left(\frac{1}{\lambda^2} - \frac{\kappa^2}{\sigma^2}\right) = \nabla^2\left(\frac{1}{\lambda^2} + \frac{\kappa^2}{\sigma^2}\right) = 0, \tag{3.8}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$.

In summary, the method of spinorial geometry is used to find IWP analogue solutions in four-dimensional Einstein–anti-Maxwell theory admitting Killing spinors. The analysis of the Killing spinor equation reveals the existence of a Killing vector and a total differential which switch roles when one changes the coupling of the Maxwell field. The phantom solutions found admit a space-like Killing vector and constitute the time-dependent analogues of the IWP metrics of the canonical Einstein–Maxwell theory. The solutions are expressed in terms of a complex function satisfying the wave equation in a flat $(2+1)$-space–time. As examples, electric and magnetic Kasner spaces can be constructed by specializing to solutions that depend only on the time coordinate. The Kasner exponent sum rules of the vacuum Kasner solution get modified in the presence of a phantom $U(1)$ gauge field. Phantom Euclidean solutions are also presented. In the Euclidean case, the phantom metric is the same as in the ordinary Einstein–Maxwell theory but with the roles of F and $*F$ interchanged. Our analysis can be extended to theories with anti-scalars and anti-vector multiplets in ungauged and gauged supergravity models in various dimensions. Work in this direction is in progress.

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