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# Higher-Dimensional Tree Structures

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Various generalizations of tree-characterization theorems are developed for $n$-dimensional complexes. In particular, generalizations of three conditions satisfied by trees $T$ are studied: $T$ is connected, $T$ is acyclic, $|V(T)| - |E(T)| = 1$, where $V(T)$ and $E(T)$ denote the vertex and edge sets of $T$, respectively.

Earlier work by Beineke and Pippert is extended in generalizing these conditions and studying which combinations of such conditions yield characterizations of the $n$-dimensional trees treated here.

There are many equivalent definitions of the concept of tree in Graph Theory; see for example [3]. Many such definitions include conditions like being connected or having no circuits or that the equation $p - q = 1$ be satisfied, where $p$ and $q$ are the number of vertices and edges, respectively, in the graph.

In this paper, trees are generalized to a kind of $n$-dimensional complex called an "$(m, n)$-tree." The definition is purely inductive and some of the above definitions of tree are generalized to characterizations of $(m, n)$-trees. In some of these characterizations, the role of the equation $p - q = 1$ is played by a set of $m$ equations, a few low-order examples of which were previously investigated by Beineke and Pippert [1].

In Section 1 below are given a number of definitions required in this paper. Each of the most familiar characterizations of trees $T$ involve some of the conditions listed below:

(a) $T$ is connected,
(b) $T$ has no circuits,
(c) $p - q = 1$,

where $p$ and $q$ represent the number of vertices and edges, respectively, of $T$. Each of these conditions is generalized below. In Section 2 the generalized conditions (a) and (b) are employed, along with a third condition, to characterize $(m, n)$-trees. In Section 3 the generalized conditions (a) and (b) are also employed to characterize $(m, n)$-trees. The latter characterization also generalizes some results of Beineke and Pippert.

## 1.

Let $K$ be a collection of subsets $x$ called *simplexes* of a finite set $V(K)$ of *vertices*. Then $K$ is called a *complex* if every subset of every simplex in $K$ is a simplex in $K$ and if $V(K) = \bigcup_{x \in K} x$. This is equivalent to the definition of a "finite abstract simplicial complex," as given in [4, p. 41]. The *dimension of a simplex* is the number $|x| - 1$. The *dimension of a complex* $K$ is the maximum dimension of its simplexes. One frequently speaks of an *n-complex* (*n-simplex*) in place of a complex (simplex) having dimension $n$.

A *graph* is an $n$-complex for which $n \leqslant 1$.

The complex $L$ is a *subcomplex* of the complex $K$ if $L \subseteq K$. Two complexes $K$ and $L$ are *isomorphic*, denoted by $K \simeq L$, if there is a 1–1 onto mapping $f\colon V(K) \to V(L)$ such that $x = \{v_0, v_1, ..., v_k\}$ is a simplex in $K$ if and only if $f(x) = \{f(v_0), f(v_1), ..., f(v_k)\}$ is a simplex in $L$. Let $S$ be an arbitrary subset of a complex $K$. The *closure of* $S$ in $K$ is the subcomplex

$$\bar{S} = \{x \in K\colon x \subseteq y \text{ for some } y \in S\}.$$

If $S = \{y\}$, we denote $\bar{S}$ merely by $\bar{y}$.

A $n$-complex $K$ having $p$ vertices is called *complete* if every $(n + 1)$-subset of $V(K)$ is present as a simplex in $K$. Such a complex will be written $K_p{}^n$.

Let $x_1, y_1, x_2, y_2, ..., x_{r-1}, y_{r-1}, x_r$ be an alternating sequence of $m$-simplexes and $n$-simplexes in a complex $K$ and suppose that $x_i$, $x_{i+1} \subseteq y_i$ for each $i = 1, 2, ..., r - 1$. Such a sequence is called a $(m, n)$-*path sequence* if all the simplexes appearing in the sequence are distinct. It is called an $(m, n)$-*circuit sequence* if $x_1 = x_r$ while all other simplexes are distinct. The subcomplex $\bar{S}$ of $K$, where $S = \{y_1, y_2, ..., y_{r-1}\}$, is called an $(m, n)$-*path* (respectively, $(m, n)$-*circuit*). The *length* of an $(m, n)$-path sequence or $(m, n)$-circuit sequence $x_1, y_1, x_2, y_2, ..., x_{r-1}, y_{r-1}, x_r$ is the number $r - 1$. The same terminology applies to $(m, n)$-paths and $(m, n)$-circuits. A complex $K$ is $(m, n)$-*connected* if for any two $m$-simplexes $x$, $x'$ in $K$, there is an $(m, n)$-path sequence in which $x$ and $x'$ are the first and last members, respectively. A complex $K$ is $(m, n)$-*simple* if for every $(m, n)$-path sequence $x_1, y_1, x_2, y_2, ..., x_r$ in $K$, the existence of a vertex $v$ in $y_i$ and $y_j$, $i < j$, implies that $v$ lies in all of $y_i$, $y_i, y_{i+1}, ..., y_j$. An $n$-complex is *pure* if every simplex of $K$ lies in some $n$-simplex of $K$.

Given a pure $n$-complex $K$ and an $n$-simplex $y$ in $K$, the *attachment of* $y$

*in K*, denoted by $A(y, K)$, is the subcomplex $\overline{K - \bar{y}} \cap \bar{y}$, i.e., the set of all simplexes of $K$ which are contained both in $y$ and in some other $n$-simplex of $K$. Let $\mathscr{S}$ be a collection of complexes and let $y_1, y_2, \ldots, y_s$ be an ordering of the $n$-simplexes of some pure $n$-complex $K$. Denote by $K_i$ the subcomplex $\overline{\{y_1, y_2, \ldots, y_i\}}$ of $K$. Then $K$ will be said to have an $\mathscr{S}$*-ordering* if for each $i = 2, 3, \ldots, s$, $A(y_i, K_i)$ is isomorphic to a member of $\mathscr{S}$. Suppose now that $\mathscr{S}$ has a single member $K^m_{m+1}$. If $K$ has an $\mathscr{S}$-ordering in this case, then $K$ will be said to have a $K^m_{m+1}$*-ordering*. A pure $n$-complex having a $K^m_{m+1}$-ordering will be called an *$(m, n)$-tree*.

In closing this section, we observe that a graph-theoretic structure in many ways equivalent to an $(m, n)$-tree could be defined. Such a graph could be obtained from an $(m, n)$-tree by deleting all its simplexes having dimension greater than 1.

## 2.

One of the simplest characterizations of a tree in the Theory of Graphs states that it is connected and contains no circuits. Since every 1-complex is (0, 1)-simple, it will be seen that the following theorem generalizes this characterization to $(m, n)$-trees.

THEOREM 1. *A pure n-complex K is an (m, n)-tree if and only if the following conditions hold*:

(i) *K is (m, n)-connected*;
(ii) *K contains no (m, n)-circuits*;
(iii) *K is (m, n)-simple.*

*Proof.* If $K$ is an $(m, n)$-tree, then the inductive arguments that $K$ has properties (i), (ii), and (iii) are obvious.

Assume then that $K$ is a pure $n$-complex having these properties. If $K$ has just one $n$-simplex, then $K$ is obviously an $(m, n)$-tree. Assume as an induction hypothesis that all pure $n$-complexes having $k - 1$ $n$-simplexes and having properties (i), (ii), and (iii) are $(m, n)$-trees. Suppose now that $K$ has $k$ $n$-simplexes, $k > 1$. It will be shown that $K$ has at least one $n$-simplex $y$ such that $A(y, K) \cong K^m_{m+1}$: since $K$ is $(m, n)$-connected, the attachment $A(y, K)$ of every $n$-simplex $y$ of $K$ contains at least one $m$-simplex. If every such attachment contains two $m$-simplexes, then an $(m, n)$-circuit can readily be found in $K$, thus violating condition (ii). Therefore, for some $n$-simplex of $K$, say $y_1$, $A(y_1, K)$ contains just one $m$-simplex, say $x$. Suppose that $A(y_1, K)$ contains a vertex $v$ not in $x$. Then there must be an $n$-simplex $y_2$ of $K$ different from $y_1$ and also containing $v$. For $i = 1, 2,$

let $x_i$ be an $n$-simplex of $K$ incident with $y_i$ and containing $v$. By condition (i), there is an $(m, n)$-path sequence $p$ from $x_1$ to $x_2$ in $K$. The first three terms of $p$ must clearly be $x_1$, $y_1$, $x$. Since $v \in x_2$ but $v \notin x$, $x_2 \neq x$, and so $x$ is not the last term of $p$: let the fourth term of $p$ be $y$.

Since $K$ is $(m, n)$-simple, $v \in y$. But in this case $x \cup \{v\} \subseteq y$ whence $y_1$ and $y$ contain at least $\binom{m+2}{m+1}$ common $m$-simplexes, and these, of course, lie in $A(y_1, K)$, a contradiction. It follows that $A(y_1, K)$ contains no vertices not in $x$. In other words, $A(y_1, K) \cong K^m_{m+1}$, as required.

The inductive step will now be taken. Denote by $K'$ the $n$-complex $\overline{K - \bar{y}_1}$ and observe that $K'$ satisfies conditions (i), (ii), and (iii). By the induction hypothesis, $K'$ is an $(m, n)$-tree and, therefore, has a $K^m_{m+1}$-ordering. By adding $y_1$ to this ordering, a $K^m_{m+1}$-ordering for $K$ is obtained. This completes the proof.

In Fig. 1 below are shown two 2-complexes $K$ and $K'$. Both complexes satisfy conditions (i) and (ii) above when $m = 1$ and $n = 2$. However, the complex $K$ is a (1, 2)-tree while $K'$ is not. This example shows that condition (iii) is not redundant in the above theorem.

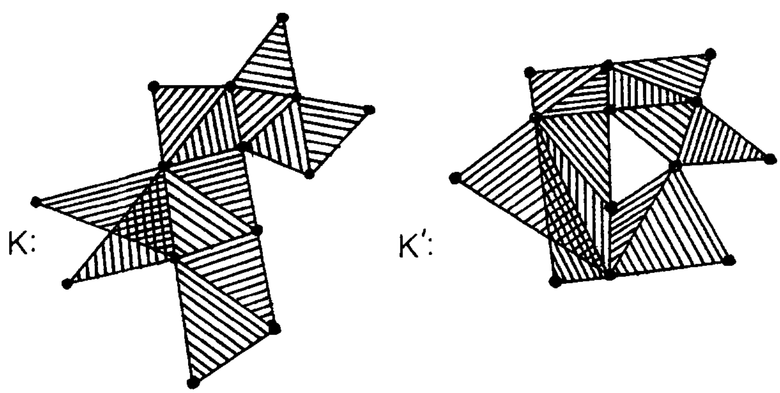


FIG. 1. A (1, 2)-tree and a non-(1, 2)-tree.

## 3.

Beineke and Pippert in [2] have defined a "$k$-tree" as a graph made up of complete graphs on $k$ vertices in a certain fashion. In their definition, these complete graphs play the same role as $n$-simplexes in an $(m, n)$-tree with $m = n - 1$. In an earlier paper [1, p. 267], Beineke and Pippert characterize "2-trees" in several ways. Due to the formal equivalence of "2-trees" with (1, 2)-trees, their principal result is stated below as a theorem about (1, 2)-trees. For any complex $K$, denote by $\alpha_k(K)$ the number of $k$-simplexes in $K$.

THEOREM 2 (Beineke and Pippert). *For a pure 2-complex, the following conditions are equivalent*:

(i) *$K$ is a $(1, 2)$-tree*;
(ii) *$K$ is $(1, 2)$-connected and $\alpha_1(K) = 2 \cdot \alpha_0(K) - 3$*;
(iii) *$K$ is $(1, 2)$-connected and $\alpha_2(K) = \alpha_0(K) - 2$*;
(iv) *$K$ has no $(1, 2)$-circuits, $\alpha_1(K) = 2 \cdot \alpha_0(K) - 3$ and $\alpha_2(K) = \alpha_0(K) - 2$.*

The equations involving $\alpha_0(K)$, $\alpha_1(K)$, and $\alpha_2(K)$ are of some interest. They extend the well-known basic relation $\alpha_1(K) = \alpha_0(K) - 1$ when $K$ is a $(0, 1)$-tree. The following set of equations contain generalizations of all these equations:

$$\alpha_k(K) = \frac{\alpha_0(K) - m - 1}{n - m}\binom{n+1}{k+1} - \frac{\alpha_0(K) - n - 1}{n - m}\binom{m+1}{k+1}, \qquad k = 1, 2, \ldots, n.$$

We will call these equations the *Beineke–Pippert equations* for an $n$-complex $K$. For $n = 2$, the complete set of these equations appears in condition (iv) of Theorem 2. Denote the right-hand side of the $k$th equation by $B_{m,n}(k, K)$. When $k > m$, one takes the usual convention of setting $\binom{m+1}{k+1} = 0$.

It will be established first that $(m, n)$-trees satisfy the appropriate set of Beineke–Pippert equations. Then after a sequence of three lemmas and a theorem, it will be shown that the following condition may be used to replace properties (ii) and (iii) in Theorem 1.

THEOREM 3. *If $K$ is an $(m, n)$-tree, then $\alpha_k(K) = B_{m,n}(k, K)$, $k = 1, 2, \ldots, n$.*

*Proof.* This result is established by induction an $\alpha_n(K)$. If $\alpha_n(K) = 1$, then $\alpha_k(K) = \binom{n+1}{k+1}$. Setting $\alpha_0(K) = n + 1$ in the $k$th Beineke–Pippert equation yields the same result. Suppose that any $(m, n)$-tree having $\alpha - 1$ $n$-simplexes satisfies the $k$th Beineke–Pippert equation. Let $K$ be an $(m, n)$-tree for which $\alpha_n(K) = \alpha$. By definition, $K$ has a $K^m_{m+1}$-ordering $y_1, y_2, \ldots, y_\alpha$. The $n$-complex $K_{\alpha-1} = \overline{\{y_1, y_2, \ldots, y_{\alpha-1}\}}$ is also an $(m, n)$-tree and satisfies the $k$th Beineke–Pippert equation by the induction hypothesis. The number of new $k$-simplexes added in going from $K_{\alpha-1}$ to $K_\alpha$ is clearly $\binom{n+1}{k+1} - \binom{m+1}{k+1}$ and, therefore,

$$\alpha_k(K) = \frac{\alpha_0(K_{\alpha-1}) - m - 1 + (n - m)}{n - m}\binom{n+1}{k+1} - \frac{\alpha_0(K_{\alpha-1}) - n - 1 + (n - m)}{n - m}\binom{m+1}{k+1}.$$

However, $\alpha_0(K) = \alpha_0(K_{\alpha-1}) + (n - m)$, and thus $K$ satisfies the $k$th Beineke–Pippert equation.

LEMMA 4a. *If $k, m, r, n$ are integers, and if $0 \leqslant k \leqslant m \leqslant r \leqslant n$, then*

$$(n-m)\binom{n}{k} - \binom{r}{k} \geqslant (n-r)\left(\binom{n}{k} - \binom{m}{k}\right), \qquad \text{(i)}$$

*and this inequality is strict if $0 < k \leqslant m < r < n$.*

*Proof.* If $k = 0$ or $m = r$ or $r = n$, the inequality is obvious. Assume that $k > 0$ and that $m < r < n$. Since $\binom{t+1}{k} - \binom{t}{k} = \binom{t}{k-1}$ for every positive integer $t \geqslant k$,

$$\begin{aligned}\binom{n}{k} - \binom{r}{k} &= \sum_{t=r}^{n-1} \binom{t}{k-1} \\ &\geqslant (n-r)\binom{r}{k-1} \\ &= \frac{n-r}{r-m}(r-m)\binom{r}{k-1} \\ &> \frac{n-r}{r-m}\sum_{t=m}^{r-1}\binom{t}{k-1} \\ &= \frac{n-r}{r-m}\left(\binom{r}{k} - \binom{m}{k}\right).\end{aligned}$$

Therefore, $(r-m)(\binom{n}{k} - \binom{r}{k}) > (n-r)(\binom{r}{k} - \binom{m}{k})$. Adding $(n-r)(\binom{n}{k} - \binom{r}{k})$ to each side of this last inequality yields the inequality (i).

LEMMA 4b. *Let $K$ be an $(m, n)$-tree, and let $K_p{}^k$ be a complete $k$-subcomplex of $K$ with $p$ vertices, where $1 \leqslant k \leqslant n$, $p \leqslant n+1$. Then there is some $n$-simplex $y$ of $K$ such that $K_p{}^n \subseteq \bar{y}$.*

*Proof.* The lemma is clearly true when $K$ has just one $n$-simplex. Let $K$ have $r$ $n$-simplexes, and let $y_1, y_2, \ldots, y_r$ be a $K^m_{m+1}$-ordering of $K$. Assume that the lemma holds for all $(m, n)$-trees having $r - 1$ $n$-simplexes, and let $K_p{}^k$ be a subcomplex of $K = K_r$. If $K_p{}^k \subseteq K_{r-1} = \overline{\{y_1, y_2, \ldots, y_{r-1}\}}$, then by the induction hypothesis, $K_{r-1}$ (and thus $K_r$) contains an $n$-simplex $y$ such that $K_p{}^k \subseteq \bar{y}$. Otherwise $K_p{}^k$ has at least one vertex $v$ in $y_r$ which is not in $K_{r-1}$. If $K_p{}^k \nsubseteq \bar{y}_r$, then $K_p{}^k$ also has a vertex $u$ in $K_{r-1}$, which is not in $y_r$. Thus the 1-simplex $\{u, v\}$ is not incident with $y_r$. But in this case, $\{u, v\}$ must lie in $K_{r-1}$, which implies that $v$ lies in $K_{r-1}$, a contradiction. This proves the lemma.

Let $S_1{}^m$, $S_{\geqslant 1}$ denote the set of all $m$-complexes containing one, respectively at least one, $m$-simplex.

THEOREM 4. *If* $1 \leqslant k \leqslant m$ *and if a pure n-complex K has an* $S_{\geqslant 1}$-*ordering* $y_1, y_2, \ldots, y_\alpha$, *then* $\alpha_k(K) \geqslant B_{m,n}(k, K)$, *and this inequality is strict unless* $y_1, y_2, \ldots, y_\alpha$ *is a* $K^m_{m+1}$-*ordering.*

*Proof.* Let $r$ be the largest integer $i$ such that $y_1, y_2, \ldots, y_i$ is a $K^m_{m+1}$-ordering of $K_i = \overline{\{y_1, y_2, \ldots, y_i\}}$. Since $K_r$ is an $(m, n)$-tree, $\alpha_k(K_r) = B_{m,n}(k, K_r)$ by Theorem 3. Thus, if $r = \alpha$, the theorem follows. Suppose that $r < \alpha$.

Denote by $\alpha_{k,i}$ the number of $k$-simplexes in $A(y_{i+1}, K_{i+1})$, $1 \leqslant i < \alpha$. Since $A(y_{i+1}, K_{i+1})$ has $n + 1 - (\alpha_0(K_{i+1}) - \alpha_0(K_i))$ vertices,

$$\alpha_{k,i} \leqslant \binom{n + 1 - (\alpha_0(K_{i+1}) - \alpha_0(K_i))}{k + 1}. \tag{i}$$

On the other hand, $B_{m,n}(k, K_{i+1}) - B_{m,n}(k, K_i)$ may be written

$$\frac{\alpha_0(K_{i+1}) - \alpha_0(K_i)}{n - m}\left(\binom{n+1}{k+1} - \binom{m+1}{k+1}\right).$$

Since $0 \leqslant k + 1 \leqslant m + 1 \leqslant n + 1 - (\alpha_0(K_{i+1}) - \alpha_0(K)) \leqslant n + 1$, we may substitute these last four quantities for $k$, $m$, $r$, and $n$, respectively, in Lemma 4a to obtain

$$\binom{n+1}{k+1} - \binom{n + 1 - (\alpha_0(K_{i+1}) - \alpha_0(K_i))}{k + 1}$$
$$\geqslant \frac{\alpha_0(K_{i+1}) - \alpha_0(K_i)}{n - m}\left(\binom{n+1}{k+1} - \binom{m+1}{k+1}\right). \tag{ii}$$

Hence

$$\binom{n+1}{k+1} - \alpha_{k,i} \geqslant B_{m,n}(k, K_{i+1}) - B_{m,n}(k, K_i). \tag{iii}$$

The inequality (iii) is now examined when $i = r$. By Lemma 4a, if $m + 1 < n + 1 - (\alpha_0(K_{r+1}) - \alpha_0(K_r)) < n + 1$, then the inequality (ii) is strict and, therefore, so is (iii). Since $y_1, y_2, \ldots, y_\alpha$ is an $S^m_{\geqslant 1}$-ordering, and since

$$A(y_{r+1}, K_{r+1}) \ncong K^m_{m+1},$$

it follows that $m + 1 < n + 1 - (\alpha_0(K_{r+1}) - \alpha_0(K_r))$. Suppose that

$n+1-(\alpha_0(K_{r+1})-\alpha_0(K_r))=n+1$. If the inequality (i) is strict when $i=r$, then so is the inequality (iii). Suppose also that

$$\alpha_{k,r}=\binom{n+1-(\alpha_0(K_{r+1})-\alpha_0(K_r))}{k+1}$$
$$=\binom{n+1}{k+1}.$$

Then $A(y_{r+1}, K_{r+1})$ contains a complete subcomplex $K_{n+1}^k$. Therefore, $K_r$ contains $K_{n+1}^k$, and by Lemma 4b there is an $n$-simplex $y_j$ in $K_r$ such that $K_{n+1}^k \subseteq \bar{y}_j$. But this implies that $|y_j \cap y_i| = n+1$ while $y_j \neq y_i$, a contradiction. We conclude that the inequality (iii) is strict when $i=r$.

Now add the inequalities (iii) for $i = 1, 2, ..., \alpha-1$. The sum of the left-hand sides is $\alpha_k(K) - \binom{n+1}{k+1}$ and the sum of the right-hand sides is $B_{m,n}(k, K) - \binom{n+1}{k+1}$. Since $1 \leqslant r \leqslant \alpha - 1$, and since the $r$th inequality is strict, one obtains

$$\alpha_k(K) > B_{m,n}(k, K).$$

COROLLARY. *If $K$ is a pure $(m, n)$-connected $n$-complex, then $\alpha_k(K) \geqslant B_{m,n}(k, K)$, $k = 1, 2, ..., m$.*

*Proof.* $K$ is $(m, n)$-connected if and only if $K$ has an $S_{\geqslant 1}^m$-ordering.

LEMMA 5a. *A pure $n$-complex $K$ has an $S_1{}^m$-ordering if and only if the following properties hold for $K$:*

(i) *$K$ is $(m, n)$-connected;*
(ii) *$K$ has no $(m, n)$-circuits.*

*Proof.* Let $K$ be a pure $n$-complex having an $S_1{}^m$-ordering. Then $K$ is shown to have properties (i) and (ii) above by the obvious inductive arguments.

Let $K$ be a pure $n$-complex having properties (i) and (ii). Suppose that $\alpha_n(K) = r$ and that $r = 1$. Then $K$ certainly has an $S_1{}^m$-ordering. Assume now that $r > 1$ and that every pure $n$-complex having $r - 1$ $n$-simplexes and properties (i) and (ii) also has an $S_1{}^m$-ordering. If every $n$-simplex of $K$ has an attachment containing at least two distinct $m$-simplexes, then obviously an $(m, n)$-circuit may be found in $K$. Since $r > 1$ and since $K$ satisfies condition (i), every $n$-simplex of $K$ has an attachment containing at least two distinct $m$-simplexes. It now follows that $K$ has at least one $n$-simplex $y$ such that $A(y, K)$ contains exactly one $n$-simplex. Let $K' = \overline{k - \bar{y}}$ and observe that $K'$ also has properties (i) and (ii). But $\alpha_n(K) = r - 1$, and, therefore, by the induction hypothesis, $K'$ has an $S_1{}^m$-ordering

$y_1, y_2, ..., y_{r-1}$. Set $y_r = y$ and observe that $y_1, y_2, ..., y_{r-1}, y_r$ is an $S_1{}^m$-ordering for $K$, completing the proof.

THEOREM 5. *A pure n-complex is an (m, n)-tree if and only if it has the following two properties*:

(i) *$K$ is $(m, n)$-connected*;
(ii) *$\alpha_k(K) = B_{m,n}(k, K)$ for at least one $k$ such that* $1 \leqslant k \leqslant m$.

*Proof.* Let $K$ be an $(m, n)$-tree. Then $K$ has a $K_{m+1}^m$-ordering, and this is also an $S_1{}^m$-ordering whence $K$ is $(m, n)$-connected by Lemma 5a. By Theorem 3, $K$ has property (ii) as well.

Suppose now that $K$ is a pure $n$-complex having properties (i) and (ii). Since $K$ is $(m, n)$-connected, $K$ has an $S_{\geqslant 1}^m$-ordering $y_1, y_2, ..., y_\alpha$, and by Theorem 4 the inequality $\alpha_k(K) \geqslant B_{m,n}(k, K)$ is strict unless $y_1, y_2, ..., y_\alpha$ is also a $K_{m+1}^m$-ordering. Since $\alpha_k(K) = B_{m,n}(k, K)$, $y_1, y_2, ..., y_\alpha$ is, in fact, a $K_{m+1}^m$-ordering and theorem is proved.

Although conditions (i) and (ii) of Theorem 1 are not by themselves sufficient to characterize $(m, n)$-trees when $m > 0$ or $n > 1$, conditions (i) and (ii) of Theorem 5 are sufficient for this purpose. Thus in the presence of condition (i), a pure $n$-complex $K$ containing no $(m, n)$-circuits and being $(m, n)$-simple is "worth" at least one of the Beineke–Pippert equations holding.

## 4.

The question remains whether the following statement is true.

6. *A pure n-complex is an (m, n)-tree if and only if it has the following two properties*:

(i) *$K$ has no $(m, n)$-circuits*;
(ii) *$\alpha_k(K) = B_{m,n}(k, K)$ for at least on $k$ such that* $1 \leqslant k \leqslant m$.

If it is true, then either one of the two equations in part (iv) of Theorem 2 can be deleted. If it is not true, then perhaps statement 6 would be true with condition (ii) replaced by

$$\text{(ii)}' \quad \alpha_k(K) = B_{m,n}(k, K) \qquad \text{for all} \quad k = 1, 2, ..., m.$$

It seems reasonable to conjecture that statement 6 altered in this manner would be true, but the results developed in this paper have not helped so far in my investigation of this conjecture.

In any event, it is interesting to speculate that if the altered statement is true and the original statement is false, then the Beineke–Pippert equations give us a kind of "currency" with which to measure the relative "worth" of other conditions. For in this case, if a pure $n$-complex $K$ is $(m, n)$-connected, then we require the addition of only one Beineke–Pippert equation to ensure that $K$ is an $(m, n)$-tree. On the other hand, if $K$ has no $(m, n)$-circuits, then we require the addition of $m$ Beineke–Pippert equations to ensure that $K$ is an $(m, n)$-tree. Thus being $(m, n)$-connected would be "worth" somewhat more than having no $(m, n)$-circuits.

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